Quasisymmetric modification of metrics on self-similar sets

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Abstract

Using the notions of scales and their gauge functions associated with self-similar sets, we give a necessary and sufficient condition for two metrics on a self-similar set being quasisymmetric to each other. As an application, we construct metrics on the Sierpinski carpet which is quasisymmetric with respect to the Euclidean metrics and obtain an upper estimate of the conformal dimension of the Sierpinski carpet.

1 Introduction

The main purpose of this paper is to give a characterization of quasisymmetry for self-similar sets in terms of scales and related notions introduced in [5]. As an application, we will construct a series of metrics on the Sierpinski carpet which are quasisymmetric to the restriction of the Euclidean metric and give an upper estimate of the quasiconformal dimension of the Sierpinski carpet.

Quasisymmetric maps have been introduced by Tukia and Väisälä in [8] as a generalization of quasiconformal mappings in the complex plane.

Definition 1.1 (Quasisymmetry)**.** (1) Let (X, d) and (X, ρ) be metric spaces. ρ is said to be quasisymmetric, or QS for short, with respect to d if and only if there exists a homeomorphism h from $[0, +\infty)$ to itself such that $h(0) = 0$ and, for any $t > 0$, $\rho(x, z) < h(t)\rho(x, y)$ whenever $d(x, z) < td(x, y)$. We write $\rho \underset{\text{QS}}{\sim} d$ if ρ is quasisymmetric with respect to d.
(2) Let (X, d) be a metric space. A homeomorphism $f : X \to X$ is called quasisymmetric if and only if $d \underset{\text{QS}}{\sim} d_f$, where $d_f(x, y)$ is defined by $d_f(x, y) = d(f(x), f(y))$.

The above definition immediately imply the following facts.

Proposition 1.2. *Let (X, d) and (X, ρ) be metric spaces.*
(1) *If $\rho \underset{\text{QS}}{\sim} d$, then the identity may of X is a homeomorphism from (X, d) to (X, ρ).*

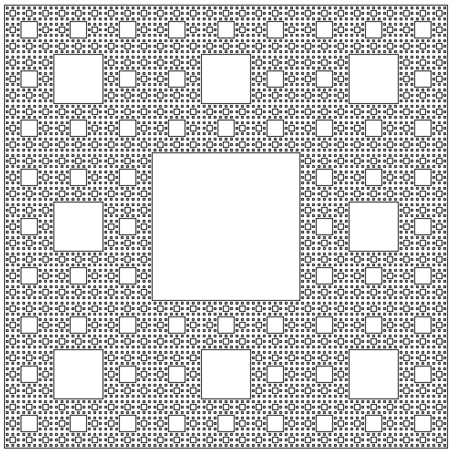


Figure 1: the Sierpinski carpet

(2) *The relation* $\underset{\text{QS}}{\sim}$ *is an equivalence relation among metrics on* X*. In particular,* $\rho \underset{\text{QS}}{\sim} d$ *if and only if* $d \underset{\text{QS}}{\sim} \rho$*.*

Associated with the notion of quasisymmetry, the quasiconformal dimension of a metric space has been introduced by Pansu in [7] as an invariant under quasisymmetric modification of a metric.

Definition 1.3 (Quasiconformal dimension)**.** Let (X, d) be a metric space. We define the conformal dimension of (X, d), $\dim_{\mathcal{C}}(X, d)$, by

$$\dim_{\mathcal{C}}(X, d) = \inf\{\dim_H(X, \rho) | \rho \text{ is a metric on } X \text{ and } d \underset{\text{QS}}{\sim} \rho\},$$

where $\dim_H(X, \rho)$ is the Hausdorff dimension of (X, ρ).

Quasisymmetric maps on self-similar sets have been paid much attentions in recent years as well as their conformal dimensions. For example, Bonk and Merenkov have shown that any quasisymmetric homeomorphism from the Sierpinski carpet to itself is a composition of rotations and reflections in [1]. About the conformal dimensions, Tyson and Wu have proven that the conformal dimension of the Sierpinski gasket is one in [9]. For the Sierpinski carpet, it is known that

$$1 + \frac{\log 3}{\log 2} \leq \dim_{\mathcal{C}}(\mathrm{SC}, d_E) < \dim_H(\mathrm{SC}, d_E) = \frac{\log 8}{\log 3}, \tag{1.1}$$

where SC is the Sierpinski carpet and d_E is the restriction of the Euclidean metric. The strict inequality between the Hausdorff and the quasiconformal dimensions in (1.1) has shown by Keith and Laakso [2]. See [6] for details.

The first problem we are going to study is to obtain a verifiable characterization of quasisymmetric metrics. It will turn out that scales and related notions introduced in [5] are useful in dealing with such a problem. Let K be a connected self-similar set associated with the family of contractions $\{F_1, \ldots, F_N\}$, i. e. $K = F_1(K) \cup \ldots \cup F_N(K)$. Define $F_{w_1 \ldots w_m} = F_{w_1} \circ \ldots \circ F_{w_m}$ and $K_{w_1 \ldots w_m} = F_{w_1 \ldots w_m}(K)$ for any $w_1, \ldots, w_m \in \{1, \ldots, N\}$. The notion of scales has been introduced in order to study how to find a metric under which the contraction mappings $\{F_1, \ldots, F_N\}$ have prescribed values of contraction ratios. A scale essentially gives "diameters" of $K_{w_1 \ldots w_m}$'s and induces a family of assumed "balls" $U_s(x)$ around $x \in K$ with radius $s > 0$. See Section 2 for precise definitions. In the language of scales, we are going to present an equivalent condition in Theorem 3.4 for metrics being quasisymmetric to each other which is easy to verify for concrete examples, in particular, in the case of "self-similar" metrics.

As an application, we will present a systematic way of constructing a self-similar metric on the Sierpinski carpet which is quasisymmetric to d_E and Ahlfors regular. The main idea is to find an "invisible" set introduced in Section 4. Roughly speaking, an invisible set is a collection of places where the shortest paths between two separated boundary points will not visit. (We define the "boundary" of the Sierpinski carpet by the union of four line segments, namely, the most upper, lower, right and left line segments of the square which is the convex hull of the Sierpinski carpet.) Putting an arbitrary weight on an invisible set, we will obtain a self-similar metric having the desired properties mentioned above with an explicit formula for its Hausdorff dimension in Theorem 5.3. Constructing series of invisible sets and taking advantage of the associated metrics, we will show that

$$\dim_{\mathcal{C}}(\mathrm{SC}, d_E) \leq \frac{\log\left(\frac{9+\sqrt{41}}{2}\right)}{\log 3} = 1.858183... < \frac{\log 8}{\log 3} = 1.892789....$$

in Section 6. Note that the conformal dimension in the above inequality can be replaced by the Ahlfors regular conformal dimension since our metrics are Ahlfors regular. See [6] for the definition of the Ahlfors regular conformal dimension.

The following is a convention in notations in this paper.
Let f and g be functions with variables $x_1, \ldots, x_n$. We use "$f \asymp g$ for any $(x_1, \ldots, x_n) \in A$" if and only if there exist positive constants c_1 and c_2 such that

$$c_1 f(x_1, \ldots, x_n) \leq g(x_1, \ldots, x_n) \leq c_2 f(x_1, \ldots, x_n)$$

for any $(x_1, \ldots, x_n) \in A$.

2 Basic Notions

This section is devoted to introducing fundamental notions and results regarding scales and self-similar sets and scales.

The following is the standard definitions on (finite and infinite) sequences of finite symbols.

Definition 2.1. Let S be a finite set. For $m \geq 0$, define $W_m(S) = S^m = \{w|w = w_1 \dots w_m, w_i \in S\}$, where $W_0(S) = \{\emptyset\}$. Define $W_* = \cup_{m\geq 0} W_m$. Also $\Sigma(S) = S^{\mathbb{N}} = \{\omega|\omega = \omega_1\omega_2 \dots, \omega_i \in S\}$. For $w = w_1 \dots w_m \in W_*(S)$, the length $|w|$ of w is defined by $|w| = m$. For $w = w_1 \dots w_m$ and $v = v_1 \dots v_n \in W_*(S)$, we define $w \cdot v$ (or wv for short) by $w \cdot v = w_1 \dots w_m v_1 \dots v_n$. For a subseteq $A, B \in W_*(S)$, $A \cdot B$ (or AB for short) is defined by $A \cdot B = \{wv|w \in A, v \in B\}$.

Remark. The notion of "gauge function" given in the above definition is not related to the notion of "conformal gauge" which is commonly used in literatures concerning the conformal dimension, for example, [6].

With the product topology, $\Sigma(S)$ is compact, perfect and totally disconnected. In other words, $\Sigma(S)$ is a Cantor set. A scale is defined by a gauge function which assign a "diameter" to every $w \in W_*(S)$.

Definition 2.2 (Scale)**.** Let S be a finite set.
(1) A function $g : W_*(S) \to (0, 1]$ is called a gauge function if and only if $g(\emptyset) = 1, g(w_1 \dots w_m) \leq g(w_1 \dots w_{m-1})$ and $\max_{w \in W_m(S)} g(w) \to 0$ as $m \to 0$. A gauge function g is said to be elliptic if and only if there exists $c \in (0, 1)$ and n such that $g_{wi} \geq cg(w)$ for any $i \in S$ and any $w \in W_*(S)$ and $g_{wv} \leq cg(w)$ for any $w \in W_*(S)$ and $v \in W_n$.
(2) Let g be a gauge function. Define

$$\Lambda_s^g = \{w = w_1 \dots w_m | g(w_1 \dots w_{m-1}) \geq s > g(w_1 \dots w_m)\}$$

We call $\mathcal{S}^g = \{\Lambda_s^g\}_{s \in (0,1]}$ a scale on Σ associated with the gauge function g.

If no confusion may occur, we omit S in $W_m(S), W_*(S)$ and $\Sigma(S)$ and simply write W_m, W_* and Σ respectively.

The notion of self-similar structure describes topological feature of self-similar sets.

Definition 2.3. $(K, S, \{F_i\}_{i \in S})$ is called a self-similar structure if the following four conditions (S1), (S2), (S3) and (S4) are satisfied:
(S1) K is a compact metrizable set.
(S2) S is a finite set.
(S3) $F_s : K \to K$ is continuous for any $s \in S$.
(S4) There exists a continuous surjection $\pi : \Sigma(S) \to K$ such that $F_s \circ \pi = \pi \circ \sigma_s$ for any $s \in K$, where $\sigma_s : \Sigma(S) \to \Sigma(S)$ is defined by $\sigma_s(\omega_1\omega_2 \dots) = s\omega_1\omega_2 \dots$.

Hereafter in this paper, $(K, S, \{F_s\}_{s \in S})$ is always a self-similar structure.

Notation. Define $F_{w_1 \dots w_m} = F_{w_1} \circ \dots \circ F_{w_m}$ and $K_w = F_w(K)$. Moreover, define $K(A) = \cup_{w \in A} K_w$ for a subset $A \subseteq W_*$.

A scale $\mathcal{S}$ on $\Sigma(S)$ induces a family of "balls" $U^{(n)}(x, s)$ around $x \in X$ with "radius" s. One of the main concerns is the existence of a metric under which those "balls" are really balls, in other words, the existence of adapted metric according to the following definition.

Definition 2.4. Let $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ be a scale on Σ associated with a gauge function g.
(1) For $x \in K$, define $\Lambda_{s,x}^{(n)}$ and $U^{(n)}(x,s)$ inductively by

$$\begin{aligned}\Lambda_{s,x}^{(0)} &= \{w | w \in \Lambda_s, x \in K_w\}\\ U^{(n)}(x,s) &= K(\Lambda_{s,x}^{(n)})\\ \Lambda_{s,x}^{(n)} &= \{w | w \in \Lambda_s, K_w \cap U^{(n-1)}(x,s) \neq \emptyset\}\end{aligned}$$

(2) A metric d on K is said to be adapted to the scale $\mathcal{S}$ if and only if there exist $\alpha, \beta > 0$ and $n \geq 1$ such that

$$B_d(x, \alpha s) \subseteq U^{(n)}(x,s) \subseteq B_d(x, \beta s)$$

for any $x \in K$ and any s.

The notion of gentleness between scales is introduced in [5] as a part of the equivalence condition for a measure being volume doubling with respect to a scale. Roughly, if two scales are gentle with respect to each other, then the transition to one scale to the other is smooth.

Definition 2.5. Let $\mathcal{S}^g$ and $\mathcal{S}^l$ be scales on Σ associated with gauge functions g and l respectively. We say $\mathcal{S}^l$ is gentle with respect to $\mathcal{S}^g$ if and only if there exists $c > 0$ such that $l(w) \leq cl(v)$ whenever $w, v \in \Lambda_s$ for some $s > 0$ and $K_w \cap K_v \neq \emptyset$. We write $\mathcal{S}^g \underset{\mathrm{GE}}{\sim} \mathcal{S}^l$ if $\mathcal{S}^l$ is gentle with respect to $\mathcal{S}^g$.

Proposition 2.6. *Among elliptic scales, i.e. scales whose gauge functions are elliptic,* $\underset{\mathrm{GE}}{\sim}$ *is an equivalent relation. In particular, if g and l are elliptic, then* $\mathcal{S}^g \underset{\mathrm{GE}}{\sim} \mathcal{S}^l$ *implies* $\mathcal{S}^l \underset{\mathrm{GE}}{\sim} \mathcal{S}^g$.

There exists a natural "pseudo"metric associated with a scale which is defined by the infimum of the "length" of paths between two points.

Definition 2.7. (1) A sequence $(w(1), \dots, w(n))$ is called a path in K if and only if $w(1), \dots, w(n) \in W_*$, $K_{w(i)} \cap K_{w(i+1)} \neq \emptyset$ for any $i = 1, \dots, n-1$. The collection of all the paths is denoted by $\mathcal{CH}$. For $U, V \subseteq K$, a path $(w(1), \dots, w(n))$ is called a path between U and V if and only if $K_{w(1)} \cap U \neq \emptyset$ and $K_{w(n)} \cap V \neq \emptyset$. We use $\mathcal{CH}(U,V)$ to denote the collection of paths between U and V. For two paths $\mathbf{p}_1 = (w(1), \dots, w(n))$ and $\mathbf{p}_2 = (v(1), \dots, v(m))$, if $K_{w(n)} \cap K_{v(1)} \neq \emptyset$, we define $\mathbf{p}_1 \vee \mathbf{p}_2 \in \mathcal{CH}$ by $\mathbf{p}_1 \vee \mathbf{p}_2 = (w(1), \dots, w(n), v(1), \dots, v(m))$.
(2) Let $\mathcal{S}$ be a scale on Σ associated with a gauge function g. For any $x, y \in K$, we define

$$D_{\mathcal{S}}(x,y) = \inf\{\sum_{i=1}^{n} g(w(i)) | (w(1), \dots, w(n)) \in \mathcal{CH}(x,y)\}.$$

Remark. We identify a point $x \in X$ and a set $\{x\}$ if no confusion may occur.

Remark. We often use D_g instead of $D_{\mathcal{S}}$ if $\mathcal{S}$ is the scale associated with a gauge function g.

Proposition 2.8. *$D_{\mathcal{S}}$ is a pseudometric, i.e. $D_{\mathcal{S}}(x,y) = D_{\mathcal{S}}(y,x)$, $D_{\mathcal{S}}(x,y) \geq 0$, $D_{\mathcal{S}}(x,x) = 0$ and $D_{\mathcal{S}}(x,y) \leq D_{\mathcal{S}}(x,z) + D_{\mathcal{S}}(z,y)$.*

By [5, Lemma 2.3.10], we have the following theorem, which says that a metric adapted to a scale $\mathcal{S}$, if such a metric exists at all, is essentially $D_{\mathcal{S}}$.

Theorem 2.9. *Let $\mathcal{S}$ be a scale. There exists a metric d on K such that d is adapted to $\mathcal{S}$ if and only if $D_{\mathcal{S}}$ is a metric on K which is adapted to $\mathcal{S}$.*

3 Quasisymmetric metrics and scales

In this section, we give an equivalent condition for two metrics on a self-similar set being quasisymmetric in terms of scales and related notions introduced in Section BNS.

Let $(K, S, \{F_i\}_{i\in S})$ be a self-similar structure. Assume that $K \neq \overline{V_0}$. Hereafter in this section, every metric on K is assumed to satisfy the following two properties:
(1) It produces the same topology as the original topology of K.
(2) The diameter of K under it equals one.

The next lemma can be verified immediately by the definitions in the previous section.

Lemma 3.1. *Let $\mathcal{S}_1 = \{\Lambda^1_s\}$ and $\mathcal{S}_2 = \{\Lambda^2_s\}$ be scales. If $\mathcal{S}_1 \underset{\text{GE}}{\sim} \mathcal{S}_2$, then for any $n \geq 1$, there exists $c_n \in (0,1)$ such that*

$$U_1^{(n)}(x, c_n t) \subseteq U_2^{(n)}(x, s) \subseteq U_1^{(n)}(x, t/c_n)$$

for any $x \in K$, any (s,t) with $w \in \Lambda^1_{t,x} \cap \Lambda^2_{s,y}$.

First we define a scale associated with a metric.

Definition 3.2. Let d be a metric on K with $\mathrm{diam}(K,d) = 1$. Define $\mathcal{S}^d = \{\Lambda^d_s\}$ be the scale with the gauge function $d_w = \mathrm{diam}(K_w, d)$.

Lemma 3.3. *Let $\mathcal{S} = \{\Lambda_s\}$ be an elliptic scale and let d be a metric on K which is adapted to $\mathcal{S}$. Let $l(w)$ be the gauge function of $\mathcal{S}$. Then*
(1) *$d_w \asymp l(w)$ for any $w \in W_*$.*
(2) *The pseudometric $D_{\mathcal{S}}$ associated with $\mathcal{S}$ is a metric and $\mathcal{D}_{\mathcal{S}}(x,y) \asymp d(x,y)$ for any $x, y \in K$.*
(3) *$\mathcal{S}^d$ is elliptic and d is adapted to $\mathcal{S}^d$.*

Proof. Write $U^{(n)}(x,r) = U^{(n)}_{\mathcal{S}}(x,r)$. Since d is adapted to $\mathcal{S}$, we have

$$U^{(n)}(x, \beta s) \subseteq B_d(x,s) \subseteq U^{(n)}(x, \alpha r) \qquad (3.2)$$

(1) For $w \in W_*$, $U^{(n)}(x, l(w)) \subseteq B_d(x, \alpha l(w))$. Hence $d_w \leq \alpha l(w)$. Now by [5, Lemma 1.3.12], there exists $y \in K_w$ and $\gamma \in (0,1)$ such that $U^{(n)}(y, \gamma l(w)) \subseteq K_w$. Hence $B_d(x, \beta\gamma l(w)) \subseteq K_w$. Since K is connected, we have $\beta\gamma l(w) \leq d_w$.
(2) This is immediate from Theorem 2.9.
(3) These claims are immediate from (1) and Lemma 3.1. □

Now we present the main theorem of this paper.

Theorem 3.4. *Let d be a metric on K and let $\mathcal{S} = \{\Lambda_s\}$ be an elliptic scale. Assume that d is adapted to $\mathcal{S}$. Let ρ be a metric on K. Then $d \underset{\mathrm{QS}}{\sim} \rho$ if and only if $\mathcal{S}^\rho$ is elliptic, $\mathcal{S} \underset{\mathrm{GE}}{\sim} \mathcal{S}^\rho$ and ρ is adapted to $\mathcal{S}^\rho$.*

The rest of this section is devoted to the proof of Theorem 3.4.

Proof. First we show $\Rightarrow$. Assume $d \underset{\mathrm{QS}}{\sim} \rho$. By Lemma 3.3, we may regard the gauge function of $\mathcal{S}$ is d_w and hence $\mathcal{S} = \mathcal{S}^d$.
By the results in [3, Part 2], $d \underset{\mathrm{QS}}{\sim} \rho$ is equivalent to the facts that there exists $\delta \in (0,1)$ such that

$$\begin{aligned} B_d(x,r) &\supseteq B_\rho(x, \delta\overline{\rho}_d(x,s)) \\ B_\rho(x,r) &\supseteq B_d(x, \delta\overline{d}_\rho(x,r)) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \overline{\rho}_d(x, r/2) &\geq \delta\overline{\rho}_d(x,r) \\ \overline{d}_\rho(x, r/2) &\geq \delta\overline{d}_\rho(x,r), \end{aligned} \tag{3.4}$$

where $\overline{\rho}_d(x,r) = \sup_{y \in B_d(x,r)} \rho(x,y)$ and $\overline{d}_d(x,r) = \sup_{y \in B_\rho(x,r)} d(x,y)$.
First we show the following claim.
Claim 1 Let $w \in \Lambda_s^d$. Then there exists $z \in K_w$ such that $\rho_w \geq c\overline{\rho}_d(z,s)$, where c is a constant which is independent of w and s.
Proof of Claim 1: By [5, Lemma 1.3.12] and (3.3), we may find $z \in K_w$ such that

$$K_w \supseteq U_d^{(n)}(z, \gamma s) \supseteq B_d(z, \gamma s/\alpha) \supseteq B_\rho(z, \delta\overline{\rho}_d(z, \gamma s/\alpha))$$

Hence by (3.4)

$$\rho_w \geq \delta\overline{\rho}_d(z, \gamma s/\alpha) \geq c\overline{\rho}_d(z,s).$$

□

Step 1: $\mathcal{S}^\rho$ is elliptic.
Proof of "$\rho_{wi} \geq c\rho_w$ for any $w \in W_*$ and any $i \in S$":
By Claim 1, it follows that

$$\rho_{wi} \geq c'\overline{\rho}_d(z, d_{wi}) \geq c'\overline{\rho}_d(z, 2d_{wi}). \tag{3.5}$$

for some $z \in K_{wi}$. On the other hand,

$$K_w \subseteq B_d(z, 2d_w) \subseteq B_\rho(z, \rho_d(x, 2d_w)).$$

Hence

$$\rho_w \leq \overline{\rho}_d(z, 2d_w).$$

This with (3.5) suffices.
Proof of "there exists $c \in (0,1)$ and m such that $\rho_{wv} \leq c\rho_w$ for any $w \in W_*$ and any $v \in W_m$.
Since $K_{wv} \subseteq B_\rho(x, \overline{\rho}_d(x, 2d_{wv}))$, we have

$$\rho_{wv} \leq \overline{\rho}_d(x, 2d_{wv}) \leq \delta\overline{\rho}_d(x, d_{wv}), \tag{3.6}$$

where $x \in K_{wv}$. On the other hand, by [5, Lemma 1.3.12], there exists $z \in K_w$ such that

$$K_w \supseteq U_d^{(n)}(x, \gamma d_w) \supseteq B_\rho(x, \delta\overline{\rho}_d(x, \gamma d_w)).$$

Hence

$$\rho_w \geq \delta\overline{\rho}_d(x, \gamma d_w) \geq \delta'\overline{\rho}_d(x, d_w) \tag{3.7}$$

Now, since $\mathcal{S}^d$ is elliptic, there exists $a \in (0,1)$ such that

$$d_{wv} \leq ca^{|v|}d_w$$

for any w and v. Hence by (3.6) and (3.7), the uniform decay of ρ with respect to d, (See [3, Proposition 10.7]),

$$\rho_{wv} \leq \delta\overline{\rho}_d(x, d_{wv}) \leq \delta\overline{\rho}_d(x, ca^{|v|}d_w) \leq cb^{|v|}\overline{\rho}_d(x, d_w) \leq c'b^{|v|}\rho_w,$$

where $b \in (0,1)$. Hence choosing sufficiently large $m = |v|$, we obtain the desired inequality.
Thus we have shown that $\mathcal{S}^\rho$ is elliptic.
Step 2: $\mathcal{S} \underset{\text{GE}}{\sim} \mathcal{S}^\rho$
Let $w, v \in \Lambda_s^d$ with $K_w \cap K_v \neq \emptyset$. Choose $x \in K_w$ and $y \in K_v$. Then $d(x,y) \leq 2s$ and hence $B_d(x, 3s) \supseteq B_d(y, s)$. This implies $\overline{\rho}_d(x, 3s) \geq \overline{\rho}_d(y, s)$. By (3.4),

$$\overline{\rho}_d(x, s) \asymp \overline{\rho}_d(y, s).$$

By Claim 1, choosing $y \in K_v$ properly, we see that $\rho_v \geq c\overline{\rho}_d(y, s)$. Since $\overline{\rho}_d(x, 2s) \geq \rho_w$, (3.4) shows that $\mathcal{S}^d \underset{\text{GE}}{\sim} \mathcal{S}$.
Step 3: ρ is adapted to $\mathcal{S}^\rho$.
Let $x \in K$ and let $w \in \Lambda_{r,x}^d \cap \Lambda_{s,x}^\rho$. Then by Lemma 3.1, (3.3) and (3.4),

$$\begin{aligned}U_\rho^{(n)}(x, cs) \supseteq U_d^{(n)}(x, r) \supseteq B_d(x, r/\alpha) \supseteq B_\rho(x, \delta\overline{\rho}_d(x, r/\alpha))&\\ \supseteq B_\rho(x, \delta'\overline{\rho}_d(x, 2r)) \supseteq B_\rho(x, \delta'\rho_w) \supseteq B_\rho(x, \delta''s)&.\end{aligned}$$

On the other hand, let $x \in K$ and let $w \in \Lambda_s^\rho \cap \Lambda_t^d$. Then

$$B_\rho(x, s) \supseteq B_d(x, \delta\overline{d}_\rho(x, s)) \supseteq U_d^{(n)}(x, \beta\delta\overline{d}_\rho(x, s)) \supseteq U_\rho^{(n)}(x, c'r), \tag{3.8}$$

where $wv \in \Lambda_{\beta\delta\overline{d}_\rho(x,s),x}^d \cap \Lambda_{r,x}^\rho$. Since $B_\rho(x, 2s) \supseteq K_w$, we see that $\overline{d}_\rho(x, 2s) \geq d_w$. Hence $\overline{d}_\rho(x, s) \geq c_1 d_w$. Consequently, $d_{wv} \geq c_2 d_w$, where c_2 is independent of w and v. This implies that $|v|$ is uniformly bounded. Since $\mathcal{S}^\rho$ is elliptic, $\rho_{wv} \geq c_3\rho_w$. This implies$U_\rho^{(n)}(x, c'r) \supseteq U_\rho^{(n)}(x, c_4 s)$. By (3.8), it follows that $B_\rho(x, s) \supseteq U_\rho^{(n)}(x, c_5 s)$. Thus we have shown that ρ is adapted to $\mathcal{S}^\rho$.

This concludes the proof of $\Rightarrow$. $\square$

To show the converse direction of Theorem 3.4, we need the following lemma.

Lemma 3.5. *Assume that d is adapted to $\mathcal{S}^d$. Then, for any n and k, there exists $\lambda \in (0, 1)$ such that*

$$U_d^{(n)}(x, r) \supseteq U_d^{(n+k)}(x, \lambda r)$$

for any $x \in K$ and any r.

Proof. Since d is adapted to $\mathcal{S}^d$, there exists $c > 0$ such that $U_d^{(n)}(x, r) \supseteq B_d(x, cr)$. Then $B_d(x, cr) \supseteq U_d^{(n+k)}(x, cr/(n + k + 2))$. $\square$

Proof of $\Leftarrow$ of Theorem 3.4. Since d and ρ are adapted to $\mathcal{S}^d$ and $\mathcal{S}^\rho$ respectively,

$$U_d^{(n)}(x, \beta_1 r) \subseteq B_d(x, r) \subseteq U_d^{(n)}(x, \alpha_1 r)$$
$$U_\rho^{(m)}(x, \beta_2 r) \subseteq B_\rho(x, r) \subseteq U_\rho^{(m)}(x, \alpha_2 r).$$

First we show (3.3). By Lemma 3.1,

$$B_d(x, r) \subseteq U_d^{(n)}(x, \alpha_1 r) \subseteq U_\rho^{(n)}(x, c\rho_w), \tag{3.9}$$

where $w \in \Lambda^d_{\alpha_1 r, x}$. Using Lemma 3.5 if necessary, we obtain

$$B_d(x, r) \subseteq U_\rho^{(m)}(x, c_1\rho_w) \subseteq B_\rho(x, c_2\rho_w).$$

Hence $\overline{\rho}_d(x, r) \leq c_2\rho_w$. Now by Lemma 3.1,

$$B_d(x, r) \supseteq U_d^{(n)}(x, \beta_1 r) \supseteq U_\rho^{(n)}(x, c'\rho_{wv}), \tag{3.10}$$

where $wv \in \Lambda^d_{\beta_1 r, x}$. By making use of Lemma 3.5 if necessary, we have

$$B_d(x, r) \supseteq U_\rho^{(m)}(x, c''\rho_{wv}) \supseteq B_\rho(x, c''\beta_2\rho_{wv}).$$

Since $\mathcal{S}^d$ is elliptic, the fact that $w \in \Lambda^d_{\alpha_1 r, x}$ and $wv \in \Lambda^d_{\beta_1 r, x}$ implies that $|v|$ is uniformly bounded with respect to x and r. Since $\mathcal{S}^\rho$ is also elliptic, we see that $\rho_w v \geq c_3\rho_w \geq c_4\overline{\rho}_d(x, r)$. Hence (3.3) holds. (By exchanging ρ and d, we also obtain the other one.)

Next we show (3.4). By (3.9),

$$\overline{\rho}_d(x, r) \leq c(n + 1)\rho_w,$$

where $w \in \Lambda^d_{\alpha_1 r}$. Replacing r by λr for $\lambda \in (0, 1)$ in (3.10), we have

$$B_d(x, \lambda r) \supseteq U^{(n)}(x, c'\rho_{wv}),$$

where $wv \in \Lambda^d_{\lambda\beta_1 r,x}$. This implies $\overline{\rho}_d(x, \lambda r) \geq c'\rho_{wv}$. The same arguments as above show that $|v|$ is uniformly bounded and $\rho_{wv} \geq c\rho_w$. Combining all these, we obtain

$$\overline{\rho}_d(x, \lambda r) \geq c'\rho_{wv} \geq c''\rho_w \geq c'''\overline{\rho}_d(x, r).$$

Again the other one is obtained by exchanging d and ρ. Thus we have obtained (3.4). $\square$

4 Sierpinski carpet and its invisible sets

In this and the following sections, we are going to apply Theorem 3.4 to the Sierpinski carpet. First we give the definition of the Sierpinski carpet.

Definition 4.1. Let $S = \{\swarrow, \downarrow, \searrow, \rightarrow, \nearrow, \uparrow, \nwarrow, \leftarrow\}$. Define $p_{\swarrow} = -1 - \sqrt{-1}$, $p_{\downarrow} = -\sqrt{-1}$, $p_{\searrow} = 1-\sqrt{-1}$, $p_{\rightarrow} = 1$, $p_{\nearrow} = 1+\sqrt{-1}$, $p_{\uparrow} = \sqrt{-1}$, $p_{\nwarrow} = -1+\sqrt{-1}$ and $p_{\leftarrow} = -1$. Moreover, define $F_s : \mathbb{C} \to \mathbb{C}$ for $s \in S$ by

$$F_s(z) = \frac{(z - p_s)}{3} + p_s.$$

The Sierpinski carpet K is the unique non-empty compact set which satisfies

$$K = \bigcup_{s \in S} F_s(K).$$

Let d_E be the restriction of the Euclidean metric on the Sierpinski carpet K.

We consider d_E as the standard metric on K and are going to construct metrics which is quasisymmetric with respect to d_E. Obviously, the scale $\mathcal{S}_{d_E}$ associated with d_E is elliptic and d_E is adapted to the scale $\mathcal{S}_{d_E}$. In fact, the gauge function associated with d_E is given by $3^{-|w|}$ for any $w \in W_*$.

Next we introduce notions and notations regarding the boundary of the Sierpinski carpet.

Definition 4.2. (1) Define $L = K \cap \{z|\mathrm{Re}\, z = -1\}$, $R = K \cap \{z|\mathrm{Re}\, z = 1\}$, $T = K \cap \{z|\mathrm{Im}\, z = 1\}$ and $B = K \cap \{z|\mathrm{Im}\, z = -1\}$. Let $H_w = F_w(H)$ for any $w \in W_*$ and any $H \in \{L, R, T, B\}$. Moreover define $\partial_m = \{L_w, R_w, T_w, B_w | w \in W_m\}$.
(2) Define $L^m = \{\swarrow, \leftarrow, \nwarrow\}^m, R^m = \{\searrow, \rightarrow, \nearrow\}^m, T^m = \{\nwarrow, \uparrow, \nearrow\}^m$, $B^m = \{\swarrow, \downarrow, \searrow\}^m$ and $\delta_m = L^m \cup R^m \cup T^m \cup B^m$.

Remark. Recall that $K(A) = \cup_{w \in A} K_w$ for a subset $A \subseteq W_*$. The map $A \to K(A)$ can be regarded as a map from the subsets of W_* to the subsets of K. In the case of the Sierpinski carpet, this map is injective, i.e. if $A \neq B$, then $K(A) \neq K(B)$. Therefore, if no confusion may occur, we identify $A \subseteq W_*$ with $K(A) \subseteq K$.

Note that $D_{d_E}(x, y) \geq 1$ for any $(x, y) \in (L \times R) \cup (T \times B)$. This fact may remain true even if you put 0 as weights (length) of some pieces of w's. Such a collection of w's is called an invisible set, whose precise definition is given below.

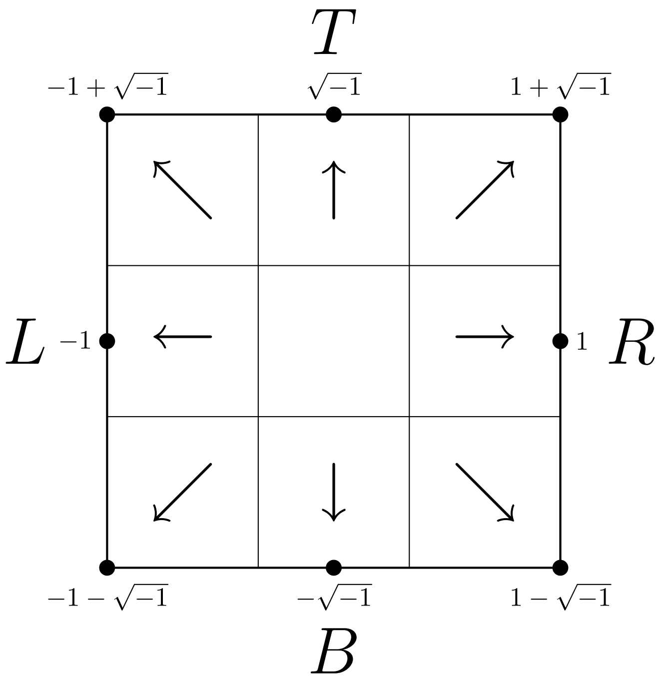


Figure 2: Generation of the Sierpinski Carpet

Definition 4.3. (1) Let

$$\mathcal{CH}_m = \{(w(1), \ldots, w(n)) | (w(1), \ldots, w(n)) \in \mathcal{CH}, w(i) \in W_m\}$$

and let $\mathcal{CH}_m(U, V) = \mathcal{CH}(U, V) \cap \mathcal{CH}_m$ for $U, V \subseteq K$.
(2) Let $A \subseteq W_m$. For $\mathbf{p} = (w(1), \ldots, w(n)) \in \mathcal{CH}_m$, define

$$\ell_A(\mathbf{p}) = \frac{\#\{i | i = 1, \ldots, n, w(i) \notin A\}}{3^m}$$

(3) Let $A \subseteq W_m$. A is said to be an invisible set if and only if

$$\inf_{\mathbf{p} \in \mathcal{CH}_m(L,R) \cup \mathcal{CH}_m(T,B)} \ell_A(\mathbf{p}) \geq 1$$

(4) Let $A \subseteq W_m$. A is said to be $+$-invariant if and only if $K(A)$ is symmetric with respect to both the real and imaginary axes.

Since L^m, R^m, T^m and B_m are the shortest paths, we have the following proposition.

Proposition 4.4. *Let $A \subseteq W_m$. If A is invisible, then $A \cap \delta_m = \emptyset$.*

The next theorem is one of the fundamental property of an invisible set. It will play a key role in constructing a metric associated with an invisible set in the next section.

Theorem 4.5. *Let $A \subseteq W_m$ be an invisible set and let $X \subseteq W_n$ be an invisible and $+$-invariant set. Then $AW_n \cup W_m X$ is an invisible set.*

The rest of this section is devoted to the proof of Theorem 4.5.

Definition 4.6. (1) Let $A \subseteq W_m$. Define $\partial_m A = \{F | F \in \partial_m, F \subseteq K(A) \cap \overline{K \backslash K(A)}\}$.
(2) Define $f_{m,\rightarrow}(z) = z + 3^{-m}$, $f_{m,\leftarrow}(z) = z - 3^{-m}$, $f_{m,\uparrow}(z) = z + 3^{-m}\sqrt{-1}$ and $f_{m,\downarrow}(z) = z - 3^{-m}\sqrt{-1}$. Moreover, let $f_{m,\swarrow} = f_{m,\downarrow} \circ f_{m,\leftarrow}$, $f_{m,\searrow} = f_{m,\downarrow} \circ f_{m,\rightarrow}$, $f_{m,\nwarrow} = f_{m,\uparrow} \circ f_{m,\leftarrow}$ and $f_{m,\nearrow} = f_{m,\uparrow} \circ f_{m,\rightarrow}$.
(3) Let $w \in W_m$. For $s \in S$, if there exists $w' \in W_m$ such that $f_{m,s}(K_w) = K_{w'}$, then define $(w)_s = w'$. Otherwise define $(w)_s = \%$, where % is used as the symbol which represents non-existence.

Lemma 4.7. *Let $F \in \partial_m$ and let $G \in \partial_m(W_m(F))$. If $X \subseteq W_n$ is invisible and $+$-invariant, then $\ell_{W_m X}(\mathbf{p}) \geq 3^{-m}$ for any $\mathbf{p} \in \mathcal{CH}_{m+n}(F, G)$.*

Proof. Note that $\#(W_m(F)) \leq 6$. Up to parallel translations, the reflections in the real and the imaginary axes and the $\pi/2$-rotation, we may assume that $F = B_w$ for some $w \in W_m$. Then $W_m(F) \subseteq \{w, (w)_\leftarrow, (w)_\swarrow, (w)_\downarrow, (w)_\searrow, (w)_\rightarrow\}$, where some of them may be %. In fact there are 7 cases. (See Figure 4.)
Case 1 $\#(W_m(F)) = 6$.
Case 2 $\#(W_m(F)) = 5$ and $(w)_\searrow = \%$.
Case 3 $\#(W_m(F)) = 5$ and $(w)_\downarrow = \%$.
Case 4 $\#(W_m(F)) = 4$ and $(w)_\downarrow = (w)_\searrow = \%$.
Case 5 $\#(W_m(F)) = 3$ and $(w)_\downarrow = (w)_\searrow = (w)_\swarrow = \%$.
Case 6 $\#(W_m(F)) = 3$ and $(w)_\leftarrow = (w)_\swarrow = (w)_\searrow = \%$.
Case 7 $\#(W_m(F)) = 2$ and $(w)_\downarrow = (w)_\swarrow = (w)_\searrow = (w)_\leftarrow = \%$.

We consider the first case. The other cases can be treated by the similar discussion. If $D = \cup_{U \in \partial_m(W_m(F))} U$, then $D = \partial K(W_m(F))$. Let $\mathbf{p} = (w(1), \ldots, w(k)) \in \mathcal{CH}_{m+n}(F, G)$. The reflection in the line containing F induces a natural bijection from $W_m(F) \cdot W_n$ to itself, which is denoted by η. Define $\theta : W_m(F) \cdot W_n \to \{(w)_\leftarrow, w, (w)_\rightarrow\} \cdot W_n$ by

$$\theta(uv) = \begin{cases} uv & \text{if } u \in \{(w)_\leftarrow, w, (w)_\rightarrow\} \text{ and } v \in W_n, \\ \eta(uv) & \text{if } u \in \{(w)_\swarrow, (w)_\downarrow, (w)_\searrow\} \text{ and } v \in W_n. \end{cases}$$

Define $v(i) = \theta(w(i))$ and $\widetilde{\mathbf{p}} = (v(1), \ldots, v(k))$. Then the $+$-invariant property of X implies that $\widetilde{\mathbf{p}} \in \mathcal{CH}_{m+n}(F, D_1)$, where $D_1 = L_{(w)_\leftarrow} \cup T_{(w)_\leftarrow} \cup T_w \cup T_{(w)_\rightarrow} \cup R_{(w)_\rightarrow}$, and

$$\ell_{W_m X}(\mathbf{p}) = \ell_{W_m X}(\widetilde{\mathbf{p}}),$$

If $v(k) \cap L_{(w)_\leftarrow} \neq \emptyset$, then there exists j such that $(v(j), v(j+1), \ldots, v(k)) \in \mathcal{CH}_m(R_{(w)_\leftarrow}, L_{(w)_\leftarrow})$ and $K_{v(i)} \subseteq (w)_\leftarrow \cdot W_n$ for any $i \in \{j, j+1, \ldots, k\}$. Since X is invisible, it follows that

$$\ell_{W_m X}(\mathbf{p}) \geq \ell_{W_m X}((v(j), \ldots, v(k))) \geq 3^{-m}.$$

The same discussion shows that $\ell_{W_m X}(\mathbf{p}) \geq 3^{-m}$ if $K_{v(k)} \cap R_{(w)_\rightarrow} \neq \emptyset$.

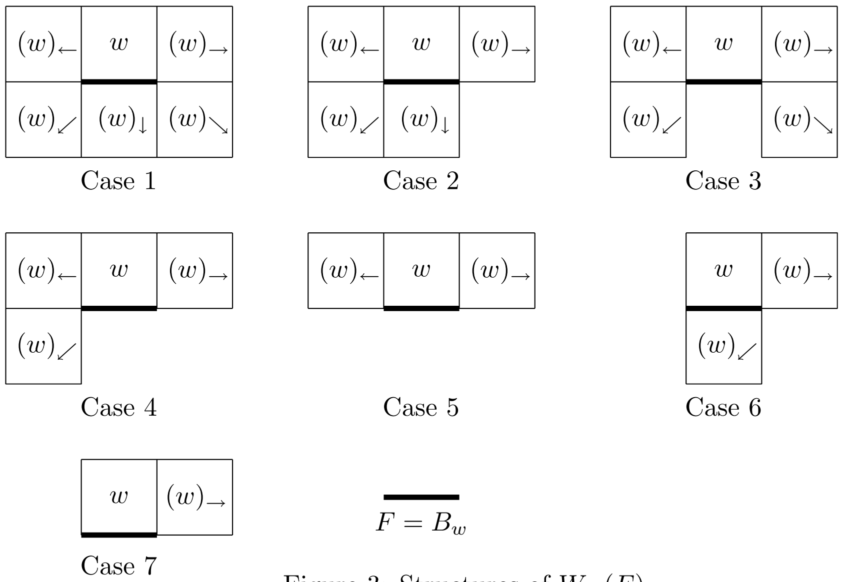


Figure 3: Structures of $W_m(F)$

Next suppose $v(K) \cap (T_{(w)_\leftarrow} \cup T_w \cup T_{(w)_\rightarrow}) \neq \emptyset$. Then using the reflections in the lines containing L_w and R_w, we may construct $(u(1), \dots, u(k)) \in \mathcal{CH}_{m+n}(B_w, T_w)$ which satisfies $u(i) \in w \cdots W_n$ for any i and $\ell_{W_m X}(\mathbf{p}) = \ell_{W_m X}((u(1), \dots, u(k)))$. Since X is invisible, it follows that $\ell_{W_m X}(\mathbf{p}) \geq 3^{-m}$. $\square$

Lemma 4.8. *Let $F, G \in \partial_m$ with $F \cap G = \emptyset$ and let $\mathbf{p} = (w(1), \dots, w(k)) \in \mathcal{CH}_{m+n}(F, G)$. If $\{w(i)\}_{i=1}^k \cap AW_n = \emptyset$, then there exists $\mathbf{p}_* \in \mathcal{CH}_m(F, G)$ such that $\ell_A(\mathbf{p}_*) \leq \ell_{AW_n \cup W_m X}(\mathbf{p})$.*

Proof. Let $k_1 = \max\{j | \{w(i)\}_{i=1}^j \subseteq W_{m+n}(F)\}$. Define $v(1) = [w(k_1)]_m$. Note that $v(1) \notin A$. There exists a unique $F_1 \in \{L_{v(1)}, R_{v(1)}, T_{v(1)}, B_{v(1)}\} \cap \partial_m(W_m(F))$ such that $K_{w(k_1)} \subseteq F_1$. By Lemma 4.7,

$$\ell_{AW_n \cup W_m X}((w(1), \dots, w(k_1))) \geq 3^{-m} = \ell_A((v(1))).$$

Now, if $F_1 \cap G \neq \emptyset$, $(v(1)) \in \mathcal{CH}_m(F, G)$ and $\ell_{AW_n \cup W_m X}(\mathbf{p}) \geq \ell_A((v(1))$. Hence we have constructed $\mathbf{p}_* = (v(1))$. Otherwise, replacing $(w(1), \dots, w(k))$ and F by $(w(k_1), \dots, w(k))$ and F_1 respectively, we repeat the same procedure as above and obtain k_2, $v(2)$ and F_2. Inductively, we have $\mathbf{p}_* = (v(1), \dots, v(l))$ with the desired properties. $\square$

Lemma 4.9. *Let $F, G \in \partial_m$ with $F \cap G = \emptyset$. Then for any $\mathbf{p} \in \mathcal{CH}_{m+n}(F, G)$, there exists $\mathbf{p}_* \in \mathcal{CH}_m(F, G)$ such that $\ell_A(\mathbf{p}_*) \leq \ell_{AW_n \cup W_m X}(\mathbf{p})$.*

Proof. Let $\mathbf{p} = (w(1), \dots, w(k))$. If $w(i) \notin AW_n$ for any i, then Lemma 4.8 suffices. Hence we assume that there exists i such that $w(i) \in AW_n$.

Claim 1: Without loss of generality, we may assume that there exists $p_1 \geq 1$ and $G_1 \in \partial_m$ such that $w(1), \ldots, w(p_1) \in W_{m+n} \backslash AW_n$, $w(p_1+1) \in AW_n$, $G_1 \cap F = \emptyset$, $G_1 \subseteq K_{[w(p_1+1)]_m}$ and $(w(1), \ldots, w(p_1)) \in \mathcal{CH}_{m+n}(F, G_1)$.
Proof of Claim 1. Case 1: $F \cap K(A) = \emptyset$
In this case, define

$$p_1 = \min\{i | w(i) \in AW_n\} - 1$$

and choose $G_1 \in \partial_m$ so that $G_1 \cap K_{w(p_1)} \cap K_{w(p_1+1)} \neq \emptyset$ and $G_1 \subseteq K_{[w(p_1)]_m}$.
Case 2: $F \cap K(A) \neq \emptyset$
In this case, F intersects at most two connected components of $K(A)$. Let C_1 and C_2 be those connected components of $K(A)$. (It is possible that $C_1 = C_2$.)
Case 2.1: $\{i | K_{w(i)} \subseteq C_1 \cup C_2\} = \emptyset$.
Define p_1 and choose G_1 as in Case 1. Then p_1 and G_1 satisfies the desired property.
Case 2.2: $\{i | K_{w(i)} \subseteq C_1 \cup C_2\} \neq \emptyset$.
Define

$$q = \max\{i | K_{w(i)} \in C_1 \cup C_2\}.$$

We may choose $F_0 \in \partial_m$ so that $F_0 \cap K_{w(q)} \cap K_{w(q+1)} \neq \emptyset$ and $F_0 \subseteq K_{[w(q)]_m}$. Moreover, we may choose $\mathbf{p}^0 = (v(1), \ldots, v(k_0)) \in \mathcal{CH}_m(F, F_0)$ so that $v(i) \in AW_n$ for any $i = 1, \ldots, k_0$ and $v(k_0) = [w(q)]_m$. Note that $\ell_A(\mathbf{p}^0) = 0$. If $F_0 \cap G \neq \emptyset$, then $K_{v(k_0)} \cap G \neq \emptyset$ and $\mathbf{p}^0 \in \mathcal{CH}_m(F, G)$. Hence letting $\mathbf{p}_* = \mathbf{p}^0$, we have constructed $\mathbf{p}_*$ which satisfies all the conditions. Assume that $F_0 \cap G = \emptyset$. Since $(w(1), \ldots, w(q)) \in \mathcal{CH}_{m+n}(F, F_0)$ corresponds $\mathbf{p}^0 \in \mathcal{CH}_m(F, F_0)$, it is enough to show the statement of the lemma in the case where F and $\mathbf{p}$ are replaced by F_0 and $(w(q+1), \ldots, w(k))$ respectively. In this situation, the counterpart of Case 2.1 holds and so does Claim 1. (End of Proof of Claim 1)
Claim 2: Without loss of generality, we may assume that there exists k_* and $F_* \in \partial_m$ such that $w(k_*), \ldots, w(k) \in W_{m+n} \backslash AW_n$, $w(k_* - 1) \in AW_n$, $F_* \cap G = \emptyset$, $F_* \subseteq K_{w(k_*)}$ and $(w(k_*), \ldots, w(k)) \in \mathcal{CH}_{m+n}(F_*, G)$.
Proof of Claim 2. By considering the chain $(w(k), w(k-1), \ldots, w(1)) \in \mathcal{CH}_{m+n}(G, F)$, the same argument as in the proof of Claim 1 yields this claim. (End of Proof of Claim 2)
Now under Claim 1 and Claim 2, we may choose $p_1, \ldots, p_{j+1}$ and $q_0, q_1, \ldots, q_j$ which satisfy the following conditions (A), (B), (C) and (D):
(A) $q_0 = 0$, $p_{j+1} = k$, $q_i < p_{i+1} < q_{i+1}$ for any i.
(B) $\{(w(q_{i-1}+1), \ldots, w(p_i)\} \cap AW_n = \emptyset$ for any $i = 1, \ldots, j+1$
(C) $K_{w(p_i+1)}$ and $K_{w(q_i)}$ belong to the same connected component of $K(A)$ for any $i = 1, \ldots, j$.
(D) $K_{w(q_i)}$ and $K_{w(p_{i+1}+1)}$ belong to the different connected components of $K(A)$ for any $i = 1, 2, \ldots, j-1$

Let $\mathbf{p}_i = (w(q_{i-1}+1), \ldots, w(p_i))$ for $i = 1, \ldots, j+1$. Define $F_1 = F$. For $i \geq 2$, we may choose $F_i \in \partial_m$ so that $F_i \cap K_{w(q_{i-1})} \cap K_{w(q_{i-1}+1)} \neq \emptyset$ and $F \subseteq K_{[w(q_{i-1})]_m}$. Moreover, for $i = 1, \ldots, j$, we may choose $G_i \in \partial_m$ so that $G_i \cap K_{w(p_i)} \cap K_{w(p_i+1)} \neq \emptyset$ and $G_i \subseteq K_{[w(p_i+1)]_m}$. Also let $F_{j+1} = G$. By the condition (D), $F_i \cap G_i = \emptyset$ for any $i = 1, \ldots, j+1$. Hence letting $F = F_i, G = G_i$

and $\mathbf{p} = \mathbf{p}_i$ and applying Lemma 4.8, we obtain $\mathbf{p}_{*,i} = (v(i,1), \ldots, v(i,k_i)) \in \mathcal{CH}_m(F_i, G_i)$ which satisfies $\ell_A(\mathbf{p}_{*,i}) \leq \ell_{AW_n \cup W_m X}(\mathbf{p}_i)$.

Note that G_i and F_i belong to the same connected component of $K(A)$ by the condition (C). Hence there exists $\mathbf{p}_i^1 = (u(i,1), \ldots, u(i,l_i)) \in \mathcal{CH}_m(G_i, F_i)$ such that $u(i,1), \ldots, u(i,l_i) \in A$.

Finally let $\mathbf{p}_* = (\mathbf{p}_{*,1}, \mathbf{p}_1^1, \mathbf{p}_{*,2}, \ldots, \mathbf{p}_j^1, \mathbf{p}_{*,j+1})$. Then $\mathbf{p}_* \in \mathcal{CH}_m(F, G)$ and $\ell_A(\mathbf{p}_*) \leq \ell_{AW_n \cup W_m X}(\mathbf{p})$. □

Proof of Theorem 4.5. Let $\mathbf{p} = (w(1), \ldots, w(k)) \in \mathcal{CH}_{m+n}(L, R)$. Set $F = L_{[w(1)]_m}$ and $G = R_{[w(k)]_m}$. By Lemma 4.9, there exists $\mathbf{p}_* \in \mathcal{CH}_m(F, G)$ such that $\ell_A(\mathbf{p}_*) \leq \ell_{AW_n \cup W_m X}(\mathbf{p})$. Since A is invisible, we have $\ell_A(\mathbf{p}_*) \geq 1$. Hence $\ell_{AW_n \cup W_m X}(\mathbf{p}) \geq 1$. In the same way, if $\mathbf{p}' \in \mathcal{CH}_{m+n}(T, B)$, it follows that $\ell_{AW_n \cup W_m X}(\mathbf{p}') \geq 1$. Thus $AW_n \cup W_m X$ is invisible. □

5 Metric associated with invisible set

In this section, we construct a metric associated with a +-invariant invisible set and characterize the Hausdorff dimension and the Hausdorff measure with respect to the metric.

Throughout this section, we fix a +-invariant invisible set $A \subseteq W_m$.

Notation. We write $W_{m,n} = (W_m)^n = W_{mn}$, $W_{m,*} = \cup_{n \geq 0} W_{m,n}$ and $\Sigma^{(m)} = (W_m)^{\mathbb{N}}$.

Naturally $W_{m,*}$ is regarded as a subset of W_* and $\Sigma^{(m)}$ is identified with Σ.

Definition 5.1. (1) Let $\epsilon > 0$. Define $D_\epsilon^A(w)$ for $w \in W_m$ by

$$D_\epsilon^A(w) = \begin{cases} 3^{-m} & \text{if } w \notin A, \\ \epsilon & \text{if } w \in A. \end{cases}$$

and $D_\epsilon^A(\emptyset) = 1$ for $\emptyset \in W_0$. For any $w = w^{(1)} \cdots w^{(n)} \in W_{m,n}$, where $w^{(i)} \in W_m$, define

$$D_\epsilon^A(w) = D_\epsilon^A(w^{(1)}) D_\epsilon^A(w^{(2)}) \cdots D_\epsilon^A(w^{(n)}).$$

(2) Define

$$\mathcal{CH}^{(m)} = \{(w(1), \ldots, w(k)) | \\ (w(1), \ldots, w(k)) \in \mathcal{CH}, w(i) \in W_{m,*} \text{ for any } i = 1, \ldots, k\}.$$

and $\mathcal{CH}^{(m)}(U, V) = \mathcal{CH}(U, V) \cap \mathcal{CH}^{(m)}$ for $U, V \subseteq K$. Moreover, define $\ell^{A,\epsilon}(\mathbf{p}) = \sum_{i=1}^k D_\epsilon^A(w(i))$ for any $\mathbf{p} = (w(1), \ldots, w(k)) \in \mathcal{CH}^{(m)}$ and, for $x, y \in K$,

$$d_\epsilon^A(x, y) = \inf\{\ell^{A,\epsilon}(\mathbf{p}) | \mathbf{p} \in \mathcal{CH}^{(m)}(x, y)\}.$$

$D_\epsilon^A(\cdot)$ is a gauge function on $\Sigma^{(m)}$ and d_ϵ^A is the associated pseudometric.

The next fact is obvious from the definition.

Proposition 5.2. *$d_0^A(x,y) \leq d_\epsilon^A(x,y)$ for any $x, y \in K$ and any $\epsilon > 0$.*

The next theorem shows that d_ϵ^a is really a metric and $d_\epsilon^A \underset{\mathrm{QS}}{\sim} d_E$.

Theorem 5.3. *For any $\epsilon > 0$, d_ϵ^A is a metric on K which is quasisymmetric with respect to d_E. The Hausdorff dimension of K with respect to the metric d_ϵ^A, $\dim_H(K, d_\epsilon^A)$ is given by the unique α which satisfies*

$$(8^m - \#(A))3^{-m\alpha} + \#(A)\epsilon^\alpha = 1. \tag{5.1}$$

Furthermore, let $\mathcal{H}^\alpha$ be the α-dimensional Hausdorff measure on (X, d_ϵ^A). Then the metric measure space $(X, d_\epsilon^A, \mathcal{H}^\alpha)$ is Ahlfors α-regular, i.e.

$$\mathcal{H}^\alpha(B_d(x,r)) \asymp r^\alpha$$

for any $x \in K$ and $r \in [0, \mathrm{diam}(X, d_\epsilon^A))$.

Letting $\epsilon \downarrow 0$ in (5.1), we obtain the following corollary.

Corollary 5.4.

$$\dim_{\mathcal{C}}(K, d_E) \leq \frac{\log 8}{\log 3} + \frac{1}{m \log 3} \log\Big(1 - \frac{\#(A)}{8^m}\Big).$$

In the rest of this section, we are going to prove the above theorem. Hereafter, we omit A in the notations $D_\epsilon^A(w)$, $\ell^{A,\epsilon}(\mathbf{p})$ and $d_\epsilon^A(x,y)$ and write $D_\epsilon(w)$, $\ell^\epsilon(\mathbf{p})$ and $d_\epsilon(x,y)$ respectively.

Lemma 5.5. *Define $A_n \subseteq W_{mn}$ inductively by $A_1 = A$ and*

$$A_{n+1} = AW_{mn} \cup W_m A_n.$$

Then A_n is $+$-invariant and invisible.

Proof. Letting $X = A_n$ and applying Theorem 4.5, we see inductively that A_{n+1} is $+$-invariant and invisible. □

Lemma 5.6. *$d_0^A(x,y) \geq 1$ for any $(x,y) \in (L \times R) \cup (T \times B)$.*

Proof. Define $I(\mathbf{p}) = \max_{i=1,\ldots,k} |w(i)|/m$ for any $\mathbf{p} = (w(1), \ldots, w(k)) \in \mathcal{CH}^{(m)}(L,R)$. We are going to show that $\ell^0(\mathbf{p}) \geq 1$ by an induction in $I(\mathbf{p})$. If $I(\mathbf{p}) = 0$, then $\mathbf{p} = (\emptyset)$ and $\ell^0(\mathbf{p}) = D_0(\emptyset) = 1$. Let $J = \{i | i = 1, \ldots, k, |w(i)| = I(\mathbf{p})m\}$. Then there exists $k_1, \ldots, k_l$ and $j_1, \ldots, j_l$ such that $k_i \leq j_i < k_{i+1}$ and $J = \cup_{i=1,\ldots,l}\{j | k_i \leq j \leq j_i\}$. Let $\mathbf{p}^i = (w(k_i), \ldots, w(j_i))$. Since $|w(k_i - 1)| \leq (I(\mathbf{p}) - 1)m$ and $|w(j_i + 1) \leq (I(\mathbf{p}) - 1)m$, there exist $F, G \in \partial_M$, where $M = (I(\mathbf{p}) - 1)m$, such that $F \subseteq K_{w(k_i - 1)}$, $F \cap K_{w(k_i)} \neq \emptyset$, $G \subseteq K_{w(j_i+1)}$ and $G \cap K_{w(j_i)} \neq \emptyset$. If $F \cap G = \emptyset$, then $K_{w(k_i-1)} \cap K_{w(j_i+1)} \neq \emptyset$. Hence if $\mathbf{p}' = (w(1), \ldots, w(k_i - 1), w(j_i + 1), \ldots, w(k)) \in \mathcal{CH}^{(m)}(L,R)$, then we define $\mathbf{p}_*^i$ as the empty sequence. Note that $\ell^0(\mathbf{p}) \geq \ell^0(\mathbf{p}')$. Now assume that

$F \cap G = \emptyset$. Set $X = A_M$. Lemma 5.5 shows that X is $+$-invariant and invisible. Then by Lemma 4.9, there exists $\mathbf{p}^i_* = (v(1), \ldots, v(l)) \in \mathcal{CH}_M(F, G)$ such that $\ell_{A_M}(\mathbf{p}^i_*) \le \ell_{A_M W_m \cup W_M A}(\mathbf{p}^i)$. Note that $A_M W_m \cup W_M A = A_{I(\mathbf{p})m}$, that $\ell_{A_M}(\mathbf{p}^i_*) = \ell^0(\mathbf{p}^i_*)$ and that $\ell_{A_M W_m \cup W_M A}(\mathbf{p}^i) = \ell^0(\mathbf{p}^i)$. Let $\mathbf{p}_*$ be the chain where $\mathbf{p}^i$ is replaced by $\mathbf{p}^i_*$ for all i. Then $\mathbf{p}_* \in \mathcal{CH}^{(m)}(L, R)$, $I(\mathbf{p}_*) < I(\mathbf{p})$ and $\ell^0(\mathbf{p}) \ge \ell^0(\mathbf{p}_*)$. Now we have $\ell^0(\mathbf{p}) \ge \ell^0(\mathbf{p}_*) \ge 1$ by induction hypothesis.

Now, $d^A_0(x, y) \ge \inf\{\ell^0(\mathbf{p}) | \mathbf{p} \in \mathcal{CH}^{(m)}(x, y)\} \ge 1$ for any $x \in L$ and any $y \in R$. In the same manner, it follows that $d^A_0(x, y) \ge 1$ for any $x \in T$ and any $y \in B$ as well. $\square$

Lemma 5.7. *$d^A_\epsilon(\cdot, \cdot)$ is a metric on K for any $\epsilon > 0$.*

Proof. Let $x, y \in K$ with $x \ne y$. Then $\mathrm{Re}\, x \ne \mathrm{Re}\, y$ or $\mathrm{Im}\, x \ne \mathrm{Im}\, y$. Suppose $\mathrm{Re}\, x < \mathrm{Re}\, y$. Then there exist n and $i \in \{0, 1, \ldots, 3^{mn} - 1\}$ such that $\mathrm{Re}\, x \le (2i - 3^{mn})3^{-mn} < (2i + 2 - 3^{mn})3^{-mn} \le \mathrm{Re}\, y$.
Claim: $d^A_\epsilon(x, y) \ge \min\{D^A_\epsilon(w) | w \in W_{m,k}, k = 0, 1, \ldots, n\}$.
Proof of Claim. Define $W^i_{m,n} = \{w | w \in W_{m,n}, K_w \subseteq \{z | (2i - 3^{mn})3^{-mn} \le \mathrm{Re}\, z \le (2i + 2 - 3^{mn})3^{-mn}\}$. Let $D_{mn,i} = \min\{D_\epsilon(w) | w \in W^i_{m,n}\}$. We also define $L_{mn,i} = \cup_{w \in W^i_{m,n}} L_w$ and $R_{mn,i} = \cup_{w \in W^i_{m,n}} R_w$. Let $\mathbf{p} = (w(1), \ldots, w(k)) \in \mathcal{CH}^{(m)}(x, y)$. If $|w(i)| \le mn$ for some i, then the claim is trivial. Hence assume that $|w(i)| < mn$ for any $i = 1, \ldots, k$. Then $\mathbf{p}$ contains $(w(p), w(p+1), \ldots, w(q)) \in \mathcal{CH}(L_{mn,i}, R_{mn,i})$ which satisfies $w(i) \in \cup_{w \in W^i_{m,n}} w W_{m,*}$. Let $w(i) = u(i)v(i)$ for $i = p, \ldots, q$, where $u(i) \in W^i_{m,n}$ and $v(i) \in W_{m,*}$. It follows that

$$\ell^{A,\epsilon}(\mathbf{p}) \ge \ell^{A,\epsilon}((w(p), w(p+1), \ldots, w(q))) \ge D_{mn,i} \sum_{i=p}^{q} D_\epsilon(v(i)). \tag{5.2}$$

Now the reflection ψ in the real axis induces a natural bijection $\varphi_\leftrightarrow : W_* \to W_*$ defined by $\psi(K_w) = K_{\varphi_\leftrightarrow(w)}$ which satisfies $\varphi_\leftrightarrow(\varphi_\leftrightarrow(w)) = w$. Hereafter in this section, we write $\varphi = \varphi_\leftrightarrow$. There exist $p_1, p_2, \ldots p_l$ such that $p_1 = p, p_l = q + 1$, $p_i < p_{i+1}$, $u(p_i) = u(p_i + 1) = \ldots = u(p_{i+1} - 1)$ and $u(p_i) \ne u(p_{i+1})$ for any i. Let $\bar{v}(j) = \varphi^i(v(j))$ for $j = p_i, p_i + 1, \ldots, p_{i+1} - 1$, where φ^j is the j-th iteration of φ. Then $(\bar{v}(p), \bar{v}(p+1), \ldots, \bar{v}(q)) \in \mathcal{CH}^{(m)}(L, R)$. Since A is $+$-invariant, $\sum_{i=p}^{q} D_\epsilon(v(i)) = \sum_{i=p}^{q} D_\epsilon(\bar{v}(i))$. Hence Lemma 5.6 implies that

$$\sum_{i=p}^{q} D_\epsilon(v(i)) = \sum_{i=p}^{q} D_\epsilon(\bar{v}(i)) \ge \sum_{i=p}^{q} D_0(\bar{v}(i)) \ge 1.$$

Combining this with (5.2), we have $\ell^{A,\epsilon}(\mathbf{p}) \ge D_{mn,i}$. Hence the claim holds. (End of Proof of Claim)
The claim shows that $d^A_\epsilon(x, y) > 0$ if $\mathrm{Re}\, x \ne \mathrm{Re}\, y$. Similar discussion implies $d^A_\epsilon(x, y) > 0$ if $\mathrm{Im}\, x \ne \mathrm{Im}\, y$. $\square$

Proof of Theorem 5.3. Let $\mathcal{S}^{(m)}(A, \epsilon) = \{\Lambda^{(m)}_s(A, \epsilon)\}_{s \in (0,1]}$ be the scale on $\Sigma^{(m)}$ whose gauge function is D^A_ϵ and let $\mathcal{S}^{(m)}$ by the scale on $\Sigma^{(m)}$ whose gauge function g is given by $g(w(1)\dots w(k)) = 3^{-mk}$ for $w(1)\dots w(k) \in W_{m,*}$ with $w(1),\dots,w(k) \in W_m$. Obviously $\Sigma^{(m)}$ is adapted to the Euclidean metric on K. Also since $\mathcal{S}^{(m)}(A,\epsilon)$ and $\mathcal{S}^{(m)}$ are self-similar, they are elliptic.

Note that $(K, W_m, \{F_w\}_{w\in W_m})$ is a rationally ramified self-similar structure. (See [5, Section 1.5] for the definition of rationally ramified self-similar structures.) In fact, define $h : L^1 \to R^1$ by $h(\nwarrow) = \nearrow, h(\leftarrow) = \rightarrow, h(\swarrow) = \searrow$ and $g : T^1 \to B^1$ by $g(\nwarrow) = \swarrow, g(\uparrow) = \downarrow, g(\nearrow) = \searrow$. Then define $h_m : L^m \to R^m$ by $h_m(w_1\dots w_m) = h(w_1)\dots h(w_m)$ for $w_1\dots w_m \in L^m$ and $g_m : T^m \to B^m$ by $g_m(w_1\dots w_m) = g(w_1)\dots g(w_m)$ for $w_1\dots w_m \in T^m$. Then a relation set $\mathcal{R}_m$ of $(K, W_m, \{F_w\}_{w\in W_m})$ is given by

$$\mathcal{R}_m = \{(L^m, R^m, h_m, w, v) | w, v \in W_m, L_w = R_v\} \cup \{(T^m, B^m, g_m, w, v) | w, v \in W_m, T_w = B_v\}$$

By Proposition 4.4, $D^A_\epsilon(w) = 3^{-m}$ for any $w \in L^m \cup R^m \cup T^m \cup B^m$. Using [5, Theorem 1.6.6], we see that $\mathcal{S}^{(m)}(A,\epsilon) \underset{\mathrm{GE}}{\sim} \mathcal{S}^{(m)}$.

Theorems 1.6.1 and 2.2.7 in [5] imply that $\mathcal{S}^{(m)}(A,\epsilon)$ is intersection type finite. Since d^A_ϵ is a metric on K by Lemma 5.7, we may apply [5, Theorem 2.3.16] and show that d^A_ϵ is adapted to the scale $\mathcal{S}^{(m)}(A,\epsilon)$. Thus we have obtained all the conditions in Theorem 3.4 and hence shown that d^A_ϵ is quasisymmetric with respect to the Euclidean metric.

The Hausdorff dimension and Ahlfors regularity of the Hausdorff measure of (K, d^A_ϵ) are immediately obtained by [4, Theorem 1.5.7]. $\square$

6 Construction of invisible sets

Under the existence of an invisible set, we have constructed a corresponding metric which is quasisymmetric with respect to d_E and characterized the associated Hausdorff dimension in the previous two sections. In this section, it is shown that invisible sets do exist. In fact, we construct a series of invisible sets inductively.

Definition 6.1. Let $\psi_\updownarrow$ and $\psi_\leftrightarrow$ be the reflections in the real and complex axes respectively. Then $\psi_\updownarrow$ induces a natural bijection $\varphi_\updownarrow$ from W_* to itself defined by $\psi_\updownarrow(K_w) = K_{\varphi_\updownarrow(w)}$. In the same way, we define a bijection $\varphi_\leftrightarrow$ from W_* to itself by $\psi_\leftrightarrow(K_w) = K_{\varphi_\leftrightarrow(w)}$. Moreover, let R be the $\pi/2$-rotation around the origin 0 and let $\rho : W_* \to W_*$ be the bijection defined by $R(K_w) = K_{\rho(w)}$.

The idea to have invisible sets is to divide the notion of a invisible set into a vertically invisible set and a horizontally invisible set. The final existence of invisible sets are established by taking intersections of vertically invisible set and horizontally invisible set in Theorem 6.4.

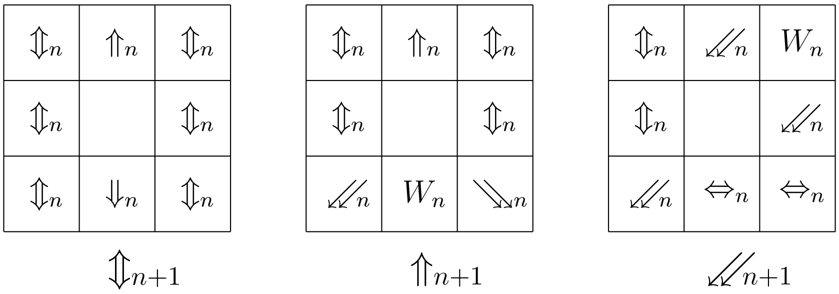


Figure 4: Construction of $\Updownarrow_n$, $\Uparrow_n$ and $\Swarrow_n$

Definition 6.2. Define $\Updownarrow_n, \Uparrow_n$ and $\Swarrow_n$ as subsets of W_n inductively by

$$\Updownarrow_{n+1} = \{\nwarrow, \leftarrow, \swarrow, \nearrow, \rightarrow, \searrow\}\cdot \Updownarrow_n \cup \uparrow \cdot \Uparrow_n \cup \downarrow \cdot \Downarrow_n \tag{6.1}$$
$$\Uparrow_{n+1} = \{\nearrow, \leftarrow, \nwarrow, \rightarrow\}\cdot \Updownarrow_n \cup \uparrow \cdot \Uparrow_n \cup \swarrow \cdot \Swarrow_n \cup \downarrow \cdot W_n \cup \searrow \cdot \Searrow_n \tag{6.2}$$
$$\Swarrow_{n+1} = \{\nwarrow, \leftarrow\}\cdot \Updownarrow_n \cup \{\downarrow, \searrow\}\cdot \Leftrightarrow_n \cup \{\uparrow, \rightarrow, \swarrow\}\cdot \Swarrow_n \cup \nearrow \cdot W_n \tag{6.3}$$

and $\Updownarrow_0 = \Uparrow_0 = \Swarrow_0 = \emptyset$, where $\Downarrow_n = \varphi_{\updownarrow}(\Uparrow_n)$, $\Searrow_n = \varphi_{\leftrightarrow}(\Swarrow_n)$ and $\Leftrightarrow_n = \rho(\Updownarrow_n)$.

Lemma 6.13 will show that $\Updownarrow_n$ and $\Leftrightarrow_n$ are vertically and horizontally invisible sets respectively.

Lemma 6.3.

$$\#(\Updownarrow_n) = 8^n - \frac{7+\sqrt{41}}{2\sqrt{41}}\left(\frac{9+\sqrt{41}}{2}\right)^n + \frac{7-\sqrt{41}}{2\sqrt{41}}\left(\frac{9-\sqrt{41}}{2}\right)^n$$

Proof. Write $a_n = \#(\Updownarrow_n)$, $b_n = \#(\Uparrow_n)$ and $c_n = \#(\Swarrow_n)$. By (6.1), (6.2) and (6.3), it follows that

$$\begin{aligned} a_{n+1} &= 6a_n + 2b_n \\ b_{n+1} &= 4a_n + b_n + 2c_n + 8^n \\ c_{n+1} &= 4a_n + 3c_n + 8^n. \end{aligned}$$

Solving these with $a_0 = b_0 = c_0 = 0$, we obtain a_n as in the statement of the lemma. $\square$

Now we have the main theorem of this section.

Theorem 6.4. *Let $A_n = \Updownarrow_n \cap \Leftrightarrow_n$. Then A_n is a $+$-invariant invisible set and*

$$\alpha_n \le 8^n - \#(A_n) \le 2\alpha_n,$$

where

$$\alpha_n = \frac{7+\sqrt{41}}{2\sqrt{41}}\left(\frac{9+\sqrt{41}}{2}\right)^n - \frac{7-\sqrt{41}}{2\sqrt{41}}\left(\frac{9-\sqrt{41}}{2}\right)^n.$$

Example 6.5. $A_0 = A_1 = A_2 = A_3 = \emptyset$.

$$A_4 = \{\uparrow\downarrow\leftarrow\rightarrow, \uparrow\downarrow\rightarrow\leftarrow, \downarrow\uparrow\leftarrow\rightarrow, \downarrow\uparrow\leftarrow\rightarrow, \leftarrow\rightarrow\uparrow\downarrow, \leftarrow\rightarrow\downarrow\uparrow, \rightarrow\leftarrow\uparrow\downarrow, \rightarrow\leftarrow\downarrow\uparrow\}.$$

Applying Corollary 5.4 and letting $n \to \infty$, we obtain the following upper estimate of the conformal dimension of the Sierpinski carpet.

Corollary 6.6.

$$\dim_{\mathcal{C}}(K, d_E) \leq \frac{\log\left(\frac{9+\sqrt{41}}{2}\right)}{\log 3} = 1.858183... < \frac{\log 8}{\log 3} = 1.892789....$$

Remark. The known lower bound of $\dim_{\mathcal{C}}(K, d_E)$ given in (1.1) is $\frac{\log 6}{\log 3} = 1.630929....$

The rest of this section is devoted to proving Theorem 6.4.

Lemma 6.7. (1) $\varphi_{\leftrightarrow}(\Updownarrow_n) = \Updownarrow_n$ *and* $\varphi_{\updownarrow}(\Updownarrow_n) = \Updownarrow_n$.
(2) $\varphi_{\leftrightarrow}(\Uparrow_n) = \Uparrow_n$.
(3) $\varphi_{\leftrightarrow} \circ \rho(\Swarrow_n) = \Swarrow_n$.

Definition 6.8. Define the vertical index $I^n_{\updownarrow} : W_n \to \{1, \dots, 3^n\}$ by

$$I^n_{\updownarrow}(w) = \frac{3^n(\mathrm{Im}\,(F_w(\sqrt{-1})) + 1)}{2}$$

For $H \in \{L, R\}$, define $w^n_H(i)$ for $i = 1, \dots, 3^n$ as the unique $w \in H^n$ which satisfies $I^n_{\updownarrow}(w) = i$. Moreover, for $w, v \in W_n$, define $\mathbf{p}^n_H(w, v) \in \mathcal{CH}_n$ by

$$\mathbf{p}^n_H(w, v) = \begin{cases} (w^n_H(I^n_{\updownarrow}(w)), w^n_H(I^n_{\updownarrow}(w)+1), \dots, w^n_H(I^n_{\updownarrow}(v))) & \text{if } I^n_{\updownarrow}(w) \leq I^n_{\updownarrow}(v), \\ (w^n_H(I^n_{\updownarrow}(w)), w^n_H(I^n_{\updownarrow}(w)-1), \dots, w^n_H(I^n_{\updownarrow}(v))) & \text{if } I^n_{\updownarrow}(v) \leq I^n_{\updownarrow}(w). \end{cases}$$

In the same way, we define the horizontal index $I^n_{\leftrightarrow} : W_n \to \{1, \dots, 3^n\}$, $w^n_T(i)$, $w^n_B(i)$, $\mathbf{p}^n_T(w, v)$ and $\mathbf{p}^n_B(w, v)$.

Lemma 6.9. *Let $A \subseteq W_n$. Assume that*

$$\inf\{\ell_A(\mathbf{p}_*) | \mathbf{p}_* \in \mathcal{CH}_n(T, p_{\swarrow})\} \geq 1 \tag{6.4}$$

Let $\mathbf{p} = (w(1), \dots, w(k)) \in \mathcal{CH}_n$. If $(w(1), w(k)) \in (T^n \cup L^n) \times L^n$, then

$$\ell_A(\mathbf{p}) \geq \frac{|I^n_{\updownarrow}(w(1)) - I^n_{\updownarrow}(w(k))| + 1}{3^n} = \ell_A(\mathbf{p}^n_L(w(1), w(k))). \tag{6.5}$$

Remark. Using the symmetries, we may exchange $(T, L, p_{\swarrow})$ in the statement of Lemma 6.9 by $(T, R, p_{\searrow})$, $(B, L, p_{\nwarrow})$ and $(B, R, p_{\nearrow})$.

Proof. Since $\ell_A((w_L^n(i))_{i=1,\ldots,3^n}) \geq 1$, it follows that $\{w_L^n(i)|i = 1,\ldots,3^n\} \cap A = \emptyset$. Let $\mathbf{p} = (w(1),\ldots,w(k)) \in \mathcal{CH}_n$.

Suppose that $(w(1), w(k)) \in T^n \times L^n$. Note that $w(k) = w_L^n(i)$ for some i. Then $\mathbf{p}_L^n(w(1), w(k)) = (w_L^n(3^n), w_L^n(3^n-1),\ldots,w_L^n(i))$ and $\ell_A(\mathbf{p}_L^n(w(1), w(k)))$ $= 1 - (i-1)/3^n$. Since $\mathbf{p} \vee \mathbf{p}_L^n(w_L^n(i-1), w_L^n(1)) \in \mathcal{CH}_n(T, p_\swarrow)$, (6.4) implies

$$\ell_A(\mathbf{p}) + \ell_A(\mathbf{p}_L^n(w_L^n(i-1), w_L^n(1))) = \ell_A(\mathbf{p} \vee \mathbf{p}_L^n(w_L^n(i-1), w_L^n(1))) \geq 1.$$

This shows (6.5) in this case.

Suppose that $(w(1), w(k)) \in L^n \times L^n$. Set $w(1) = w_L^n(j)$ and $w(k) = w_L^n(i)$. If $j < i$, then we consider $(w(k),\ldots,w(1))$ in place of $(w(1),\ldots,w(k))$. In this way, we may assume that $j \geq i$ without loss of generality. Let $\widetilde{\mathbf{p}} = \mathbf{p}_L^n(w_L^n(3^n), w_L^n(j+1)) \vee \mathbf{p} \vee \mathbf{p}_L^n(w_L^n(j-1), w_L^n(1))$. Since $\widetilde{\mathbf{p}} \in \mathcal{CH}_n(T, p_\swarrow)$, we have

$$\frac{3^n - j}{3^n} + \ell_A(\mathbf{p}) + \frac{i-1}{3^n} = \ell_A(\widetilde{\mathbf{p}}) \geq 1$$

This immediately implies (6.5) in this case. $\square$

Lemma 6.10. *Let $X, Y \subseteq W_n$. Assume that*

$$\inf\{\ell_X(\mathbf{p})|\mathbf{p} \in \mathcal{CH}_n(T, B)\} \geq 1$$

and that

$$\inf\{\ell_Y(\mathbf{p})|\mathbf{p} \in \mathcal{CH}_n(T, p_\swarrow)\} \geq 1.$$

Define $A = \nwarrow \cdot X \cup \uparrow \cdot Y$. If $\mathbf{p} = (w(1),\ldots,w(k)) \in \mathcal{CH}_{n+1}(T, B_\nwarrow)$ and $\{w(1),\ldots,w(k)\} \subseteq \{\nwarrow, \uparrow\} \cdot W_n$, then

$$\ell_A(\mathbf{p}) \geq \frac{1}{3}.$$

Proof. Let $w(i) = s(i)v(i)$, where $s(i) \in \{\nwarrow, \uparrow\}$ and $v(i) \in W_n$.

First assume that $s(1) = \nwarrow$. Then there exist $j_1, j_2, \ldots, j_{2p+2}$ which satisfies the following three conditions (C1), (C2) and (C3):

(C1) $j_1 = 1$, $j_{2p+2} = k+1$ and $j_1 < j_2 < \ldots < j_{2p+2}$

(C2) $s(i) = \nwarrow$ for $i = j_{2q-1},\ldots,j_{2q}-1$ and $q = 1,\ldots,p+1$

(C3) $s(i) = \uparrow$ for $i = j_{2q},\ldots,j_{2q+1}-1$ and $q = 1,\ldots,p$.

Set $\mathbf{p}_{1,q} = (w(j_{2q-1}),\ldots,w(j_{2q}-1))$ and $\widetilde{\mathbf{p}}_{1,q} = (v(j_{2q-1}),\ldots,v(j_{2q}-1))$. Since $(v(j_{2q-1}), v(j_{2q}-1)) \in (T^n \times R^n) \cup (R^n \times R^n) \cup (R^n \times B^n)$, Lemma 6.9 and its variants explained in the remark imply

$$\ell_A(\mathbf{p}_{1,q}) = \frac{1}{3}\ell_X(\widetilde{\mathbf{p}}_{1,q}) \geq \frac{1}{3}\ell_X(\mathbf{p}_R^n(v(j_{2q-1}), v(j_{2q}-1))). \tag{6.6}$$

Set $\mathbf{p}_{2,q} = (w(j_{2q}),\ldots,w(j_{2q+1}-1))$ and $\widetilde{\mathbf{p}}_{2,q} = (v(j_{2q}),\ldots,v(j_{2q+1}-1))$. Since $(v(j_{2q}), v(j_{2q+1}-1)) \in L^n \times L^n$, Lemma 6.9 shows that

$$\ell_A(\mathbf{p}_{2,q}) = \frac{1}{3}\ell_Y(\widetilde{\mathbf{p}}_{2,q}) \geq \frac{1}{3}\ell_Y(\mathbf{p}_L^n(v(j_{2q}), v(j_{2q+1}-1))). \tag{6.7}$$

Note that for any $i = 1, \dots, 3^n$, there exists $l = 1, 2, \dots, 2q+1$ such that $I^n_{\updownarrow}(v(j_l)) \le i \le I^n_{\updownarrow}(v(j_{l+1}-1))$ or $I^n_{\updownarrow}(v(j_l)) \ge i \ge I^n_{\updownarrow}(v(j_{l+1}-1))$. Hence

$$\sum_{q=1}^{p+1} \ell_X(\mathbf{p}^n_R(v(j_{2q-1}), v(j_{2q}-1))) + \sum_{q=1}^{p} \ell_Y(\mathbf{p}^n_L(v(j_{2q}), v(j_{2q+1}-1))) \ge 1.$$

Combining this with (6.6) and (6.7), we obtain

$$\ell_A(\mathbf{p}) = \sum_{q=1}^{p+1} \ell_A(\mathbf{p}_{1,q}) + \sum_{q=1}^{p} \ell_A(\mathbf{p}_{2,q}) \ge \frac{1}{3}.$$

Thus we have shown the desired statement in the case when $s(1) = \nwarrow$.

If $s(1) = \uparrow$, slight modification of the above arguments yields the lemma as well. $\square$

Definition 6.11. Define $\pi : W_* \to W_*$ by

$$\pi(w) = \begin{cases} w & \text{if } \operatorname{Re} F_w(0) \le 0, \\ \varphi_{\leftrightarrow}(w) & \text{if } \operatorname{Re} F_w(0) > 0. \end{cases}$$

For $\mathbf{p} = (w(1), \dots, w(k)) \in \mathcal{CH}_n$, we define $\pi_n(\mathbf{p}) = (\pi(w(1)), \dots, \pi(w(k))$. Also define $\xi : W_* \to W_*$ by

$$\xi(w) = \begin{cases} w & \text{if } \operatorname{Re} F_w(0) \le \operatorname{Im} F_w(0), \\ \varphi_{\leftrightarrow}(\rho(w)) & \text{if } \operatorname{Re} F_w(0) > \operatorname{Im} F_w(0). \end{cases}$$

For $\mathbf{p} = (w(1), \dots, w(k)) \in \mathcal{CH}_n$, we define $\xi_n(\mathbf{p}) = (\xi(w(1)), \dots, \xi(w(k))$.

By the symmetry of $\Updownarrow_n$, $\Uparrow_n$ and $\swarrow\!\!\!\swarrow_n$ given in Lemma 6.7, we have the following lemma.

Lemma 6.12. (1) $\pi_n : \mathcal{CH}_n \to \mathcal{CH}_n$, *$\ell_{\Updownarrow_n}(\pi_n(\mathbf{p})) = \ell_{\Updownarrow_n}(\mathbf{p})$ and $\ell_{\Uparrow_n}(\pi_n(\mathbf{p})) = \ell_{\Uparrow_n}(\mathbf{p})$.*
(2) *$\xi_n : \mathcal{CH}_n \to \mathcal{CH}_n$ and $\ell_{\swarrow\!\!\!\swarrow_n}(\xi_n(\mathbf{p})) = \ell_{\swarrow\!\!\!\swarrow_n}(\mathbf{p})$.*

Lemma 6.13. *Suppose that*

$$\inf\{\ell_{\Updownarrow_n}(\mathbf{p}) | \mathbf{p} \in \mathcal{CH}_n(T, B)\} \ge 1 \tag{6.8}$$

and

$$\inf\{\ell_{\Uparrow_n}(\mathbf{p}) | \mathbf{p} \in \mathcal{CH}_n(T, \{p_{\swarrow}, p_{\searrow}\}) \ge 1. \tag{6.9}$$

If $\mathbf{p} = (w(1), \dots, w(k)) \in \mathcal{CH}_{n+1}(T, B_{\nwarrow} \cup B_{\nearrow})$ and $\{w(i)\}_{i=1}^k \subseteq \{\nwarrow, \uparrow, \nearrow\} \cdot W_n$, then $\ell_{\Updownarrow_{n+1}}(\mathbf{p}) \ge 1/3$.

Proof. Replacing $\mathbf{p}$ by $\pi_{n+1}(\mathbf{p})$, we may assume that $w(1), \dots, w(k) \in \{\nwarrow, \uparrow\} \cdot W_n$ and $w(k) \in \nwarrow \cdot B^n$ without loss of generality. Set $X = \Updownarrow_n$ and $Y = \Uparrow_n$. Then the assumptions (6.8) and (6.9) of Lemma 6.10 follows. Hence $\ell_{\Updownarrow_{n+1}}(\mathbf{p}) \ge 1/3$. $\square$

Lemma 6.14. *Suppose that* (6.8) *holds and that*

$$\inf\{\ell_{\swarrow\!\swarrow_n}(\mathbf{p})|\mathbf{p}\in\mathcal{CH}_n(T\cup R,p_{\swarrow})\geq 1. \tag{6.10}$$

Let $\mathbf{p}=(w(1),\ldots,w(k))\in\mathcal{CH}_{n+1}(T,B_{\swarrow})$. *If* $\{w(1),\ldots,w(k)\}\subseteq\{\nwarrow,\uparrow,\nearrow\}\cdot W_n$, *then* $\ell_{\swarrow\!\swarrow_{n+1}}(\mathbf{p})\geq 1/3$.

Proof. First assume that $\{w(1),\ldots,w(k)\}\subseteq\{\nwarrow,\uparrow\}\cdot W_n$. By (6.3), applying Lemma 6.10 with $X=\Updownarrow_n$ and $Y=\swarrow\!\!\swarrow_n$, we have $\ell_{\swarrow\!\swarrow_{n+1}}(\mathbf{p})\geq 1/3$.

Next, suppose that $w(i)\in\nearrow\cdot W_n$ for some i. Let

$$i_*=\max\{i|w(i)\in\nearrow\cdot W_n\}+1.$$

and let

$$j_*=\min\{j|w(j)\in\nwarrow\cdot W_n,j\geq i_*\}-1.$$

Then, for $i=\{i_*,\ldots,j_*\}$, there exists $v(i)\in W_n$ such that $w(i)=\uparrow\cdot v(i)$. Define

$$\mathbf{p}_*=(\uparrow\cdot\xi(v(i_*)),\uparrow\cdot\xi(v(i_*+1)),\ldots,\uparrow\cdot\xi(v(j_*)),w(j_*+1),\ldots,w(k)).$$

By (6.3) and Lemma 6.12,

$$\begin{aligned}\ell_{\swarrow\!\swarrow_{n+1}}((w(i_*),\ldots,w(j_*)))&=\frac{1}{3}\ell_{\swarrow\!\swarrow_n}(v(i_*),\ldots,v(j_*)))\\&=\frac{1}{3}\ell_{\swarrow\!\swarrow_n}(\xi(v(i_*)),\ldots,\xi(v(j_*)))=\ell_{\swarrow\!\swarrow_{n+1}}(\uparrow\cdot\xi(v(i_*)),\ldots,\uparrow\cdot\xi(v(j_*))).\end{aligned}$$

Hence $\ell_{\swarrow\!\swarrow_{n+1}}(\mathbf{p})\geq\ell_{\swarrow\!\swarrow_{n+1}}(\mathbf{p}_*)$. Let $\mathbf{p}_*=(w_*(1),w_*(2),\ldots,w_*(l))$. Then $w^*(i)\in\{\nwarrow,\uparrow\}\cdot W_n$ for any $i=1,\ldots,l$. Now replacing $\mathbf{p}$ by $\mathbf{p}_*$, we are exactly in the first case and hence the desired inequality is satisfied. □

Lemma 6.15. (6.8), (6.9) *and* (6.10) *hold for any* $n\geq 0$.

Proof. We use induction on n. Obviously (6.8), (6.9) and (6.10) holds for $n=0$ since $\Updownarrow_n,\Uparrow_n$ and $\swarrow\!\!\swarrow_n$ are the empty sets. Assume that (6.8), (6.9) and (6.10) are true for $n=m$.

First we show (6.8) holds for $n=m+1$. Let $\mathbf{p}=(w(1),\ldots,w(k))\in\mathcal{CH}_{m+1}(T,B)$. Note that by Lemma 6.7-(1), $\pi_{n+1}(\mathbf{p})\in\mathcal{CH}_{m+1}(T,B)$ and $\ell_{\Updownarrow_{m+1}}(\mathbf{p})=\ell_{\Updownarrow_{m+1}}(\pi_{n+1}(\mathbf{p}))$. Hence replacing $\mathbf{p}$ by $\pi_{n+1}(\mathbf{p})$, we may assume that $w(i)\in\{\nwarrow,\uparrow,\leftarrow,\swarrow,\downarrow\}\cdot W_m$ for any $i=1,\ldots,k$ without loss of generality. Set $w(i)=s(i)v(i)$, where $s(i)\in\{\nwarrow,\uparrow,\leftarrow,\swarrow,\downarrow\}$ and $v(i)\in W_m$. We may choose i_1,i_2,i_3 and i_4 which satisfies $i_1<i_2<i_3<i_4$ and the following tree conditions (a1), (b1) and (c1):
(a1) $s(1),\ldots,s(i_1)\in\{\nwarrow,\uparrow\}$, $(v(1),\ldots,v(i_1))\in\mathcal{CH}_m(T,B_{\swarrow})$,
(b1) $s(i)=\leftarrow$ for $i=i_2,\ldots,i_3$, $(v(i_2),\ldots,v(i_3))\in\mathcal{CH}_m(T,B)$,
(c1) $s(i_4),\ldots,s(k)\in\{\swarrow,\downarrow\}$, $w_*(i_4)\in\swarrow\cdot T^m$.

Let $\mathbf{p}_1=(w(1),\ldots,w(i_1))$. Then by the induction hypothesis, we may apply Lemma 6.13 and see that $\ell_{\Updownarrow_{m+1}}(\psi_1)\geq 1/3$.

Let $\mathbf{p}_2 = (w(i_2), \dots, w(i_3))$. Since $(v(i_2), \dots, v(i_3)) \in \mathcal{CH}_m(T, B)$, the induction hypothesis implies

$$\ell_{\Updownarrow_{m+1}}(\mathbf{p}_2) = \frac{1}{3}\ell_{\Updownarrow_m}((v(i_2), \dots, v(i_3))) \geq \frac{1}{3}.$$

Set $\mathbf{p}_3 = (w(i_3), \dots, w(k))$ and $\widetilde{\mathbf{p}}_3 = (\varphi_{\updownarrow}(w(k)), \varphi_{\updownarrow}(w(k-1)), \dots, \varphi_{\updownarrow}(w(i_3)))$. Then $\ell_{\Updownarrow_{m+1}}(\mathbf{p}_3) = \ell_{\Updownarrow_{m+1}}(\widetilde{\mathbf{p}}_3)$. As $\mathbf{p}_1$, we may apply Lemma 6.13 to $\widetilde{\mathbf{p}}_3$ and obtain $\ell_{\Updownarrow_{m+1}}(\widetilde{\mathbf{p}}_3) \geq 1/3$. Combining all the estimates on $\ell_{\Updownarrow_{m+1}}(\mathbf{p}_i)$ for $i = 1, 2, 3$, we have $\ell_{\Updownarrow_{m+1}}(\mathbf{p}) \geq 1$.

Secondly, we show that (6.9) holds for $n = m+1$. Let $\mathbf{p} = (w(1), \dots, w(k)) \in \mathcal{CH}_{m+1}(T, \{p_{\swarrow}, p_{\searrow}\})$. As in the first case, we may assume that $w(i) \in \{\nwarrow, \uparrow, \leftarrow, \swarrow, \downarrow\}$ for any $i = 1, \dots, k$ without loss of generality. Set $w(i) = s(i)v(i)$, where $s(i) \in \{\nwarrow, \uparrow, \leftarrow, \swarrow, \downarrow\}$ and $v(i) \in W_m$. We may choose i_1, i_2, i_3 and i_4 which satisfies $i_1 < i_2 < i_3 < i_4$ and the following tree conditions (a2), (b2) and (c2):

(a2) $s(1), \dots, s(i_1) \in \{\nwarrow, \uparrow\}$, $(v(1), \dots, v(i_1)) \in \mathcal{CH}_m(T, B_{\swarrow})$,

(b2) $s(i) = \leftarrow$ for $i = i_2, \dots, i_3$, $(v(i_2), \dots, v(i_3)) \in \mathcal{CH}_m(T, B)$,

(c2) $s(i) = \swarrow$ for $i = i_4, \dots, k$. $(v(i_4), \dots, v(k)) \in \mathcal{CH}_m(T \cup R, p_{\swarrow})$.

Define $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_3$ as in the first case. Then using the same discussion as in the first case, we obtain $\ell_{\Uparrow_{m+1}}(\mathbf{p}_j) \geq 1/3$ for $j = 1, 2$. Since $(v(i_4), \dots, v(k)) \in \mathcal{CH}_m(T \cup R, p_{\swarrow})$, The induction hypothesis and Lemma 6.14 yield that $\ell_{\swarrow\!\!\swarrow_m}((v(i_4), \dots, v(k))) \geq 1$. By (6.2), it follows that

$$\ell_{\Uparrow_{m+1}}(\mathbf{p}_3) = \frac{1}{3}\ell_{\swarrow\!\!\swarrow_m}((v(i_4), \dots, v(k))) \geq 1/3.$$

Thus, we have shown that $\ell_{\Uparrow_{m+1}}(\mathbf{p}) \geq 1$.

Finally we show that (6.10) holds for $n = m+1$. Let $\mathbf{p} = (w(1), \dots, w(k)) \in \mathcal{CH}_{m+1}(T \cup R, p_{\swarrow})$. Note that $\xi_{m+1}(\mathbf{p}) \in \mathcal{CH}_{m+1}(T, p_{\swarrow})$ and $\ell_{\swarrow\!\!\swarrow_{m+1}}(\mathbf{p}) = \ell_{\swarrow\!\!\swarrow_{m+1}}(\xi_{m+1}(\mathbf{p}))$. Hence replacing $\mathbf{p}$ by $\xi_{m+1}(\mathbf{p})$, we may assume that $w(i) \in \{\nwarrow, \uparrow, \nearrow, \leftarrow, \swarrow\} \cdot W_m$ for any $i = 1, \dots, k$ without loss of generality. Set $w(i) = s(i)v(i)$, where $s(i) \in \{\nwarrow, \uparrow, \nearrow, \leftarrow, \swarrow\}$ and $v(i) \in W_m$. We may choose i_1, i_2, i_3 and i_4 which satisfies $i_1 < i_2 < i_3 < i_4$ and the following tree conditions (a3), (b3) and (c3):

(a3) $s(1), \dots, s(i_1) \in \{\nwarrow, \uparrow\}$, $(v(1), \dots, v(i_1)) \in \mathcal{CH}_m(T, B_{\swarrow})$,

(b3) $s(i) = \leftarrow$ for $i = i_2, \dots, i_3$, $(v(i_2), \dots, v(i_3)) \in \mathcal{CH}_m(T, B)$,

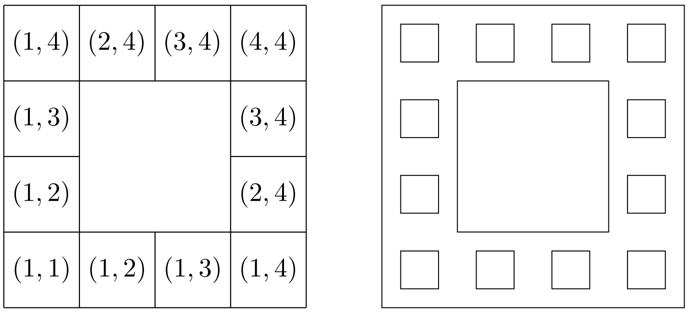
(c3) $s(i) = \swarrow$ for $i = i_4, \dots, k$. $(v(i_4), \dots, v(k)) \in \mathcal{CH}_m(T \cup R, p_{\swarrow})$.

Define $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_3$ as in the above two cases. Then by the induction hypothesis and (6.3), it follows that $\ell_{\swarrow\!\!\swarrow_{m+1}}(\mathbf{p}_j) \geq 1/3$ for $j = 2, 3$. Furthermore, Lemma 6.14 implies $\ell_{\swarrow\!\!\swarrow_{m+1}}(\mathbf{p}_1) \geq 1/3$. Hence we have $\ell_{\swarrow\!\!\swarrow_{m+1}}(\mathbf{p}) \geq 1$.

Thus we have obtained (6.8), (6.9) and (6.10) for $n = m+1$. □

Proof of Theorem 6.4. Since $A_n \subseteq \Updownarrow_n$, $\ell_{A_n}(\mathbf{p}) \geq 1$ for any $\mathbf{p} \in \mathcal{CH}_n(T, B)$. By the fact that $\Leftrightarrow_n = \rho(\Updownarrow_n)$, it follows that $\ell_{\Leftrightarrow_n}(\mathbf{p}) \geq 1$ for any $\mathbf{p} \in \mathcal{CH}_n(L, R)$. Hence $\ell_{A_n}(\mathbf{p}) \geq 1$ for any $\mathbf{p} \in \mathcal{CH}_n(L, R)$. Thus A_n is invisible. By Lemma 6.7-(1), it follows that A_n is $+$-invariant.

Lemma 6.3 shows that $8^n - \#(\Updownarrow_n) = \#(W_n \setminus \Updownarrow_n) = \alpha_n$. Since $W_n \setminus \Updownarrow_n \subseteq W_n \setminus A_n \subseteq (W_n \setminus \Updownarrow_n) \cup (W_n \setminus \Leftrightarrow_n)$, we have $\alpha_n \leq 8^n - \#(A_n) \leq 2\alpha_n$. □



$F_s([-1,1]^2)$ for $s \in S^{(4)}$ $\qquad$ $\bigcup_{w \in W_2} F_w([-1,1]^2)$

Figure 5: Construction of $K^{(4)}$

7 Generalized Sierpinski carpet

The idea of invisible sets can be exploited for the generalized Sierpinski carpets. We will present results for a special class of the generalized Sierpinski carpet in this section. We fix $N \geq 3$. The complex plane $\mathbb{C}$ is identified with $\mathbb{R}^2$ in the usual manner.

Definition 7.1. (1) For any $(i,j) \in \{1,\ldots,N\}^2$, we define $J_{(i,j)} = [-1 + 2(i-1)/N, -1 + 2i/N] \times [-1 + 2(j-1)/N, -1 + 2j/N]$ and $F_{(i,j)} : \mathbb{R}^2 \to \mathbb{R}^2$ by $F_{(i,j)}(x,y) = (x/N + a_{(i,j)}, y/N + b_{(i,j)})$, where $a_{(i,j)} = -1 + (2i-1)/N$ and $b_{(i,j)} = -1 + (2j-1)/N$.
(2) Define $S^{(N)} = \{(i,j) | (i,j) \in \{1,\ldots,N\}^2, i \in \{1,N\} \text{ or } j \in \{1,N\}\}$. Let $K^{(N)}$ be the unique compact set which satisfies

$$K^{(N)} = \bigcup_{(i,j) \in S^{(N)}} F_{(i,j)}\big(K^{(N)}\big).$$

When $N = 3$, $K^{(3)}$ is the Sierpinski carpet.

Proposition 7.2. *$\#(S^{(N)}) = 4N - 4$ and $\dim_H (K^{(N)}, d_E) = \dfrac{\log(4N-4)}{\log N}$, where d_E is the restriction of the Euclidean metric.*

In the following, we occasionally omit N in $S^{(N)}$ and $K^{(N)}$ and write them S and K respectively. Also we use W_m, W_* and Σ in place of $W_m(S^{(N)}), W_*(S^{(N)})$ and $\Sigma(S^{(N)})$.

Definition 7.3. Let $A \subseteq W_m$.
(1) Let $A \subseteq W_m$. For $\mathbf{p} = (w(1), \ldots, w(k)) \in \mathcal{CH}_m$, define

$$\ell_A(\mathbf{p}) = \frac{\#(\{i | i = 1, \ldots, k, w(i) \notin A\})}{N^m}.$$

(2) A is called an invisible set if and only if

$$\inf_{\mathbf{p}\in\mathcal{CH}_m(T,B)\cup\mathcal{CH}_m(L,R)} \ell_A(\mathbf{p}) \geq 1,$$

where T, B, L and R are the same as in the last tree sections.

We also define the notion of $+$-invariance exactly same as in the previous sections. Then the analogous results as Theorems 4.5 and 5.3 hold. As a consequence we have the following statement.

Theorem 7.4. *Let $A \subset W_m$ be a $+$-invariant invisible set. Then*

$$\dim_{\mathcal{C}}(K^{(N)}, d_E) \leq \frac{\log\left((4N-4)^m - \#(A)\right)}{m\log N}.$$

A procedure which is similar to that in Section 6 produces a sequence of invisible sets. We assume $N \geq 4$ hereafter. The maps $\varphi_{\leftrightarrow}$, $\varphi_{\updownarrow}$ and ρ from W_* to itself associated with symmetries can be defined in the same way as in the last section.

Definition 7.5. Define $\Updownarrow_n \subseteq W_n$ and $\Searrow_n \subseteq W_n$ inductively by

$$\begin{aligned}\Updownarrow_{n+1} = &\{(i,j)|(i,j)\in S. i\in\{1,N\}\}\cdot \Updownarrow_n \\ &\cup (2,1)\cdot \Searrow_n \cup (2,N)\cdot \Nearrow_n \cup (N-1,1)\cdot \Swarrow_n \cup (N-1,N)\cdot \Nwarrow_n \\ &\cup \{(i,j)|(i,j)\in S, j\in\{1,N\}, i\notin\{1,2,N-1,N\}\}\cdot W_n,\end{aligned}$$

$$\begin{aligned}\Searrow_{n+1} = &\{(1,N),(2,1),(N-1,N)\}\cdot \Searrow_n \\ &\cup \{(1,j)|j=1,\ldots,N-1\}\cdot \Updownarrow_n \cup \{(i,N)|i=2,\ldots,N\}\cdot \Leftrightarrow_n \\ &\cup \{(1,j)|j=3,\ldots,N\}\cdot W_n \cup \{(i,N)|i=1,\ldots,N-2\}\cdot W_n,\end{aligned}$$

$\Updownarrow_0 = \emptyset$ and $\Searrow_0 = \emptyset$, where $\Leftrightarrow_n = \rho(\Updownarrow_n)$, $\Swarrow_n = \varphi_{\leftrightarrow}(\Searrow_n)$, $\Nearrow_n = \varphi_{\updownarrow}(\Searrow_n)$ and $\Nwarrow_n = \varphi_{\leftrightarrow}(\Nearrow_n)$.

By the above definition, it follows that

$$\begin{aligned}x_{n+1} &= 2Nx_n + 4y_n + 2(N-4)(4N-4)^n \\ y_{n+1} &= 2(N-1)x_n + 3y_n + (2N-5)(4N-4)^n,\end{aligned}$$

where $x_n = \#(\Updownarrow_n)$ and $y_n = \#(\Searrow_n)$. Define

$$\tau_N = \sqrt{4N^2 + 20N - 23}.$$

Then we have

$$x_n = \\ (4N-4)^n - \Big(\frac{2N+5}{2\tau_N} + \frac{1}{2}\Big)\Big(\frac{2N+3+\tau_N}{2}\Big)^n + \Big(\frac{2N+5}{2\tau_N} - \frac{1}{2}\Big)\Big(\frac{2N+3-\tau_N}{2}\Big)^n$$

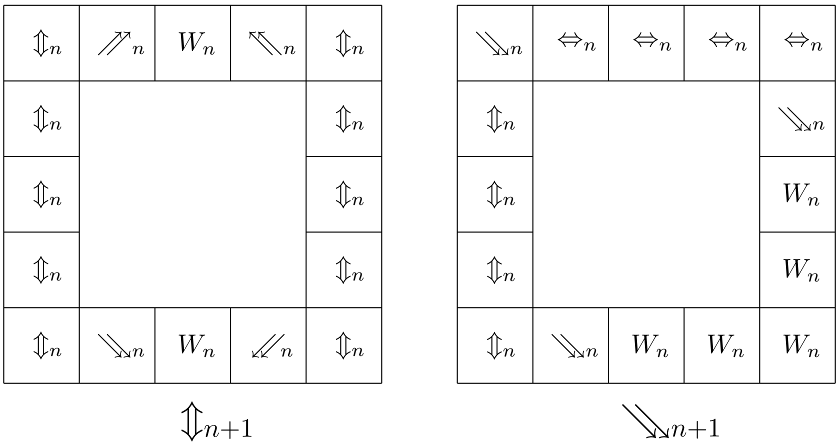


Figure 6: Construction of $\Updownarrow_n$ and $\searrow\!\!\searrow_n$ for $N = 5$

The same discussion as in the last section shows

$$\inf_{\mathbf{p} \in \mathcal{CH}_n(T,B)} \ell_{\Updownarrow_n}(\mathbf{p}) \geq 1.$$

Hence we obtain the counterpart of Theorem 6.4.

Theorem 7.6. *Let $A_n = \Updownarrow_n \cap \Leftrightarrow_n$. Then A_n is $+$-invariant invisible set and there exist $c_1, c_2 > 0$ such that*

$$c_1 \Big(\frac{2N + 3 + \tau_N}{2} \Big)^n \leq (4N - 4)^n - \#(A_n) \leq c_2 \Big(\frac{2N + 3 + \tau_N}{2} \Big)^n$$

for sufficiently large n.

As an corollary, we obtain the following estimate of the conformal dimension of $K^{(N)}$. The lower estimate is shown by applying [6, Example 4.1.9].

Corollary 7.7.

$$\frac{\log (2N)}{\log N} \leq \dim_{\mathcal{C}}(K^{(N)}, d_E) \leq \frac{\log \frac{2N+3+\tau_N}{2}}{\log N} < \frac{\log (4N - 4)}{\log N} = \dim_H (K^{(N)}, d_E).$$

Remark.

$$2N + 3 \leq \frac{2N + 3 + \tau_n}{2} < 2N + 4.$$

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