Completely Symmetric Resistance Forms on the Stretched Sierpinski Gasket

P. Alonso Ruiz* U. Freiberg† J. Kigami‡

Abstract

The stretched Sierpinski gasket, SSG for short, is the space obtained by replacing every branching point of the Sierpinski gasket by an interval. It has also been called "deformed Sierpinski gasket" or "Hanoi attractor". As a result, it is the closure of a countable union of intervals and one might expect that a diffusion on SSG is essentially a kind of gluing of the Brownian motions on the intervals. In fact, there have been several works in this direction. There still remains, however, "reminiscence" of the Sierpinski gasket in the geometric structure of SSG and the same should therefore be expected for diffusions. This paper shows that this is the case. In this work, we identify all the completely symmetric resistance forms on SSG. A completely symmetric resistance form is a resistance form whose restriction to every contractive copy of SSG in itself is invariant under all geometrical symmetries of the copy, which constitute the symmetry group of the triangle. We prove that completely symmetric resistance forms on SSG can be sums of the Dirichlet integrals on the intervals with some particular weights, or a linear combination of a resistance form of the former kind and the standard resistance form on the Sierpinski gasket.

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1 Introduction

A major area of research interest in mathematical physics deals with the modelling of heat and wave propagation in branching media. One way to tackle this problem consists in approximating the object under consideration by unions of one-dimensional segments, and studying the combination of the corresponding equations on the segments. This approach has been extensively investigated under different names, for instance "quantum graphs" in mathematical physics [9] and "cable systems" in stochastic analysis [3].

Nevertheless, these models can fail to capture the essential structure of the media they are supposed to describe. The main message of the present paper is that reducing the analysis on an object to one-dimensional analysis on a union of lines can ignore a significant part of its intrinsic structure and therefore give a far too simple, hence

*Institute of Stochastics, Ulm University, Helmholtzstr. 20, 89081 Ulm, Germany; email: patricia.alonso@uni-ulm.de

†Institute of Stochastics and Applications, University of Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany; email: Uta.Freiberg@mathematik.uni-stuttgart.de

‡Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan; email: kigami@i.kyoto-u.ac.jp

incomplete, framework to investigate analytical questions on it. We aim to furnish last statement by studying here what we call the *stretched Sierpinski gasket*, SSG for short, in $\mathbb{R}^2$. This space has also been called "deformed Sierpinski gasket" [10] or "Hanoi attractor" [1, 2] and it is obtained from the classical Sierpinski gasket SG by replacing each branching point of the SG by an interval (see Figure 1).

As a result, SSG is the closure of a countable union of one-dimensional intervals. One could thus think of constructing and analysing diffusion processes on it via quantum graphs/cable systems, an approach that has actually been considered in several works [5, 2].

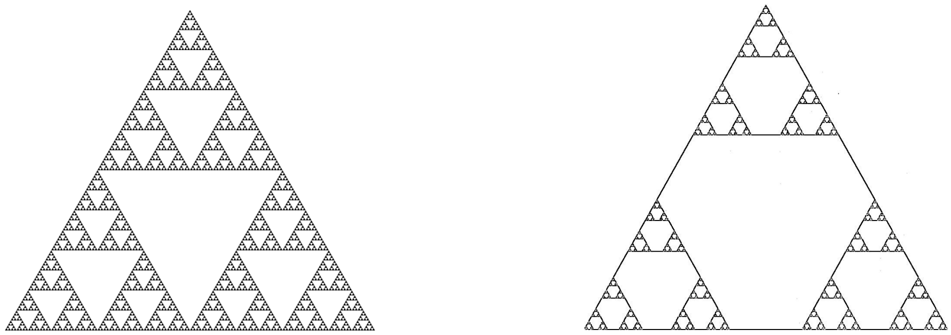

Figure 1: The Sierpinski gasket (SG) and the stretched Sierpinski gasket (SSG).

Let us give a rough definition of a cable system/quantum graph, leaving details to [3, 9]. Starting from a weighted graph (V, E, C) with vertex set V, edge set $E \subseteq \{(p, q) \mid p, q \in V\}$ and edge conductances/weights $C = \{C_{pq} \mid (p, q) \in E\}$, each edge $(p, q) \in E$ is identified with the line segment parametrized by $\xi_{p,q}(t) = (1-t)p + tq$, $t \in [0, 1]$, and equipped with the Dirichlet energy $\mathcal{D}_{pq}$ on the line segment pq given by

$$\mathcal{D}_{pq}(\cdot, \cdot) = \int_0^1 \frac{d(\cdot \circ \xi_{p,q})}{dt} \frac{d(\cdot \circ \xi_{p,q})}{dt} dt.$$

The consequent energy form $\mathcal{E}$ on the whole graph is thus defined as

$$\mathcal{E}(u, v) = \sum_{pq \in E} C_{pq} \mathcal{D}_{pq}(u, v),$$

where the domain of $\mathcal{E}$ consists of all continuous functions on the whole graph whose restriction to each edge pq belongs to the Sobolev space $H^1(\xi_{p,q}([0,1]), dx)$. In a natural way, this quadratic form $\mathcal{E}$ induces a diffusion process on the graph that behaves like one-dimensional Brownian motion on each edge.

Following this direction, a diffusion on SSG might be expected to consist basically in gluing the different Brownian motions on each interval. However, in considering SSG as a union of one-dimensional lines, one overlooks the "reminiscence" of SG in the geometric structure of SSG. In fact, the cable system/quantum graph approach disregards the underlying geometry of the space in the sense that it ignores the considerable role played by the arrangement of the vertices in space. Furthermore, classical quantum graph theory requires some finiteness condition that makes them inapplicable to cases such as fractals or infinite trees.

Indeed, we show in this paper that the geometric "reminiscence" of the Sierpinski gasket also appears in the diffusion on SSG, a fact that stays hidden when using cable systems/quantum graphs.

The diffusion processes considered here will be associated with a Dirichlet form induced by a *completely symmetric resistance form.* The theory of resistance forms was introduced in [7] and further developed in particular to study analysis on "low-dimensional" fractals from an intrinsic point of view, see [8] and references therein. Their most representative property is that, unlike Dirichlet forms, they are defined without requiring any measure on the underlying space. In our case, a completely symmetric resistance form $(\mathcal{E},\mathcal{F})$ on SSG is a resistance form whose restriction to every contractive copy in itself is invariant under all geometrical symmetries of the copy. More precisely, let X be a subset of SSG which is similar to SSG itself and let $G : SSG \to X$ be the associated contractive similitude. If we denote by $\mathcal{E}_X$ the part of the original form $\mathcal{E}$ associated with X, then

$$\mathcal{E}_X(u \circ G^{-1}, v \circ G^{-1})$$

is again a form on SSG. We say that $(\mathcal{E},\mathcal{F})$ is completely symmetric if $\mathcal{E}_X(u \circ G^{-1}, v \circ G^{-1})$ is invariant under any isometry of the regular triangle. (See Section 4 for the exact definition.)

As a key step towards the study of such diffusion processes, the present paper is devoted to establishing the existence of completely symmetric resistance forms on SSG. Even more, we provide a full characterization of all possible forms of this type by showing in Theorem 4.7 that any completely symmetric resistance form on SSG can be written as

$$a\mathcal{E}^*(\cdot,\cdot) + b\mathcal{D}^I_\eta(\cdot,\cdot)$$

for some $a \geq 0$ and $b > 0$. The forms $\mathcal{E}^*$ and $\mathcal{D}^I_\eta$ are briefly explained below. Conversely, we will show that any linear combination of $\mathcal{E}^*$ and $\mathcal{D}^I_\eta$ as above with $a \geq 0$ and $b > 0$ can be realized as a resistance form on SSG.

On the one hand, $\mathcal{D}^I_\eta$ arises as a limit of sums of standard Dirichlet energies and it is defined as follows. Let $\eta = \{\eta_m\}_{m\geq 1} \subseteq (0,\infty)$ satisfy $\sum_{m\geq 1}(\frac{5}{3})^{m-1}\eta_m = 1$ and let $\mathcal{D}^I_k$ be the sum of the Dirichlet integrals over the line segments that appear in the kth approximation step of SSG for the first time, i.e.

$$\mathcal{D}^I_k(u,v) = \sum_{pq \in \{k\text{-th approximation}\}\setminus\{(k-1)\text{-th approximation}\}} \mathcal{D}_{pq}(u,v).$$

The quadratic form $\mathcal{D}^I_\eta$ is defined as the weighted sum of the $\mathcal{D}^I_k$'s whose weights are given by $\eta = \{\eta_m\}_{m\geq 1}$, i.e.

$$\mathcal{D}^I_\eta(u,v) = \sum_{k\geq 1} \frac{1}{\eta_k}\mathcal{D}^I_k(u,v).$$

It resembles the cable system/quantum graph approach in this setting. In particular, the special case $a = 0$ has been called "fractal quantum graph" in [2], where the authors have shown that $\mathcal{D}^I_\eta$ is a resistance form for some limited choices of η.

On the other hand, the form $\mathcal{E}^*$ corresponds to the standard resistance form on SG. (See Definition 3.3 and [7] for further details about this form.) Notice that any function on SG can be thought of as a function on SSG by making its value constant on each line segment. In this manner, we can regard the standard resistance form on SG as a quadratic form on SSG, see Definition 4.4 for a precise formulation. This part of $\mathcal{E}$, which may be called the "fractal part", had remained unseen in the previous works [5, 2] because there, only limits of quantum graphs were considered.

In conclusion, this paper reveals that SSG is more than just the combination of a countably infinite number of line segments, not only from a geometric, but also from an analytic point of view, since the reminiscence of the Sierpinski gasket in SSG remains essentially present in both of them.

We will begin our exposition by discussing the geometry of SSG in Section 2, providing a detailed construction as well as some of its most relevant intrinsic geometric properties. Section 3 reviews the construction of the standard resistance form on SG and establishes a first link between functions on SG and on SSG. Completely symmetric resistance forms on SSG are rigorously introduced in Section 4 and the main classification result of this paper is stated in Theorem 4.7. The forthcoming sections develop the machinery to prove this theorem: Section 5 proceeds with the construction of resistance forms on SSG by means of compatible sequences based on sequences of what we call *matching pairs* of resistances. We will see in Section 6 that any completely symmetric resistance form on SSG actually corresponds to a constant multiple of a resistance form on SSG derived from a sequence of matching pairs. Once this correspondence is settled, Section 7 establishes a preliminary classification result for resistance forms $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$ derived from matching pairs displayed in Theorem 7.4. At this point, any such form $\mathcal{E}_{\mathcal{R}}$ becomes the sum of an *SG part* and a *line part*. In this way, the reminiscence of SG in SSG comes to light. In Section 8, the previous theorem is enhanced through a projection mapping onto the resistance forms having only line part. Section 9 is the core of the paper: in Theorem 9.1, the domain of the completely symmetric resistance forms on SSG is fully described, and the SG part and the line part get their corresponding expression as the aforementioned forms $\mathcal{E}^*$, $\mathcal{D}^I_\eta$ respectively. This characterization will finally lead to the classification provided by Theorem 4.7.

2 Geometry of K

In this section, we set up the geometric construction of SSG in $\mathbb{R}^2$ and fix the corresponding notation that will be carried throughout the paper.

Let $S = \{1, 2, 3\}$ and let $\{p_1, p_2, p_3\}$ be the collection of vertices of a regular triangle in $\mathbb{R}^2$. For the purpose of normalization, we assume that $p_1+p_2+p_3 = 0$ and $|p_i-p_j| = 1$ for any $i \neq j$.

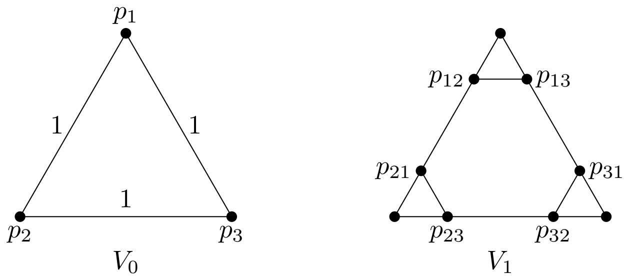

Figure 2: Geometric construction

Definition 2.1. For each $i \in S$, define $G_i : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$G_i(x) = \frac{1-\alpha}{2}(x - p_i) + p_i,$$

where $0 \leq \alpha \leq 1$. Moreover, set $p_{ij} = G_i(p_j)$ for $i \neq j$ and denote by e_{ij} the line segment $p_{ij}p_{ji}$.

If $\alpha = 0$, then $p_{ij} = p_{ji}$ for any $i \neq j$ and hence $e_{ij} = \{p_{ij}\}$. Notice that G_i, p_{ij} and e_{ij} actually depend on α. However, we will see in Proposition 2.4 that the sets K_α are homeomorphic to each other for $\alpha \in (0, 1)$ and therefore we do not write α explicitly in the notation.

Definition 2.2. Let $W_0 = \{\emptyset\}$ and define

$$W_m = S^m = \{w \mid w = w_1 \dots w_m, w_i \in S \text{ for any } i = 1, \dots, m\}$$

for $m \geq 1$, as well as $W_* = \cup_{m \geq 0} W_m$. Moreover, for any $w = w_1 \dots w_m \in W_*$, define $G_w : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$G_w = G_{w_1} \circ G_{w_2} \circ \cdots \circ G_{w_m}.$$

Finally, set $V_0 = \{p_1, p_2, p_3\}$ and

$$V_m = \bigcup_{w \in W_m} G_w(V_0)$$

for $m \geq 1$.

Notation. From now on, we denote by $B = \{(1, 2), (2, 3), (3, 1)\}$, where B stands for the word "Bond", and write $e^w_{ij} = G_w(e_{ij})$ for any $(w, (i, j)) \in W_* \times B$.

Proposition 2.3. *For any $0 \leq \alpha \leq 1$, there exists a unique compact set $K_\alpha \subseteq \mathbb{R}^2$ such that*

$$K_\alpha = G_1(K_\alpha) \cup G_2(K_\alpha) \cup G_3(K_\alpha) \cup e_{12} \cup e_{23} \cup e_{31}.$$

Furthermore,

$$K_\alpha = \Sigma_\alpha \cup \bigcup_{(w,(i,j)) \in W_* \times B} e^w_{ij},$$

where Σ_α is the self-similar set associated with $\{G_1, G_2, G_3\}$, i.e. Σ_α is the unique nonempty compact set satisfying

$$\Sigma_\alpha = G_1(\Sigma_\alpha) \cup G_2(\Sigma_\alpha) \cup G_3(\Sigma_\alpha). \tag{2.1}$$

Moreover, $\cup_{m \geq 0} V_m$ is a dense subset of Σ_α.

Proof. This follows from [6, Section 4, Theorem 1] since G_1, G_2, G_3 are $\frac{1-\alpha}{2}$-contractions. □

Remark. Σ_α is a Cantor set for any $0 < \alpha < 1$.

Notice that K_0 coincides with the Sierpinski gasket while K_1 is the union of the three line segments p_1p_2, p_2p_3 and p_3p_1. Whenever $\alpha \in (0,1)$, we can refer to any of K_α as *the* stretched Sierpinski gasket SSG in view of the next proposition.

Proposition 2.4. *The sets K_α, $\alpha \in (0,1)$, are pairwise homeomorphic.*

Proof. Consider $0 < \alpha < \beta < 1$ and denote by G_i^α, $i \in S$, the map in Definition 2.1. The mapping $\varphi_{\alpha\beta}\colon K_\alpha \to K_\beta$ that interchanges G_i^α and G_i^β, i.e. $\varphi_{\alpha\beta}(x) = \varphi_{\alpha\beta}(G_w^\alpha(y)) = G_w^\beta(y)$ is a homeomorphism because both it and its inverse $\varphi_{\beta\alpha}$ are bijective and open. □

Since resistance forms on K_α only depend on the topologic structure of K_α, which is the same for any $\alpha \in (0,1)$ due to the previous proposition, we will omit α in the definition given by Proposition 2.3 and write $K = K_\alpha$ and $\Sigma = \Sigma_\alpha$ as long as $\alpha \in (0,1)$. Moreover, we will consider d_E to be the restriction of the Euclidean metric to $K_{1/2}$ and regard d_E as the canonical metric on K.

In view of (2.1), there exists a canonical map $\iota\colon S^{\mathbb{N}} \to \Sigma$ defined by $\iota(\omega_1\omega_2\ldots) = \bigcap_{m\geq 1} G_{\omega_1\ldots\omega_m}(\Sigma)$. Through this map ι, we identify Σ with $S^{\mathbb{N}}$ hereafter in this paper.

3 The Sierpinski gasket

As already mentioned, if $\alpha = 0$ in Definition 2.1, then K_α is the Sierpinski gasket, $p_{ij} = p_{ji}$ and $e_{ij} = \{p_{ij}\}$ for any $(i,j) \in B$. In this case, we will denote G_i and K_α by F_i and K_* respectively. We explain in this section how to view continuous functions on the Sierpinski gasket K_* as continuous functions on the stretched Sierpinski gasket K and review the construction of the standard resistance form on K_*. Further details and proofs can be found e.g. in [7].

Definition 3.1. Let $\pi\colon \Sigma \to K_*$ be the canonical coding map given by $\pi(\omega_1\omega_2\ldots) = \big\{\bigcap\limits_{m\geq 0} F_{\omega_1\ldots\omega_m}(K_*)\big\}$ and define $\pi_*\colon K \to K_*$ by

$$\pi_*|_\Sigma = \pi$$

and

$$\pi_*(e_{ij}^w) = \pi(wi(j)^\infty) = \pi(wj(i)^\infty)$$

for any $(w,(i,j)) \in W_* \times B$. Furthermore, define $\pi^*\colon C(K_*) \to C(K)$ by $\pi^*(u) = u \circ \pi_*$.

From this definition it follows that $u \in \pi^*(C(K_*))$ if and only if $u \in C(K)$ and $u|_{e_{ij}^w}$ is constant for each $(w,(i,j)) \in W_* \times B$, a fact stated in the next proposition. Moreover, π^* is injective and it preserves the supremum norm. We will thus identify $C(K_*)$ with $\pi^*(C(K_*))$ and think of $C(K_*)$ as a subset of $C(K)$ in this manner. Thus we have the following proposition.

Proposition 3.2.

$$C(K_*) = \{u \mid u \in C(K),\ u|_{e_{ij}^w} \textit{ is constant for any } (w,(i,j)) \in W_* \times B\}.$$

We finish this paragraph with some classical definitions and results concerning the standard resistance form on K_* that will become relevant to state our main theorem.

Notation. For any set V we use the standard notation $\ell(V) = \{u \mid u\colon V \to \mathbb{R}\}$.

Definition 3.3. Let $V_0^* = \{p_1, p_2, p_3\}$ and define V_m^* inductively by $V_{m+1}^* = \cup_{i=1}^3 F_i(V_m^*)$ for $m \geq 0$. Furthermore, let the quadratic form $\mathcal{E}_m^*(\cdot,\cdot)$ on $\ell(V_m^*)$ be defined as

$$Q_0(u,u) = \sum_{(i,j)\in B} (u(p_i) - u(p_j))^2$$

for $m = 0$, and

$$\mathcal{E}_m^*(u,u) = \left(\frac{5}{3}\right)^m \sum_{w\in W_m} Q_0(u\circ F_w, u\circ F_w)$$

for $m \geq 1$.

Proposition 3.4. *For any $u : K_* \to \mathbb{R}$ and any $m \geq 0$,*

$$\mathcal{E}_m^*(u|_{V_m^*}, u|_{V_m^*}) \leq \mathcal{E}_{m+1}^*(u|_{V_{m+1}^*}, u|_{V_{m+1}^*})$$

and $\lim\limits_{m\to\infty} \mathcal{E}_m^*(u|_{V_m^*}, u|_{V_m^*}) = 0$ if and only if u is constant on K_*.

Proof. This follows directly from Definition 3.3. □

Theorem 3.5. *Define*

$$\mathcal{F}^* = \{u \mid u \in C(K_*), \lim_{m\to\infty} \mathcal{E}_m^*(u|_{V_m^*}, u|_{V_m^*}) < +\infty\}$$

and

$$\mathcal{E}_*(u,u) = \lim_{m\to\infty} \mathcal{E}_m^*(u|_{V_m^*}, u|_{V_m^*})$$

for $u \in \mathcal{F}^*$. Then $\mathcal{F}^* \subseteq C(K_*)$ and $(\mathcal{E}_*, \mathcal{F}^*)$ is a resistance form on K_*.

Proof. See e.g. [8, Theorem 3.13]. □

Analogously to $C(K_*)$, we will identify $\mathcal{F}^*$ with $\pi^*(\mathcal{F}^*)$ and thus regard $\mathcal{F}^*$ as a subset of $C(K)$.

4 Completely symmetric resistance forms

This section is devoted to giving a rigorous definition of completely symmetric resistance forms on SSG and presenting the main theorem of this paper, Theorem 4.7, which provides their complete characterization and classification by means of the forms $\mathcal{E}^*$ and $\mathcal{D}_\eta^I$. The proof of Theorem 4.7 will require a suitable combination of the results obtained in the succeeding sections and it will therefore be presented at the end of Section 9. We start by introducing some auxiliary notation and definitions. Recall that we write $K = K_\alpha$ for any $\alpha \in (0,1)$.

Definition 4.1. (1) Let $H^1([0,1])$ denote the Sobolev space

$$H^1([0,1]) = \left\{u \;\middle|\; u\colon [0,1] \to \mathbb{R},\ \frac{du}{dx} \in L^2([0,1], dx),\right.$$
$$\left.\text{where } \frac{du}{dx} \text{ is the derivative of } u \text{ in the sense of distributions}\right\}.$$

(2) For any $p, q \in \mathbb{R}^2$, let pq denote the line segment with extreme points p and q, and let $\xi_{p,q}\colon [0,1] \to pq$ be given by $\xi_{p,q}(t) = (1-t)p + tq$. We define

$$H^1(pq) = \{u \mid u\colon pq \to \mathbb{R},\ u \circ \xi_{p,q} \in H^1([0,1])\}$$

and

$$\mathcal{D}_{pq}(u,v) = \int_0^1 \frac{d(u\circ\xi_{p,q})}{dx}\frac{d(v\circ\xi_{p,q})}{dx}dx$$

for any $u, v \in H^1(pq)$.
(3) Define

$$\widetilde{\mathcal{F}} = \{u \mid u \in C(K) \text{ and } u|_{e_{ij}^w} \in H^1(e_{ij}^w) \text{ for any } (w,(i,j)) \in W_* \times B\}$$

as well as

$$\mathcal{D}_m^I(u,v) = \sum_{(w,(i,j))\in W_{m-1}\times B} \mathcal{D}_{e_{ij}^w}(u|_{e_{ij}^w}, v|_{e_{ij}^w})$$

for any $u, v \in \widetilde{\mathcal{F}}$ and $m \geq 1$.

We introduce now the family of completely symmetric resistance forms on K that play the central role in the classification theorem.

Basic definitions and notation concerning resistance forms are reviewed in Section 10, see also [8]. First of all, consider the set of all linear mappings under which K is invariant, i.e.

$$\mathcal{G}_K = \{\varphi \mid \varphi\colon \mathbb{R}^2 \to \mathbb{R}^2 \text{ linear and such that } \varphi(K) = K\}.$$

Notice that this is in fact the dihedral group of symmetries of the triangle.

Definition 4.2. (1) Let $\mathcal{RF}_S^{(0)}$ be the collection of resistance forms $(\mathcal{E}, \mathcal{F})$ on K satisfying the following three conditions (a), (b) and (c):

(a) $\mathcal{F} \subseteq C(K)$ and $u \in \mathcal{F}$ if and only if $u|_{G_i(K)} \in \mathcal{F}|_{G_i(K)}$ for any $i \in S$, and $u|_{e_{ij}} \in H^1(e_{ij})$ for any $(i,j) \in B$.

(b) Let R be the resistance metric associated with $(\mathcal{E}, \mathcal{F})$. Then the identity map from (K, d_E) to (K, R) is a homeomorphism.

(c) For any $\varphi \in \mathcal{G}_K$ and $u \in \mathcal{F}$, $u\circ\varphi \in \mathcal{F}$ and

$$\mathcal{E}(u\circ\varphi, u\circ\varphi) = \mathcal{E}(u,u).$$

(2) Define $\mathcal{RF}_S$ to be the collection of resistance forms $(\mathcal{E}, \mathcal{F}) \in \mathcal{RF}_S^{(0)}$ with the following property:

There exist a sequence $\{(\mathcal{E}_m, \mathcal{F}_m)\}_{m\geq 0} \subseteq \mathcal{RF}_S^{(0)}$ and a sequence $\{\eta_m\}_{m\geq 1} \subseteq (0,\infty)$ such that $(\mathcal{E}_0, \mathcal{F}_0) = (\mathcal{E}, \mathcal{F})$, $\mathcal{F}_m = \{u \circ G_i \mid u \in \mathcal{F}_{m-1}\}$ for any $m \geq 1$ and $i \in S$, and

$$\mathcal{E}_{m-1}(u,v) = \sum_{i=1}^{3} \mathcal{E}_m(u\circ G_i, v\circ G_i) + \frac{1}{\eta_m}\mathcal{D}_1^I(u,v) \tag{4.1}$$

for any $m \geq 1$ and $u, v \in \mathcal{F}_{m-1}$. The sequence $\{(\mathcal{E}_m, \mathcal{F}_m, \eta_m)\}_{m\geq 0} \subseteq \mathcal{RF}_S^{(0)} \times (0,\infty)$ is called the *resolution of* $(\mathcal{E}, \mathcal{F})$.

Remark. Although η_0 is not needed in the previous definition, we will always set $\eta_0 = 1$ for the sake of formality.

Applying (4.1) repeatedly, one immediately obtains the following proposition.

Proposition 4.3. *Let* $(\mathcal{E}, \mathcal{F}) \in \mathcal{RF}_S$. *If* $\{(\mathcal{E}_m, \mathcal{F}_m, \eta_m)\}_{m\geq 0} \subseteq \mathcal{RF}_S^{(0)} \times (0,\infty)$ *is the resolution of* $(\mathcal{E}, \mathcal{F})$, *then*

$$\begin{aligned}\mathcal{F} = \{u \mid\ & u\circ G_w \in \mathcal{F}_m \text{ for any } w \in W_m \text{ and} \\ & u|_{e_{ij}^w} \in H^1(e_{ij}^w) \text{ for any } (w,(i,j)) \in \big(\cup_{k=0}^{m-1} W_k\big) \times B\}.\end{aligned} \tag{4.2}$$

Moreover, for any $m \geq 1$ *and any* $u, v \in \mathcal{F}$,

$$\mathcal{E}(u,v) = \sum_{w\in W_m} \mathcal{E}_m(u\circ G_w, v\circ G_w) + \sum_{k=1}^{m} \frac{1}{\eta_k}\mathcal{D}_k^I(u,v). \tag{4.3}$$

The next quadratic form resembles the classical resistance form $(\mathcal{E}_*, \mathcal{F}^*)$ on K_* of Theorem 3.5 and it will be precisely the "fractal part" missed by the cable system/quantum graph approach discussed in the introduction.

Definition 4.4. Let the quadratic form Q_m^Σ on $\ell(V_m)$ be given by

$$Q_0^\Sigma(u,u) = \sum_{(i,j)\in B} (u(p_i) - u(p_j))^2$$

for any $u \in \ell(V_0)$, and by

$$Q_m^\Sigma(u,u) = \sum_{w\in W_m} Q_0(u\circ G_w, u\circ G_w)$$

for $m \geq 1$ and any $u \in \ell(V_m)$. Moreover, define

$$\mathcal{F}^\Sigma = \left\{u \;\middle|\; u \in C(K) \text{ and } \left\{\left(\frac{5}{3}\right)^m Q_m^\Sigma(u,u)\right\}_{m\geq 0} \text{ is a Cauchy sequence}\right\},$$

as well as

$$\mathcal{E}^*(u,u) = \lim_{m\to\infty} \left(\frac{5}{3}\right)^m Q_m^\Sigma(u,u)$$

for $u \in \mathcal{F}^\Sigma$.

Definition 4.5. (1) For any $m \geq 1$, let

$$\widetilde{\mathcal{F}}_m = \{u \mid u \in \widetilde{\mathcal{F}},\ u|_{G_w(K)} \text{ is constant for any } w \in W_m\},$$

and

$$\widetilde{\mathcal{F}}_\infty = \cup_{m\geq 1}\widetilde{\mathcal{F}}_m.$$

(2) For any $m \geq 1$, let

$$\mathcal{F}^*_m = \{u \mid u \in \widetilde{\mathcal{F}},\ u \circ G_w \in \mathcal{F}^* \text{ for any } w \in W_m\}$$

and

$$\mathcal{F}^*_\infty = \cup_{m\geq 1}\mathcal{F}^*_m.$$

Remark. Notice that $\widetilde{\mathcal{F}}_m \subseteq \mathcal{F}^*_m \subseteq \mathcal{F}^\Sigma$ and $\mathcal{F}^* \subseteq C(K_*) \subseteq \widetilde{\mathcal{F}}$.

Finally, we introduce the quadratic form $\mathcal{D}^I_\eta$ as the weighted sum of Dirichlet integrals whose weights are given by sequences $\{\eta_m\}_{m\geq 1} \subseteq (0,\infty)$. This form is the part that mirrors the cable system/quantum graph approach of an energy form on SSG.

Definition 4.6. Let $\eta = \{\eta_m\}_{m\geq 1}$ be a sequence of positive numbers and for any $u \in \widetilde{\mathcal{F}}$, let

$$\mathcal{D}^I_\eta(u,u) = \sum_{m=1}^{\infty} \frac{1}{\eta_m}\mathcal{D}^I_m(u,u).$$

(Note that $\mathcal{D}^I_\eta(u,u)$ is well-defined if we allow the value ∞.) Moreover, define

$$\begin{aligned}\mathcal{F}_\eta = \{u \mid u \in \widetilde{\mathcal{F}},\ \mathcal{D}^I_\eta(u,u) < +\infty \text{ and there exists } \{u_n\}_{n\geq 1} \subseteq \widetilde{\mathcal{F}}_\infty \text{ such that}\\ \lim_{n\to\infty} \mathcal{D}^I_\eta(u-u_n,u-u_n) = 0 \text{ and } \lim_{n\to\infty} u_n(x) = u(x) \text{ for any } x \in K\},\end{aligned}$$

as well as

$$\begin{aligned}\mathcal{F}^*_\eta = \{u \mid u \in \widetilde{\mathcal{F}} \cap \mathcal{F}^\Sigma,\ \mathcal{D}^I_\eta(u,u) < +\infty \text{ and there exists } \{u_n\}_{n\geq 1} \subseteq \mathcal{F}^*_\infty\\ \text{such that } \lim_{n\to\infty} \mathcal{E}^*(u-u_n,u-u_n) = \lim_{m\to\infty} \mathcal{D}^I_\eta(u-u_n,u-u_n) = 0\\ \text{and } \lim_{n\to\infty} u_n(x) = u(x) \text{ for any } x \in K\}.\end{aligned}$$

Our main result fully characterizes and identifies all resistance forms in $\mathcal{RF}_S$ by showing the correspondence between resistance forms on SSG that belong to $\mathcal{RF}_S$ and linear combinations of the forms $\mathcal{E}^*$ and $\mathcal{D}^I_\eta$.

Theorem 4.7. (1) $(\mathcal{E},\mathcal{F}) \in \mathcal{RF}_S$ *if and only if there exist* $a \geq 0, b > 0$ *and a sequence* $\eta = \{\eta_m\}_{m\geq 1} \subseteq (0,\infty)$ *such that*

$$\sum_{m=1}^{\infty} \left(\frac{5}{3}\right)^{m-1} \eta_m = 1, \tag{4.4}$$

$$\mathcal{F} = \begin{cases} \mathcal{F}_\eta & \text{if } a = 0, \\ \mathcal{F}^*_\eta & \text{if } a > 0, \end{cases} \tag{4.5}$$

and

$$\mathcal{E}(u,v) = a\mathcal{E}^*(u,v) + b\mathcal{D}^I_\eta(u,v)$$

for any $u, v \in \mathcal{F}$.
(2) *If* $\eta = \{\eta_m\}_{m \geq 1} \subseteq (0, \infty)$ *satisfies* (4.4), *then* $\mathcal{F}_\eta \subseteq \mathcal{F}^*_\eta$ *and*

$$\mathcal{F}_\eta = \{u \mid u \in \mathcal{F}^*_\eta,\ \mathcal{E}^*(u, u) = 0\}.$$

Remark. As we mentioned in the introduction, the case $a = 0$ was treated in [2] for a restricted type of sequences η. We would also like to emphasize that, even though at first sight one might want to apply the abstract result in that paper [2, Theorem 8.1] in order to obtain (part of) Theorem 7.4, it is not possible to do so in this setting since in particular the resistance metric associated to $(\mathcal{E}, \mathcal{F})$ does not lead to a geodesic metric on SSG.

5 Construction of resistance forms on K

In this section, we explain how to construct resistance forms on K by means of compatible sequences in a natural way that takes in fully consideration the intrinsic symmetry of K.

Proposition 5.1. $(Q_0^\Sigma, \ell(V_0))$ *is a resistance form on* V_0.

Proof. Since V_0 is a finite set, all properties of a resistance form (see Definition 10.1) are immediately fulfilled. □

Definition 5.2. For each $m \geq 1$, define the quadratic form $Q_m^I(\cdot, \cdot)$ on $\ell(V_m)$ by

$$Q_1^I(u, u) = \sum_{(i,j) \in B} (u(p_{ij}) - u(p_{ji}))^2$$

for any $u \in \ell(V_1)$, and by

$$Q_m^I(u, u) = \sum_{w \in W_{m-1}} Q_1^I(u \circ G_w, u \circ G_w)$$

for $m \geq 2$ and any $u \in \ell(V_m)$.

Note that neither $Q_m^\Sigma(\cdot, \cdot)$ defined in Definition 4.4 nor $Q_m^I(\cdot, \cdot)$ are resistance forms if $m \geq 1$. However, we show in the next lemma that any weighted combination of them actually yields a resistance form on V_m for any $m \geq 1$.

Lemma 5.3. *For any* $m \geq 1$, *let* $\delta, \gamma_1, \ldots, \gamma_m$ *be positive numbers. If*

$$Q(u, u) = \frac{1}{\delta} Q_m^\Sigma(u, u) + \sum_{k=1}^m \frac{1}{\gamma_k} Q_k^I(u, u)$$

for any $u \in \ell(V_m)$, *then* $(Q, \ell(V_m))$ *is a resistance form on* V_m.

Proof. Again, the conditions in Definition 10.1 are fulfilled because V_m is finite. □

As a first step in order to construct resistance forms on K, we consider compatible sequences of resistance forms on the sets V_m. To this purpose, we introduce the concept of *matching pairs* of resistances.

Definition 5.4. A pair $(r,\rho)\in(0,\infty)^2$ is said to be matching if and only if

$$\frac{5}{3}r+\rho=1.$$

The collection of all matching pairs of resistances will be denoted by $\mathcal{MP}$.

The next lemma displays the nature of the definition of matching pairs and it follows from a straightforward application of the Δ-Y transform as illustrated in Figure 3. For details on the Δ-Y transform see [7, Lemma 2.1.15].

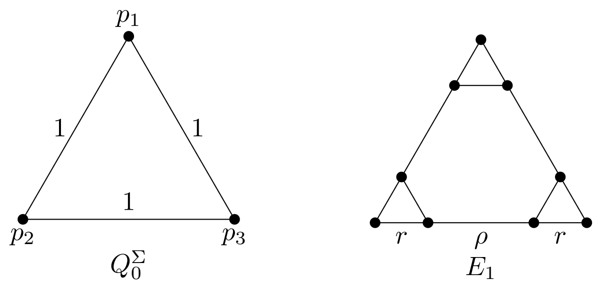

Figure 3: Renormalization of resistances

Lemma 5.5. *Let r and ρ be positive real numbers and define the resistance form $(E_1,\ell(V_1))$ on V_1 by*

$$E_1(u,u)=\frac{1}{r}Q_1^\Sigma(u,u)+\frac{1}{\rho}Q_1^I(u,u)$$

for any $u\in\ell(V_1)$. Then, $(E_1,\ell(V_1))$ on V_1 is compatible with $(Q_0^\Sigma,\ell(V_0))$ on V_0 if and only if (r,ρ) is matching.

This result is the basis leading to the relationship between sequences of matching pairs and compatible sequences of resistance forms.

Theorem 5.6. *Define*

$$E_0(u,u)=Q_0^\Sigma(u,u)$$

for any $u\in\ell(V_0)$ and

$$E_m(u,u)=\frac{1}{\delta_m}Q_m^\Sigma(u,u)+\sum_{k=1}^{m}\frac{1}{\gamma_k}Q_k^I(u,u)$$

for any $u\in\ell(V_m)$ and $m\geq1$. Then, $\{(E_m,\ell(V_m))\}_{m\geq0}$ is a compatible sequence if and only if there exists a sequence of matching pairs $\{(r_m,\rho_m)\}_{m\geq1}$ such that

$$\delta_m=r_1\cdots r_m \quad \textit{and} \quad \gamma_k=r_1\cdots r_{k-1}\rho_k$$

for any $m\geq1$ and any $k\geq1$.

Proof. By definition, $\{(E_m,\ell(V_m))\}_{m\geq0}$ is compatible if and only if $(E_m,\ell(V_m))$ is compatible with $(E_{m+1},\ell(V_{m+1}))$ for all $m\geq0$.

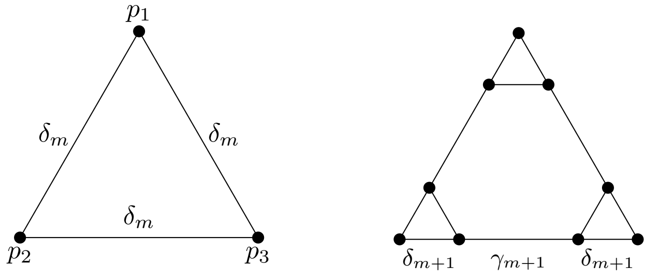

Figure 4: Renormalization of resistances

By means of the Δ-Y transform, this is the case if and only if the networks in Figure 4 are also compatible, i.e. if and only if $\frac{5}{3}\delta_{m+1} + \gamma_{m+1} = \delta_m$. Setting $r_m = \frac{\delta_{m+1}}{\delta_m}$ and $\rho_m = \frac{\gamma_{m+1}}{\delta_m}$, we have that (r_m, ρ_m) is matching and for all $m \geq 0$,

$$\delta_{m+1} = r_m \delta_m \qquad \text{and} \qquad \gamma_{m+1} = \rho_m \delta_m.$$

Applying these equalities recursively leads to the desired statement. □

Notation. We denote by $\mathcal{MP}^{\mathbb{N}}$ the collection of sequences of matching pairs of resistances, i.e.

$$\mathcal{MP}^{\mathbb{N}} = \big\{\{(r_m, \rho_m)\}_{m\geq 1} \mid r_m, \rho_m \in (0,\infty),\ \frac{5}{3}r_m + \rho_m = 1 \text{ for any } m \geq 1\big\}.$$

Definition 5.7. Let $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$ and define the quadratic form $E_{\mathcal{R},m}$ on $\ell(V_m)$ to be E_m as given in Theorem 5.6. Moreover, define

$$\widehat{\mathcal{F}}_{\mathcal{R}} = \{u \mid u \in \ell(V_*), \lim_{m\to\infty} E_{\mathcal{R},m}(u|_{V_m}, u|_{V_m}) < \infty\}$$

and

$$\widehat{\mathcal{E}}_{\mathcal{R}}(u,v) = \lim_{m\to\infty} E_{\mathcal{R},m}(u|_{V_m}, v|_{V_m})$$

for any $u, v \in \widehat{\mathcal{F}}_{\mathcal{R}}$.

In view of Theorem 5.6 and Theorem 10.6, $(\widehat{\mathcal{E}}_{\mathcal{R}}, \widehat{\mathcal{F}}_{\mathcal{R}})$ is a resistance form on $\ell(V_*)$ for any $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$. Note that if $\{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$, then $r_m \leq \frac{3}{5}$ for any $m \geq 1$ and hence $\delta_m \leq (\frac{3}{5})^m$ and $\gamma_m \leq (\frac{3}{5})^{m-1}$.

Lemma 5.8. *Let $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$ and let R_m denote the resistance metric on V_m associated with $(E_{\mathcal{R},m}, \ell(V_m))$. Then,* $\operatorname{diam}(V_m, R_m) \leq 4$ *for any $m \geq 1$, where* $\operatorname{diam}(X,d)$ *is the diameter of the metric space (X,d) given by* $\sup_{x,y\in X} d(x,y)$.

Proof. Let $q = G_{w_1\ldots w_m}(p_i)$. Define $q_k = G_{w_1\ldots w_k}(p_{w_k})$ for $k = 1,\ldots,m$ and set $q_{m+1} = q$. Since $G_{w_k}(p_{w_k}) = p_{w_k}$, we have that $q_k = G_{w_1\ldots w_{k-1}}(p_{w_k})$ and in particular $q_1 = G_{w_1}(p_{w_1}) = p_{w_1}$. Since $\{(E_m, V_m)\}_{m\geq 0}$ is compatible, it holds that

$$\begin{aligned} R_m(q_k, q_{k+1}) &= R_m(G_{w_1\ldots w_k}(p_{w_k}), G_{w_1\ldots w_k}(p_{w_{k+1}})) \\ &= R_k(G_{w_1\ldots w_k}(p_{w_k}), G_{w_1\ldots w_k}(p_{w_{k+1}})) \leq \delta_k. \end{aligned}$$

Therefore,

$$R_m(p_{w_1}, q) = R_m(q_1, q_{m+1}) \leq \sum_{k=1}^{m} R_m(q_k, q_{k+1}) \leq \sum_{k=1}^{m} \delta_k \leq \sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^k = \frac{3}{2}.$$

Thus if $x = G_{w_1 \ldots w_m}(p_i)$ and $y = G_{v_1 \ldots v_m}(p_j)$, then

$$R_m(x, y) = R_m(x, p_{w_1}) + R_m(p_{w_1}, p_{v_1}) + R_m(p_{v_1}, y) \leq \frac{3}{2} + 1 + \frac{3}{2} = 4.$$

□

The resistance form $(\widehat{\mathcal{E}}_{\mathcal{R}}, \widehat{\mathcal{F}}_{\mathcal{R}})$ possesses every symmetry and similarity (which is inhomogeneous with respect to m) required by the definition of completely symmetric resistance forms, although a function u in the domain $\widehat{\mathcal{F}}_{\mathcal{R}}$ is not a function on K but on V_*.

Lemma 5.9. *Let $\mathcal{R} = \{(r_m, \rho_m)\}_{m \geq 1} \in \mathcal{MP}^{\mathbb{N}}$. If $\mathcal{R}^{(n)} = \{(r_{n+m}, \rho_{n+m})\}_{m \geq 1}$, then $u \circ G_w \in \widehat{\mathcal{F}}_{\mathcal{R}^{(m)}}$ for any $m \geq 1$, $u \in \widehat{\mathcal{F}}_{\mathcal{R}}$ and $w \in W_m$. Moreover,*

$$\widehat{\mathcal{E}}_{\mathcal{R}}(u, v) = \sum_{w \in W_m} \frac{1}{\delta_m} \widehat{\mathcal{E}}_{\mathcal{R}^{(m)}}(u \circ G_w, v \circ G_w) + \sum_{k=1}^{m} \frac{1}{\gamma_k} Q_k^I(u, v)$$

for any $u, v \in \widehat{\mathcal{F}}_{\mathcal{R}}$ and $m \geq 1$.

Proof. Let $n, m \geq 1$. For any $u \in \widehat{\mathcal{F}}_{\mathcal{R}}$ it holds that

$$\begin{aligned} \widehat{\mathcal{E}}_{\mathcal{R}, n+m}(u, u) &= \frac{1}{\delta_{n+m}} Q_{n+m}^{\Sigma}(u, u) + \sum_{k=m+1}^{n+m} \frac{1}{\gamma_k} Q_k^I(u, u) + \sum_{k=1}^{m} \frac{1}{\gamma_k} Q_k^I(u, u) \\ &= \sum_{w \in W_m} \frac{1}{\delta_m} \widehat{\mathcal{E}}_{\mathcal{R}^{(m)}, n}(u \circ G_w, u \circ G_w) + \sum_{k=1}^{m} \frac{1}{\gamma_k} Q_k^I(u, u). \end{aligned}$$

Letting $n \to \infty$ in both sides of the equality leads to the desired result, which implies that $u \circ G_w \in \widehat{\mathcal{F}}_{\mathcal{R}^{(m)}}$ for any $u \in \widehat{\mathcal{F}}_{\mathcal{R}}$ and $w \in W_m$. □

Lemma 5.10. *Let $\mathcal{R} = \{(r_m, \rho_m)\}_{m \geq 1} \in \mathcal{MP}^{\mathbb{N}}$ and let R be the resistance metric on V_* associated with $(\widehat{\mathcal{E}}_{\mathcal{R}}, \widehat{\mathcal{F}}_{\mathcal{R}})$. If $\overline{V_*}$ is the completion of V_* with respect to R, then the identity map $\iota : V_* \to V_*$ is extended to a homeomorphism from $(\overline{V_*}, R)$ to (Σ, d_E).*

By the above lemma and Theorem 10.6, the resistance form $(\widehat{\mathcal{E}}_{\mathcal{R}}, \widehat{\mathcal{F}}_{\mathcal{R}})$ is naturally regarded as a resistance form on Σ and $\widehat{\mathcal{F}}_{\mathcal{R}}$ is thought of as a subset of $C(\Sigma)$.

Proof. Let $w \in W_*$ and let $x, y \in V_*$. Set $p = G_w(x)$ and $q = G_w(y)$. By Lemma 5.8 and Lemma 5.9,

$$\frac{|u(p) - u(q)|^2}{\mathcal{E}_{\mathcal{R}}(u, u)} \leq \delta_m \frac{|u(G_w(x)) - u(G_w(y))|^2}{\mathcal{E}_{\mathcal{R}^{(m)}}(u \circ G_w, u \circ G_w)} \leq 4\delta_m$$

holds for any $u \in \widehat{\mathcal{F}}_{\mathcal{R}}$. Thus, $\mathrm{diam}(G_w(V_*), R) \leq 4\delta_m$. Let $\{x_n\}_{n \geq 1}$ be a Cauchy sequence in V_* with respect to d_E. Then, there exists $x \in \Sigma$ such that $d_E(x, x_n) \to 0$ as $n \to \infty$. If $x = \iota(\omega_1 \omega_2 \ldots)$, then $x_n \in G_{\omega_1 \ldots \omega_m}(V_*)$ for sufficiently large n and therefore

$R(x_k, x_l) \leq 4\delta_m$ for sufficiently large k and l. Hence, $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (V_*, R) as well.

On the other hand, if $w, v \in W_m$ and $w \neq v$, there exists $u \in \widehat{\mathcal{F}}_{\mathcal{R}}$ such that $u|_{G_w(V_*)} \equiv 1$ and $u|_{G_w(V_*)} \equiv 0$. For any $x \in G_w(V_*)$ and $y \in G_v(V_*)$, we thus have that

$$1 = |u(x) - u(y)|^2 \leq \widehat{\mathcal{E}}_{\mathcal{R}}(u, u) R(x, y),$$

which shows that $\inf\{R(x, y) \mid x \in G_w(V_*), y \in G_v(V_*)\} > 0$. Hence,

$$\min\left\{ \inf\{R(x, y) \mid x \in G_w(V_*), y \in G_v(V_*)\} \;\Big|\; w, v \in W_m, w \neq v \right\} > 0.$$

If $\{y_n\}_{n\geq 1}$ is a Cauchy sequence in (V_*, R), then for any $m \geq 0$, there exists $w \in W_m$ such that $y_n \in G_w(V_*)$ for sufficiently large n. Thus, $\{y_n\}_{n\geq 1}$ is a Cauchy sequence in (V_*, d_E) as well.
Consequently, the identity map from V_* to V_* is extended to a homeomorphism between $\overline{V_*}$ and Σ. □

In order to obtain a resistance form on K from a sequence of matching pairs of resistances, we need to replace $Q_k^I(u, u)$ by a sum of H^1-inner products on the line segments e_{ij}^w. The bilinear form that arises in this way preserves many properties of the former and it is defined for functions in $\widetilde{\mathcal{F}}$.

Definition 5.11. Let $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$. For each $m \geq 1$, define

$$\mathcal{E}_{\mathcal{R},m}(u, v) = \frac{1}{\delta_m} Q_m^{\Sigma}(u, v) + \sum_{k=1}^{m} \frac{1}{\gamma_k} \mathcal{D}_k^I(u, v)$$

for any $u, v \in \widetilde{\mathcal{F}}$, where $\delta_m = r_1 \cdots r_m$ and $\gamma_m = \delta_{m-1}\rho_m$.

We start by establishing some relations between the forms $E_{\mathcal{R},m}$ and $\mathcal{E}_{\mathcal{R},m}$.

Lemma 5.12. *For any $u \in \widetilde{\mathcal{F}}$ and $m \geq 1$,*

$$Q_m^I(u, u) \leq \mathcal{D}_m^I(u, u) \tag{5.1}$$

and

$$E_{\mathcal{R},m}(u, u) \leq \mathcal{E}_{\mathcal{R},m}(u, u). \tag{5.2}$$

Proof. For any $u \in H^1([0, 1])$,

$$(u(1) - u(0))^2 \leq \int_0^1 \left(\frac{du}{dx}\right)^2 dx.$$

Applying this to every e_{ij}^w, we obtain (5.1). Consequently,

$$\begin{aligned} E_{\mathcal{R},m}(u, u) &= \frac{1}{\delta_m} Q_m^{\Sigma}(u, u) + \sum_{k=1}^{m} \frac{1}{\gamma_k} Q_k^I(u, u) \\ &\leq \frac{1}{\delta_m} Q_m^{\Sigma}(u, u) + \sum_{k=1}^{m} \frac{1}{\gamma_k} \mathcal{D}_k^I(u, u) = \mathcal{E}_{\mathcal{R},m}(u, u). \end{aligned}$$

□

Lemma 5.13. *Assume that $(r, \rho) \in \mathcal{MP}$. Then for any $u \in \widetilde{\mathcal{F}}$,*

$$Q_0^{\Sigma}(u,u) \leq \frac{1}{r} Q_1^{\Sigma}(u,u) + \frac{1}{\rho} \mathcal{D}_1^I(u,u). \tag{5.3}$$

Proof. By Lemma 5.5, we see that

$$Q_0^{\Sigma}(u,u) \leq \frac{1}{r} Q_1^{\Sigma}(u,u) + \frac{1}{\rho} Q_1^I(u,u).$$

Combining this with Lemma 5.12, we obtain (5.3). □

We can now use these properties in order to show that for any $u \in \widetilde{\mathcal{F}}$, the sequence $\{\mathcal{E}_{\mathcal{R},m}(u,u)\}_{m \geq 1}$ is monotonically non-decreasing.

Lemma 5.14. *If $\{(r_m, \rho_m)\}_{m \geq 1} \in \mathcal{MP}^{\mathbb{N}}$, then*

$$\mathcal{E}_{\mathcal{R},m}(u,u) \leq \mathcal{E}_{\mathcal{R},m+1}(u,u)$$

for any $u \in \widetilde{\mathcal{F}}$.

Proof. By Lemma 5.13, it follows that

$$\sum_{w \in W_m} Q_0^{\Sigma}(u \circ G_w, u \circ G_w) \leq \sum_{w \in W_m} \Big(\frac{1}{r_{m+1}} Q_1^{\Sigma}(u \circ G_w, u \circ G_w) + \frac{1}{\rho_{m+1}} \mathcal{D}_1^I(u \circ G_w, u \circ G_w) \Big).$$

Multiplying by $(\delta_m)^{-1}$ and adding $\sum_{k=1}^m \frac{1}{\delta_{k-1}\rho_k} \mathcal{D}_k^I(u,u)$ on both sides of the inequality, we verify the desired statement. □

In view of this lemma, $\{\mathcal{E}_{\mathcal{R},m}(u,u)\}_{m \geq 1}$ converges to a non-negative real number or infinity as $m \to \infty$. Therefore, the following definition makes sense.

Definition 5.15. Let $\mathcal{R} = \{(r_m, \rho_m)\}_{m \geq 1} \in \mathcal{MP}^{\mathbb{N}}$. Define

$$\mathcal{F}_{\mathcal{R}} = \{u \mid u \in \widetilde{\mathcal{F}}, \lim_{m \to \infty} \mathcal{E}_{\mathcal{R},m}(u,u) < \infty\}$$

and

$$\mathcal{E}_{\mathcal{R}}(u,v) = \lim_{m \to \infty} \mathcal{E}_{\mathcal{R},m}(u,v)$$

for any $u, v \in \mathcal{F}_{\mathcal{R}}$.

Next theorem is the main result of this section. It shows that resistance forms on K constructed from a sequence of matching pairs $\mathcal{R}$ are completely symmetric resistance forms. In addition, it provides an explicit expression of their corresponding resolution.

Theorem 5.16. *Let $\mathcal{R} = \{(r_m, \rho_m)\}_{m \geq 1} \in \mathcal{MP}^{\mathbb{N}}$. Then, $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}}) \in \mathcal{RF}_S$. More precisely, if $\mathcal{R}^{(n)} = \{(r_{n+m}, \rho_{n+m})\}_{m \geq 1} \in \mathcal{MP}^{\mathbb{N}}$ for any $n \geq 0$, then the resolution of $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$ is given by $\{((\delta_m)^{-1}\mathcal{E}_{\mathcal{R}^{(m)}}, \mathcal{F}_{\mathcal{R}^{(m)}}, \gamma_m)\}_{m \geq 0}$, where $\delta_m = r_1 \cdots r_m$ and $\gamma_m = \delta_{m-1}\rho_m$ for any $m \geq 0$.*

In order to show this theorem, we need several lemmas.

Lemma 5.17. *Let* $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$. *For any* $x \neq y \in K$, *there exists* $u \in \widetilde{\mathcal{F}}_\infty$ *such that* $u(x) \neq u(y)$.

Proof. If either x or y belong to $K\backslash\Sigma$, for instance $x \in K\backslash\Sigma$, then there exists $w \in W_*$ such that $x \in G_w(e_{ij}\backslash\{p_{ij}, p_{ji}\})$. In this case, there exists $u|_{e^w_{ij}} \in H^1(e^w_{ij})$ such that $u|_{e^w_{ij}}(x) = 1$ and $u|_{e^w_{ij}}(G_w(p_{ij})) = u|_{e^w_{ij}}(G_w(p_{ji})) = 0$. Letting $u(z) = 0$ for any $z \in K\backslash e^w_{ij}$, we obtain the desired function $u \in \widetilde{\mathcal{F}}_{|w|+1}$.

If $x, y \in \Sigma$, then there exist $m \geq 1$, $w \in W_m$ and $v \in W_m$ such that $x \in G_w(K)$, $y \in G_v(K)$ and $G_w(K) \cap G_v(K) = \emptyset$. Now, there is a function $u \in \widetilde{\mathcal{F}}_m$ such that $u|_{G_w(K)} = 1$ and $u|_{G_v(K)} = 0$. □

The following lemma is straightforward from the definition of $\mathcal{E}_{\mathcal{R},m}$, $\mathcal{F}_\mathcal{R}$ and $\mathcal{E}_\mathcal{R}$.

Lemma 5.18. *Let* $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$ *and for any* $n \geq 0$, *let* $\mathcal{R}^{(n)} = \{(r_{m+n}, \rho_{m+n})\}_{m\geq 1}$. *Then*

$$\mathcal{F}_\mathcal{R} = \{u \mid u\colon K \to \mathbb{R},\ u\circ G_w \in \mathcal{F}_{\mathcal{R}^{(n)}} \text{ for any } w \in W_n,\ u|_{e^w_{ij}} \in H^1(e^w_{ij}) \text{ for any } (w,(i,j)) \in \left(\cup_{k=0}^{n-1} W_k\right) \times B\}$$

and for any $u \in \mathcal{F}_\mathcal{R}$,

$$\mathcal{E}_\mathcal{R}(u,u) = \frac{1}{\delta_n}\sum_{w\in W_n} \mathcal{E}_{\mathcal{R}^{(n)}}(u\circ G_w, u\circ G_w) + \sum_{k=1}^{n} \frac{1}{\gamma_k}\mathcal{D}^I_k(u,u).$$

Lemma 5.19. *Let* $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$. *If* $w \in W_m$, *then*

$$|u(x) - u(y)|^2 \leq 16\delta_m \mathcal{E}_\mathcal{R}(u,u) \tag{5.4}$$

for any $u \in \mathcal{F}_\mathcal{R}$ *and* $x, y \in G_w(K)$.

Proof. First we show the case when $m = 0$ and $w = \emptyset$, namely

$$|u(x) - u(y)|^2 \leq 16\mathcal{E}_\mathcal{R}(u,u) \tag{5.5}$$

for any $u \in \mathcal{F}_\mathcal{R}$ and $x, y \in K$. Let $x, y \in B_* := \bigcup_{(w,(i,j))\in W_*\times B} e^w_{ij}$. Then, $x \in G_w(e_{ij}), y \in G_v(e_{kl})$ for some $w, v \in W_*$ and $(i,j),(k,l) \in B$. Set $p = G_w(p_i)$ and $q = G_w(p_k)$. Since $\gamma_n = \delta_{n-1}\rho_n \leq 1$ for any $n \geq 1$, it follows that

$$|u(x) - u(p)|^2 \leq \gamma_{|w|+1}\frac{1}{\gamma_{|w|+1}}\mathcal{D}_{e^w_{ij}}(u,u) \leq \gamma_{|w|+1}\mathcal{E}_{\mathcal{R},|w|+1}(u,u) \leq \mathcal{E}_\mathcal{R}(u,u).$$

In the same way, we obtain $|u(y) - u(q)|^2 \leq \mathcal{E}_\mathcal{R}(u,u)$. Setting $m = \max\{|w|, |v|\}$, Lemma 5.8 and (5.2) yield

$$|u(p) - u(q)|^2 \leq 4E_{\mathcal{R},m}(u,u) \leq 4\mathcal{E}_{\mathcal{R},m}(u,u) \leq 4\mathcal{E}_\mathcal{R}(u,u).$$

Combining these inequalities, we have

$$|u(x) - u(y)|^2 \leq (|u(x) - u(p)| + |u(p) - u(q)| + |u(q) - u(y)|)^2 \leq 16\mathcal{E}_\mathcal{R}(u,u).$$

Since $\widetilde{\mathcal{F}} \subseteq C(K)$ and B_* is dense in K with respect to the Euclidean metric, (5.5) holds for any $x, y \in K$.

Consider now $w \in W_m$ with $m \geq 1$, and set $x = G_w(x')$ and $y = G_w(y')$. For any $u \in \mathcal{F}_{\mathcal{R}}$, Lemma 5.18 implies that $u \circ G_w \in \mathcal{F}_{\mathcal{R}^{(m)}}$. Applying (5.5) to $(\mathcal{E}_{\mathcal{R}^{(m)}}, \mathcal{F}_{\mathcal{R}^{(m)}})$ and using again Lemma 5.18, we see that

$$|u(x)-u(y)|^2 = |u(G_w(x'))-u(G_w(y'))|^2 \leq 16\mathcal{E}_{\mathcal{R}^{(m)}}(u \circ G_w, u \circ G_w) \leq 16\delta_m \mathcal{E}_{\mathcal{R}}(u,u).$$

□

Proof of Theorem 5.16. We start by showing that $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$ is a resistance form on K.
(RF1): By definition, $\mathcal{F}_{\mathcal{R}} \subseteq C(K)$ and $\mathcal{E}_{\mathcal{R}}$ is a non-negative quadratic form on $\mathcal{F}_{\mathcal{R}}$. Moreover, if $\mathcal{E}_{\mathcal{R}}(u,u) = 0$, then $\mathcal{E}_{\mathcal{R},m}(u,u) = 0$ for any $m \geq 0$. This implies that u is constant on e_{ij}^w and $G_w(V_0)$ for any $(w,(i,j)) \in W_m \times B$. Therefore, u is constant on K and (RF1) holds.
(RF2): It suffices to prove that $(\mathcal{F}_{\mathcal{R},0}, \mathcal{E}_{\mathcal{R}})$ is complete, where $\mathcal{F}_{\mathcal{R},0} = \{u | u \in \mathcal{F}_{\mathcal{R}}, u(p_1) = 0\}$. Let $\{u_n\}_{n\geq 1}$ be a Cauchy sequence in $(\mathcal{F}_{\mathcal{R},0}, \mathcal{E}_{\mathcal{R}})$. By (5.4),

$$|u_n(x) - u_m(x)|^2 = |(u_n - u_m)(p_1) - (u_n - u_m)(x)|^2 \leq 16\mathcal{E}_{\mathcal{R}}(u_n - u_m, u_n - u_m).$$

This implies that $\{u_n\}_{n\geq 1}$ converges uniformly on K as $n \to \infty$. Let u be its limit. Then $u_n|_{e_{ij}^w}$ converges to $u|_{e_{ij}^w}$ in the sense of $H^1(e_{ij}^w)$ and hence $u|_{e_{ij}^w} \in H^1(e_{ij}^w)$. If $m \geq n$,

$$\mathcal{E}_{\mathcal{R},k}(u_n - u_m, u_n - u_m) \leq \mathcal{E}_{\mathcal{R}}(u_n - u_m, u_n - u_m) \leq \sup_{m \geq n} \mathcal{E}_{\mathcal{R}}(u_n - u_m, u_n - u_m).$$

Letting first $m \to \infty$ and afterwards $k \to \infty$, we see that $u \in \mathcal{F}_{\mathcal{R}}$ and

$$\mathcal{E}_{\mathcal{R}}(u_n - u, u_n - u) \leq \sup_{m \geq n} \mathcal{E}_{\mathcal{R}}(u_n - u_m, u_n - u_m).$$

Letting $n \to \infty$, we finally verify $\mathcal{E}_{\mathcal{R}}(u_n - u, u_n - u) \to 0$ as $n \to \infty$. Thus, $(\mathcal{F}_{\mathcal{R},0}, \mathcal{E}_{\mathcal{R}})$ is complete.
(RF3) follows from Lemma 5.17.
(RF4) is immediate by Lemma 5.19.
(RF5): Note that for any $u \in \mathcal{F}_{\mathcal{R}}$ and any $m \geq 1$,

$$Q_m^{\Sigma}(\overline{u}, \overline{u}) \leq Q_m^{\Sigma}(u,u) \quad \text{and} \quad \mathcal{D}_m^I(\overline{u}, \overline{u}) \leq \mathcal{D}_m^I(u,u). \tag{5.6}$$

This implies that $\mathcal{E}_{\mathcal{R},m}(\overline{u}, \overline{u}) \leq \mathcal{E}_{\mathcal{R},m}(u,u)$ for any $m \geq 1$, hence $\overline{u} \in \mathcal{F}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}(\overline{u}, \overline{u}) \leq \mathcal{E}_{\mathcal{R}}(u,u)$.

Thus we have shown that $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$ is a resistance form on K. Let us prove next that the identity map from (K, d_E) to (K, R) is continuous. Assume that $d_E(x_n, x) \to 0$ as $n \to \infty$. For the moment, we consider the following two cases (I) and (II):
(I) There exists $(w,(i,j)) \in W_* \times B$ such that $\{x_n\}_{n\geq 1} \subseteq e_{ij}^w$ and $x \in e_{ij}^w$.
(II) There exists $\{w(n)\}_{n\geq 1} \subseteq W_*$ such that $x, x_n \in G_{w(n)}(K)$ and $\lim_{n\to\infty} |w(n)| = \infty$.
Assume (I). Since for any $u \in \mathcal{F}_{\mathcal{R}}$

$$\frac{|u(x_n) - u(x)|^2}{\mathcal{E}_{\mathcal{R}}(u,u)} \leq \frac{|u(x_n) - u(x)|^2}{\gamma_{|w|}{}^{-1}\mathcal{D}_{e_{ij}^w}(u,u)} \leq \gamma_{|w|} \frac{d_E(x_n, x)}{d_E(G_w(p_{ij}), G_w(p_{ji}))},$$

it follows that $R(x_n, x) \leq \gamma_m d_E(x_n, x)/d_E(G_w(p_{ij}), G_w(p_{ji}))$ and hence $R(x_n, x) \to 0$ as $n \to \infty$.
Assume (II). Then Lemma 5.19 yields $R(x_n, x) \leq \delta_{|w(n)|}$, which immediately implies

that $R(x_n, x) \to 0$ as $n \to \infty$.
Let us now consider general cases. If $x \in K \backslash B_*$, then there exists $w_1 w_2 \ldots \in S^{\mathbb{N}}$ such that $x = \bigcap_{m \geq 1} G_{w_1 \ldots w_m}(K)$ and x belongs to the interior of $G_{w_1 \ldots w_m}(K)$ for any $m \geq 1$. Thus, if $d_E(x_n, x) \to 0$ as $n \to 0$, then we have case (II). If x belongs to the interior of e^w_{ij} for some $(w, (i, j)) \in W_* \times B$, then we have case (I). Finally, if $x = G_w(p_{ij})$ and $d_E(x_n, x) \to 0$ as $n \to 0$, then we can decompose $\{x_n\}_{n \geq 1}$ into $\{x_n \mid x_n \in e^w_{ij}\}$ and $\{x_n \mid x_n \in K \backslash e^w_{ij}\}$ (either one may be empty). Applying case (I) and case (II) to the first part and the second part respectively, we verify that $R(x_n, x) \to 0$ as $n \to \infty$. Thus, we have shown that the identity from (K, d_E) to (K, R) is continuous. Since (K, d_E) is compact, so is (K, R) and the inverse is continuous as well. Therefore, R gives the same topology as d_E. Notice that by definition, $\mathcal{E}_{\mathcal{R},m}(u, u) = \mathcal{E}_{\mathcal{R},m}(u \circ \varphi, u \circ \varphi)$ for any $\varphi \in \mathcal{G}_K$ and hence the same holds for $\mathcal{E}_{\mathcal{R}}$.

Finally, applying Lemma 5.18, we conclude that $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}}) \in \mathcal{RF}_S$ and its resolution is $\{((\delta_m)^{-1}\mathcal{E}_{\mathcal{R}^{(m)}}, \mathcal{F}_{\mathcal{R}^{(m)}}, \gamma_m)\}_{m \geq 0}$. □

6 Identification of $\mathcal{RF}_S$ with the resistance forms from matching pairs

In the previous section, we proved that any resistance form $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$ derived from a sequence of matching pairs $\mathcal{R}$ is completely symmetric. This section focuses on the converse statement by proving in Theorem 6.11 that, up to multiplication by a constant, any completely symmetric resistance form can be obtained from a sequence of matching pairs.

First of all, notice that since $(\mathcal{E}, \mathcal{F}) \in \mathcal{RF}^{(0)}_S$ is symmetric and V_0 has only three points, one immediately arrives to the following fact.

Lemma 6.1. *For any $(\mathcal{E}, \mathcal{F}) \in \mathcal{RF}^{(0)}_S$, there exists $r_0 > 0$ such that*

$$\mathcal{E}_{V_0}(u, v) = \frac{1}{r_0} Q^{\Sigma}_0(u, v)$$

for any $u, v \in \ell(V_0)$, where $\mathcal{E}_{V_0}$ is the trace of $(\mathcal{E}, \mathcal{F})$ on V_0.

See Proposition 10.8 for the definition of trace of a resistance form.

Proof. Since $(\mathcal{E}, \mathcal{F}) \in \mathcal{RF}^{(0)}_S$, the trace $\mathcal{E}_{V_0}$ should have the same symmetry as the equilateral triangle $p_1 p_2 p_3$. Therefore, $\mathcal{E}_{V_0}$ must be a constant multiple of Q^{Σ}_0. □

Resistance forms whose trace on V_0 coincides with Q^{Σ}_0 will play a special role in the forthcoming discussion.

Definition 6.2. For $(\mathcal{E}, \mathcal{F}) \in \mathcal{RF}^{(0)}_S$, define $r_0(\mathcal{E}, \mathcal{F})$ to be the constant r_0 given in Lemma 6.1. Furthermore, define

$$\mathcal{RF}^N_S = \{(\mathcal{E}, \mathcal{F}) \mid (\mathcal{E}, \mathcal{F}) \in \mathcal{RF}_S,\ r_0(\mathcal{E}, \mathcal{F}) = 1\}.$$

Lemma 6.3. *Let $(\mathcal{E}, \mathcal{F}) \in \mathcal{RF}_S$ and let $\{(\mathcal{E}_m, \mathcal{F}_m, \eta_m)\}_{m \geq 0} \subseteq \mathcal{RF}^{(0)}_S \times (0, \infty)$ be the resolution of $(\mathcal{E}, \mathcal{F})$. If $\delta_m = r_0(\mathcal{E}_m, \mathcal{F}_m)$ for $m \geq 0$, then $(\delta_m/\delta_{m-1}, \eta_m/\delta_{m-1}) \in \mathcal{MP}$.*

Proof. From Definition 4.2, we know that $\mathcal{F}_m = \{u \circ G_i \mid u \in \mathcal{F}_{m-1}\}$ for any $i \in S$ and $\{u|_{e_{ij}} : u \in \mathcal{F}_{m-1}\} = H^1(e_{ij})$ for any $(i,j) \in B$. Therefore, equality (4.1) yields

$$\mathcal{E}_{m-1}|_{V_1}(u,v) = \frac{1}{\delta_m} Q_1^{\Sigma}(u,v) + \frac{1}{\eta_m} Q_1^I(u,v)$$

for any $u, v \in \mathcal{F}_{m-1}|_{V_1} = \ell(V_1)$, where $(\mathcal{E}_{m-1}|_{V_1}, \mathcal{F}_{m-1}|_{V_1})$ is the trace of $(\mathcal{E}_{m-1}, \mathcal{F}_{m-1})$ on V_1 (see Definition 10.7). On the other hand,

$$\mathcal{E}_{m-1}|_{V_0}(u,v) = \frac{1}{\delta_{m-1}} Q_0^{\Sigma}(u,v)$$

for any $u, v \in \mathcal{F}_{m-1}|_{V_0} = \ell(V_0)$. Since $(\mathcal{E}_{m-1}|_{V_1}, \ell(V_1))$ and $(\mathcal{E}_{m-1}|_{V_0}, \ell(V_0))$ are compatible, Lemma 5.5 shows that $(\delta_m/\delta_{m-1}, \eta_m/\delta_{m-1}) \in \mathcal{MP}$. $\square$

Due to the above lemma, it is possible to associate a sequence of matching pairs to any $(\mathcal{E}, \mathcal{F}) \in \mathcal{RF}_S$.

Definition 6.4. Let $(\mathcal{E}, \mathcal{F}) \in \mathcal{RF}_S$ and let $\{(\mathcal{E}_m, \mathcal{F}_m, \eta_m)\}_{m \geq 0} \subseteq \mathcal{RF}_S^{(0)} \times (0, \infty)$ be the resolution of $(\mathcal{E}, \mathcal{F})$. We define $\mathcal{R}_{(\mathcal{E},\mathcal{F})} \in \mathcal{MP}^{\mathbb{N}}$ by $\mathcal{R}_{(\mathcal{E},\mathcal{F})} = \{(\delta_m/\delta_{m-1}, \eta_m/\delta_{m-1})\}_{m \geq 1}$, where $\delta_m = r_0(\mathcal{E}_m, \mathcal{F}_m)$.

We show next that for each $(\mathcal{E}, \mathcal{F}) \in \mathcal{RF}_S$ and $m \geq 1$, multiplied by $r_0(\mathcal{E}, \mathcal{F})$, its trace $\mathcal{E}|_{V_m}$ on V_m coincides with the resistance form introduced in Definition 5.7 associated with the sequence of matching pairs $\mathcal{R}_{(\mathcal{E},\mathcal{F})}$.

Lemma 6.5. *For any $(\mathcal{E}, \mathcal{F}) \in \mathcal{RF}_S$ and any $m \geq 1$, $r_0(\mathcal{E}, \mathcal{F})\mathcal{E}|_{V_m} = E_{\mathcal{R}_{(\mathcal{E},\mathcal{F})}, m}$.*

Proof. Let $\{(\mathcal{E}_m, \mathcal{F}_m, \eta_m)\}_{m \geq 0}$ be the resolution of $(\mathcal{E}, \mathcal{F})$ and set $\delta_m = r_0(\mathcal{E}_m, \mathcal{F}_m)$. By Proposition 4.3, $\mathcal{F}_m = \{u \circ G_w \mid u \in \mathcal{F}\}$, and $\mathcal{F}|_{e_{ij}^w} = H^1(e_{ij}^w)$ for any $m \geq 1$ and any $(w, (i,j)) \in W_m \times B$. Hence, it follows from (4.3) that

$$\mathcal{E}|_{V_m}(u,v) = \sum_{w \in W_m} \frac{1}{\delta_m} Q_0^{\Sigma}(u \circ G_w, u \circ G_w) + \sum_{k=1}^{m} \frac{1}{\eta_k} Q_k^I(u,v) = \frac{1}{\delta_0} E_{\mathcal{R},m}(u,v) \tag{6.1}$$

for any $u, v \in \ell(V_m)$. $\square$

Lemma 6.6. *Let $(\mathcal{E}, \mathcal{F}) \in \mathcal{RF}_S$ and let $u_* \in \mathcal{F}|_{\Sigma}$. If u is the Σ-harmonic function with respect to $(\mathcal{E}, \mathcal{F})$ with boundary value u_*, then*

$$u|_{e_{ij}^w}((1-t)G_w(p_{ij}) + tG_w(p_{ji})) = (1-t)u_*(G_w(p_{ij})) + tu_*(G_w(p_{ji}))$$

for any $t \in [0,1]$ and any $(w, (i,j)) \in W_* \times B$.

Proof. Since the restriction of a Σ-harmonic function to a line segment e_{ij}^w is a harmonic function with respect to the Dirichlet integral, it must be an affine function. $\square$

The harmonic functions determined in the previous lemma provide the definition of the trace of $(\mathcal{E}, \mathcal{F})$ on Σ. We will denote the subspace of Σ-harmonic functions by $\mathcal{H}_{(\mathcal{E},\mathcal{F})}(\Sigma)$ and its counterpart by $\mathcal{F}(\Sigma)$. Last one is characterized in the following lemma.

Lemma 6.7. *Let* $(\mathcal{E},\mathcal{F}) \in \mathcal{RF}_S$ *and let* $\{(\mathcal{E}_m,\mathcal{F}_m,\eta_m)\}_{m\geq 0} \subseteq \mathcal{RF}_S^{(0)} \times (0,\infty)$ *be the resolution of* $(\mathcal{E},\mathcal{F})$*. If* $\mathcal{F}(\Sigma) = \{u \mid u \in \mathcal{F}, u|_\Sigma = 0\}$*, then*

$$\mathcal{F}(\Sigma) = \{u \mid u\colon K \to \mathbb{R},\ u|_{e_{ij}^w} \in H^1(e_{ij}^w) \text{ for any } (w,(i,j)) \in W_* \times B,\ u|_\Sigma \equiv 0,\ \sum_{k=1}^\infty \frac{1}{\eta_k}\mathcal{D}_k^I(u,u) < +\infty\}. \quad (6.2)$$

Moreover, for any $u \in \mathcal{F}(\Sigma)$,

$$\mathcal{E}(u,u) = \sum_{k=1}^\infty \frac{1}{\eta_k}\mathcal{D}_k^I(u,u). \quad (6.3)$$

Proof. By Proposition 4.3, if $u \in \mathcal{F}(\Sigma)$, then u belongs to the set on the right-hand side of (6.2). Conversely, suppose that u belongs to the set on the right-hand side of (6.2). Define $u_n : K \to \mathbb{R}$ as

$$u_n(x) = \begin{cases} u(x) & \text{if } x \in \displaystyle\bigcup_{(w,(i,j))\in\left(\cup_{k=0}^{n-1} W_k\right)\times B} e_{ij}^w, \\ 0 & \text{otherwise.} \end{cases}$$

In view of (4.2), $u_n \in \mathcal{F}$, and if $m \geq n$, it follows from (4.3) that

$$\mathcal{E}(u_n - u_m, u_n - u_m) = \sum_{k=n+1}^m \frac{1}{\eta_k}\mathcal{D}_k^I(u,u).$$

Because $\sum_{k=1}^\infty \frac{1}{\eta_k}\mathcal{D}_k^I(u,u) < \infty$, the sequence $\{u_n\}_{n\geq 1}$ is Cauchy in $(\mathcal{E},\mathcal{F}_{p_1})$, where $\mathcal{F}_{p_1} = \{u \mid u \in \mathcal{F}, u(p_1) = 0\}$ and since $(\mathcal{E},\mathcal{F}_{p_1})$ is complete, there exists $\tilde{u} \in \mathcal{F}_{p_1}$ such that $\mathcal{E}(\tilde{u} - u_n, \tilde{u} - u_n) \to 0$ as $n \to \infty$. Moreover, for any $x \in K$,

$$|u_n(x) - \tilde{u}(x)|^2 = |u_n(x) - \tilde{u}(x) - (u_n(p_1) - \tilde{u}(p_1))|^2 \leq \mathcal{E}(u_n - \tilde{u}, u_n - \tilde{u})R(x,p_1),$$

where $R(\cdot,\cdot)$ is the resistance metric associated with $(\mathcal{E},\mathcal{F})$. This implies that $u_n(x) \to \tilde{u}(x)$ as $n \to \infty$. On the other hand, for any $x \in K$, $u_n(x) \to u(x)$ as $n \to \infty$. Hence, $u = \tilde{u} \in \mathcal{F}$ and

$$\mathcal{E}(u,u) = \lim_{n\to\infty} \mathcal{E}(u_n,u_n) = \sum_{k=1}^\infty \frac{1}{\eta_k}\mathcal{D}_k^I(u,u).$$

□

We see next that the subspaces $\mathcal{H}_{(\mathcal{E},\mathcal{F})}(\Sigma)$ and $\mathcal{F}(\Sigma)$ actually provide an orthogonal decomposition of the domain of the resistance forms $(\mathcal{E},\mathcal{F}) \in \mathcal{RF}_S$.

Theorem 6.8. *Let* $(\mathcal{E},\mathcal{F}) \in \mathcal{RF}_S$ *and let* $\{(\mathcal{E}_m,\mathcal{F}_m,\eta_m)\}_{m\geq 0} \subseteq \mathcal{RF}_S^{(0)} \times (0,\infty)$ *be the resolution of* $(\mathcal{E},\mathcal{F})$*. Then,*

$$\mathcal{F} = \mathcal{H}_{(\mathcal{E},\mathcal{F})}(\Sigma) \oplus \mathcal{F}(\Sigma) \quad (6.4)$$

and for any $u \in \mathcal{F}$,

$$\mathcal{E}(u,u) = \mathcal{E}|_\Sigma(u|_\Sigma, u|_\Sigma) + \sum_{k=1}^\infty \frac{1}{\eta_k}\mathcal{D}_k^I(u - h_\Sigma(u), u - h_\Sigma(u)), \quad (6.5)$$

where $h_\Sigma(u)$ *denotes the* Σ*-harmonic extension of* $u|_\Sigma$.

Proof. The direct sum decomposition (6.4) follows from Proposition 10.10. Combining Proposition 10.10 and Lemma 6.7 we immediately verify (6.5). □

Corollary 6.9. *Let* $(\mathcal{E},\mathcal{F}),(\mathcal{E}',\mathcal{F}')\in\mathcal{RF}_S$*. Then, the following conditions are equivalent:*

(E1) $(\mathcal{E},\mathcal{F})=(\mathcal{E}',\mathcal{F}')$.

(E2) $(\mathcal{E}|_{V_m},\ell(V_m))=(\mathcal{E}'|_{V_m},\ell(V_m))$ *for any* $m\geq 0$.

(E3) $(\mathcal{E}|_\Sigma,\mathcal{F}|_\Sigma)=(\mathcal{E}'|_\Sigma,\mathcal{F}'|_\Sigma)$.

(E4) $\mathcal{R}_{(\mathcal{E},\mathcal{F})}=\mathcal{R}_{(\mathcal{E}',\mathcal{F}')}$ *and* $r_0(\mathcal{E},\mathcal{F})=r_0(\mathcal{E}',\mathcal{F}')$.

Proof. By Theorem 10.11, (E2) implies (E3). Since $(\mathcal{E}|_\Sigma)|_{V_m}=\mathcal{E}|_{V_m}$ and $(\mathcal{E}'|_\Sigma)|_{V_m}=\mathcal{E}'|_{V_m}$, we see that (E3) implies (E2). In view of Lemma 6.5, (E4) implies (E2). Conversely, assume that (E2) holds. Since $\mathcal{E}|_{V_0}=\mathcal{E}'|_{V_0}$, then $r_0(\mathcal{E},\mathcal{F})=r_0(\mathcal{E}',\mathcal{F}')$ and by Lemma 6.5 we obtain $E_{\mathcal{R}_{(\mathcal{E},\mathcal{F})},m}=E_{\mathcal{R}_{(\mathcal{E}',\mathcal{F}')},m}$ for any $m\geq 1$. Therefore, it follows that $\mathcal{R}_{(\mathcal{E},\mathcal{F})}=\mathcal{R}_{(\mathcal{E}',\mathcal{F}')}$ and hence we have (E4). Moreover, (E1) immediately implies (E3) and it only remains to verify that (E3) implies (E1).
Let us assume (E2), (E3) and (E4). By Lemma 6.6, if $h_\Sigma\colon\mathcal{F}|_\Sigma\to\mathcal{F}$ and $h'|_\Sigma\colon\mathcal{F}'|_\Sigma\to\mathcal{F}'$ are the Σ-harmonic extension maps associated with $(\mathcal{E},\mathcal{F})$ and $(\mathcal{E}',\mathcal{F}')$ respectively, then $h_\Sigma=h'_\Sigma$ and hence $\mathcal{H}_{(\mathcal{E},\mathcal{F})}(\Sigma)=\mathcal{H}_{(\mathcal{E}',\mathcal{F}')}(\Sigma)$. If $\{(\mathcal{E}_m,\mathcal{F}_m,\eta_m)\}_{m\geq 0}$ and $\{(\mathcal{E}'_m,\mathcal{F}'_m,\eta'_m)\}_{m\geq 0}$ are the resolutions of $(\mathcal{E},\mathcal{F})$ and $(\mathcal{E}',\mathcal{F}')$ respectively, then (6.1) implies that $\eta_m=\eta'_m$. Lemma 6.7 thus yields $\mathcal{F}(\Sigma)=\mathcal{F}'(\Sigma)$ and by (6.4) it follows that $\mathcal{F}=\mathcal{F}'$. Finally, since $(\mathcal{E}|_\Sigma,\mathcal{F}|_\Sigma)=(\mathcal{E}'|_\Sigma,\mathcal{F}'|_\Sigma)$ and $\eta_m=\eta'_m$ for any $m\geq 0$, (6.5) shows that $\mathcal{E}(u,u)=\mathcal{E}'(u,u)$ for any $u\in\mathcal{F}=\mathcal{F}'$, hence $(\mathcal{E},\mathcal{F})=(\mathcal{E}',\mathcal{F}')$. □

Resistance forms on K constructed by means of sequences of matching pairs as explained in Section 5 have the property of belonging to $\mathcal{RF}_S^N$.

Lemma 6.10. *Let* $\mathcal{R}\in\mathcal{MP}^{\mathbb{N}}$*. Then* $r_0(\mathcal{E}_\mathcal{R},\mathcal{F}_\mathcal{R})=1$.

Proof. Let $f:V_0\to\mathbb{R}$ and let u be the V_0-harmonic function with respect to $(\widehat{\mathcal{E}}_\mathcal{R},\widehat{\mathcal{F}}_\mathcal{R})$ with boundary value f. Since $(\widehat{\mathcal{E}}_\mathcal{R}|_{V_0},\widehat{\mathcal{F}}_\mathcal{R}|_{V_0})=(Q_0^\Sigma,\ell(V_0))$, we have

$$\widehat{\mathcal{E}}_\mathcal{R}(u,u)=Q_0^\Sigma(f,f).$$

By Lemma 5.10, $\widehat{\mathcal{F}}_\mathcal{R}\subseteq C(\Sigma)$ and hence $u\in C(\Sigma)$. For any $(w,(i,j))\in W_*\times B$, we extend the domain of u to each e_{ij}^w by defining

$$\varphi|_{e_{ij}^w}((1-t)G_w(p_{ij})+tG_w(p_{ji}))=(1-t)u(G_w(p_{ij}))+tu(G_w(p_{ji}))$$

for any $t\in[0,1]$ and $\varphi|_\Sigma=u$. In this manner, u is extended to a function φ on K such that $\varphi\in\widetilde{\mathcal{F}}$. Since for any $m\geq 1$

$$Q_0^\Sigma(f,f)=E_{\mathcal{R},m}(\varphi,\varphi)=\mathcal{E}_{\mathcal{R},m}(\varphi,\varphi),$$

$\varphi\in\mathcal{F}_\mathcal{R}$ and $\mathcal{E}_\mathcal{R}(\varphi,\varphi)=Q_0^\Sigma(f,f)$. Now, if $v\in\mathcal{F}_\mathcal{R}$ and $v|_{V_0}=f$, (5.2) yields

$$\mathcal{E}_\mathcal{R}(\varphi,\varphi)=Q_0^\Sigma(f,f)\leq E_{\mathcal{R},m}(v,v)\leq\mathcal{E}_{\mathcal{R},m}(v,v)\leq\mathcal{E}_\mathcal{R}(v,v).$$

Therefore, φ is the V_0-harmonic function with respect to $(\mathcal{E}_\mathcal{R},\mathcal{F}_\mathcal{R})$ with boundary value f and hence $\mathcal{E}_\mathcal{R}|_{V_0}=Q_0^\Sigma$. This shows $r_0(\mathcal{E}_\mathcal{R},\mathcal{F}_\mathcal{R})=1$. □

We conclude this section by showing that $\mathcal{RF}_S = \{(\delta\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}}) \mid \mathcal{R} \in \mathcal{MP}^{\mathbb{N}}, \delta > 0\}$.

Theorem 6.11. (1) $\mathcal{R}_{(\mathcal{E}_{\mathcal{R}},\mathcal{F}_{\mathcal{R}})} = \mathcal{R}$ *for any* $\mathcal{R} \in \mathcal{MP}^{\mathbb{N}}$.
(2) $(\mathcal{E}_{\mathcal{R}_{(\mathcal{E},\mathcal{F})}}, \mathcal{F}_{\mathcal{R}_{(\mathcal{E},\mathcal{F})}}) = (r_0(\mathcal{E},\mathcal{F})\mathcal{E}, \mathcal{F})$ *for any* $(\mathcal{E},\mathcal{F}) \in \mathcal{RF}_S$.
In particular, the map $\mathcal{R} \in \mathcal{MP}^{\mathbb{N}} \to (\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}}) \in \mathcal{RF}_S^N$ *is bijective.*

Proof. (1) Let $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1}$. By Theorem 5.16, the resolution of $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$ is given by $\{((\delta_m)^{-1}\mathcal{E}_{\mathcal{R}^{(m)}}, \mathcal{F}_{\mathcal{R}^{(m)}}, \gamma_m)\}_{m\geq 0}$, where $\delta_m = r_1 \cdots r_m$ and $\gamma_m = \delta_{m-1}\rho_m$. By Lemma 6.10, $r_0((\delta_m)^{-1}\mathcal{E}_{\mathcal{R}^{(m)}}, \mathcal{F}_{\mathcal{R}^{(m)}}) = \delta_m$ and therefore $\mathcal{R}_{(\mathcal{E}_{\mathcal{R}},\mathcal{F}_{\mathcal{R}})} = \mathcal{R}$.
(2) Without loss of generality, we may assume that $r_0(\mathcal{E},\mathcal{F}) = 1$. Set $\mathcal{R} = \mathcal{R}(\mathcal{E},\mathcal{F})$. Applying (1), $\mathcal{R}_{(\mathcal{E}_{\mathcal{R}},\mathcal{F}_{\mathcal{R}})} = \mathcal{R}$ and by Lemma 6.10, $r_0(\mathcal{E}_{\mathcal{R}},\mathcal{F}_{\mathcal{R}}) = 1 = r_0(\mathcal{E},\mathcal{F})$. Thus, condition (E4) in Corollary 6.9 is satisfied and hence $(\mathcal{E}_{\mathcal{R}},\mathcal{F}_{\mathcal{R}}) = (\mathcal{E},\mathcal{F})$. □

Remark. This result reveals that $\mathcal{RF}_S^N$ is in fact the set of fixed points of the mapping

$$\begin{aligned} \Phi\colon \mathcal{RF}_S &\longrightarrow \mathcal{RF}_S \\ (\mathcal{E},\mathcal{F}) &\mapsto (r_0(\mathcal{E},\mathcal{F})^{-1}\mathcal{E}_{\mathcal{R}_{(\mathcal{E},\mathcal{F})}}, \mathcal{F}_{\mathcal{R}_{(\mathcal{E},\mathcal{F})}}). \end{aligned}$$

7 Classification of resistance forms derived from matching pairs

In the previous section we have identified any complete symmetric resistance form on SSG with a resistance form $(\mathcal{E}_{\mathcal{R}},\mathcal{F}_{\mathcal{R}})$ derived from a sequence of matching pairs $\mathcal{R}$ up to multiplication by a constant. The present section analyzes the detailed structure of $(\mathcal{E}_{\mathcal{R}},\mathcal{F}_{\mathcal{R}})$ by decomposing it into two parts, called the *SG part* in allusion to the reminiscence of SG in SSG, and the *line part* that corresponds to the cable system/quantum graph approach. In Theorem 7.4 we use a certain property of the sequence $\mathcal{R}$ to determine when the SG part is non-trivial and therefore captures the reminiscence of the SG in the geometric structure of SSG.

Let $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$ and set $\delta_m = r_1 \cdots r_m$ and $\gamma_m = \delta_{m-1}\rho_m$. For any $u \in \mathcal{F}_{\mathcal{R}}$ and $m \geq 1$,

$$\sum_{k=1}^{m} \frac{1}{\gamma_k} \mathcal{D}_k^I(u,u) \leq \mathcal{E}_{\mathcal{R}}(u,u)$$

and the left-hand side is monotonically increasing with respect to m. We can thus define $\mathcal{E}_{\mathcal{R}}^I(u,u)$ as

$$\mathcal{E}_{\mathcal{R}}^I(u,u) = \sum_{k=1}^{\infty} \frac{1}{\gamma_k} \mathcal{D}_k^I(u,u).$$

Moreover, we define

$$\mathcal{E}_{\mathcal{R}}^{\Sigma}(u,u) = \mathcal{E}_{\mathcal{R}}(u,u) - \mathcal{E}_{\mathcal{R}}^I(u,u)$$

and therefore

$$\mathcal{E}_{\mathcal{R}}^{\Sigma}(u,u) = \lim_{m\to\infty} \frac{1}{\delta_m} Q_m^{\Sigma}(u,u).$$

Notice that both $\mathcal{E}_{\mathcal{R}}^{\Sigma}$ and $\mathcal{E}_{\mathcal{R}}^I$ are non-negative quadratic forms on $\mathcal{F}_{\mathcal{R}}$. As a consequence, it follows that

$$\mathcal{E}_{\mathcal{R}}(u,v) = \mathcal{E}_{\mathcal{R}}^{\Sigma}(u,v) + \mathcal{E}_{\mathcal{R}}^I(u,v).$$

We call $\mathcal{E}^{\Sigma}_{\mathcal{R}}$ (resp. $\mathcal{E}^{I}_{\mathcal{R}}$) the SG part (resp. the line part) of $\mathcal{E}_{\mathcal{R}}$. It is easy to see that, in order for $\mathcal{E}_{\mathcal{R}}$ to be a resistance form, the line part should be non-zero. On the contrary, the SG part may vanish, as we will see in the the course of our discussion.

The following useful lemma is an exercise of undergraduate calculus.

Lemma 7.1. *Let* $\{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$ *and define*

$$\kappa_m = -\log(1-\rho_m).$$

The following three conditions are equivalent:

(a) $\sum_{m=1}^{\infty} \kappa_m < +\infty$,

(b) $\sum_{m=1}^{\infty} \rho_m < +\infty$,

(c) *There exists* $C > 0$ *such that*

$$C\Big(\frac{3}{5}\Big)^m \leq r_1 r_2 \cdots r_m$$

for any $m \geq 1$.

Note that $\kappa_m = \log\frac{3}{5} - \log r_m$ for each $m \geq 1$. Thus if $\sum_{m=1}^{\infty}\rho_m < +\infty$ and we set $\kappa = \sum_{m=1}^{\infty}\kappa_m$, then

$$r_1 r_2 \cdots r_m = e^{-\kappa} e^{\sum_{i\geq m+1}\kappa_i}\Big(\frac{3}{5}\Big)^m.$$

With this equality one leads to the following lemma.

Lemma 7.2. *Let* $\{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$. *There exist* $C_* > 0$ *and a sequence* $\{c_m\}_{m\geq 1}$ *such that* $\lim_{m\to\infty} c_m = 0$ *and*

$$\frac{1}{r_1 r_2 \cdots r_m} = C_*\Big(\frac{5}{3}\Big)^m (1 - c_m)$$

if and only if $\sum_{m=1}^{\infty}\rho_m < \infty$. *Moreover, if* $\sum_{m=1}^{\infty}\rho_m < \infty$, *then* $C_* = \prod_{m=1}^{\infty}(1-\rho_m)^{-1}$, $c_m \geq 0$ *for any* $m \geq 0$, *and* $\{c_m\}_{m\geq 1}$ *is monotonically decreasing.*

Definition 7.3. Define $\mathcal{F}_{\mathcal{R},*} = \mathcal{F}_{\mathcal{R}} \cap C(K_*)$.

As announced at the beginning of the section, next theorem reveals the sufficient condition for the survival of $\mathcal{E}^{\Sigma}_{\mathcal{R}}$, that is $\sum_{m=1}^{\infty}\rho_m < \infty$. In the next section, this condition will turn out to be necessary as well.

Theorem 7.4. *Let* $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$. $\mathcal{F}_{\mathcal{R},*}$ *consists of constants if and only if* $\sum_{m=1}^{\infty}\rho_m = \infty$. *Furthermore, if* $\sum_{m=1}^{\infty}\rho_m < \infty$, *then* $\mathcal{F}^* \subseteq \mathcal{F}_{\mathcal{R},*}$ *and*

$$\mathcal{E}_{\mathcal{R}}(u,u) = \mathcal{E}^{\Sigma}_{\mathcal{R}}(u,u) = C_*\mathcal{E}^*(u,u)$$

for any $u \in \mathcal{F}^*$, *where* $C_* = \prod_{m=1}^{\infty}(1-\rho_m)^{-1}$.

Proof. For each $m \geq 1$, set $\delta_m = r_1 \cdots r_m$. If $\sum_{m=1}^{\infty}\rho_m < \infty$, then Lemma 7.2 yields

$$\mathcal{E}_{\mathcal{R},m}(u,u) = \frac{1}{\delta_m}Q^{\Sigma}_m(u,u) = C_*(1-c_m)\Big(\frac{5}{3}\Big)^m Q^{\Sigma}_m(u,u)$$

for any $u \in \mathcal{F}^*$. By Theorem 3.5 we now have that

$$\mathcal{E}_{\mathcal{R}}(u,u) = \lim_{m\to\infty} \frac{1}{\delta_m} Q_m^{\Sigma}(u,u) = C_* \mathcal{E}^*(u,u)$$

and hence $\mathcal{F}^* \subseteq \mathcal{F}_{\mathcal{R},*}$. On the other hand, if $\sum_{m=1}^{\infty} \rho_m = \infty$, then for any $u \in C(K_*)$,

$$\mathcal{E}_{\mathcal{R},m}(u,u) = \frac{1}{\delta_m} Q_m^{\Sigma}(u,u) = \frac{1}{\delta_m}\Big(\frac{3}{5}\Big)^m \cdot \Big(\frac{5}{3}\Big)^m Q_m^{\Sigma}(u,u).$$

From Lemma 7.1, it follows that $\frac{1}{\delta_m}\big(\frac{3}{5}\big)^m$ is unbounded as $m \to \infty$ and by Proposition 3.4, $\lim_{m\to\infty}\big(\frac{5}{3}\big)^m Q_m^{\Sigma}(u,u) > 0$ unless u is constant. Therefore, $u \in \mathcal{F}_{\mathcal{R}}$ if and only if u is constant. □

We conclude this section with some preliminary results concerning the domains of the SG part and the line part.

Definition 7.5. Let $\mathcal{R} \in \mathcal{MP}^{\mathbb{N}}$. Define

$$\mathcal{F}_{\mathcal{R}}^{I} = \{u \mid u \in \mathcal{F}_{\mathcal{R}},\ \mathcal{E}_{\mathcal{R}}^{\Sigma}(u,u) = 0\},$$
$$\mathcal{F}_{\mathcal{R}}^{\Sigma} = \{u \mid u \in \mathcal{F}_{\mathcal{R}},\ \mathcal{E}_{\mathcal{R}}^{I}(u,u) = 0\}.$$

Lemma 7.6. *Let $\mathcal{R} \in \mathcal{MP}^{\mathbb{N}}$. Then,*

$$\mathcal{F}_{\mathcal{R}}^{I} = \{u \mid u \in \mathcal{F}_{\mathcal{R}},\ \mathcal{E}_{\mathcal{R}}^{\Sigma}(u,v) = 0 \text{ for any } v \in \mathcal{F}_{\mathcal{R}}\}$$

and

$$\mathcal{F}_{\mathcal{R}}^{\Sigma} = \{u \mid u \in \mathcal{F}_{\mathcal{R}},\ \mathcal{E}_{\mathcal{R}}^{I}(u,v) = 0 \text{ for any } v \in \mathcal{F}_{\mathcal{R}}\}.$$

Proof. Let us consider first $\mathcal{F}_{\mathcal{R}}^{I}$. Applying the Cauchy-Schwartz inequality,

$$\mathcal{E}_{\mathcal{R}}^{\Sigma}(u,v)^2 \le \mathcal{E}_{\mathcal{R}}^{\Sigma}(u,u)\mathcal{E}_{\mathcal{R}}^{\Sigma}(v,v)$$

for any $u, v \in \mathcal{F}_{\mathcal{R}}$. Therefore, if $u \in \mathcal{F}_{\mathcal{R}}^{I}$, then $\mathcal{E}_{\mathcal{R}}^{\Sigma}(u,v) = 0$ for any $v \in \mathcal{F}_{\mathcal{R}}$. The converse direction is obvious. The argument for $\mathcal{F}_{\mathcal{R}}^{\Sigma}$ works verbatim. □

Lemma 7.7. *Let $\mathcal{R} \in \mathcal{MP}^{\mathbb{N}}$. Then, $\mathcal{F}_{\mathcal{R}}^{\Sigma} = \mathcal{F}_{\mathcal{R}} \cap C(K_*)$.*

Proof. If $u \in \mathcal{F}_{\mathcal{R}} \cap C(K_*)$, then u is constant on every e_{ij}^{w}. Therefore, $\mathcal{E}_{\mathcal{R}}^{I}(u,u) = 0$ and hence $u \in \mathcal{F}_{\mathcal{R}}^{\Sigma}$. Conversely, assume that $u \in \mathcal{F}_{\mathcal{R}}^{\Sigma}$. Then, $\mathcal{D}_{e_{ij}^{w}}(u|_{e_{ij}^{w}}, u|_{e_{ij}^{w}}) = 0$ for every e_{ij}^{w} and hence u is constant on e_{ij}^{w}. Thus, $u \in C(K_*)$. □

8 Projection to the line part

Let us define $(\mathcal{MP}^{\mathbb{N}})^{I} \subseteq \mathcal{MP}^{\mathbb{N}}$ by

$$(\mathcal{MP}^{\mathbb{N}})^{I} = \{\mathcal{R} \mid \mathcal{R} \in \mathcal{MP}^{\mathbb{N}},\ (\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}}) = (\mathcal{E}_{\mathcal{R}}^{I}, \mathcal{F}_{\mathcal{R}}^{I})\}.$$

In this section, we are going to introduce a natural projection $\mathcal{L}\colon \mathcal{MP}^{\mathbb{N}} \to (\mathcal{MP}^{\mathbb{N}})^{I}$ and through an explicit expression of this mapping, it will be shown in Theorem 8.9 that

$$(\mathcal{MP}^{\mathbb{N}})^{I} = \big\{\mathcal{R} \;\big|\; \mathcal{R} = \{(r_m, \rho_m)\}_{m\ge 1} \in \mathcal{MP}^{\mathbb{N}}, \sum_{m=1}^{\infty} \rho_m = \infty\big\}.$$

In other words, the converse of Theorem 7.4 holds, i.e. $\sum_{m=1}^{\infty} \rho_m = \infty$ if and only if $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}}^I$, or equivalently $\mathcal{E}_{\mathcal{R}}^{\Sigma} = 0$.

Before doing so, we present some results concerning a general theory that is not confined to resistance forms on K and is applicable in very abstract settings.

Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a set X and let R be the associated resistance metric. Assume that there exist non-negative symmetric quadratic forms $\mathcal{E}^{(1)}(\cdot,\cdot)$ and $\mathcal{E}^{(2)}(\cdot,\cdot)$ on $\mathcal{F} \times \mathcal{F}$ such that

$$\mathcal{E}(u,v) = \mathcal{E}^{(1)}(u,v) + \mathcal{E}^{(2)}(u,v)$$

for any $u, v \in \mathcal{F}$. Define $\mathcal{F}^{(2)} = \{u \mid u \in \mathcal{F},\ \mathcal{E}^{(1)}(u,u) = 0\}$ and note that

$$\mathcal{E}(u,v) = \mathcal{E}^{(2)}(u,v)$$

for any $u, v \in \mathcal{F}^{(2)}$. As in Definition 10.1, for any $u, v \in \mathcal{F}$ we define $u \sim v$ if and only if $u - v$ is constant.

Lemma 8.1. *$(\mathcal{F}^{(2)}/\!\sim, \mathcal{E})$ is a closed subspace of $(\mathcal{F}/\!\sim, \mathcal{E})$.*

Proof. Let $x \in X$ and set $\mathcal{F}_x^{(2)} = \{u | u \in \mathcal{F}^{(2)}, u(x) = 0\}$ and $\mathcal{F}_x = \{u | u \in \mathcal{F}, u(x) = 0\}$. It suffices to show that $(\mathcal{F}_x^{(2)}, \mathcal{E})$ is a closed subspace of $(\mathcal{F}_x, \mathcal{E})$. Let $\{u_n\}_{n\geq 1} \in \mathcal{F}_x^{(2)}$ and suppose that $\mathcal{E}(u_n - u, u_n - u) \to 0$ as $n \to \infty$ for some $u \in \mathcal{F}_x$. Then,

$$\mathcal{E}^{(1)}(u - u_n, u - u_n) = \mathcal{E}^{(1)}(u_n, u_n) - 2\mathcal{E}^{(1)}(u_n, u) + \mathcal{E}^{(1)}(u,u) \to 0$$

as $n \to \infty$. Since $\mathcal{E}^{(1)}(u_n, u_n) = 0$ and $\mathcal{E}^{(1)}(u_n, u)^2 \leq \mathcal{E}^{(1)}(u_n, u_n)\mathcal{E}^{(1)}(u,u)$, it follows that $\mathcal{E}^{(1)}(u,u) = 0$. Thus, $(\mathcal{F}_x^{(2)}, \mathcal{E})$ is closed and so is $(\mathcal{F}^{(2)}/\!\sim, \mathcal{E})$. □

Using the above lemma, we may easily verify the following statement.

Theorem 8.2. *$(\mathcal{E}, \mathcal{F}^{(2)})$ is a resistance form on X if the following two conditions are satisfied:*

(1) *For any $x \neq y \in X$, there exists $u \in \mathcal{F}^{(2)}$ such that $u(x) \neq u(y)$.*

(2) *For any $u \in \mathcal{F}$, $\mathcal{E}^{(1)}(\overline{u}, \overline{u}) \leq \mathcal{E}^{(1)}(u,u)$ and $\mathcal{E}^{(2)}(\overline{u}, \overline{u}) \leq \mathcal{E}^{(2)}(u,u)$.*

Proof. (RF1) and (RF4) hold because $\mathcal{F}^{(2)} \subseteq \mathcal{F}$ and $(\mathcal{E}, \mathcal{F})$ is a resistance form. (RF3) is condition (1) and (RF5) is condition (2). To prove (RF2), notice that $(\mathcal{F}/\!\sim, \mathcal{E})$ is complete because $(\mathcal{E}, \mathcal{F})$ is a resistance form. By Lemma 8.1, $(\mathcal{F}^{(2)}/\!\sim, \mathcal{E})$ is a closed subspace of $(\mathcal{F}/\!\sim, \mathcal{E})$ and hence also complete. □

Back to resistance forms on SSG, we start by constructing the projection $\mathcal{L}$ from $\mathcal{MP}^{\mathbb{N}}$ onto $(\mathcal{MP}^{\mathbb{N}})^I$.

Theorem 8.3. *Let $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$. If*

$$\mathcal{F}_{\mathcal{R}}^I = \{u \mid u \in \mathcal{F}_{\mathcal{R}}, \mathcal{E}_{\mathcal{R}}^{\Sigma}(u,u) = 0\},$$

then $(\mathcal{E}_{\mathcal{R}}^I, \mathcal{F}_{\mathcal{R}}^I) \in \mathcal{RF}_S$ and its resolution is given by $\{((\delta_m)^{-1}\mathcal{E}_{\mathcal{R}^{(m)}}^I, \mathcal{F}_{\mathcal{R}^{(m)}}^I, \gamma_m)\}_{m\geq 0}$, where $\delta_m = r_1 \cdots r_m$ and $\gamma_m = \delta_{m-1}\rho_m$ for any $m \geq 1$. Moreover, $r_0(\mathcal{E}_{\mathcal{R}}^I, \mathcal{F}_{\mathcal{R}}^I) \leq 1$ and there exists a unique $\mathcal{R}' \in \mathcal{MP}^{\mathbb{N}}$ such that $(r_0(\mathcal{E}_{\mathcal{R}}^I, \mathcal{F}_{\mathcal{R}}^I)\mathcal{E}_{\mathcal{R}}^I, \mathcal{F}_{\mathcal{R}}^I) = (\mathcal{E}_{\mathcal{R}'}, \mathcal{F}_{\mathcal{R}'})$.

To prove this theorem, we need the following lemma.

Lemma 8.4.

$$\mathcal{F}^I_{\mathcal{R}} = \{u \mid u \in C(K),\ u\circ G_i \in \mathcal{F}^I_{\mathcal{R}^{(1)}} \text{ for any } i \in S,\ u|_{e_{ij}} \in H^1(e_{ij}) \text{ for any } (i,j) \in B\}$$

and

$$\mathcal{E}^I_{\mathcal{R}}(u,u) = \sum_{i\in S} \frac{1}{r_1}\mathcal{E}^I_{\mathcal{R}^{(1)}}(u\circ G_i, u\circ G_i) + \frac{1}{\rho_1}\mathcal{D}^I_1(u,u) \tag{8.1}$$

for any $u \in \mathcal{F}^I_{\mathcal{R}}$.

Proof. Note that for any $m \geq 1$ and $u \in \mathcal{F}^I_{\mathcal{R}}$,

$$\frac{1}{\delta_m}Q^{\Sigma}_m(u,u) = \frac{1}{\delta_m}\sum_{i\in S} Q^{\Sigma}_{m-1}(u\circ G_i, u\circ G_i).$$

By definition, $u \in \mathcal{F}^I_{\mathcal{R}}$ if and only if $u \in \mathcal{F}_{\mathcal{R}}$ and

$$\lim_{m\to\infty} \frac{1}{\delta_{m+1}} Q^{\Sigma}_m(u\circ G_i, u\circ G_i) = 0$$

for any $i \in S$, i.e. $u \in \mathcal{F}^I_{\mathcal{R}}$ if and only if $u \in \mathcal{F}_{\mathcal{R}}$ and $u\circ G_i \in \mathcal{F}^I_{\mathcal{R}^{(1)}}$. This immediately implies the desired equivalence. $\{((\delta_m)^{-1}\mathcal{E}_{\mathcal{R}^{(m)}}, \mathcal{F}_{\mathcal{R}^{(m)}}, \gamma_m)\}_{m\geq 0}$ is the resolution of $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$, we obtain (8.1). □

Proof of Theorem 8.3. Applying Theorem 8.2 to $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$ with $\mathcal{E}^{(1)} = \mathcal{E}^{\Sigma}_{\mathcal{R}}$ and $\mathcal{E}^{(2)} = \mathcal{E}^I_{\mathcal{R}}$, we see that $(\mathcal{E}^I_{\mathcal{R}}, \mathcal{F}^I_{\mathcal{R}})$ is a resistance form on K.
Moreover, by Lemma 8.4, $u \in \mathcal{F}^I_{\mathcal{R}}$ if and only if $u|_{G_i(K)} \in \mathcal{F}^I_{\mathcal{R}}|_{G_i(K)}$ for any $i \in S$ and $u|_{e_{ij}} \in H^1(e_{ij})$. If $R(\cdot,\cdot)$ and $R^I(\cdot,\cdot)$ are the resistance metrics associated with $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$ and $(\mathcal{E}^I_{\mathcal{R}}, \mathcal{F}^I_{\mathcal{R}})$ respectively, then

$$R^I(x,y) = \sup\Big\{\frac{|u(x)-u(y)|^2}{\mathcal{E}_{\mathcal{R}}(u,u)} \;\Big|\; u \in \mathcal{F}^I_{\mathcal{R}}, u(x) \neq u(y)\Big\} \leq R(x,y) \tag{8.2}$$

for any $x, y \in K$. Hence, the identity map ι from (K,R) to (K,R^I) is continuous and since (K,R) is compact, the map ι is a homeomorphism. Furthermore, $\mathcal{E}^I_{\mathcal{R}}$ is invariant under all geometric symmetries of K because $\mathcal{E}_{\mathcal{R}}$ is. Combining these previous facts, we conclude that $(\mathcal{E}^I_{\mathcal{R}}, \mathcal{F}^I_{\mathcal{R}}) \in \mathcal{RF}^{(0)}_S$. Applying (8.1) to $(\mathcal{E}^I_{\mathcal{R}^{(m)}}, \mathcal{F}^I_{\mathcal{R}^{(m)}})$ repeatedly, we get that $(\mathcal{E}^I_{\mathcal{R}}, \mathcal{F}^I_{\mathcal{R}}) \in \mathcal{RF}_S$ and its resolution is $\{((\delta_m)^{-1}\mathcal{E}^I_{\mathcal{R}^{(m)}}, \mathcal{F}^I_{\mathcal{R}^{(m)}}, \gamma_m)\}_{m\geq 0}$. Finally,

$$r_0(\mathcal{E}^I_{\mathcal{R}}, \mathcal{F}^I_{\mathcal{R}}) = \frac{3}{2}R^I(p_1,p_2) \leq \frac{3}{2}R(p_1,p_2) = 1$$

and the existence of $\mathcal{R}' \in \mathcal{MP}^{\mathbb{N}}$ follows immediately from Theorem 6.11. □

Definition 8.5. For any $\mathcal{R} \in \mathcal{MP}^{\mathbb{N}}$, $\mathcal{L}(\mathcal{R}) \in \mathcal{MP}^{\mathbb{N}}$ is defined as $\mathcal{R}'$ given in Theorem 8.3.

Lemma 8.6. *Let* $\mathcal{R} \in \mathcal{MP}^{\mathbb{N}}$. *If* $\mathcal{L}(\mathcal{R}) = \{(s_m, \sigma_m)\}_{m\geq 1}$, *then*

$$\sum_{m=1}^{\infty} \sigma_m = \infty. \tag{8.3}$$

Proof. Notice that for any $u \in \mathcal{F}^* \cap \mathcal{F}_{\mathcal{L}(\mathcal{R})}$, u is constant on every e^w_{ij} and hence $\mathcal{E}_{\mathcal{L}(\mathcal{R})}(u,u) = r_0(\mathcal{E}^I_{\mathcal{R}}, \mathcal{F}^I_{\mathcal{R}})\mathcal{E}^I_{\mathcal{R}}(u,u) = 0$. Since $(\mathcal{E}_{\mathcal{L}(\mathcal{R})}, \mathcal{F}_{\mathcal{L}(\mathcal{R})})$ is a resistance form, u is constant on K. By Theorem 7.4, we see that $\sum_{m=1}^{\infty} \sigma_m = \infty$. □

Next lemma gives an explicit expression of $\mathcal{L}(\mathcal{R})$, which plays an essential role in the rest of the section.

Lemma 8.7. *Let $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$ and let $\rho_0 = r_0(\mathcal{E}^I_{\mathcal{R}}, \mathcal{F}^I_{\mathcal{R}})$. If $\mathcal{L}(\mathcal{R}) = \{(s_m, \sigma_m)\}_{m\geq 1}$, then*

$$\rho_0 \prod_{i=1}^{m-1}(1-\sigma_i) = \prod_{i=1}^{m-1}(1-\rho_i) - (1-\rho_0) \tag{8.4}$$

for any $m \geq 1$. In particular,

$$\sigma_m = \frac{\prod_{i=1}^{m-1}(1-\rho_i)}{\prod_{i=1}^{m-1}(1-\rho_i) - (1-\rho_0)}\rho_m \tag{8.5}$$

and

$$\rho_m = \frac{\rho_0 \prod_{i=1}^{m-1}(1-\sigma_i)}{\rho_0 \prod_{i=1}^{m-1}(1-\sigma_i) + (1-\rho_0)}\sigma_m. \tag{8.6}$$

Proof. For $(w,(i,j)) \in W_* \times B$, choose $u \in \mathcal{F}_{\mathcal{R}}$ so that $u(x) = 0$ for any $x \notin e^w_{ij}$ and $\mathcal{E}_{\mathcal{R}}(u,u) > 0$. Then, $u \in \mathcal{F}^I_{\mathcal{R}}$ and $\rho_0 \mathcal{E}^I_{\mathcal{R}}(u,u) = \mathcal{E}_{\mathcal{R}'}(u,u) = \mathcal{E}^I_{\mathcal{R}'}(u,u)$. Hence, we get

$$r_1 r_2 \cdots r_{m-1}\rho_m = \rho_0 s_1 s_2 \ldots s_{m-1}\sigma_m \tag{8.7}$$

for any $m \geq 1$, which yields

$$\rho_m \prod_{i=1}^{m-1}(1-\rho_i) = \rho_0 \sigma_m \prod_{i=1}^{m-1}(1-\sigma_i). \tag{8.8}$$

By induction, we obtain (8.4). □

Lemma 8.8. *Let $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$ and let $\rho_0 = r_0(\mathcal{E}^I_{\mathcal{R}}, \mathcal{F}^I_{\mathcal{R}})$. If $\mathcal{L}(\mathcal{R}) = \{(s_m, \sigma_m)\}_{m\geq 1}$, then*

$$\sigma_m = \frac{1-\alpha_m}{\rho_0 - \alpha_m}\rho_m \tag{8.9}$$

for any $m \geq 1$, where $\alpha_m = 1 - \prod_{i=1}^{m-1}(1-\rho_i)$. In particular,

$$\rho_0 \geq \lim_{m\to\infty} \alpha_m.$$

Proof. The equality follows directly from Lemma 8.7, which also implies that for any $m \geq 1$,

$$\rho_0 = \rho_0 \prod_{i=1}^{m-1}(1-\sigma_i) + 1 - \prod_{i=1}^{m-1}(1-\rho_i) \geq 1 - \prod_{i=1}^{m-1}(1-\rho_i) = \alpha_m$$

and therefore $\rho_0 \geq \lim_{m\to\infty} \alpha_m$. □

Remark. $\{\alpha_n\}_{n\geq 1}$ is monotonically increasing and $\alpha_n \uparrow \alpha$ as $n \to \infty$ for some $\alpha \in (0,1]$.

Finally, we present the main theorem of this section. It characterizes $(\mathcal{MP}^{\mathbb{N}})^I$ and essentially says that the SG part $\mathcal{E}_{\mathcal{R}}^{\Sigma}$ truly exists if and only if $\sum_{m=1}^{\infty} \rho_m < \infty$.

Theorem 8.9. *Let $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$. Then, $\mathcal{L}(\mathcal{R}) = \mathcal{R}$ if and only if $\sum_{m=1}^{\infty} \rho_m = \infty$. In particular, $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}}^I$ if and only if $\sum_{m=1}^{\infty} \rho_m = \infty$.*

Proof. Assume that $\sum_{m=1}^{\infty} \rho_m = \infty$. Then $\alpha = 1$, which implies that $\rho_0 \geq 1$ and therefore $\rho_0 = 1$. In view of (8.5), we have that $\rho_m = \sigma_m$ for any $m \geq 1$, hence $\mathcal{R} = \mathcal{R}'$. Thus we have shown that $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}'} = \mathcal{E}_{\mathcal{R}}^I$. Conversely, if $\mathcal{R} = \mathcal{R}'$, then Lemma 8.6 shows that $\sum_{m=1}^{\infty} \rho_m = \sum_{m=1}^{\infty} \sigma_m = \infty$. □

As a consequence of this theorem,

$$\mathcal{L}(\mathcal{L}(\mathcal{R})) = \mathcal{L}(\mathcal{R})$$

for any $\mathcal{R} \in \mathcal{MP}^{\mathbb{N}}$ and $\mathcal{L}(\mathcal{MP}^{\mathbb{N}}) = (\mathcal{MP}^{\mathbb{N}})^I$. Thus, we may regard $\mathcal{L}$ as a projection onto $(\mathcal{MP}^{\mathbb{N}})^I$.

We finish this section with several useful equalities leading to an explicit expression of $r_0(\mathcal{E}_{\mathcal{R}}^I, \mathcal{F}_{\mathcal{R}}^I)$ in terms of the elements of $\mathcal{R}$.

Lemma 8.10. *Let $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$. Then,*

(1) *For any $m \geq 1$,*

$$\sum_{i=1}^{m} \left(\frac{5}{3}\right)^{i-1} \gamma_i + \left(\frac{5}{3}\right)^{m} \delta_m = 1,$$

where $\delta_m = r_1 \cdots r_m$ and $\gamma_i = \delta_{i-1}\rho_i$.

(2) *$\sum_{m=1}^{\infty} \rho_m = \infty$ if and only if $\lim_{m\to\infty} \left(\frac{5}{3}\right)^m \delta_m = 0$.*

Proof. Let $\gamma_m = r_1 \cdots r_{m-1}\rho_m$. Since $\frac{5}{3} r_m + \rho_m = 1$, we have

$$\begin{aligned}\left(\frac{5}{3}\right)^{m-1} \gamma_m &= (1-\rho_1)\cdots(1-\rho_{m-1})\rho_m \\ &= (1-\rho_1)\cdots(1-\rho_{m-1}) - (1-\rho_1)\cdots(1-\rho_m)\end{aligned}$$

and hence

$$\sum_{i=1}^{m} \left(\frac{5}{3}\right)^{i-1} \gamma_i = 1 - \prod_{i=1}^{m}(1-\rho_i) = 1 - \left(\frac{5}{3}\right)^{m} \delta_m.$$

This proves (1). Assertion (2) follows immediately from the fact that $\prod_{i=1}^{m}(1-\rho_i) = \left(\frac{5}{3}\right)^m \delta_m$. □

Proposition 8.11. *Let $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$. Then,*

$$r_0(\mathcal{E}_{\mathcal{R}}^I, \mathcal{F}_{\mathcal{R}}^I) = 1 - \prod_{m=1}^{\infty}(1-\rho_m) = \sum_{m=1}^{\infty}\left(\rho_m \prod_{i=1}^{m-1}(1-\rho_i)\right). \tag{8.10}$$

In particular, $r_0(\mathcal{E}_{\mathcal{R}}^I, \mathcal{F}_{\mathcal{R}}^I) < 1$ if and only if $\sum_{m=1}^{\infty} \rho_m < \infty$.

Proof. Set $\rho_0 = r_0(\mathcal{E}^I_{\mathcal{R}}, \mathcal{F}^I_{\mathcal{R}})$ and $\mathcal{L}(\mathcal{R}) = \{(s_m, \sigma_m)\}_{m\geq 1}$. If $\sum_{m=1}^{\infty} \rho_m = \infty$, then we have already shown in the proof of Theorem 8.9 that $\rho_0 = 1$. Since $\prod_{m=1}^{\infty}(1-\rho_m) = 0$, Lemma 8.10 implies (8.10).

Suppose that $\sum_{m=1}^{\infty} \rho_m < \infty$ and set $\alpha = 1 - \prod_{m=1}^{\infty}(1-\rho_m)$. If $\rho_0 > \alpha$, then (8.9) and Lemma 8.10-(1) lead to

$$\sum_{m=1}^{\infty} \sigma_m \leq \frac{1}{\rho_0 - \alpha} \sum_{m=1}^{\infty} (1-\alpha_m)\rho_m \leq \frac{1}{\rho_0 - \alpha} < \infty.$$

This contradicts (8.3), hence $\rho_0 = \alpha$. Applying Lemma 8.10 again, we immediately obtain (8.10). □

9 Domain of resistance forms given by infinite sequences of matching pairs

The results obtained in previous sections come together in the present one to prove the main theorem of this paper, Theorem 4.7. In fact, Theorem 6.11 and Theorem 8.9 already identify any completely symmetric resistance form on SSG as the sum of its line part and its SG part, whenever the latter survives. This identification is now completed by giving a full description of the domains of these forms. This characterization of the domains in the next theorem is the key step to show Theorem 4.7.

Theorem 9.1. *Let* $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$ *and set* $R_* = \prod_{m=1}^{\infty}(1-\rho_m)$. *Moreover, define*

$$\eta_m = \frac{r_1 \cdots r_{m-1}\rho_m}{1 - R_*}$$

for any $m \geq 1$ *and* $\eta = \{\eta_m\}_{m\geq 1}$.

(1) *If* $\sum_{m=1}^{\infty} \rho_m = \infty$, *then* $R_* = 0$, $\mathcal{F}_{\mathcal{R}} = \mathcal{F}_\eta$ *and*

$$\mathcal{E}_{\mathcal{R}}(u,v) = \mathcal{D}^I_\eta(u,v)$$

for any $u, v \in \mathcal{F}_{\mathcal{R}}$.

(2) *If* $\sum_{m=1}^{\infty} \rho_m < \infty$, *then* $R_* \in (0,1)$, $\mathcal{F}_{\mathcal{R}} = \mathcal{F}^*_\eta$ *and*

$$\mathcal{E}_{\mathcal{R}}(u,v) = \frac{1}{R_*}\mathcal{E}^*(u,v) + \frac{1}{1-R_*}\mathcal{D}^I_\eta(u,v)$$

for any $u, v \in \mathcal{F}_{\mathcal{R}}$.

The idea to prove this theorem will be to show that the restriction of $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$ to $\mathcal{F}_\eta$ in the case $\sum_{m=1}^{\infty} \rho_m = \infty$, respectively $\mathcal{F}^*_\eta$ in the case $\sum_{m=1}^{\infty} \rho_m < \infty$, is again completely symmetric and derived from the same matching pair as $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$. Developing this strategy requires some effort and consists in several steps shown in the subsequent lemmas.

We start with two remarks.

Remark. (i) By Lemma 8.10-(1),

$$\sum_{m=1}^{\infty} \left(\frac{5}{3}\right)^{m-1} \gamma_m = 1 - \prod_{m=1}^{\infty}(1-\rho_m) \tag{9.1}$$

and therefore

$$\sum_{m=1}^{\infty}\left(\frac{5}{3}\right)^{m-1}\eta_m = 1.$$

(ii) By Proposition 8.11,

$$1 - R_* = r_0(\mathcal{E}^I_{\mathcal{R}}, \mathcal{F}^I_{\mathcal{R}}) = \rho_0.$$

Definition 9.2. For each $\mathcal{R} = \{(r_m,\rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$ and $n \geq 0$, define $\mathcal{R}^{(n)} = \{(r_{m+n},\rho_{m+n})\}_{m\geq 1}$, $R^{(n)}_* = \prod_{m=1}^{\infty}(1-\rho_{m+n})$,

$$\eta^{(n)}_m = \frac{r_{n+1}\cdots r_{n+m-1}\rho_{n+m}}{1-R^{(n)}_*}$$

for $m \geq 1$, and $\eta^{(n)} = \{\eta^{(n)}_m\}_{m\geq 1}$. Moreover, for each $n \geq 0$, define

$$\mathcal{F}^{(n)} = \begin{cases} \mathcal{F}_{\eta^{(n)}} & \text{if } \sum_{m=1}^{\infty}\rho_m = \infty,\\ \mathcal{F}^*_{\eta^{(n)}} & \text{if } \sum_{m=1}^{\infty}\rho_m < \infty,\end{cases}$$

with $\mathcal{F}_{\eta^{(n)}}$ and $\mathcal{F}_{\eta^{(n)}}$ as in Definition 4.6.

Lemma 9.3. *Let $\mathcal{R} = \{(r_m,\rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$ and define for each $n \geq 0$ $\mathcal{E}^{(n)}_{p_1}(u,v) = \mathcal{E}_{\mathcal{R}^{(n)}}(u,v) + u(p_1)v(p_1)$ for any $u,v \in \mathcal{F}_{\mathcal{R}^{(n)}}$.*

(1) *If $\sum_{m=1}^{\infty}\rho_m = \infty$, $\mathcal{F}^{(n)}$ is the closure of $\widetilde{\mathcal{F}}_\infty$ with respect to the inner product $\mathcal{E}^{(n)}_{p_1}$.*

(2) *If $\sum_{m=1}^{\infty}\rho_m < \infty$, $\mathcal{F}^{(n)}$ is the closure of $\mathcal{F}^*_\infty$ with respect to the inner product $\mathcal{E}^{(n)}_{p_1}$.*

In either case, $\mathcal{F}^{(n)} \subseteq \mathcal{F}_{\mathcal{R}^{(n)}}$.

Proof. (1) It suffices to show the case $n = 0$. Let us assume first that $\sum_{m=1}^{\infty}\rho_m = \infty$. Then, $R_* = 0$ and $\mathcal{E}_{\mathcal{R}} = \mathcal{E}^I_{\mathcal{R}} = \mathcal{D}^I_\eta$. Consider now $u \in \mathcal{F}_\eta$, i.e. $u \in \widetilde{\mathcal{F}}$, $\mathcal{D}^I_\eta(u,u) < \infty$ and there exists $\{u_n\}_{n\geq 1} \subseteq \widetilde{\mathcal{F}}_\infty$ such that $\lim_{n\to\infty}\mathcal{D}^I_\eta(u-u_n,u-u_n) = 0$ and $\lim_{n\to\infty}u_n(x) = u(x)$ for any $x \in K$. Then, $\{u_n\}_{n\geq 1}$ is a Cauchy sequence in $(\mathcal{F}_{\mathcal{R}},\mathcal{E}_{p_1})$. Since $(\mathcal{D}^I_\eta,\mathcal{F}_{\mathcal{R}})$ is a resistance form, there exists $\tilde{u} \in \mathcal{F}_{\mathcal{R}}$ such that $\mathcal{E}_{p_1}(\tilde{u}-u_n,\tilde{u}-u_n) \to 0$ and $u_n(x) \to \tilde{u}(x)$ as $n \to \infty$ for any $x \in K$. Therefore, $u = \tilde{u} \in \mathcal{F}_{\mathcal{R}}$ and hence it belongs to the closure of $\widetilde{\mathcal{F}}_\infty$ with respect to the inner product $\mathcal{E}_{p_1}$. Conversely, it is easy to see that the closure of $\widetilde{\mathcal{F}}_\infty$ with respect to $\mathcal{E}_{p_1}$ is a subset of $\mathcal{F}_\eta$. Thus, $\mathcal{F}_\eta$ is the closure of $\widetilde{\mathcal{F}}_\infty$ with respect to the inner product $\mathcal{E}_{p_1}$ and in particular $\mathcal{F}_\eta \subseteq \mathcal{F}_{\mathcal{R}}$. If $\sum_{m=1}^{\infty}\rho_m < \infty$, it follows from Theorem 7.4 that

$$\mathcal{E}_{\mathcal{R}}(u,v) = \frac{1}{R_*}\mathcal{E}^*(u,v) + \frac{1}{1-R_*}\mathcal{D}^I_\eta(u,v) \tag{9.2}$$

for any $u,v \in \mathcal{F}_{\mathcal{R}}$. Consider now $u \in \mathcal{F}^*_\eta$, i.e. $u \in \widetilde{\mathcal{F}}\cap\mathcal{F}^\Sigma$, $\mathcal{D}^I_\eta(u,u) < \infty$ and there exists $\{u_n\}_{n\geq 1} \subseteq \mathcal{F}^*_\infty$ such that $\lim_{n\to\infty}\mathcal{E}^*(u-u_n,u-u_n) = \lim_{m\to\infty}\mathcal{D}^I_\eta(u-u_n,u-u_n) = 0$ and $\lim_{n\to\infty}u_n(x) = u(x)$ for any $x \in K$. Similar arguments as the previous case imply that u belongs to $\mathcal{F}_{\mathcal{R}}$ and $\mathcal{E}_{p_1}(u-u_n,u-u_n) \to 0$ as $n \to \infty$, hence $\mathcal{F}^{(0)}$ is a subset of the closure of $\mathcal{F}^*_\infty$ with respect to $\mathcal{E}_{p_1}$. The converse inclusion is straightforward and the desired statement follows. □

Definition 9.4. For each $\mathcal{R} = \{(r_m,\rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$ and any $n \geq 1$, define $\mathcal{E}^{(n)} = \mathcal{E}_{\mathcal{R}^{(n)}}|_{\mathcal{F}^{(n)}\times\mathcal{F}^{(n)}}$.

Lemma 9.5. *Let $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$. For any $n \geq 0$ and any $i \in S$,*

$$\{u\circ G_i \mid u \in \mathcal{F}^{(n)}\} = \mathcal{F}^{(n+1)}.$$

$$\mathcal{F}^{(n)} = \{u \mid u \in C(K),\ u\circ G_i \in \mathcal{F}^{(n+1)} \text{ for any } i \in S,\ u|_{e_{ij}} \in H^1(e_{ij}) \text{ for any } (i,j) \in B\},$$

and

$$\mathcal{E}^{(n)}(u,v) = \sum_{i\in S} \frac{1}{r_{n+1}} \mathcal{E}^{(n+1)}(u\circ G_i, v\circ G_i) + \frac{1}{\rho_{n+1}} \mathcal{D}_1^I(u,v)$$

for any $u, v \in \mathcal{F}^{(n)}$.

Proof. From Theorem 5.16 we know that $\{((\delta_n)^{-1}\mathcal{E}_{\mathcal{R}^{(n)}}, \mathcal{F}_{\mathcal{R}^{(n)}}, \gamma_n)\}_{n\geq 0}$, where $\delta_n = r_1 \cdots r_n$ and $\gamma_n = \delta_{n-1}\rho_n$, is the resolution of $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$. This directly implies the last equality of the lemma because $\mathcal{F}^{(n)} \subseteq \mathcal{F}_{\mathcal{R}^{(n)}}$ by Lemma 9.3. In view of that equality, if $u \in \mathcal{F}^{(n)}$, then $u|_{e_{ij}} \in H^1(e_{ij})$ for any $(i,j) \in B$. In addition, if $\{u_k\}_{k\geq 1}$ is the sequence that approximates u, then $\{u_k\circ G_i\}_{k\geq 1}$ approximates $u\circ G_i$ in such a way that $u\circ G_i \in \mathcal{F}^{(n+1)}$. On the other hand, consider $u \in C(K)$ such that $u\circ G_i \in \mathcal{F}^{(n+1)}$ for any $i \in S$ and $u|_{e_{ij}} \in H^1(e_{ij})$ for all $(i,j) \in B$. Our aim is to prove that $u \in \mathcal{F}^{(n)}$. Since $u\circ G_i \in \mathcal{F}^{(n+1)}$, $\mathcal{E}^{(n+1)}(u\circ G_i, u\circ G_i) < \infty$ and by Lemma 9.3 there exists $\{u_{k,i}\}_{k\geq 1} \subseteq \widetilde{\mathcal{F}}_\infty$ (resp. $\mathcal{F}^*_\infty$) such that

$$\lim_{k\to\infty} \mathcal{E}^{(n+1)}(u\circ G_i - u_{k,i}, u\circ G_i - u_{k,i}) = 0 \quad \text{and} \quad \lim_{k\to\infty} u_{k,i}(x) = u\circ G_i(x).$$

for any $x \in G_i(K)$. For each $k \geq 1$ define $v_k : K \to \mathbb{R}$ by

$$v_k(x) := \begin{cases} u_{k,i}\circ G_i^{-1}(x) & \text{if } x \in G_i(K), \\ u(x) + \varphi_k^{ij}(x) & \text{if } x \in e_{ij}, (i,j) \in B, \end{cases}$$

where φ_k^{ij} is an affine function on e_{ij} chosen so that $v_k \in C(K)$. Since $\lim_{k\to\infty} \varphi_k^{ij}(p_{ij}) = \lim_{k\to\infty} \varphi_k^{ij}(p_{ji}) = 0$, we have $\mathcal{D}_{e_{ij}}(\varphi_k^{ij}, \varphi_k^{ij}) \to 0$. By construction, $v_k \in C(K)$ and $v_k \in \widetilde{\mathcal{F}}_\infty$ (resp. $\mathcal{F}^*_\infty$) for any $k \geq 1$. Furthermore, $\mathcal{D}_1^I(u - v_n, u - v_n) \to 0$ as $n \to \infty$ and hence $\mathcal{E}^{(n)}(u - v_k, u - v_k) \to 0$ as $k \to \infty$. Moreover, $\lim_{k\to\infty} v_k(p_1) = \lim_{k\to\infty} v_k(G_1(p_1)) = \lim_{k\to\infty} u_{k,1}(p_1) = u(p_1)$ and therefore $u \in \mathcal{F}^{(n)}$.

It remains to prove that $\{u\circ G_i \mid u \in \mathcal{F}^{(n)}\} = \mathcal{F}^{(n+1)}$. On the one hand, it follows from the previous discussion that if $u \in \mathcal{F}^{(n)}$, then $u\circ G_i \in \mathcal{F}^{(n+1)}$. On the other hand, consider $u \in \mathcal{F}^{(n+1)}$. By Lemma 9.3, $\mathcal{F}^{(n+1)} \subseteq \mathcal{F}_{\mathcal{R}^{(n+1)}} = \{v\circ G_i \mid v \in \mathcal{F}_{\mathcal{R}^{(n)}}\}$ and we can pick $v \in \mathcal{F}_{\mathcal{R}^{(n)}}$ such that $v\circ G_i = u$ for any $i \in S$. In particular, $v \in C(K)$, $v\circ G_i \in \mathcal{F}^{(n+1)}$ and $v_{|e_{ij}} \in H^1(e_{ij})$ for all $(i,j) \in B$, so that $v \in \mathcal{F}^{(n)}$. □

Lemma 9.6. *Let $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$. Then, $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)}) \in \mathcal{RF}_S$ and its resolution is $\{((\delta_m)^{-1}\mathcal{E}^{(m)}, \mathcal{F}^{(m)}, \gamma_m)\}_{m\geq 0}$, where $\delta_m = r_1 \cdots r_m$ and $\gamma_m = \delta_{m-1}\rho_m$.*

Proof. We start by showing that $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$ is a resistance form. Condition (RF1) is obvious. Condition (RF2) follows immediately from Lemma 9.3. Moreover, since $\widetilde{\mathcal{F}}_\infty$ already has the property (RF3) and $\widetilde{\mathcal{F}}_\infty \subseteq \mathcal{F}^{(0)}$, (RF3) is also fulfilled. Condition (RF4) holds because $\mathcal{F}_\eta \subseteq \mathcal{F}_{\mathcal{R}}$, and

$$\sup\left\{\frac{|u(x) - u(y)|^2}{\mathcal{E}(u,u)} \mid u \in \mathcal{F}_\eta,\ \mathcal{E}_{\mathcal{R}}(u,u) \neq 0\right\} \leq R(x,y) \tag{9.3}$$

for any $x, y \in K$.

It remains to prove (RF5). Suppose first that $\sum_{m=1}^{\infty} \rho_m = \infty$. Obviously, $\widetilde{\mathcal{F}}_\infty$ has the Markov property. Now, let μ be a Borel regular probability measure on K that satisfies $\mu(O) > 0$ for any non-empty open set O and $\mu(A) = 0$ for any finite set A. Define

$$\mathcal{E}_\mu(u, v) = \mathcal{E}_\mathcal{R}(u, v) + \int_K |u(x)|^2 \mu(dx)$$

for any $u, v \in \mathcal{F}_\mathcal{R}$. Due to the fact that

$$|u(x) - u(p_1)|^2 \leq \mathcal{E}_\mathcal{R}(u, u) R(x, p_1) \leq C \mathcal{E}_\mathcal{R}(u, u),$$

where $C = \sup_{x \in K} R(x, p_1)$, we can find $C' > 0$ such that

$$\frac{1}{C'} \mathcal{E}_{p_1}(u, u) \leq \mathcal{E}_\mu(u, u) \leq C' \mathcal{E}_{p_1}(u, u)$$

for any $u \in \mathcal{F}_\mathcal{R}$. Therefore, by Lemma 9.3, $\mathcal{F}^{(0)}$ is the closure of $\widetilde{\mathcal{F}}_\infty$ with respect to $\mathcal{E}_\mu$ and [4, Theorem 3.1.1] implies that $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$ is a Dirichlet form on $L^2(K, \mu)$. In particular, $\mathcal{F}^{(0)}$ has the Markov property and hence (RF5) holds in this case. Suppose now that $\sum_{m=1}^{\infty} \rho_m < \infty$. Replacing $\widetilde{\mathcal{F}}_\infty$ by $\mathcal{F}^*_\infty$, the previous arguments show that (RF5) holds again. Thus, $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$ is a resistance form.

Let $R^{(0)}$ be the resistance metric on K associated with $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$ that equals the left-hand side of (9.3). In view of (9.3), the identity map from (K, R) to $(K, R^{(0)})$ is continuous and since (K, R) is homeomorphic to (K, d_E), it is compact. Therefore, the identity map from (K, R) to $(K, R^{(0)})$ is a homeomorphism. The rest of the statement follows immediately from Lemma 9.5. □

We finally show Theorem 9.1 by making use of these preliminary lemmas to prove that any completely symmetric resistance form $(\mathcal{E}_\mathcal{R}, \mathcal{F}_\mathcal{R})$ actually coincides with the resistance form $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$ introduced in Definition 9.2. The representation of $\mathcal{E}^{(0)}$ as linear combination of $\mathcal{E}^*$ and $\mathcal{D}^I_\eta$ appears in the proof of Lemma 9.3, while the domain $\mathcal{F}^{(0)}$ is explicitly given in Definition 4.6.

Proof of Theorem 9.1. Set $\xi_m = r_0(\mathcal{E}^{(m)}, \mathcal{F}^{(m)})$. Then $r_0((\delta_m)^{-1}\mathcal{E}^{(m)}, \mathcal{F}^{(m)}) = \delta_m \xi_m$. By Lemma 9.6, the resolution of $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$ is $\{((\delta_m)^{-1}\mathcal{E}^{(m)}, \mathcal{F}^{(m)}, \gamma_m)\}_{m \geq 0}$ and the results in Section 6, in particular Definition 6.4 and Theorem 6.11, yield

$$\mathcal{R}_{(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})} = \left\{ \left(\frac{\delta_m \xi_m}{\delta_{m-1} \xi_{m-1}}, \frac{\gamma_m}{\delta_{m-1} \xi_{m-1}} \right) \right\}_{m \geq 1} = \left\{ \left(r_m \frac{\xi_m}{\xi_{m-1}}, \frac{\rho_m}{\xi_{m-1}} \right) \right\}_{m \geq 1}. \tag{9.4}$$

Thus, for any $m \geq 1$,

$$\frac{5}{3} r_m \frac{\xi_m}{\xi_{m-1}} + \frac{\rho_m}{\xi_{m-1}} = 1. \tag{9.5}$$

Since $r_m = \frac{3}{5}(1 - \rho_m)$, (9.5) yields

$$(1 - \xi_m)(1 - \rho_m) = 1 - \xi_{m-1} \tag{9.6}$$

for any $m \geq 1$, and therefore

$$\xi_m = \frac{\xi_0 - 1}{(1 - \rho_1) \cdots (1 - \rho_m)} + 1 \tag{9.7}$$

for any $m \geq 1$. Now, it suffices to show that $\xi_m = 1$ for any $m \geq 0$.
Case 1: Assume that $\sum_{m=1}^{\infty} \rho_m = \infty$. Since $\xi_m > 0$, we have

$$1 - \xi_0 < (1 - \rho_1) \cdots (1 - \rho_m) \tag{9.8}$$

for any $m \geq 1$. The limit of the right-hand side of (9.8) as $m \to \infty$ is 0, hence $\xi_0 \geq 1$. On the other hand, it follows from (9.3) that $\xi_0 = r_0(\mathcal{E}^{(0)}, \mathcal{F}^{(0)}) \leq r_0(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}}) = 1$. Therefore, $\xi_0 = 1$ and (9.7) implies that $\xi_m = 1$ for any $m \geq 1$. Thus, $\mathcal{R}_{(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})} = \mathcal{R}$. Moreover, $r_0(\mathcal{E}^{(0)}, \mathcal{F}^{(0)}) = \xi_0 = 1 = r_0(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$ and Corollary 6.9 yields $(\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}}) = (\mathcal{E}^{(0)}, \mathcal{F}^{(0)})$.
Case 2: Assume that $\sum_{m=1}^{\infty} \rho_m < \infty$. By (9.2) we have that

$$\mathcal{E}_{\mathcal{R}_{(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})}}(u, v) = \xi_0 \mathcal{E}^{(0)}(u, v) = \frac{\xi_0}{R_*} \mathcal{E}^*(u, v) + \frac{\xi_0}{1 - R_*} \mathcal{D}^I_\eta(u, v)$$

for any $u, v \in \mathcal{F}^{(0)}$. Since $\mathcal{F}^* \subseteq \mathcal{F}^{(0)}$, Theorem 7.4 and (9.4) yield

$$\sum_{m=1}^{\infty} \frac{\rho_m}{\xi_{m-1}} < \infty$$

as well as

$$\frac{R_*}{\xi_0} = \prod_{m=1}^{\infty} \Big(1 - \frac{\rho_m}{\xi_{m-1}}\Big). \tag{9.9}$$

On the one hand, in view of (9.7), we have that $\{\xi_m\}_{m \geq 1}$ converges as $m \to \infty$. Set $\xi = \lim_{m \to \infty} \xi_m$. Now, (9.6) leads to

$$1 - \frac{\rho_m}{\xi_{m-1}} = \frac{\xi_m}{\xi_{m-1}}(1 - \rho_m),$$

hence by (9.9),

$$\frac{R_*}{\xi_0} = \frac{R_*}{\xi_0} \xi$$

and therefore $\xi = 1$. On the other hand, it follows from (9.7) that

$$\xi = \frac{\xi_0 - 1}{R_*} + 1.$$

This implies $\xi_0 = 1$ and thus $\mathcal{R}_{(\mathcal{E}^{(0)}, \mathcal{F}^{(0)})} = \mathcal{R}$, which shows $(\mathcal{E}^{(0)}, \mathcal{F}^{(0)}) = (\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$. □

The final step to prove Theorem 4.7 consists in showing that any (positive) linear combination of $\mathcal{E}^*$ and $\mathcal{D}^I_\eta$ can be realized as a completely symmetric resistance form on SSG.

Proof of Theorem 4.7. (1) If $(\mathcal{E}, \mathcal{F}) \in \mathcal{RF}_S$, then Theorem 6.11 implies that $(\mathcal{E}, \mathcal{F}) = (c\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}})$ for some $c > 0$ and $\mathcal{R} \in \mathcal{MP}^{\mathbb{N}}$. By Theorem 9.1, there exist $a \geq 0, b > 0$ and $\eta = \{\eta_m\}_{m \geq 1} \subseteq (0, \infty)$ such that η satisfies (4.4) and $\mathcal{E}(u, v) = a\mathcal{E}^*(u, v) + b\mathcal{D}^I_\eta(u, v)$ for any $u, v \in \mathcal{F}$ with $\mathcal{F}$ as in (4.5).

Conversely, let $\eta = \{\eta_m\}_{m \geq 1} \subseteq (0, 1)$ satisfy (4.4). Inductively we may construct a sequence $\{\sigma_m\}_{m \geq 1}$ such that

$$\begin{aligned}\eta_m = \Big(\frac{3}{5}\Big)^{m-1} &(1 - \sigma_1) \cdots (1 - \sigma_{m-1}) \sigma_m \\ &= \Big(\frac{3}{5}\Big)^{m-1} \big((1 - \sigma_1) \cdots (1 - \sigma_{m-1}) - (1 - \sigma_1) \cdots (1 - \sigma_m)\big)\end{aligned}$$

for any $m \geq 1$. In view of (4.4), it follows that $\prod_{m=1}^{\infty}(1-\sigma_m) = 0$, hence $\sum_{m=1}^{\infty} \sigma_m = \infty$. Defining $s_m = \frac{3}{5}(1-\sigma_m)$ for any $m \geq 1$, $\mathcal{R}_* = \{(s_m, \sigma_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$ and Theorem 9.1 yields

$$\mathcal{E}_{\mathcal{R}_*}(u, v) = \mathcal{D}_\eta^I(u, v)$$

for any $u, v \in \mathcal{F}_{\mathcal{R}_*} = \mathcal{F}_\eta$. Thus, for any $b > 0$, $(b\mathcal{D}_\eta^I, \mathcal{F}_\eta) = (b\mathcal{E}_{\mathcal{R}_*}, \mathcal{F}_{\mathcal{R}_*}) \in \mathcal{RF}_S$ and the case $a = 0$ of Theorem 4.7-(1) is proven.

In order to prove the case $a > 0$, choose $\rho_0 \in (0, 1)$ arbitrarily and define ρ_m for $m \geq 1$ by (8.6). Then, $\rho_m \in (0, 1)$ for any $m \geq 1$. Taking $r_m = \frac{3}{5}(1 - \rho_m)$, we have that $\mathcal{R} = \{(r_m, \rho_m)\}_{m\geq 1} \in \mathcal{MP}^{\mathbb{N}}$. Now, set $A_m = \rho_0 \prod_{i=1}^{m}(1-\sigma_i) + (1-\rho_0)$ and notice that (8.6) leads to

$$1 - \rho_m = \frac{A_m}{A_{m-1}},$$

hence

$$R_* = \prod_{m=1}^{\infty}(1 - \rho_m) = \lim_{m\to\infty} A_m = 1 - \rho_0 > 0.$$

Moreover, by (8.8),

$$r_1 \cdots r_{m-1}\rho_m = \left(\frac{3}{5}\right)^{m-1}(1 - \rho_1)\cdots(1 - \rho_{m-1})\rho_m = \left(\frac{3}{5}\right)^{m-1}\rho_0\sigma_m \prod_{i=1}^{m-1}(1 - \sigma_i) = \eta_m\rho_0$$

and Theorem 9.1 yields

$$\mathcal{E}_{\mathcal{R}}(u, v) = \frac{1}{1 - \rho_0}\mathcal{E}^*(u, v) + \frac{1}{\rho_0}\mathcal{D}_\eta^I(u, v)$$

for any $u, v \in \mathcal{F}_{\mathcal{R}} = \mathcal{F}_\eta^*$. Since $\rho_0 \in (0, 1)$ is arbitrary, for every pair $(a, b) \in (0, \infty) \times (0, \infty)$ in the statement of Theorem 4.7-(1), we find $(a\mathcal{E}^* + b\mathcal{D}_\eta^I, \mathcal{F}_\eta^*) = (c\mathcal{E}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}}) \in \mathcal{RF}_S$ by setting $\rho_0 = a/(a + b)$ and $c = ab/(a + b)$.
(2) Let $\eta = \{\eta_m\}_{m\geq 1}$ satisfy (4.4). Choose any $\rho_0 \in (0, 1)$ and construct $\mathcal{R}_*$ and $\mathcal{R}$ as in (1). Then, it follows that $\mathcal{L}(\mathcal{R}) = \mathcal{R}_*$. Note that $\mathcal{F}_\eta = \mathcal{F}_{\mathcal{R}_*}$ and $\mathcal{F}_\eta^* = \mathcal{F}_{\mathcal{R}}$, hence Theorem 8.3 yields $\mathcal{F}_\eta = \mathcal{F}_{\mathcal{R}_*} = \mathcal{F}_{\mathcal{R}}^I = \{u \mid u \in \mathcal{F}_\eta^*, \mathcal{E}_{\mathcal{R}}^\Sigma(u, u) = 0\}$. Since $\mathcal{E}_{\mathcal{R}}^\Sigma = \frac{1}{R_*}\mathcal{E}^*$, we finally obtain (2). $\square$

10 Basics on resistance forms

For convenience of the reader, we give in this last section a summary of definitions and basic facts from the theory of resistance forms used within the paper. A detailed and more extensive exposition of this theory can be found e.g. in [7, 8].

Definition 10.1. Let X be a set. A pair $(\mathcal{E}, \mathcal{F})$ is called a resistance form on X if it satisfies the following conditions (RF1) through (RF5):
(RF1) $\mathcal{F}$ is a linear subspace of $\ell(X) = \{u | u : X \to \mathbb{R}\}$ containing constants and $\mathcal{E}$ is a non-negative symmetric quadratic form on $\mathcal{F}$. $\mathcal{E}(u, u) = 0$ if and only if u is constant on X.
(RF2) Let $\sim$ be the equivalence relation on $\mathcal{F}$ defined by $u \sim v$ if and only if $u - v$ is constant on X. Then, $(\mathcal{F}/\sim, \mathcal{E})$ is a Hilbert space.

(RF3) If $x \neq y$, then there exists $u \in \mathcal{F}$ such that $u(x) \neq u(y)$.
(RF4) For any $p, q \in X$,

$$\sup\left\{\frac{|u(p)-u(q)|^2}{\mathcal{E}(u,u)} \mid u \in \mathcal{F}, \mathcal{E}(u,u) > 0\right\}$$

is finite. The above supremum is denoted by $R_{(\mathcal{E},\mathcal{F})}(p,q)$ and it is called the resistance metric on X associated with the resistance form $(\mathcal{E}, \mathcal{F})$.
(RF5) For any $u \in \mathcal{F}$, $\overline{u} \in \mathcal{F}$ and $\mathcal{E}(\bar{u}, \bar{u}) \leq \mathcal{E}(u,u)$, where $\overline{u}$ is defined by

$$\overline{u}(p) = \begin{cases} 1 & \text{if } u(p) \geq 1, \\ u(p) & \text{if } 0 < u(p) < 1, \\ 0 & \text{if } u(p) \leq 0. \end{cases}$$

Proposition 10.2. *If $(\mathcal{E}, \mathcal{F})$ is a resistance form on a set X, then the associated resistance metric $R_{(\mathcal{E},\mathcal{F})}(\cdot,\cdot)$ is a distance on X.*

If the set X is finite, any resistance form on X is a non-negative quadratic form on $\ell(X) \times \ell(X)$ that satisfies several conditions stated in the following lemma.

Lemma 10.3. *Let V be a finite set. Then $(\mathcal{E}, \ell(V))$ is a resistance form on V if and only if there exists $(C_{pq})_{p,q \in V}$ such that for any $p \neq q \in V$, $C_{pq} = C_{qp} \geq 0$ and there exist $m \geq 0$ and $(p_0, p_1, \ldots, p_m) \in V^{m+1}$ such that $p_0 = p, p_m = q$ and $C_{p_i p_{i+1}} > 0$ for any $i = 0, \ldots, m-1$ and*

$$\mathcal{E}(u,v) = \frac{1}{2} \sum_{p,q \in V} C_{pq}(u(p)-u(q))(v(p)-v(q))$$

for any $u \in \ell(V)$.

If the set X is infinite, in many cases a resistance form on X is constructed by means of a suitable sequence of resistance forms on finite sets that approximate X as Theorem 10.6 indicates.

Definition 10.4. Let V and U be finite sets satisfying $V \subseteq U$ and let $(\mathcal{E}_V, \ell(V))$ and $(\mathcal{E}_U, \ell(U))$ be resistance forms on V and U respectively. We write $(\mathcal{E}_V, \ell(V)) \leq (\mathcal{E}_U, \ell(U))$ if and only if

$$\mathcal{E}_V(u,u) = \min\{\mathcal{E}_U(v,v) \mid v \in \ell(U),\ v|_V = u\}$$

for any $u \in \ell(V)$. Let V_m be a finite set and let $(\mathcal{E}_m, \ell(V_m))$ be a resistance form on V_m for every $m \geq 0$. A sequence of resistance forms $\{(\mathcal{E}_m, \ell(V_m))\}_{m \geq 0}$ is called compatible if and only if $V_m \subseteq V_{m+1}$ and $(\mathcal{E}_m, \ell(V_m)) \leq (\mathcal{E}_{m+1}, \ell(V_{m+1}))$ for any $m \geq 0$.

Note that if $\{(\mathcal{E}_m, \ell(V_m))\}_{m \geq 0}$ is a compatible sequence, then, for any function $u\colon \cup_{m \geq 0} V_m \to \mathbb{R}$, the sequence $\mathcal{E}_m(u|_{V_m}, u|_{V_m})$ is monotonically non-decreasing. By this fact, the following definition makes sense.

Definition 10.5. Let V_m be a finite set and let $(\mathcal{E}_m, \ell(V_m))$ be a resistance form on V_m for every $m \geq 0$. If $\mathcal{S} = \{(\mathcal{E}_m, \ell(V_m))\}_{m \geq 0}$ is a compatible sequence, then we define

$$\mathcal{F}_{\mathcal{S}} = \{u \mid u \in V_*, \lim_{m \to \infty} \mathcal{E}_m(u|_{V_m}, u|_{V_m}) < \infty\},$$

where $V_* = \cup_{m \geq 0} V_m$, and for any $u, v \in \mathcal{F}_{\mathcal{S}}$,

$$\mathcal{E}_{\mathcal{S}}(u,v) = \lim_{m \to \infty} \mathcal{E}_m(u|_{V_m}, v|_{V_m}).$$

Theorem 10.6 (Theorem 3.13 of [8]). *Let V_m be a finite set and let $(\mathcal{E}_m, \ell(V_m))$ be a resistance form on V_m for every $m \geq 0$. If $\mathcal{S} = \{(\mathcal{E}_m, \ell(V_m))\}_{m\geq 0}$ is a compatible sequence, then $(\mathcal{E}_\mathcal{S}, \mathcal{F}_\mathcal{S})$ is a resistance form on V_*. Furthermore, let $R_\mathcal{S}$ be the associated resistance metric on V_* and let (X, R) be the completion of $(V_*, R_\mathcal{S})$. Then, there exists a unique resistance form $(\mathcal{E}, \mathcal{F})$ on X such that for any $u \in \mathcal{F}$, u is continuous on (X, R), $u|_{V_*} \in \mathcal{F}_\mathcal{S}$ and $\mathcal{E}(u, u) = \mathcal{E}_\mathcal{S}(u|_{V_*}, u|_{V_*})$. In particular, R is the resistance metric associated with $(\mathcal{E}, \mathcal{F})$.*

An important concept is the notion of trace of a resistance form. This corresponds, roughly speaking, to the restriction of a resistance form to a subset of the original domain.

Definition 10.7. Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a set X. For any $Y \subseteq X$, define

$$\mathcal{F}|_Y = \{u|_Y \ : \ u \in \mathcal{F}\}$$

and

$$\mathcal{F}_0(Y) = \{u \mid u \in \mathcal{F},\ u|_Y \equiv 0\}.$$

Proposition 10.8 (Lemma 8.2 and Theorem 8.4 of [8]). *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a set X and let $Y \subseteq X$ be non-empty. Then, for any $u_* \in \mathcal{F}|_Y$, there exists a unique $u \in \mathcal{F}$ such that $u|_Y = u_*$ and*

$$\mathcal{E}(u, u) = \min\{\mathcal{E}(v, v) \mid v \in \mathcal{F}, v|_Y = u_*\}.$$

Moreover, if we denote $u = h_Y(u_*)$, then the map $h_Y : \mathcal{F}|_Y \to \mathcal{F}$ is linear. If we define $\mathcal{E}|_Y(u, v) = \mathcal{E}(h_Y(u), h_Y(v))$ for any $u, v \in \mathcal{F}|_Y$, then $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ is a resistance form on Y and the associated resistance metric R_Y is the restriction onto $Y \times Y$ of the resistance metric associated with $(\mathcal{E}, \mathcal{F})$.

Definition 10.9. Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a set X and let $Y \subseteq X$ be non-empty. The map $h_Y : \mathcal{F}|_Y \to \mathcal{F}$ is called the Y-harmonic extension map associated with $(\mathcal{E}, \mathcal{F})$ and $h_Y(u_*)$ is called the Y-harmonic function with boundary value u_* associated with $(\mathcal{E}, \mathcal{F})$. We define $\mathcal{H}_{(\mathcal{E},\mathcal{F})}(Y) = h_Y(\mathcal{F}|_Y)$. The resistance form $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ on Y is called the trace of $(\mathcal{E}, \mathcal{F})$ on Y.

By [8, Lemma 8.5] and the discussion after it, the domain of a resistance form admits the orthogonal decomposition presented below.

Proposition 10.10. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a set X and let $Y \subseteq X$ be non-empty. Then,*

$$\mathcal{F} = \mathcal{H}_{(\mathcal{E},\mathcal{F})}(Y) \oplus \mathcal{F}(Y),$$

where $\oplus$ represents the direct sum. Moreover, for any $u \in \mathcal{F}$, the projection of u onto $\mathcal{H}_{(\mathcal{E},\mathcal{F})}(Y)$ associated with the above direct sum is given by $h_Y(u|_Y)$ and

$$\mathcal{E}(u, u) = \mathcal{E}(h_Y(u), h_Y(u)) + \mathcal{E}(u - h_Y(u), u - h_Y(u)).$$

Finally, Theorem 10.6 along with [8, Theorem 3.14] leads to the following result.

Theorem 10.11. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a set X and let R be the associated resistance metric on X. If $\{V_m\}_{m\geq 0}$ is an increasing sequence of finite subsets of X, i.e. $V_m \subseteq V_{m+1} \subseteq X$ for any $m \geq 0$, then $\mathcal{S} = \{(\mathcal{E}|_{V_m}, \ell(V_m))\}_{m\geq 0}$ is a compatible sequence of resistance forms. If A is the closure of V_* with respect to R, then for any $u \in \mathcal{F}|_A$, $u|_{V_*} \in \mathcal{F}$ and $\mathcal{E}|_A(u, u) = \mathcal{E}_\mathcal{S}(u|_{V_*}, u|_{V_*})$.*

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