Resistance forms, quasisymmetric maps and heat kernel estimates

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Contents

1 Introduction

Originally, the main purpose of this paper is to give answers to the following two questions on heat kerenels associated with resistance forms or, in other words, strongly recurrent Hunt processes.

(I) When and how can we find a metric which is suitable for describing asymptotic behaviors of a given heat kernel?

(II) What kind of requirement for jumps of a process is necessary to ensure a good asymptotic behavior of the heat kernel associated with the process?

Eventually we are going to make these questions more precise. For the moment, let us explain what a heat kernel is. Assume that we have a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, \mu)$, where X is a metric space and μ is a Borel regular measure on X. Let L be the "Laplacian" associated wiht this Dirichlet form, i.e. Lv is characterized by the unique element in $L^2(X, \mu)$ which satisfies

$$\mathcal{E}(u, v) = -\int_X u(Lu)d\mu$$

for any $u \in \mathcal{F}$. A nonnegative measurable functon $p(t,x,y)$ on $(0,\infty)\times X\times X$ is called the heat kernel associated with the Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(X,\mu)$ if

$$u(t,x) = \int_X p(t,x,y)u(y)\mu(dy)$$

for any initial value $u \in L^2(X,\mu)$, where $u(t,x)$ is the solution of the heat equation associated with the Laplacian L:

$$\frac{\partial u}{\partial t} = Lu.$$

The heat kernel may not exist in general. However, it is know to exist in many cases like the Browinan motions on the Euclidean spaces, Reimannian manifolds and certain classes of fractals.

If the Dirichlet form $(\mathcal{E},\mathcal{F})$ has the local property, in other words, the corrresponding stochastic process is diffusion, then one of the preferable goals on an asymptotic estimate of a heat kernel is to show the so-called Li-Yau type (sub-)Gaussian estimate, which is

$$p(t,x,y) \asymp \frac{c_1}{V_d(x,t^{1/\beta})}\exp\left(-c_2\left(\frac{d(x,y)^\beta}{t}\right)^{1/(\beta-1)}\right), \tag{1.1}$$

where d is a metric on X, $V_d(x,r)$ is the volume of a ball $B_d(x,r)=\{y|d(x,y)<r\}$ and $\beta \geq 2$ is a constant. It is well-known that the heat kernel of the Brownian motion on $\mathbb{R}^n$ is Gaussian which is a special case of (1.1) with $d(x,y)=|x-y|$, $\beta = 2$ and $V_d(x,r) = r^n$. Li and Yau have shown in [38] that, for a complete Riemannian manifold with non-negative Ricci curvature, (1.1) holds with $\beta = 2$, where d is the geodesic metric and $V_d(x,r)$ is the Riemannian volume. In this case, (1.1) is called the Li-Yau type Gaussian estimate. Note that $V_d(x,t^{1/\beta})$ may have inhomogeneity with respect to x in this case. On the other hand, for fractals, Barlow and Perkins have shown in [9] that the Brownian motion on the Sierpinski gasket satisfies sub-Gaussian estimate, that is, (1.1) with $d(x,y) = |x-y|$, $\beta = \log 5/\log 2$ and $V_d(x,r) = r^\alpha$, where $\alpha = \log 3/\log 2$ is the Hausdorff dimension of the Sierpinski gasket. Note that $V_d(x,r)$ is homogeneous in this particular case. Full generality of (1.1) is realized, for example, by a certain time change of the Brownian motion on $[0,1]$, whose heat kernel satisfies (1.1) with $\beta > 2$ and inhomogeneous $V_d(x,r)$. See [35] for details.

There have been extensive studies on the conditions which is equivalent to (1.1). For Riemannian manifolds, Gregor'yan [19] and Saloff-Coste [42] have independently shown that the Li-Yau type Gaussian esitmate is equivalent to the Poincaré inequality and the voulme doubling property. For random walks on weighted graphs, Grigor'yan and Telcs have obtained several equivalent conditions for general Li-Yau type sub-Gaussian estimate, for example, the combination of the volume doubling property, the elliptic Harnack inequality and the Poincaré inequality in [21, 22]. Similar results have been obtained for diffusions. See [27] and [10] for example.

The importance of the Li-Yau type (sub-)Gaussian estimate (1.1) is that it describe the asymptotic behavior of analytical object, namely, the heat kernel $p(t, x, y)$ in terms of the geometrical objects like the metric d and the volume of a ball $V_d(x, r)$. Such an interpaly of analysis and geometry makes the study of heat kernels interesting. In this paper, we have resistance forms on the side of analysis and quasisymmetric maps on the side of geometry. To establish a foundation in studying heat kernel estimates, we first need to do considarable works on both sides, i.e. resistance forms and quasisymmetric maps. Those two subjects come to the other main parts of this paper as a consequence.

The theory of resistance forms has been developed to study analysis on "low-dimensional" fractals including the Sierpinski gasket, the 2-dimensional Sierpinski carpet, random Sierpinski gaskets and so on. Roughly, a non-negative quadratic form $\mathcal{E}$ on a subspace $\mathcal{F}$ of real-valued functions on a set X is called a resistance form on X if it has the Markov property and

$$\min\{\mathcal{E}(u, u) | u \in \mathcal{F}, u(x) = 1 \text{ and } u(y) = 0\}$$

exists and is positive for any $x \neq y \in X$. The reciprocal of the above minimum, denoted by $R(x, y)$, is known to be a metric (distance) and is called the resistance metric associated with $(\mathcal{E}, \mathcal{F})$. See [33] for details. In Part I, we are going to establish fundamental notions on resistance forms, for instance, the existence and properties of the Green function with an infinite set as a boundary, regularity of a resistance form, traces, the existence and continuity of heat kernels. More precisely, let μ be a Borel regular measure on (X, R). In Section 8, $(\mathcal{E}, \mathcal{F})$ is shown to be a regular Dirichlet form on $L^2(X, \mu)$ under weak assumptions. We also prove that the associated heat kernel $p(t, x, y)$ exists and is continuous on $(0, \infty) \times X \times X$ in Section 9.

The notion of quasisymmetric maps has been introduced by Tukia and Väisälä in [45] as a generalization of qusiconformal mappings in the complex plane. Soon its importance has been recognized in wide area of analysis and geometry. There have been many works on quasisymmetric maps since then. See Heinonen [28] and Semmes [43] for references. In this paper, we are going to modify the resistance metric R quasisymmetrically to obtain a new metric which is more suitable for describing an asymptotic behavior of the heat kernel. The key will be to realize the following relation:

$$\text{Resistance} \times \text{Volume} \asymp (\text{Distance})^{\beta}, \tag{1.2}$$

where "Volume" is the volume of a ball and "Distance" is the distance with repsect to the new metric. Quasisymmetric modification of a metric has many advantages. For example, it preserves the volume doubling property of a measure. In Part II, we will study quasisymmetric homeomorphisms on a metric space. In particular, we are going to establish relations between properties such as (1.2) concerning the original metric D, the quasisymmetrically modified metric d and the volume of a ball $V_d(x, r) = \mu(B_d(x, r))$ and show how to construct a metric d which is quasisymmetric to original metric D and satisfy a required property like (1.2).

Let us return to question (I). We will confine ourselves to the case of diffusion processes for simplicity. The lower part of the Li-Yau type (sub-)Gaussian estimate (1.1) is known to hold only when the distance is geodesic, i.e. any two points are connected by a geodesic curve. This is not the case for most of general metric spaces. So, we use an adequate substitute called near diagonal lower estimate, $(\mathrm{NDL})_{\beta,\mathrm{d}}$ for short. We say that $(\mathrm{NDL})_{\beta,\mathrm{d}}$ holds if and only if

$$\frac{c_3}{V_d(x, t^{1/\beta})} \leq p(t, x, y) \tag{1.3}$$

when $d(x, y)^\beta \leq c_4 t$. For upper estimate, the Li-Yau type (sub-)Gaussian upper estimate of order β, $(\mathrm{LYU})_{\beta,\mathrm{d}}$ for short, is said to hold if and only if

$$p(t, x, y) \leq \frac{c_5}{V_d(x, t^{1/\beta})} \exp\left(- c_6 \left(\frac{d(x, y)^\beta}{t} \right)^{1/(\beta-1)} \right). \tag{1.4}$$

Another important property is the doubling property of a heat kernel, (KD) for short, that is,

$$p(t, x, x) \leq c_7 p(2t, x, x). \tag{1.5}$$

Note that $p(t, x, x)$ is monotonically decreasing with respect to t. If is known that the Li-Yau type (sub-)Gaussian heat kernel estimate togother with the volume doublling property implies (KD). Let $p(t, x, y)$ be the heat kernel associated with a diffusion process. Now, the question (I) can be rephrased as follows:

Question When and how can we find a metric d under which $p(t, x, y)$ satisfy $(\mathrm{LYU})_{\beta,\mathrm{d}}$, $(\mathrm{NDL})_{\beta,\mathrm{d}}$ and (KD)?

In Corollary 14.12, we are going to answer this if the Dirichlet form associated with the diffusion process is derived from a resistance form. Roughly speaking, our answer is the following.

Answer The underlying measure μ has the volume doubling property with respect to the resistance metric R if and only if there exist $\beta > 1$ and a metric d which is quaisymmetric with respect to R such that $(\mathrm{LYU})_{\beta,\mathrm{d}}$, $(\mathrm{NDL})_{\beta,\mathrm{d}}$ and (KD) hold.

Of course, one can ask the same question for general diffusion process with a heat kernel. Such a problem is very interesting. In this paper, however, we only consider the case where the process is associate with a resistance form.

Next, we are going to explain the second problem, the question (II). Recently, there have been many results on an asymtotic behavior of a heat kernel associated with a jump process. See [11, 13, 14, 5] for example. They have dealt with a specific class of jump processes and studied a set of conditions which is equivalent to certain kind of (off-diagonal) heat kernel estimate. For example, in [13], they have shown the existence of jointly continuous heat kernel for an generalization of α-stable process on an Ahlfors regular set and give a condition for best possible off-diagonal heat kernel estimate. In this paper, we will only consider the following Li-Yau type on-diagonal estimate, $(\mathrm{LYD})_{\beta,\mathrm{d}}$ for short,

$$p(t, x, x) \asymp \frac{1}{V_d(x, t^{1/\beta})} \tag{1.6}$$

which is the diagonal part of (1.1). Our question is

Question When and how can we find a metric d with $(\mathrm{LYD})_{\beta,\mathrm{d}}$ for a given (jump) process which posesses a heat kernel?

In this case, the "when" part of the question includes the study of the requirement on jumps. In this paper, again we confine our selves to the case when a process is associated with a resistance form. Our proposal for a condition on jumps is the annulus comparable condition, (ACC) for short, which says that the resistance between a point and the complement of a ball is comparable with the resistance between a point and an annulus. More exactly, (ACC) is formulated as

$$R(x, B_R(x,r)^c) \asymp R(x, A_R(x, r, (1+\epsilon)r)) \tag{1.7}$$

for some $\epsilon > 0$, where R is a resistance metric, $B_R(x,r)$ is a resistance ball and $A_R(x, r, (1+\epsilon)r) = \overline{B_R(x,(1+\epsilon)r)} \backslash B_R(x,r)$ is an annulus. If the process in question has no jump, i.e. is a diffusion process, then the quantities in the both sides of (1.7) coincide and hence (ACC) holds. As our answer to the above question, we obtain the following statement in Theorem 14.11:

Theorem 1.1. *The following three conditions are equivalent:*
(C1) *The underlying measure μ has the volume doubling property with respect to R and* (ACC) *holds.*
(C2) *The underlying measure μ has the volume doubling property with respect to R and the so-called "Einstein relation":*

$$\text{Resistance} \times \text{Volume} \asymp \textit{Average escape time}$$

holds for the resistance metric.
(C3) (ACC) *and* (KD) *is satisfied and there exist $\beta > 1$ and a metric d which is quasisymmetric with respect to R such that* $(\mathrm{LYD})_{\beta,\mathrm{d}}$ *holds.*

See [22, 44] on the Einstein relation, which is known to be implied by the Li-Yau type (sub-)Gaussian heat kernel estimate.

Our work on heat kernel estimates is largely inspired by the previous two papers [6] and [37]. In [6], the strongly recurrent random walk on infinite graph has been studied by using two different metrics, one is the shortest path metric d on a graph and the other is the resistance metric R. It has shown that the condition $\mathrm{R}(\beta)$, that is,

$$R(x,y)V_d(x, d(x,y)) \asymp d(x,y)^\beta$$

is essentially equivalent to the random walk version of (1.1). Note the resemblance between (1.2) and $\mathrm{R}(\beta)$. The metric d is however fixed in their case. In [37], Kumagai has studied the (strongly recurrent) diffusion process associated with a resistance form using the resistance metric R. He has shown that the uniform doubling property with respect to R is equivalent to the combination of natural extensions of $(\mathrm{LYU})_{\beta,\mathrm{d}}$ and $(\mathrm{NDL})_{\beta,\mathrm{d}}$ with respect to R. Examining those results carefully from geometrical view point, we have realized that quasisymmetric change of metrics (implicitly) plays an important role. In this respect, this paper can be though of an extension of those works.

There is another closely related work. In [30], a problem which is very similar to our question (I) has been investigated for a heat kernel associated with a self-similar Dirichlet form on a self-similar set. The result in [30] is also quite similar to ours. It has been shown that the volume doubling property of the underlying measure is equivalent to the existence of a metric with $(\mathrm{LYD})_{\beta,\mathrm{d}}$. Note that the results in [30] include higher dimensional Sierpinski carpets where the self-similar Dirichlet forms are not resistance forms. The processes studied in [30], however, have been all diffusions

Finally, we present one application of our results to an α-stable process on $\mathbb{R}$ for $\alpha \in (1, 2]$. Define

$$\mathcal{E}^{(\alpha)}(u,u) = \int_{\mathbb{R}^2} \frac{(u(x)-u(y))^2}{|x-y|^{1+\alpha}} dxdy$$

and $\mathcal{F} = \{u|\mathcal{E}^{(\alpha)}(u,u) < +\infty\}$ for $\alpha \in (1,2)$ and $(\mathcal{E}^{(2)}, \mathcal{F}^{(2)})$ is the ordinary Dirichlet form associated with the Brownian motion on $\mathbb{R}$. Then $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ is a resistance form for $\alpha \in (1,2]$ and the associated resistance metric is $c|x-y|^{\alpha-1}$. If $\alpha \neq 2$, then the corresponding process is not a diffusion but has jumps. If $p^{(\alpha)}(t,x,x)$ is the associated heat kernel, it is well known that $p^{(\alpha)}(t,x,x) = ct^{1/\alpha}$. Let $(\mathcal{E}^{(\alpha)}|_K, \mathcal{F}^{(\alpha)}|_K)$ be the trace of $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ onto the ternary Cantor set K. Let $p_K^{(\alpha)}(t,x,y)$ be the heat kernel associated with the Dirichlet form $(\mathcal{E}^{(\alpha)}|_K, \mathcal{F}^{(\alpha)}|_K)$ on $L^2(K,\nu)$, where ν is the normalized Hausdorff measure of K. By Theorem 14.13, we may confirm that (ACC) holds and obtain

$$p_K^{(\alpha)}(t,x,x) \asymp t^{-\eta},$$

where $\eta = \frac{\log 2}{(\alpha-1)\log 3 + \log 2}$. See Section 15 for details.

This paper consists of four parts. In Part I, we will develop basic theory of resistance forms regarding the Green function, trace of a form, regularity and heat kerenels. This part is the foundation of the discussion in Part III. Part II is devoted to studying quasisymmetric homeomorphisms. This is another foundation of the discussion in Part III. After preparing those basics, we will consider heat kernel estimates in Part III. Finally in Part IV, we consider estimates of heat kernels on random Sierpinski gaskets as an application of the theorems in Part III.

The followings are conventions in notations in this paper.

(1) Let f and g be functions with variables $x_1, \ldots, x_n$. We use "$f \asymp g$ for any $(x_1, \ldots, x_n) \in A$" if and only if there exist positive constants c_1 and c_2 such that

$$c_1 f(x_1, \ldots, x_n) \leq g(x_1, \ldots, g_n) \leq c_2 f(x_1, \ldots, x_n)$$

for any $(x_1, \ldots, x_n) \in A$.

(2) The lower case c (with or without a subscript) represents a constant which is independent of the variables in question and may have different values from place to place (even in the same line).

2 List of frequently used abbreviations

(ACC): Annulus comparable condition, Definition 6.1.
$(\mathrm{VD})_{\mathrm{d}}$: Volume doubling property with respect to d, Definition 6.5-(2).
(RES): Resistance estimate, Definition 6.9.
$(\mathrm{SQS})_{\mathrm{d}}$: Semi-quasisymmetric, Definition 10.1.
$(\mathrm{SQC})_{\mathrm{d}}$: Semi-quasiconformal, Definition 10.4-(2).
$(\mathrm{ASQC})_{\mathrm{d}}$: Annulus semi-qausiconformal, Definition 10.4-(3).
$d \underset{\mathrm{QS}}{\sim} \rho$: quasisymmetric, Definition 11.1.
(DM1): Definition 12.1-(1).
(DM2): Definition 12.1-(2).
(DM3): Definition 12.1-(3).
$(\mathrm{EIN})_{\mathrm{d}}$: the Einstein relation, Definition 14.3-(2).
$(\mathrm{DHK})_{\mathrm{g,d}}$: on-diagonal heat kernel estimate, Definition 14.9-(1).
(KD): Doubling property of the heat kernel, Definition 14.9-(2).
$(\mathrm{DM1})_{\mathrm{g,d}}$: Definition 14.9-(3).
$(\mathrm{DM2})_{\mathrm{g,d}}$: Definition 14.9-(4).
$(\mathrm{HK})_{\mathrm{g,d}}$: Theorem 14.10.
(EL): Theorem 22.2.
(GE): Theorem 22.2.

Part I
Resistance forms and heat kernels

In this part, we will establish basics of resistance forms such as the Green function, harmonic functions, traces and heat kernels. In the previous papers [33, 31, 34], we have established the notions of the Green function, harmonic functions and traces if a boundary is a finite set. One of the main subjects is to extend those results to the case where a boundary is an infinite set. In fact, this is more than a matter of extesion but at first we should determine what kind of an infinite set can be regarded as a proper boundary. Moreover, we will establish the existence of jointly continuous heat kernel associated with the Dirichlet form derived from a resistance form under several mild assumptions, which do not include the ultracontractivity.

The followings are basic notations used in this paper.

Notation. (1) For a set V, we define $\ell(V) = \{f | f : V \to \mathbb{R}\}$. If V is a finite set, $\ell(V)$ is considered to be equipped with the standard inner product $(\cdot, \cdot)_V$ defined by $(u, v)_V = \sum_{p \in V} u(p)v(p)$ for any $u, v \in \ell(V)$. Also $|u|_V = \sqrt{(u, u)_V}$ for any $u \in \ell(V)$.
(2) Let V be a finite set. The characteristic function χ_U^V of a subset $U \subseteq V$ is

defined by

$$\chi_U^V(q) = \begin{cases} 1 & \text{if } q \in U, \\ 0 & \text{otherwise.} \end{cases}$$

If no confusion can occur, we write χ_U instead of χ_U^V. If $U = \{p\}$ for a point $p \in V$, we write χ_p instead of $\chi_{\{p\}}$. If $H : \ell(V) \to \ell(V)$ is a linear map, then we set $H_{pq} = (H\chi_q)(p)$ for $p, q \in V$. For $f \in \ell(V)$, $(Hf)(p) = \sum_{q \in V} H_{pq} f(q)$.
(3) Let (X, d) be a metric space. Then

$$B_d(x, r) = \{y | y \in X, d(x, y) < r\}$$

for $x \in X$ and $r > 0$.

3 Topology associated with a subspace of functions

In this section, we will introduce the operation $B \to B^{\mathcal{F}}$ from subsets of a space X to itself associated with a linear subspace $\mathcal{F}$ of real valued functions $\ell(X)$. This operation will turn out to be essential in describing whether a set can be treated as a boundary or not.

Definition 3.1. Let $\mathcal{F}$ be a linear subspace of $\ell(X)$ for a set X. For a subset $B \subseteq X$, define

$$\mathcal{F}(B) = \{u | u \in \mathcal{F}, u(x) = 0 \text{ for any } x \in B\}.$$

and

$$B^{\mathcal{F}} = \bigcap_{u \in \mathcal{F}(B)} u^{-1}(0)$$

The following lemma is immediate from the definition.

Lemma 3.2. *Let $\mathcal{F}$ be a linear subspace of $\ell(X)$ for a set X.*
(1) *For any $B \subseteq X$, $B \subseteq B^{\mathcal{F}}$, $\mathcal{F}(B) = \mathcal{F}(B^{\mathcal{F}})$ and $(B^{\mathcal{F}})^{\mathcal{F}} = B^{\mathcal{F}}$.*
(2) *$X^{\mathcal{F}} = X$.*
(3) *$\emptyset^{\mathcal{F}} = \emptyset$ if and only if $\{u(x) | u \in \mathcal{F}\} = \mathbb{R}$ for any $x \in X$.*

The above lemma suggests that the operation $B \to B^{\mathcal{F}}$ satisfy the axiom of closure and hence it defines a topology on X. Indeed, this is the case if $\mathcal{F}$ is stable under the unit contraction.

Theorem 3.3. *Let $\mathcal{F}$ be a linear subspace of $\ell(X)$ for a set X. Assume that $\{u(x) | u \in \mathcal{F}\} = \mathbb{R}$ for any $x \in X$ and that $\overline{u} \in \mathcal{F}$ for any $u \in \mathcal{F}$, where $\overline{u}$ is defined by*

$$\overline{u}(p) = \begin{cases} 1 & \text{if } u(p) \geq 1, \\ u(p) & \text{if } 0 < u(p) < 1, \\ 0 & \text{if } u(p) \leq 0. \end{cases}$$

Then $\mathcal{C}_{\mathcal{F}} = \{B | B \subseteq X, B^{\mathcal{F}} = B\}$ satisfies the axiom of closed sets and it defines a topology of X. Moreover, the T_1-axiom of separation holds for this topology, i.e. $\{x\}$ is a closed set for any $x \in X$, if and only if, for any $x, y \in X$ with $x \neq y$, there exists $u \in \mathcal{F}$ such that $u(x) \neq u(y)$.

$\mathcal{F}$ is said to be stable under the unit contraction if $\overline{u} \in \mathcal{F}$ for any $u \in \mathcal{F}$.

Lemma 3.4. *Under the assumptions of Theorem 3.3, if $x \in X, B \in \mathcal{C}_{\mathcal{F}}$ and $x \notin B$, then there exists $u \in \mathcal{F}$ such that $u \in \mathcal{F}(B)$, $u(x) = 1$ and $0 \leq u(y) \leq 1$ for any $y \in X$.*

Proof. Since $B^{\mathcal{F}} = B$, there exists $v \in \mathcal{F}(B)$ such that $v(x) \neq 0$. Let $u = v/v(x)$. Then u satisfies the required properties. □

Proof of Theorem 3.3. First we show that $\mathcal{C}_{\mathcal{F}}$ satisfies the axiom of closed sets. Since $\mathcal{F}(X) = \{0\}$, $X^{\mathcal{F}} = 0$. Also we have $\emptyset^{\mathcal{F}} = \emptyset$ by Lemma 3.2-(3). Let $B_i \in \mathcal{C}_{\mathcal{F}}$ for $i = 1, 2$ and let $x \in (B_1 \cup B_2)^c$, where A^c is the complement of A in X, i.e. $A^c = X \backslash A$. By Lemma 3.4, there exists $u_i \in \mathcal{F}(B_i)$ such that $\overline{u_i} = u_i$ and $u_i(x) = 1$. Let $v = u_1 + u_2 - 1$. Then $v(x) = 1$ and $v(y) \leq 0$ for any $y \in B_1 \cup B_2$. If $u = \overline{v}$, then $u \in \mathcal{F}(B_1 \cup B_2)$ and $u(x) = 1$. Hence $B_1 \cup B_2 \in \mathcal{C}_{\mathcal{F}}$. Let $B_\lambda \in \mathcal{C}_{\mathcal{F}}$ for any $\lambda \in \Lambda$. Set $B = \cap_{\lambda \in \Lambda} B_\lambda$. If $x \notin B$, then there exists $\lambda_* \in \Lambda$ such that $x \notin B_{\lambda_*}$. We have $u \in \mathcal{F}(B_\lambda) \subseteq \mathcal{F}(B)$ satisfying $u(x) \neq 0$. Hence $x \notin B^{\mathcal{F}}$. This shows $B \in \mathcal{C}_{\mathcal{F}}$. Thus we have shown that $\mathcal{C}_{\mathcal{F}}$ satisfies the axiom of closed sets.

Next define $U_{x,y} = \left\{ \begin{pmatrix} f(x) \\ f(y) \end{pmatrix} \middle| f \in \mathcal{F} \right\}$. We will show that $U_{x,y} = \mathbb{R}^2$ if there exists $u \in \mathcal{F}$ such that $u(x) \neq u(y)$. Suppose that $u(x) \neq 0$. Considering $u/u(x)$, we see that $\begin{pmatrix} 1 \\ a \end{pmatrix} \in U_{x,y}$, where $a \neq 1$. Since there exists $v \in \mathcal{F}$ with $v(y) \neq 0$, it follows that $\begin{pmatrix} b \\ 1 \end{pmatrix} \in U_{x,y}$ for some $b \in \mathbb{R}$. Now we have five cases.

Case 1: Assume that $a \leq 0$. Considering the operation of $\overline{u}$ for $u \in \mathcal{F}$, we have $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in U_{x,y}$. Also $\begin{pmatrix} b \\ 1 \end{pmatrix} \in U_{x,y}$. Since $U_{x,y}$ is a linear subspace of $\mathbb{R}^2$, $U_{x,y}$ coincides with $\mathbb{R}^2$.

Case 2: Assume that $b \leq 0$. By the similar argument as Case 1, we have $U_{x,y} = \mathbb{R}^2$.

Case 3: Assume that $b \geq 1$. The $\overline{u}$-operation shows that $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in U_{x,y}$. Since $\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ a \end{pmatrix} \right)$ is independent, $U_{x,y} = \mathbb{R}^2$.

Case 4: Assume that $a \in (0, 1)$ and $b \in (0, 1)$. Then $\left(\begin{pmatrix} 1 \\ a \end{pmatrix}, \begin{pmatrix} b \\ 1 \end{pmatrix} \right)$ is independent. Hence $U_{x,y} = \mathbb{R}^2$.

Case 5: Assume that $a > 1$ and $b \in (0, 1)$. The $\overline{u}$-operation shows $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in U_{x,y}$. Then $\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} b \\ 1 \end{pmatrix} \right)$ is independent and hence $U_{x,y} = \mathbb{R}^2$.

Thus $U_{x,y} = \mathbb{R}^2$ in all the cases. Exchanging x and y, we also deduce the same conclusion even if $u(x) = 0$. In particular, the fact that $U_{x,y} = \mathbb{R}^2$ implies that $y \notin \{x\}^{\mathcal{F}}$. Hence if there exists $u \in \mathcal{F}$ such that $u(x) \neq u(y)$ for any $x, y \in X$ with $x \neq y$, then $\{x\} \in \mathcal{C}_{\mathcal{F}}$ for any $x \in X$. The converse direction is immediate. □

4 Resistance forms and the Green functions

In this section, we first introduce definition and basics on resistance forms and then study the Green function associated with an infinite set as a boundary. In the course of discussion, we will show that a set B is a suitable boundary if and only if $B^{\mathcal{F}} = B$.

Definition 4.1 (Resistance form)**.** Let X be a set. A pair $(\mathcal{E}, \mathcal{F})$ is called a resistance form on X if it satisfies the following conditions (RF1) through (RF5).
(RF1) $\mathcal{F}$ is a linear subspace of $\ell(X)$ containing constants and $\mathcal{E}$ is a non-negative symmetric quadratic form on $\mathcal{F}$. $\mathcal{E}(u, u) = 0$ if and only if u is constant on X.
(RF2) Let $\sim$ be an equivalent relation on $\mathcal{F}$ defined by $u \sim v$ if and only if $u - v$ is constant on X. Then $(\mathcal{F}/{\sim}, \mathcal{E})$ is a Hilbert space.
(RF3) If $x \neq y$, then there exists $u \in \mathcal{F}$ such that $u(x) \neq u(y)$.
(RF4) For any $p, q \in X$,

$$\sup\{\frac{|u(p) - u(q)|^2}{\mathcal{E}(u, u)} : u \in \mathcal{F}, \mathcal{E}(u, u) > 0\}$$

is finite. The above supremum is denoted by $R_{(\mathcal{E},\mathcal{F})}(p, q)$.
(RF5) $\overline{u} \in \mathcal{F}$ and $\mathcal{E}(\bar{u}, \bar{u}) \leq \mathcal{E}(u, u)$ for any $u \in \mathcal{F}$.

By (RF3) and (RF5) along with Theorem 3.3, the axiom of closed sets holds for $\mathcal{C}_{\mathcal{F}}$ and the associated topology satisfies the T_1-separation axiom.

Proposition 4.2. *Assume that $\overline{u} \in \mathcal{F}$ for any $u \in \mathcal{F}$. Then* (RF3) *in the above definition is equivalent to the following conditions:*
(RF3-1) *$F^{\mathcal{F}} = F$ for any finite subset $F \subseteq X$.*
(RF3-2) *For any finite subset $F \subset X$ and any $v \in \ell(F)$, there exists $u \in \mathcal{F}$ such that $u|_F = v$.*

Proof. (RF3) ⇒ (RF3-1) By Theorem 3.3, (RF3) implies that $\{x\}^{\mathcal{F}} = \{x\}$ for any $x \in X$. Let F be a finite subset of X. Again by Theorem 3.3, $F^{\mathcal{F}} = (\cup_{x \in F}\{x\})^{\mathcal{F}} = \cup_{x \in F}\{x\}^{\mathcal{F}} = F$.
(RF3-1) ⇒ (RF3-2) Let F be a finite subset of X. Set $F_x = F\backslash\{x\}$ for $x \in F$. Since $(F_x)^{\mathcal{F}} = F_x$, there exists $u_x \in \mathcal{F}$ such that $u_x|_{F_x} \equiv 0$ and $u_x(x) = 1$. For any $v \in \ell(F)$, define $u = \sum_{x \in F} v(x)u_x$. Then $u|_F = v$ and $u \in \mathcal{F}$.
(RF3-2) ⇒ (RF3) This is obvious. □

Remark. In the previous literatures [33, 31, 34], (RF3-2) was employed as a part of the definition of resistance forms in place of the current (RF3).

It is known that the supremum in (RF4) is the maximum for a resistance form and $R_{(\mathcal{E},\mathcal{F})}$ is a metric on X. See [33] for example. We use R to denote $R_{(\mathcal{E},\mathcal{F})}$ and call it the resistance metric associated with $(\mathcal{E},\mathcal{F})$. By (RF4), we immediately obtain the following fact.

Proposition 4.3. *Let $(\mathcal{E}, F)$ be a resistance form on X and let R be the associated resistance metric. For any $x, y \in X$ and any $u \in \mathcal{F}$,*

$$|u(x) - u(y)|^2 \leq R(x,y)\mathcal{E}(u,u). \tag{4.1}$$

In particular, $u \in F$ is continuous with respect to the resistance metric.

Next we recall the notion of Laplacians on a finite set and harmonic functions with a finite set as a boundary. See [33, Section 2.1] for details, in particular, the proofs of Proposition 4.5 and 4.6.

Definition 4.4. Let V be a non-empty finite set. Recall that $\ell(V)$ is equipped with the standard inner-product $(\cdot,\cdot)_V$. A symmetric linear operator $H : \ell(V) \to \ell(V)$ is called a Laplacian on V if it satisfies the following three conditions:

(L1) H is non-positive definite,

(L2) $Hu = 0$ if and only if u is a constant on V,

(L3) $H_{pq} \geq 0$ for all $p \neq q \in V$.

We use $\mathcal{LA}(V)$ to denote the collection of Laplacians on V.

The next proposition says that a resistance form on a finite set corresponds to a Laplacian.

Proposition 4.5. *Let V be a non-empty finite set and let H be a linear operator form $\ell(V)$ to itself. Define a symmetric bilinear form $\mathcal{E}_H$ on $\ell(V)$ by $\mathcal{E}_H(u,v) = -(u, Hv)_V$ for any $u, v \in \ell(V)$. Then, $\mathcal{E}_H$ is a resistance from on V if and only if $H \in \mathcal{LA}(V)$.*

The harmonic function with a finite set as a boundary is defined as the energy minimizing function.

Proposition 4.6. *Let $(\mathcal{E},\mathcal{F})$ be a resistance from on X and let V be a finite subset of X. Let $\rho \in \ell(V)$. Then there exists a unique $u \in \mathcal{F}$ such that $u|_V = \rho$ and u attains the following minimum:*

$$\min\{\mathcal{E}(v,v)|v \in \mathcal{F}, v|_V = \rho\}.$$

Moreover, the map from ρ to u is a linear map from $\ell(V)$ to $\mathcal{F}$. Denote this map by h_V. Then there exists a Laplacian $H \in \mathcal{LA}(V)$ such that

$$\mathcal{E}_H(\rho,\rho) = \mathcal{E}(h_V(\rho), h_V(\rho)). \tag{4.2}$$

Definition 4.7. $h_V(\rho)$ defined in Proposition 4.6 is called the V-harmonic function with the boundary value ρ. Also we denote the above $H \in \mathcal{LA}(V)$ by $H_{(\mathcal{E},\mathcal{F}),V}$.

Hereafter in this section, $(\mathcal{E},\mathcal{F})$ is always a resistance form on a set X and $R(\cdot,\cdot)$ is the resistance metric associated with $(\mathcal{E},\mathcal{F})$.

Proposition 4.8. *$B^{\mathcal{F}}$ is a closed set with respect to the resistance metric R. In other word, the topology associated with $\mathcal{C}_{\mathcal{F}}$ is weaker than that given by the resistance metric.*

Proof. Let $\{x_n\}_{n\geq 1} \subset B^{\mathcal{F}}$. Assume $\lim_{n\to\infty} R(x, x_n) = 0$. If $u \in \mathcal{F}(B)$, then $u(x) = \lim_{n\to\infty} u(x_n) = 0$ for any $u \in \mathcal{F}(B)$. Hence $x \in B^{\mathcal{F}}$. $\square$

The next theorem establishes the existence and basic properties of the Green function with an infinite set as a boundary.

Theorem 4.9. *Let $B \subseteq X$ be non-empty. Then $(\mathcal{E},\mathcal{F}(B))$ is a Hilbert space and there exists a unique $g_B : X \times X \to \mathbb{R}$ that satisfies the following condition* (GF1)*:*
(GF1) *Define $g_B^x(y) = g_B(x,y)$. For any $x \in X$, $g_B^x \in \mathcal{F}(B)$ and $\mathcal{E}(g_B^x, u) = u(x)$ for any $u \in \mathcal{F}(B)$.*

Moreover, g_B satisfies the following properties (GF2), (GF3) *and* (GF4)*:*
(GF2) *$g_B(x,x) \geq g_B(x,y) = g_B(y,x) \geq 0$ for any $x,y \in X$. $g_B(x,x) > 0$ if and only if $x \notin B^{\mathcal{F}}$.*
(GF3) *Define $R(x,B) = g_B(x,x)$ for any $x \in X$. If $x \notin B^{\mathcal{F}}$, then*

$$\begin{aligned} R(x,B) &= \big(\min\{\mathcal{E}(u,u) | u \in \mathcal{F}(B), u(x) = 1\}\big)^{-1} \\ &= \sup\left\{\frac{|u(x)|^2}{\mathcal{E}(u,u)}\middle| u \in \mathcal{F}(B), u(x) \neq 0\right\}. \end{aligned}$$

(GF4) *For any $x,y,z \in X$, $|g_B(x,y) - g_B(x,z)| \leq R(y,z)$.*

By (GF2), if $B \neq B^{\mathcal{F}}$, then $g_B^x \equiv 0$ for any $x \in B^{\mathcal{F}}\backslash B$. Such a set B is not a good boundary.

We will prove this and the next theorem at the same time.

Definition 4.10. The function $g_B(\cdot,\cdot)$ given in the above theorem is called the Green function associated with the boundary B or the B-Green function.

The next theorem assures another advantage of being $B = B^{\mathcal{F}}$. Namely, if $B = B^{\mathcal{F}}$, we may reduce B to a one point, consider the "shorted" resistance form $(\mathcal{E},\mathcal{F}^B)$ and obtain a expression of the Green function (4.3) by the "shorted" resistance metric $R_B(\cdot,\cdot)$.

Theorem 4.11. *Let $(\mathcal{E},\mathcal{F})$ be a resistance form and let $B \subseteq X$ be non-empty. Suppose that $B^{\mathcal{F}} = B$. Set*

$$\mathcal{F}^B = \{u | u \in \mathcal{F}, u \text{ is a constant on } B\}$$

and $X_B = \{B\} \cup (X \backslash B)$. *Then* $(\mathcal{E}, \mathcal{F}^B)$ *is a resistance form on* X_B. *Furthermore, if* $R_B(\cdot,\cdot)$ *is the resistance metric associated with* $(\mathcal{E}, \mathcal{F}^B)$, *then*

$$g_B(x,y) = \frac{R_B(x,B) + R_B(y,B) - R_B(x,y)}{2} \tag{4.3}$$

for any $x, y \in X$. *In particular,* $R(x,B) = R_B(x,B)$ *for any* $x \in X \backslash B$.

Remark. In [40, Section 3], V. Metz has shown (4.3) in the case where B is a one point.

The proofs of the those two theorems are divided into several parts.

Note that B is closed with respect to R if $B^{\mathcal{F}} = B$.

Proof of the first half of Theorem 4.9. Let $x \in B$. By (RF2), $(\mathcal{E}, \mathcal{F}(x))$ is a Hilbert space, where $\mathcal{F}(x) = \mathcal{F}(\{x\})$. Note that $\mathcal{F}(B) \subseteq \mathcal{F}(x)$. If $\{u_m\}_{m \geq 1}$ is a Cauchy sequence in $\mathcal{F}(B)$, there exists the limit $u \in \mathcal{F}(x)$. For $y \in B$,

$$|u_m(y) - u(y)|^2 \leq R(x,y)\mathcal{E}(u_m - u, u_m - u).$$

Letting $m \to \infty$, we see that $u(y) = 0$. Hence $u \in \mathcal{F}(B)$. This shows that $(\mathcal{E}, \mathcal{F}(B))$ is a Hilbert space. For any $z \in X$ and any $u \in \mathcal{F}(B)$, $|u(z)|^2 \leq R(x,y)\mathcal{E}(u,u)$. The map $u \to u(z)$ is continuous linear functional and hence there exists a unique $\varphi_z \in \mathcal{F}(B)$ such that $\mathcal{E}(\varphi_z, u) = u(z)$ for any $u \in \mathcal{F}(B)$. Define $g_B(z,w) = \varphi_z(w)$. Since $\mathcal{E}(\varphi_z, \varphi_w) = \varphi_z(w) = \varphi_w(z)$, we have (GF1) and $g_B(z,w) = g_B(w,z)$. If $z \in B^{\mathcal{F}}$, then $u(z) = 0$ for any $u \in \mathcal{F}(B)$. Hence $g_B(z,z) = g_B^z(z) = 0$. Conversely, assume $g_B(z,z) = 0$. Since $g_B(z,z) = \mathcal{E}(g_B^z, g_B^z)$, (RF1) implies that g_B^z is constant on X. On the other hand, $g_B^z(y) = 0$ for any $y \in B$. Hence $g_B^z \equiv 0$. For any $u \in \mathcal{F}(B)$, $u(z) = \mathcal{E}(g_B^z, u) = 0$. Therefore, $z \in B^{\mathcal{F}}$. □

Lemma 4.12. *Let* $B \subseteq X$ *be non-empty. Define* $u_*(y) = g_B^x/g_B(x,x)$ *for* $x \notin B^{\mathcal{F}}$. *Then* u_* *is the unique element which attains the following minimum:*

$$\min\{\mathcal{E}(u,u) | u \in \mathcal{F}(B), u(x) = 1\}.$$

In particular, (GF3) *holds.*

Proof. Let $u \in \mathcal{F}(B)$ with $u(x) = 1$. Since

$$\mathcal{E}(u - u_*, u_*) = \frac{\mathcal{E}(u - u_*, g_B^x)}{g_B(x,x)} = \frac{(u(x) - u_*(x))}{g_B(x,x)} = 0,$$

we have

$$\mathcal{E}(u,u) = \mathcal{E}(u - u_*, u - u_*) + \mathcal{E}(u_*, u_*)$$

Hence $\mathcal{E}(u,u) \geq \mathcal{E}(u_*, u_*)$ and if the equality holds, then $u = u_*$. Now,

$$\mathcal{E}(u_*, u_*) = \frac{\mathcal{E}(g_B^x, g_B^x)}{g_B(x,x)^2} = \frac{1}{g_B(x,x)}.$$

This suffices for (GB3). □

Definition 4.13. Let $B \subseteq X$ be non-empty. If $x \notin B^{\mathcal{F}}$, we define $\psi_x^B = g_B^x / g_B(x,x)$.

Lemma 4.14. *Let $B \subseteq X$ be non-empty. Then $g_B(x,x) \geq g_B(x,y) \geq 0$ for any $x, y \in X$.*

Proof. If $x \in B^{\mathcal{F}}$, then $g_B^x \equiv 0$. Otherwise, set $u_*(y) = g_B^x(y)/g_B(x,x)$. Define $v = \overline{u_*}$. Then by (RF5), $\mathcal{E}(u_*, u_*) \geq \mathcal{E}(v,v)$. The above lemma shows that $u_* = v$. Hence $0 \leq u_* \leq 1$. □

So far, we have obtained (GF1), (GF2) and (GF3). Before showing (GF4), we prove Theorem 4.11.

Proof of Theorem 4.11. (RF1), (RF2) and (RF5) are immediate by the definition of $\mathcal{F}^B$. To show (RF3), let x and $y \in X$ with $x \neq y$. We may assume $y \neq B$ without loss of generality. Set $B_x = B \cup \{x\}$. Since $(B_x)^{\mathcal{F}} = B_x$, there exists $u \in \mathcal{F}(B_x)$ such that $u(y) \neq 0$. Hence we obtain (RF3). To see (RF4), note that

$$\sup\left\{\frac{|u(x)-u(y)|^2}{\mathcal{E}(u,u)}\middle| u \in \mathcal{F}^B, \mathcal{E}(u,u) > 0\right\} \leq R_{(\mathcal{E},\mathcal{F})}(x,y)$$

because $\mathcal{F}^B \subseteq \mathcal{F}$. Hence we have (RF4). To prove (4.3), it is enough to show the case where B is a one point. Namely we will show that

$$g_{\{z\}}(x,y) = \frac{R(x,z) + R(y,z) - R(x,y)}{2} \tag{4.4}$$

for any $x, y, z \in X$. We write $g(x,y) = g_{\{z\}}(x,y)$. The definition of $R(\cdot,\cdot)$ along with Lemma 4.12 shows that $g(x,x) = R(x,z)$. Also by Lemma 4.12, if $u_*(y) = g(x,y)/g(x,x)$, then u_* is the $\{x,z\}$-harmonic function whose boundary values are $u_*(z) = 0$ and $u_*(x) = 1$. Let $V = \{x,y,z\}$. Then by Proposition 4.6, there exists a Laplacian $H \in \mathcal{LA}(V)$ with (4.2). Note that

$$\mathcal{E}_H(u_*|_V, u_*|_V) = \min\{\mathcal{E}_H(v,v) | v \in \ell(V), v(x) = 1, v(z) = 0\}.$$

Therefore, $(Hu_*)(y) = 0$. Set $H = (H_{pq})_{p,q \in V}$. Hereafter we assume that $H_{pq} > 0$ for any $p, q \in V$ with $p \neq q$. (If this condition fails, the proof becomes easier.) Let $R_{pq} = (H_{pq})^{-1}$. Solving $Hu_*(y) = 0$, we have

$$u_*(y) = \frac{H_{xy}}{H_{xy} + H_{yz}} = \frac{R_{yz}}{R_{xy} + R_{yz}}. \tag{4.5}$$

On the other hand, by using the δ-Y transform, if $R_x = R_{xy}Rxz/R_*, R_y = R_{yx}R_{yz}/R_*$ and $R_z = R_{zx}R_{zy}/R_*$, where $R_* = R_{xy} + R_{yz} + R_{zx}$, then $R(p,q) = R_p + R_q$ for any p and q with $p \neq q$. Hence

$$\frac{R(x,z) + R(y,z) - R(x,y)}{2} = R_z \tag{4.6}$$

Since $g(x,x) = R(x,z)$, (4.5) implies

$$g(x,y) = g(x,x)u_*(y) = R(x,z)u_*(y) = \frac{R_{xz}(R_{xy}+R_{yz})}{R_*}u_*(y) = R_z.$$

By (4.6), we have (4.4). □

Proof of (GF4) of Theorem 4.9. Let $K = B^{\mathcal{F}}$. Note that $g_B(x,y) = g_K(x,y)$. By (4.3),

$$\begin{aligned}|g_B(x,y) - g_B(x,z)| &\leq \frac{|R(y,K) - R(z,K)| + |R_K(x,y) - R_K(x,z)|}{2} \\ &\leq R_K(y,z) \leq R(y,z).\end{aligned}$$

□

In the rest of this section, we study a sufficient condition ensuring that $B^{\mathcal{F}} = B$.

Definition 4.15. Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X and let R be the associated resistance metric.
(1) For a non-empty subset of B, define

$$N(B,r) = \min\{\#(A)|A \subseteq B \subseteq \cup_{y\in A}B_R(y,r)\}$$

for any $r > 0$.
(2) For any subsets $U, V \subset X$, define

$$\underline{R}(U,V) = \inf\{R(x,y)|x \in U, y \in V\}.$$

The following theorem plays an important role in proving heat kernel estimates in Part III.

Theorem 4.16. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X. Let B be a non-empty subset of X and let $x \in X\backslash B$. If $N(B, \underline{R}(x,B)/2) < +\infty$, then $x \notin B^{\mathcal{F}}$ and*

$$\frac{\underline{R}(x,B)}{4N(B,\underline{R}(x,B)/2)} \leq R(x,B) \leq \underline{R}(x,B).$$

The key idea of the following proof has been extracted from [6, Lemma 2.4] and [37, Lemma 4.1].

Proof. Write $u_y = \psi_x^{x,y}$ for any $x, y \in X$. Then,

$$u_y(z) = u_y(z) - u_y(y) \leq \frac{R(y,z)}{R(y,x)}.$$

If $y \in B, x \in X\backslash B$ and $z \in B_R(x, \underline{R}(x,B)/2)$, then $u_y(z) \leq 1/2$. Suppose that $n = N(B, \underline{R}(x,B)/2)$ is finite. We may choose $y_1, \ldots, y_n \in B$ so that $B \subseteq \cup_{i=1}^n B_R(y_i, \underline{R}(x,B)/2)$. Define $v(z) = \min_{i=1,\ldots,n} u_{y_i}(z)$ for any $z \in X$.

Then $v \in \mathcal{F}, v(x) = 1$ and $v(z) \le 1/2$ for any $z \in B$. Letting $h = 2\overline{(v - 1/2)}$, we see that $0 \le h(z) \le 1$ for any $z \in X$, $h(x) = 1$ and $h \in \mathcal{F}(B)$. Hence $x \notin B^{\mathcal{F}}$. Moreover,

$$\mathcal{E}(h,h) \le 4\mathcal{E}(v,v) \le 4\sum_{i=1}^{n} \mathcal{E}(u_{y_i}, u_{y_i}) \le 4\sum_{i=1}^{n} \frac{1}{R(x,y_i)} \le \frac{4n}{\underline{R}(x,B)}.$$

Therefore,

$$R(x,B) = \big(\min\{\mathcal{E}(u,u)|u \in \mathcal{F}(B), u(x) = 1\}\big)^{-1} \ge \frac{\underline{R}(x,B)}{4n}.$$

□

Corollary 4.17. *Let $(\mathcal{E},\mathcal{F})$ be a resistance form on X. If B is compact with respect to the resistance metric associated with $(\mathcal{E},\mathcal{F})$, then $B^{\mathcal{F}} = B$.*

In general, $B^{\mathcal{F}}$ does not coincide with B for every closed set B. We have the following example where (X,R) is locally compact and $B^{\mathcal{F}} \ne B$ for some closed set $B \subset X$.

Example 4.18. Let $X = \mathbb{N} \cup \{0\}$ and let $V_m = \{1,\dots,m\} \cup \{0\}$. Define a linear operator $H_m : \ell(V_m) \to \ell(V_m)$ by

$$(H_m)_{ij} = \begin{cases} 2 & \text{if } |i-j| = 1 \text{ or } |i-j| = m, \\ 1 & \text{if } \{i,j\} = \{0,k\} \text{ for some } k \in \{1,\dots,m\}\backslash\{1,m\}, \\ -4 & \text{if } i = j \text{ and } i \in \{1,m\}, \\ -5 & \text{if } i = j \text{ and } i \in \{1,\dots,m\}\backslash\{1,m\}, \\ -(m+2) & \text{if } i = j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then H_m is a Laplacian on V_m and $\{(V_m,H_m)\}_{m\ge1}$ is a compatible sequence. Set $\mathcal{E}_m(u,v) = -(u,Hv)_{V_m}$ for any $u,v \in \ell(V_m)$. Define

$$\mathcal{F} = \{u|u \in \ell(X), \lim_{m\to\infty} \mathcal{E}_m(u|_{V_m}, u|_{V_m}) < \infty\}$$

and

$$\mathcal{E}(u,v) = \lim_{m\to\infty} \mathcal{E}_m(u|_{V_m}, v|_{V_m})$$

for any $u,v \in \mathcal{F}$. Then $(\mathcal{E},\mathcal{F})$ is a resistance form on X. Let R be the associated resistance metric on X. Using the fact that $R(i,j) = R_m(i,j)$ for $i,j \in V_m$, where R_m is the effective resistance with respect to $\mathcal{E}_m$, we may calculate $R(i,j)$ for any $i,j \in X$. As a result,

$$\begin{cases} R(0,j) & = \frac{1}{3} \quad \text{for any } j \ge 1, \\ R(i,j) & = \frac{2}{3}(1 - 2^{-|i-j|}) \quad \text{if } i,j \ge 1. \end{cases}$$

Since $1/3 \leq R(i,j) \leq 2/3$ for any $i,j \in X$ with $i \neq j$, any one point set $\{x\}$ is closed and open. In particular, (X,R) is locally compact. Let $B = \mathbb{N}$. Since B is the complement of a open set $\{0\}$, B is closed. Define $\psi \in \ell(X)$ by $\psi(0) = 1$ and $\psi(x) = 0$ for any $x \in B$. Since $\mathcal{E}_m(\psi|_{V_m}, \psi|_{V_m}) = m + 2 \to \infty$ as $m \to \infty$, we see that $\psi \notin \mathcal{F}$. Therefore if $u \in \mathcal{F}(B)$, then $u(0) = 0$. This shows that $B^{\mathcal{F}} = B \cup \{0\}$.

5 Regularity of resistance forms

Does a domain $\mathcal{F}$ of a resisatnce form $\mathcal{E}$ contain enough many functions? The notion of regularity of a resistance form will provide an answer to such a question. As you will see in Definition 5.2, a resistance form is regular if and only if the domain of the resistance from is large enough to approximate any continuous function with a compact support. It is notable that the operation $B \to B^{\mathcal{F}}$ plays an important role again in this section.

Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a set X and let R be the associated resistance metric on X. We assume that (X,R) is separable.

Definition 5.1. (1) Let $u : X \to \mathbb{R}$. The support of u, $\mathrm{supp}(u)$ is defined by $\mathrm{supp}(u) = \overline{\{x|u(x) \neq 0\}}$. We use $C_0(X)$ to denote the collection of continuous functions on X whose support are compact.
(2) Let K be a subset of X and let $u : X \to \mathbb{R}$. We define the supremum norm of u on K, $||u||_{\infty,K}$ by

$$||u||_{\infty,K} = \sup_{x \in K} |u(x)|.$$

We write $|| \cdot ||_{\infty} = || \cdot ||_{\infty,X}$ if no confusion can occur.

Definition 5.2. The resistance form $(\mathcal{E}, \mathcal{F})$ on X is called regular if and only if $\mathcal{F} \cap C_0(X)$ is dense in $C_0(X)$ in the sense of the supremum norm $|| \cdot ||_{\infty}$.

The regularity of a resistance form is naturally associated with that of a Dirichlet form. See Section 8 for details. The following theorem gives a simple criteria for the regularity.

Theorem 5.3. *Assume that (X,R) is locally compact. The following conditions are equivalent:*
(R1) *$(\mathcal{E}, \mathcal{F})$ is regular.*
(R2) *$B^{\mathcal{F}} = B$ for any closed subset B.*
(R3) *If B is closed and $\overline{B^c}$ is compact, then $B^{\mathcal{F}} = B$.*
(R4) *If K is a compact subset of X, U is a open subset of X, $K \subseteq U$ and $\overline{U}$ is compact, then there exists $\varphi \in \mathcal{F}$ such that $\mathrm{supp}(\varphi) \subseteq \overline{U}$, $0 \leq \varphi(y) \leq 1$ for any $y \in X$ and $\varphi|_K \equiv 1$.*

Combining the above theorem with Corollary 4.17, we obtain the following result.

Corollary 5.4. *If (X,R) is compact, then $(\mathcal{E}, \mathcal{F})$ is regular.*

In general, even if (X, R) is locally compact, $(\mathcal{E}, \mathcal{F})$ is not always regular. Recall Example 4.18.

To prove Theorem 5.3, we need the following lemma, which can be proven by direct calculation.

Lemma 5.5. *If $u, v \in \mathcal{F} \cap C_0(X)$, then $uv \in \mathcal{F} \cap C_0(X)$ and*

$$\mathcal{E}(uv, uv) \leq 2||u||_\infty^2 \mathcal{E}(v, v) + 2||v||_\infty^2 \mathcal{E}(u, u).$$

Proof of Theorem 5.3. (R1) $\Rightarrow$ (R2) Let $x \notin B$. Choose $r > 0$ so that $\overline{B(x, r)}$ is compact and $B \cap \overline{B(x, r)} = \emptyset$. Then there exists $f \in C_0(X)$ such that $0 \leq f(y) \leq 1$ for any $y \in X$, $f(x) = 1$ and $\mathrm{supp}(f) \subseteq \overline{B(x, r)}$. Since $(\mathcal{E}, \mathcal{F})$ is regular, we may find $v \in \mathcal{F} \cap C_0(X)$ such that $||v - f||_\infty \leq 1/3$. Define $u = \overline{3v - 1}$. Then $u(x) = 1$ and $u|_B \equiv 0$. Hence $x \notin B^{\mathcal{F}}$.
(R2) $\Rightarrow$ (R3) This is obvious.
(R3) $\Rightarrow$ (R4) By (R3), $(U^c)^{\mathcal{F}} = U^c$. Hence, for any $x \in K$, we may choose r_x so that $B(x, r_x) \subseteq U$ and $\psi_x^{U^c}(y) \geq 1/2$ for any $y \in B(x, r_x)$. Since K is compact, $K \subseteq \cup_{i=1}^n B(x_i, r_{x_i})$ for some $x_1, \ldots, x_n \in K$. Let $v = \sum_{i=1}^n \psi_{x_i}^{U^c}$. Then $v(y) \geq 1/2$ for any $y \in K$ and $\mathrm{supp}(v) \subseteq \overline{U}$. If $\varphi = 2\overline{v}$, then u satisfies the desired properties.
(R4) $\Rightarrow$ (R1) Let $u \in C_0(X)$. Set $K = \mathrm{supp}(u)$. Define $\Omega_K = \{u|_K : u \in \mathcal{F} \cap C_0(X)\}$. Then by (R4) and Lemma 5.5, we can verify the assumptions of the Stone Weierstrass theorem for the $|| \cdot ||_K$ closure of Ω_K. (See, for example, [46] on the Stone Weierstrass theorem.) Hence, Ω_K is dense in $C(K)$ with respect to the supremum norm on K. For any $\epsilon > 0$, there exists $u_\epsilon \in \mathcal{F} \cap C_0(X)$ such that $||u - u_\epsilon||_{\infty, K} < \epsilon$. Let $V = K \cup \{x||u_\epsilon(x)| < \epsilon\}$. Suppose that $x \in K$ and that there exists $\{x_n\}_{n=1,2,\ldots} \subseteq V^c$ such that $R(x_n, x) \to 0$ as $x \to \infty$. Then $|u_\epsilon(x_n)| \geq \epsilon$ for any n and hence $|u_\epsilon(x)| \geq \epsilon$. On the other hand, since $x_n \in K^c$, $u(x_n) = 0$ for any n and hence $u(x) = 0$. Since $x \in K$, this contradict to the fact that $||u - u_\epsilon||_{\infty, K} < \epsilon$. Therefore, V is open. We may choose a open set U so that $K \subseteq U$, $\overline{U}$ is compact and $U \subseteq V$. Let φ be the function obtained in (R4). Define $v_\epsilon = \varphi u_\epsilon$. Then by Lemma 5.5, $v_\epsilon \in \mathcal{F} \cap C_0(X)$. Also it follows that $||u - v_\epsilon||_\infty \leq \epsilon$. This shows that $\mathcal{F} \cap C_0(X)$ is dense in $C_0(X)$ with respect to the norm $|| \cdot ||_\infty$. □

6 Annulus comparable condition and local property

We can modify a given resistance form by adding a new resistor between two distinct points. The modified new resistance form has a "jump" associated with the added resistor. Such "jumps" naturally appears the associated probabilistic process. In this section, we introduce annulus comparable condition, (ACC) for short, which assures certain kind of control to such jumps, or direct connections between two distinct points. For instance, Theorems in Section 14 will show that (ACC) is necessary to get the Li-Yau type on-diagonal heat kernel estimate.

We need the following topological notion to state (ACC).

Definition 6.1. Let (X, d) be a metric space. (X, d) is said to be uniformly perfect if and only if there exists $\epsilon > 0$ such that $B_d(x, (1+\epsilon)r)\backslash B_d(x, r) \neq \emptyset$ for any $x \in X$ and $r > 0$ with $X\backslash B_d(x, r) \neq \emptyset$.

In this section, $(\mathcal{E}, \mathcal{F})$ is a regular resistance form on X and R is the associated resistance metric. We assume that (X, R) is separable and complete.

Definition 6.2. A resistance form $(\mathcal{E}, \mathcal{F})$ on X is said to satisfy the annulus comparable condition, (ACC) for short, if and only if (X, R) is uniformly perfect and there exists $\epsilon > 0$ such that

$$R(x, B_R(x, r)^c) \asymp R\big(x, \overline{B_R(x, (1+\epsilon)r)} \cap B_R(x, r)^c\big) \tag{6.1}$$

for any $x \in X$ and any $r > 0$ with $B_R(x, r) \neq X$.

Remark. It is obvious that

$$R(x, B_R(x, r)^c) \leq R\big(x, \overline{B_R(x, (1+\epsilon)r)} \cap B_R(x, r)^c\big).$$

So the essential requirement of (ACC) is the opposite inequality up to a constant multiplication.

The annulus comparable condition holds if (X, R) is uniformly perfect and $(\mathcal{E}, \mathcal{F})$ has the local property defined below.

Definition 6.3. $(\mathcal{E}, \mathcal{F})$ is said to have the local property if and only if $\mathcal{E}(u, v) = 0$ for any $u, v \in \mathcal{F}$ with $\underline{R}(\mathrm{supp}(u), \mathrm{supp}(v)) > 0$.

Proposition 6.4. *Assume that $(\mathcal{E}, \mathcal{F})$ has the local property and that $\overline{B_R(x, r)}$ is compact for any $x \in X$ and any $r > 0$. If $\overline{B_R(x, (1+\epsilon)r)} \cap B_R(x, r)^c \neq \emptyset$, then*

$$R(x, B_R(x, r)^c) = R\big(x, \overline{B_R(x, (1+\epsilon)r)} \cap B_R(x, r)^c\big).$$

In particular, we have (ACC) *if (X, R) is uniformly perfect.*

Proof. Let $K = \overline{B_R(x, (1+\epsilon)r)} \cap B_R(x, r)^c$. Recall that $\psi_x^K(y) = \frac{g_K(x,y)}{g_K(x,x)}$ and that $\mathcal{E}(\psi_x^K, \psi_x^K) = R(x, K)^{-1}$. By Theorem 5.3, there exists $\varphi \in \mathcal{F}$ such that $\mathrm{supp}(\varphi) \subseteq \overline{B_R(x, (1+\epsilon/2)r)}$, $0 \leq \varphi(y) \leq 1$ for any $y \in X$ and $\varphi(y) = 1$ for any $y \in \overline{B_R(x, r)}$. By Lemma 5.5, if $\psi_1 = \psi_x^K \varphi$ and $\psi_2 = \psi_x^K(1-\varphi)$, then ψ_1 and ψ_2 belong to $\mathcal{F}$. Since $\mathrm{supp}(\psi_2) \subseteq B_r(x, (1+\epsilon)r)^c$, the local property implies

$$\mathcal{E}(\psi_x^K, \psi_x^K) = \mathcal{E}(\psi_1, \psi_1) + \mathcal{E}(\psi_2, \psi_2) \geq \mathcal{E}(\psi_1, \psi_1).$$

Note that $\psi_1(y) = 0$ for any $y \in B_R(x, r)^c$ and that $\psi_1(x) = 1$. Hence, $\mathcal{E}(\psi_1, \psi_1) \geq \mathcal{E}(\psi_x^B, \psi_x^B)$, where $B = B(x, r)^c$. On the other hand, since $K \subseteq B$, $\mathcal{E}(\psi_x^B, \psi_x^B) \geq \mathcal{E}(\psi_x^K, \psi_x^K)$. Therefore, we have

$$R(x, B)^{-1} = \mathcal{E}(\psi_x^B, \psi_x^B) = \mathcal{E}(\psi_x^K, \psi_x^K) = R(x, K)^{-1}.$$

□

There are non-local resistance forms which satisfy (ACC), for example, the α-stable process on $\mathbb{R}$ and their traces on the Cantor set. See Sections 15. In the next section, we will show that if the original resistance form has (ACC), then so do its traces, which are non-local in general.

To study non-local cases, we need the doubling property of the space.

Definition 6.5. Let (X, d) be a metric space.
(1) (X, d) is said to have the doubling property or be the doubling space if and only if

$$\sup_{x \in X, r>0} N_d(B_d(x, r), \delta r) < +\infty \tag{6.2}$$

for any $\delta \in (0, 1)$, where

$$N_d(A, r) = \min\{\#F | F \subseteq A, A \subseteq \cup_{x \in F} B_d(x, r)\}.$$

(2) Let μ be a Borel regular measure on (X, d) which satisfies $0 < \mu(B_d(x, r)) < +\infty$ for any $x \in X$ and any $r > 0$. μ is said to have the volume doubling property with respect to d or be volume doubling with respect to d, $(\mathrm{VD})_{\mathrm{d}}$ for short, if and only if there exists $c > 0$ such that

$$\mu(B_d(x, 2r)) \le c\mu(B_d(x, r)) \tag{6.3}$$

for any $x \in X$ and any $r > 0$.

Remark. (1) (X, d) is the doubling space if (6.2) holds for some $\delta \in (0, 1)$.
(2) If μ is $(\mathrm{VD})_{\mathrm{d}}$, then, for any $\alpha > 1$, $\mu(B_d(x, \alpha r)) \asymp \mu(B_d(x, r))$ for any $x \in X$ and any $r > 0$.

One of the sufficient condition for the doubling property is the existence of a volume doubling measure. The following theorem is well-known. See [28] for example.

Proposition 6.6. *Let (X, d) be a metric space and let μ be a Borel regular measure on (X, R) with $0 < \mu(B_R(x, r)) < +\infty$ for any $x \in X$ and any $r > 0$. If μ is $(\mathrm{VD})_{\mathrm{d}}$, then (X, d) has the doubling property.*

The next proposition is straight forward from the definitions.

Proposition 6.7. *If a metric space (X, d) has the doubling property, then any bounded subset of (X, d) is totally bounded.*

By the above proposition, if the space is doubling and complete, then every bounded closed set is compact.

Now we return to (ACC). The following key lemma is a direct consequence of Theorem 4.16.

Lemma 6.8. *Assume that (X, R) has the doubling property and is uniformly perfect. Then, for some $\epsilon > 0$,*

$$R\big(x, B_R(x, r)^c \cap \overline{B_R(x, (1+\epsilon)r)}\big) \asymp r \tag{6.4}$$

for any $x \in X$ and any $r > 0$ with $B_R(x, r) \neq X$.

Proof. Set $B = \overline{B_R(x,(1+\epsilon)r)} \cap B_R(x,r)^c$. Choose ϵ so that $B \neq \emptyset$ for any $x \in X$ and any $r > 0$ with $B_R(x,r) \neq X$. Then, $r \leq \underline{R}(x,B) \leq (1+\epsilon)r$. This and the doubling property of (X,R) imply

$$N(B, \underline{R}(x,B)/2) \leq N(B, r/2) \leq N(B_R(x,(1+2\epsilon)r), r/2) \leq c_*,$$

where c_* is independent of x and r. Using Theorem 4.16, we see

$$\frac{r}{8c_*} \leq R(x,B) \leq (1+\epsilon)r.$$

□

By the above lemma, (ACC) turns out to be equivalent to (RES) defined below if (X,R) is the doubling space.

Definition 6.9. A resistance from $(\mathcal{E},\mathcal{F})$ on X is said to satisfy the resistance estimate, (RES) for short, if and only if

$$R(x, B_R(x,r)^c) \asymp r \tag{6.5}$$

for any $x \in X$ and any $r > 0$ with $B_R(x,r) \neq X$.

Theorem 6.10. *Assume that (X,R) has the doubling property. Then (X,R) is uniformly perfect and* (RES) *holds if and only if* (ACC) *holds.*

Proof of Theorem 6.10. If (ACC) holds, then (6.4) and (ACC) immediately imply (6.5). Conversely, (6.5) along with (6.4) shows (ACC). □

Corollary 6.11. *If $(\mathcal{E},\mathcal{F})$ has the local property, (X,R) has the doubling property and is uniformly perfect, then* (RES) *holds.*

7 Trace of resistance form

In this section, we introduce the notion of the trace of a resistance form on a subset of the original domain. This notion is a counterpart of that in the theory of Dirichlet form, which has been extensively studied in [17, Section 6.2], for example. In fact, if a Dirichlet form is derived from a regular resistance form, a trace of the Dirichlet form coincides with the counterpart of the resistance form.

Throughout this section, $(\mathcal{E},\mathcal{F})$ is a resistance form on X and R is the associated resistance distance. We assume that (X,R) is separable and complete.

Definition 7.1. For a non-empty subset $Y \subseteq X$, define $\mathcal{F}|_Y = \{u|_Y|\ u \in \mathcal{F}\}$.

Lemma 7.2. *Let Y be a non-empty subset of (X,R). For any $u \in \mathcal{F}|_Y$, there exists a unique $u_* \in \mathcal{F}$ such that $u_*|_Y = u$ and $\mathcal{E}(u_*,u_*) = \min\{\mathcal{E}(v,v)|v \in \mathcal{F}, v|_Y = u\}$.*

The unique u_* is thought of as the harmonic function with the boundary value u on Y.

To prove this lemma, we use the following fact which has been shown in [33, Section 2.3].

Proposition 7.3. *Let $\{V_m\}_{m\geq 1}$ be an increasing sequence of finite subsets of X. Assume that $V_* = \cup_{m\geq 1} V_m$ is dense in X. Set $H_m = H_{(\mathcal{E},\mathcal{F}),V_m}$ and define $\mathcal{E}_m(\cdot,\cdot) = \mathcal{E}_{H_m}(\cdot,\cdot)$, where $H_{(\mathcal{E},\mathcal{F}),V_m}$ is defined in Definition 4.7. Then for any $u \in \ell(V_*)$, $\mathcal{E}_m(u|_{V_m}, u|_{V_m})$ is monotonically non-decreasing. Moreover,*

$$\mathcal{F} = \{u | u \in C(X,R), \lim_{m\to\infty} \mathcal{E}_m(u|_{V_m}, u|_{V_m}) < +\infty\}$$

and

$$\mathcal{E}(u,v) = \lim_{m\to\infty} \mathcal{E}_m(u|_{V_m}, v|_{V_m})$$

for any $u, v \in \mathcal{F}$.

Proof of Lemma 7.2. Let $p \in Y$. Replacing u by $u - u(p)$, we may assume that $u(p) = 0$ without loss of generality. Choose a sequence $\{v_n\}_{n\geq 1} \subseteq \mathcal{F}$ so that $v_n|_Y = u$ and $\lim_{n\to\infty} \mathcal{E}(v_n, v_n) = \inf\{\mathcal{E}(v,v) | v \in \mathcal{F}, v|_Y = u\}$. Let $C = \sup_n \mathcal{E}(v_n, v_n)$. By Proposition 4.2, if $v = v_n$, then

$$|v(x) - v(y)|^2 \leq CR(x,y) \tag{7.1}$$

and

$$|v(x)|^2 \leq CR(x,p) \tag{7.2}$$

Let $\{V_m\}_{m\geq 1}$ be an increasing sequence of finite subsets of X. Assume that $V_* = \cup_{m\geq 1} V_m$ is dense in X. (Since (X,R) is separable, such $\{V_m\}_{m\geq 1}$ does exists.) By (7.1) and (7.2), the standard diagonal construction gives a subsequence $\{v_{n_i}\}_{i\geq 1}$ which satisfies $\{v_{n_i}(x)\}_{i\geq 1}$ is convergent as $i \to \infty$ for any $x \in V_* = \cup_{m\geq 1} V_m$. Define $u_*(x) = \lim_{i\to\infty} v_{n_i}(x)$ for any $x \in V_*$. Since u_* satisfies (7.1) and (7.2) on V_* with $v = u_*$, u_* is extended to a continuous function on X. Note that this extension also satisfies (7.1) and (7.2) on X with $v = u_*$. Set $\mathcal{E}_m(\cdot,\cdot) = \mathcal{E}_{H_{(\mathcal{E},\mathcal{F}),V_m}}$. Then, by Proposition 7.3,

$$\mathcal{E}_m(v_n, v_n) \leq \mathcal{E}(v_n, v_n) \leq C \tag{7.3}$$

for any $m \geq 1$ and any $n \geq 1$. Define $M = \inf\{\mathcal{E}(v,v) | v \in \mathcal{F}, v|_Y = u\}$. For any $\epsilon > 0$, if n is large enough, then (7.3) shows $\mathcal{E}_m(v_n, v_n) \leq M + \epsilon$ for any $m \geq 1$. Since $v_n|_{V_m} \to u_*|_{V_m}$ as $n \to \infty$, it follows that $\mathcal{E}_m(u_*, u_*) \leq M + \epsilon$ for any $m \geq 1$. Proposition 7.3 implies that $u_* \in \mathcal{F}$ and $\mathcal{E}(u_*, u_*) \leq M$.

Next assume that $u_i \in \mathcal{F}$, $u_i|_Y = u$ and $\mathcal{E}(u_i, u_i) = M$ for $i = 1, 2$. Since $\mathcal{E}((u_1+u_2)/2, (u_1+u_2)/2) \geq \mathcal{E}(u_1, u_1)$, we have $\mathcal{E}(u_1, u_2 - u_1) \geq 0$. Similarly, $\mathcal{E}(u_2, u_1 - u_2) \geq 0$. Combining those two inequalities, we obtain $\mathcal{E}(u_1 - u_2, u_1 - u_2) = 0$. Since $u_1 = u_2$ on Y, we have $u_1 = u_2$ on X. $\square$

Definition 7.4. Define $h_Y : \mathcal{F}|_Y \to \mathcal{F}$ by $h_Y(u) = u_*$, where u and u_* are the same as in Lemma 7.2. $h_V(u)$ is called the Y-harmonic function with the boundary value u. For any $u, v \in \mathcal{F}|_Y$, define $\mathcal{E}|_Y(u,v) = \mathcal{E}(h_Y(u), h_Y(v))$.

Trough the harmonic functions, we construct a resistance form on a subspace Y of X, which is called the trace.

Theorem 7.5. *Let Y be a non-empty subset of X. Then $h_Y : \mathcal{F}|_Y \to \mathcal{F}$ is linear and $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ is a resistance form on Y. The associated resistance metric equals to the restriction of R on Y. If Y is closed and $(\mathcal{E}, \mathcal{F})$ is regular, then $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ is regular.*

Definition 7.6. $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ is called the trace of the resistance form $(\mathcal{E}, \mathcal{F})$ on Y.

The following lemma is essential to prove the above theorem.

Lemma 7.7. *Let Y be a non-empty subset of X. Define*

$$\mathcal{H}_Y = \{u | u \in \mathcal{F}, \mathcal{E}(u, v) = 0 \text{ for any } v \in \mathcal{F}(Y)\}.$$

Then, for any $f \in \mathcal{F}|_Y$, $u = h_Y(f)$ if and only if $u \in \mathcal{H}_Y$ and $u|_Y = f$.

By this lemma, $\mathcal{H}_Y = \mathrm{Im}(h_Y)$ is the space of Y-harmonic functions and $\mathcal{F} = \mathcal{H}_Y \oplus \mathcal{F}(Y)$, where $\oplus$ means that $\mathcal{E}(u, v) = 0$ for any $u \in \mathcal{H}_Y$ and any $v \in \mathcal{F}(Y)$. The counter part of this fact has been know for Dirichlet forms. See [17] for details.

Proof. Let $f_* = h_Y(f)$. If $v \in \mathcal{F}$ and $v|_Y = f$, then

$$\mathcal{E}(t(v - f_*) + f_*, t(v - f_*) + f_*) \geq \mathcal{E}(f_*, f_*)$$

for any $t \in \mathbb{R}$. Hence $\mathcal{E}(v - f_*, f_*) = 0$. This implies that $f_* \in \mathcal{H}_Y$. Conversely assume that $u \in \mathcal{H}_Y$ and $u|_Y = f$. Then, for any $v \in \mathcal{F}$ with $v|_Y = f$,

$$\mathcal{E}(v, v) = \mathcal{E}((v - u) + u, (v - u) + u) = \mathcal{E}(v - u, v - u) + \mathcal{E}(u, u) \geq \mathcal{E}(u, u).$$

Hence by Lemma 7.2, $u = h_Y(f)$. □

Proof of Theorem 7.5. By Lemma 7.7, if $r_Y : \mathcal{H}_Y \to \mathcal{F}|_Y$ is the restriction on Y, then r_Y is the inverse of h_Y. Hence h_Y is linear. The conditions (RF1) through (RF4) for $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ follows immediately from the counterpart for $(\mathcal{E}, \mathcal{F})$. About (RF5),

$$\begin{aligned}\mathcal{E}|_Y(\overline{u}, \overline{u}) = \mathcal{E}(h_Y(\overline{u}), h_Y(\overline{u})) \leq \mathcal{E}(\overline{h_Y(u)}, \overline{h_Y(u)}) \\ \leq \mathcal{E}(h_Y(u), h_Y(u)) = \mathcal{E}|_Y(u, u).\end{aligned}$$

The rest of the statement is straight forward. □

In the rest of this section, the conditions (ACC) and (RES) are shown to be preserved by the traces under reasonable assumptions.

Theorem 7.8. *Let $(\mathcal{E}, \mathcal{F})$ be a regular resistance form on X and let R be the associated resistance metric. Assume that $(\mathcal{E}, \mathcal{F})$ satisfies* (RES). *If Y is a closed subset of X and $(Y, R|_Y)$ is uniformly perfect, then* (RES) *holds for the trace $(\mathcal{E}|_Y, \mathcal{F}|_Y)$.*

By Theorem 6.10, we immediately have the following corollary.

Corollary 7.9. *Let $(\mathcal{E}, \mathcal{F})$ be a regular resistance from on X and let R be the associated resistance metric. Assume that (X, R) has the doubling property. Let Y be a closed subset of X and assume that $(Y, R|_Y)$ is uniformly perfect. If* (ACC) *holds for $(\mathcal{E}, \mathcal{F})$, then so does for the trace $(\mathcal{E}|_Y, \mathcal{F}|_Y)$.*

Notation. Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X and let R be the associated resistance metric. For a non-empty subset Y of X, we use R^Y to denote the resistance metric associated with the trace $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ on Y. Also we write $B_R^Y(x, r) = B_R(x, r) \cap Y$ for any $x \in Y$ and $r > 0$.

Proof of Theorem 7.8. Note that $R^Y(x, Y \backslash B_R^Y(x, r)) = R(x, B_R(x, r)^c \cap Y)$. Hence if (RES) holds for $(\mathcal{E}, \mathcal{F})$ then,

$$R^Y(x, Y \backslash B_R^Y(x, r)) \geq R(x, B_R(x, r)^c) \geq c_1 r. \tag{7.4}$$

On the other hand, since (Y, R^Y) is uniformly perfect, there exists $\epsilon > 0$ such that $B_R^Y(x, (1+\epsilon)r) \backslash B_R^Y(x, r) \neq \emptyset$ for any $x \in Y$ and $r > 0$ with $B_R^Y(x, r) \neq Y$. Let $y \in B_R^Y(x, (1+\epsilon)r) \backslash B_R^Y(x, r)$. Then

$$(1+\epsilon)r \geq R^Y(x, y) \geq R^Y(x, Y \backslash B_R^Y(x, r)). \tag{7.5}$$

Combining (7.4) and (7.5), we obtain (RES) for $(\mathcal{E}|_Y, \mathcal{F}|_Y)$. □

8 Resistance forms as Dirichlet forms

In this section, we will present how to obtain a regular Dirichlet form from a regular resistance form and show that every single point has a positive capacity. As in the previous sections, $(\mathcal{E}, \mathcal{F})$ is a resistance form on X and R is the associated resistance metric on X. We continue to assume that (X, R) is separable, complete and locally compact.

We present how to obtain a regular Dirichlet form out of a regular resistance form. Let μ be a Borel regular measure on (X, R) which satisfies $0 < \mu(B_R(x, r)) < +\infty$ for any $x \in X$ and $r > 0$. Note that $C_0(X)$ is a dense subset of $L^2(X, \mu)$ by those assumptions on μ.

Definition 8.1. For any $u, v \in \mathcal{F} \cap L^2(X, \mu)$, define $\mathcal{E}_1(u, v)$ by

$$\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + \int_X uv d\mu.$$

By [33, Theorem 2.4.1], we have the following fact.

Lemma 8.2. *$(\mathcal{F} \cap L^2(X, \mu), \mathcal{E}_1)$ is a Hilbert space.*

Since $\mathcal{F} \cap C_0(X) \subseteq \mathcal{F} \cap L^2(X, \mu)$, the closure of $\mathcal{F} \cap C_0(X)$ is a subset of $\mathcal{F} \cap L^2(X, \mu)$.

Definition 8.3. We use $\mathcal{D}$ to denote the closure of $\mathcal{F} \cap C_0(X)$ with respect to the inner product $\mathcal{E}_1$.

Note that if (X, R) is compact, then $\mathcal{D} = \mathcal{F}$.

Theorem 8.4. *If $(\mathcal{E}, \mathcal{F})$ is regular, then $(\mathcal{E}|_{\mathcal{D}\times\mathcal{D}}, \mathcal{D})$ is a regular Dirichlet form on $L^2(X, \mu)$.*

See [17] for the definition of a regular Dirichlet form.
For ease of notation, we write $\mathcal{E}$ instead of $\mathcal{E}_{\mathcal{D}\times\mathcal{D}}$.

Proof. $(\mathcal{E}, \mathcal{D})$ is closed form on $L^2(X, \mu)$. Also, since $C_0(X)$ is dense in $L^2(X, \mu)$, the assumption that $\mathcal{F} \cap C_0(X)$ is dense in $C_0(X)$ shows that $\mathcal{D}$ is dense in $L^2(K, \mu)$. Hence $(\mathcal{E}, \mathcal{D})$ is a regular Dirichlet form on $L^2(X, \mu)$ with a core $\mathcal{F} \cap C_0(X)$. $\square$

Hereafter in this section, $(\mathcal{E}, \mathcal{F})$ is always assumed to be regular. Next we study the capacity of points associated with the Dirichlet form constructed above.

Lemma 8.5. *Let $x \in X$. Then there exists $c_x > 0$ such that*

$$|u(x)| \leq c_x \sqrt{\mathcal{E}_1(u, u)}$$

for any $u \in \mathcal{D}$. In other words, the map $u \to u(x)$ from $\mathcal{D}$ to $\mathbb{R}$ is bounded.

Proof. If the claim of the lemma is false, then there exists a sequence $\{u_n\}_{n\geq 1} \subset \mathcal{F}$ such that $u_n(x) = 1$ and $\mathcal{E}_1(u_n, u_n) \leq 1/n$ for any $n \geq 1$. By (4.1),

$$|u_n(x) - u_n(y)| \leq \frac{\sqrt{R(x, y)}}{\sqrt{n}} \leq \sqrt{R(x, y)}$$

Hence $u_n(y) \geq 1/2$ for any $y \in B(x, 1/4)$. This implies that

$$||u_n||_2^2 \geq \int_{B(x,1/4)} u(y)^2 d\mu \geq \mu(B(x, 1/4))/4 > 0$$

This contradicts to that fact that $\mathcal{E}_1(u_n, u_n) \to 0$ as $n \to \infty$. $\square$

Lemma 8.6. *If K is a compact subset of X, then the restriction map $\iota_K : \mathcal{D} \to C(K)$ defined by $\iota_K(u) = u|_K$ is a compact operator, where $\mathcal{D}$ and $C(K)$ are equipped with the norms $\sqrt{\mathcal{E}_1(\cdot, \cdot)}$ and $||\cdot||_{\infty,K}$ respectively.*

Proof. Set $D = \sup_{x,y\in K} R(x, y)$. Let $\mathcal{U}$ be a bounded subset of $\mathcal{D}$, i.e. there exists $M > 0$ such that $\mathcal{E}_1(u, u) \leq M$ for any $u \in \mathcal{U}$. Then by (4.1),

$$|u(x) - u(y)|^2 \leq R(x, y)M$$

for any $x, y \in X$ and any $u \in \mathcal{U}$. Hence $\mathcal{U}$ is equicontinuous. Choose $x_* \in K$. By Lemma 8.5 along with (4.1),

$$u(x)^2 \leq 2|u(x) - u(x_*)|^2 + 2|u(x_*)|^2 \leq 2DM + 2c_{x_*}^2 M$$

for any $u \in \mathcal{U}$ and any $x \in K$. This shows that $\mathcal{U}$ is uniformly bounded on K. By the Ascoli-Arzelà theorem, $\{u|_K\}_{u \in \mathcal{U}}$ is relatively compact with respect to the supremum norm. Hence ι_K is a compact operator. $\square$

Definition 8.7. For an open set $U \subseteq X$, define the $\mathcal{E}_1$-capacity of U, $\mathrm{Cap}(U)$ by

$$\mathrm{Cap}(U) = \inf\{\mathcal{E}_1(u,u) | u \in \mathcal{D}, u(x) \geq 1 \text{ for any } x \in U\}.$$

If $\{u | u \in \mathcal{D}, u(x) \geq 1 \text{ for any } x \in U\} = \emptyset$, we define $\mathrm{Cap}(U) = \infty$. For any $A \subseteq X$, we define $\mathrm{Cap}(A)$ by

$$\mathrm{Cap}(A) = \inf\{\mathrm{Cap}(U) | U \text{ is an open subset of } X \text{ and } A \subseteq U\}.$$

Theorem 8.8. *For any $x \in X$, $0 < \mathrm{Cap}(\{x\}) < \infty$. Moreover, if K is a compact subset of X, then $0 < \inf_{x \in K} \mathrm{Cap}(\{x\})$.*

Lemma 8.9. *For any $x \in X$, there exists a unique $g \in \mathcal{D}$ such that*

$$\mathcal{E}_1(g,u) = u(x)$$

for any $u \in \mathcal{D}$. Moreover, let $\varphi = g/g(x)$. Then, φ is the unique element in $\{u | u \in \mathcal{D}, u(x) \geq 1\}$ which attains the following minimum

$$\min\{\mathcal{E}_1(u,u) | u \in \mathcal{D}, u(x) \geq 1\}.$$

Proof. The existence of g follows by Lemma 8.5. Assume that $\mathcal{E}_1(f,u) = u(x)$ for any $u \in \mathcal{D}$. Since $\mathcal{E}_1(f-g,u) = 0$ for any $u \in \mathcal{D}$, we have $f = g$. Now, if $u \in \mathcal{D}$ and $u(x) = a > 1$, then $\mathcal{E}_1(u - a\varphi, \varphi) = u(x)/g(x) - 1/g(x) = 0$. Hence,

$$\mathcal{E}_1(u,u) = \mathcal{E}_1(u - a\varphi, u - a\varphi) + \mathcal{E}_1(a\varphi, a\varphi) \geq \mathcal{E}_1(\varphi,\varphi).$$

This immediately shows the rest of the statement. $\square$

Definition 8.10. We denote the function g and φ in Lemma 8.9 by g_1^x and φ_1^x respectively.

Proof of Theorem 8.8. Fix $x \in X$. By the above lemma, for any open set U with $x \in U$,

$$\begin{aligned}\mathrm{Cap}(U) &= \min\{\mathcal{E}_1(u,u) | u \in \mathcal{D}, u(y) \geq 1 \text{ for any } y \in U\} \\ &\geq \min\{\mathcal{E}_1(u,u) | u \in \mathcal{D}, u(x) \geq 1\} \geq \mathcal{E}_1(\varphi_1^x, \varphi_1^x) = \frac{1}{g_1^x(x)}.\end{aligned}$$

Hence $0 < 1/g_1^x(x) < \mathrm{Cap}(\{x\}) < \mathrm{Cap}(U) < +\infty$.

Let K be a compact subset of X. By Lemma 8.6, there exists $c_K > 0$ such that $||u||_{\infty,K} \leq c_K\sqrt{\mathcal{E}_1(u,u)}$ for any $u \in \mathcal{D}$. Now, for $x \in K$,

$$g_1^x(x) = \mathcal{E}_1(g_1^x, g_1^x) = \sup_{u \in \mathcal{D}, u \neq 0} \frac{\mathcal{E}_1(g_1^x, u)^2}{\mathcal{E}_1(u,u)} = \sup_{u \in \mathcal{D}, u \neq 0} \frac{u(x)^2}{\mathcal{E}_1(u,u)} \leq (c_K)^2$$

Hence $\mathrm{Cap}(\{x\}) \geq 1/g_1^x(x) \geq (c_K)^{-2}$. $\square$

Theorem 8.8 implies that every quasi continuous function is continuous and every exceptional set is empty.

Definition 8.11. A function $u : X \to \mathbb{R}$ is called quasi continuous if and only if, for any $\epsilon > 0$, there exists $V \subseteq X$ such that $\mathrm{Cap}(V) < \epsilon$ and $u|_{X\setminus V}$ is continuous.

Proposition 8.12. *Any quasi continuous function is continuous on X.*

Proof. Let u be a quasi continuous function. Let $x \in X$. Since (X, R) is locally compact, $\overline{B(x, r)}$ is compact for some $r > 0$. By Theorem 8.8, we may choose $\epsilon > 0$ so that $\inf_{y\in\overline{B(x,r)}} \mathrm{Cap}(\{y\}) > \epsilon$. There exists $V \subseteq X$ such that $\mathrm{Cap}(V) < \epsilon$ and $u|_{X\setminus V}$ is continuous. Since $V \cap \overline{B(x, r)} = \emptyset$, u is continuous at x. Hence u is continuous on X. □

9 Transition density

In this section, without ultracontractivity, we establish the existence of jointly continuous transition density (i.e. heat kernel) associated with the regular Dirichlet form derived from a resistance form.

As in the last section, $(\mathcal{E}, \mathcal{F})$ is a resistance form on X and R is the associated resistance metric. We assume that (X, R) is separable, complete and locally compact. μ is a Borel regular measure on X which satisfies $0 < \mu(B_R(x, r)) < \infty$ for any $x \in X$ and any $r > 0$. We continue to assume that $(\mathcal{E}, \mathcal{F})$ is regular. By Theorem 8.4, $(\mathcal{E}, \mathcal{D})$ is a regular Dirichlet form on $L^2(X, \mu)$, where $\mathcal{D}$ is the closure of $\mathcal{F} \cap C_0(X)$ with respect to the $\mathcal{E}_1$-inner product.

Let H be the nonnegative self-adjoint operator associated with the Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(X, \mu)$ and let T_t be the corresponding strongly continuous semigroup. Since $T_t u \in \mathcal{D}$ for any $u \in L^2(X, \mu)$, we always take the continuous version of $T_t u$. In other words, we may naturally assume that $T_t u$ is continuous.

Let $\mathbf{M} = (\Omega, \{X_t\}_{t>0}, \{P_x\}_{x\in X})$ be the Hunt process associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(X, \mu)$. Note that $\mathrm{Cap}(A) = 0$ if and only if $A = \emptyset$ by Theorem 8.8. Hence, the Hunt process $\mathbf{M}$ is determined for every $x \in X$. Moreover, by [17, Theorem 4.2.1], every exceptional set is empty. Let p_t be the transition semigroup associated with the Hunt process $\mathbf{M}$. In particular, for non-negative μ-measurable function u,

$$(p_t u)(x) = E_x(u(X_t))$$

for any $x \in X$. Let $\mathcal{B}$ be the Borel σ-algebra of (X, R). We say that u is Borel measurable, if and only if $u^{-1}((a, b]) \in \mathcal{B}$ for any $a, b \in \mathbb{R}$. Combining Proposition 8.12 and [17, Theorem 4.2.3], we have the following statement.

Proposition 9.1. *For any nonnegative $u \in L^2(X, \mu)$, $(p_t u)(x) = (T_t u)(x)$ for any $t > 0$ and any $x \in X$.*

Definition 9.2. Let U be an open subset of X. Define $\mathcal{D}_U = \{u | u \in \mathcal{D}, u|_{U^c} \equiv 0\}$. Also we define $\mathcal{E}_U = \mathcal{E}|_{\mathcal{D}_U\times\mathcal{D}_U}$.

Note that if $\overline{U}$ is compact, then $\mathcal{D}_U = \mathcal{F}(U^c)$.

Combining the results in [17, Section 4.4], we have the following facts.

Theorem 9.3. *Let μ_U be the restriction of μ on U, i.e. $\mu_U(A) = \mu(A \cap U)$ for any Borel set U. Then $(\mathcal{E}_U, \mathcal{D}_U)$ is a regular Dirichlet form on $L^2(U, \mu_U)$. Moreover, if $\mathbf{M}_U = (\Omega_U, X_t^U, P_x^U)$ be the associated Hunt process, then*

$$P_x^U(X_t^U \in A) = P(X_t \in A, t < \sigma_{U^c})$$

for any Borel set A and any $x \in U$, where σ_{U^c} is the hitting time of U^c defined by

$$\sigma_{U^c}(\omega) = \inf\{t > 0 | X_t(\omega) \in U^c\}.$$

Moreover, if p_t^U is the transition semigroup associated with $\mathbf{M}_U$, then

$$(p_t^U u)(x) = E_x^U(u(X_t^U)) = E_x(\chi_{\{t<\sigma_{U^c}\}} u(X_t))$$

for any non-negative measurable function u and any $x \in X$.

Remark. For a function $u : U \to \mathbb{R}$, we define $\epsilon_U(u) : X \to \mathbb{R}$ by $\epsilon_U(u)|_U = u$ and $\epsilon_U(u)|_{U^c} \equiv 0$. Through this extension map, $L^2(U, \mu_U)$ is regarded as a subspace of $L^2(X, \mu)$. Also, if $u \in \mathcal{D}_U$, then $\epsilon_U(u|_U) = u$ and hence we may think of $\mathcal{D}_U$ as a subset of $C(X)$ trough ϵ_U. Hereafter, we always use these conventions.

Remark. By the same reason as in the case of $\mathbf{M}$, the process $\mathbf{M}_U$ is determined for every $x \in U$.

The existence and the continuity of heat kernel have been studied by several authors. In [5], the existence of quasi-continuous version of heat kernel (i.e. transition density) has been proven under ultracontractivity. Grigor'yan has shown the corresponding result only assuming local ultracontractivity in [18]. In [15], the existence of jointly continuous heat kernel have been shown for resistance forms under ultracontractivity. The following theorem establish the existence of jointly continuous heat kernel for resistance forms without ultracontractivity and, at the same time, gives an upper diagonal estimate of the heat kernel. The main theorem of this section is the following.

Theorem 9.4. *Assume that $\overline{B_R(x, r)}$ is compact for any $x \in X$ and $r > 0$. Let U be a non-empty open subset of X. Then there exits $p_U(t, x, y) : (0, \infty) \times X \times X \to [0, +\infty)$ which satisfies the following conditions:*

(TD1) *$p_U(t, x, y)$ is continuous on $(0, \infty) \times X \times X$. Define $p_U^{t,x}(y) = p_U(t, x, y)$. Then $p_U^{t,x} \in \mathcal{D}_U$ for any $(t, x) \in (0, \infty) \times X$.*

(TD2) *$p_U(t, x, y) = p_U(t, y, x)$ for any $(t, x, y) \in (0, \infty) \times X \times X$.*

(TD3) *For any non-negative (Borel)-measurable function u and any $x \in X$,*

$$(p_t^U u)(x) = \int_X p_U(t, x, y) u(y) \mu(dy) \tag{9.1}$$

(TD4) *For any $t, s > 0$ and any $x, y \in X$,*

$$p_U(t + s, x, y) = \int_X p_U(t, x, z) p_U(s, y, z) \mu(dz). \tag{9.2}$$

Furthermore, let A be a Borel subset of X which satisfies $0 < \mu(A) < \infty$. Define $\overline{R}(x, A) = \sup_{y \in A} R(x, y)$ for any $x \in X$. Then

$$p_U(t, x, x) \leq \frac{2\overline{R}(x, A)}{t} + \frac{\sqrt{2}}{\mu(A)} \tag{9.3}$$

for any $x \in X$ and any $t > 0$.

The proof of the upper heat kernel estimate (9.3) is fairly simple. Originally, the same result has been obtained by more complicated discussion in [6] and [37]. Simplified argument, which is essentially the same as ours, for random walks can be found in [8].

Remark. In fact, we have the following inequality which is slightly better than (9.3). For any $\epsilon > 0$,

$$p_U(t, x, x) \leq \left(1 + \frac{1}{\epsilon}\right) \frac{\overline{R}(x, A)}{t} + \frac{\sqrt{1 + \epsilon}}{\mu(A)}. \tag{9.4}$$

This inequality implies that

$$\lim_{t \to \infty} p_U(t, x, x) \leq \frac{1}{\mu(X)}.$$

for any $x \in X$.

Definition 9.5. $p_U(t, x, y)$ is called the transition density and/or the heat kernel associated with the Dirichlet form $(\mathcal{E}_U, \mathcal{D}_U)$ on $L^2(X, \mu)$.

Corollary 9.6. *Assume that $\overline{B_R(x, r)}$ is compact for any $x \in X$ and any $r > 0$. Let U be a non-empty open subset of X. Then*

$$\lim_{t \downarrow 0} t p_U(t, x, x) = 0$$

for any $x \in X$.

Proof. Choose $A = B_R(x, r)$. By (9.3), it follows that $tp_U(t, x, x) \leq 3r$ for sufficiently small t. □

The rest of this section is devoted to the proof of Theorem 9.4. First we deal with the case where $\overline{U}$ is compact.

Lemma 9.7. *If $\overline{U}$ is compact, then we have $p_U(t, x, y) : (0, \infty) \times X \times X$ which satisfies* (TD1), (TD2), (TD3) *and* (TD4).

Proof. Let H_U be the non-negative self-adjoint operator on $L^2(U, \mu_U)$ associated with the Dirichlet form $(\mathcal{E}_U, \mathcal{D}_U)$. Then by Lemma 8.6, H_U has compact resolvent. Hence, there exists a complete orthonormal system $\{\varphi_n\}_{n \geq 1}$ of $L^2(U, \mu_U)$ and $\{\lambda_n\}_{n \geq 1} \subseteq [0, +\infty)$ such that $\varphi_n \in \mathrm{Dom}(H_U) \subseteq \mathcal{D}_U$, $H_U \varphi_n = \lambda_n \varphi_n$,

$\lambda_n \leq \lambda_{n+1}$ and $\lim_{n\to\infty} \lambda_n = +\infty$.
Claim 1:

$$\sum_{n\geq 1} \frac{1}{\lambda_n + 1} < +\infty \tag{9.5}$$

Proof of Claim 1: By Lemma 8.5, for any $x \in U$, there exists $g^x_{1,U} \in \mathcal{D}_U$ such that $\mathcal{E}_1(g^x_{1,U}, u) = u(x)$ for any $u \in \mathcal{D}_U$. Since

$$\varphi_n(x) = \mathcal{E}_1(g^x_{1,U}, \varphi_n) = (\lambda_n + 1)\int_U g^x_{1,U}\varphi_n d\mu_U,$$

we have $g^x_{1,U} = \sum_{n\geq 1} \frac{\varphi_n(x)}{\lambda_n+1}\varphi_n$ in $L^2(U, \mu_U)$. Hence

$$g^x_{1,U}(x) = \mathcal{E}_1(g^x_{1,U}, g^x_{1,U}) = \sum_{n\geq 1} \frac{\varphi_n(x)^2}{\lambda_n + 1} \tag{9.6}$$

On the other hand, by the same argument as in the proof of Theorem 8.8, there exists $c_U > 0$ such that

$$|\mathcal{E}_1(u, g^x_{1,U})| \leq |u(x)| \leq ||u||_{\infty,K} \leq c_U\sqrt{\mathcal{E}_1(u,u)}$$

for any $u \in \mathcal{D}_U$, where $K = \overline{U}$. This implies that $\mathcal{E}_1(g^x_{1,U}, g^x_{1,U}) \leq c_U$. Combining this with (9.6), we see that $g^x_{1,U}(x)$ is uniformly bounded on U. Hence by integrating (9.6) with respect to x, we obtain (9.5) by the monotone convergence theorem.
Claim 2: $||\varphi_n||_\infty \leq \sqrt{D\lambda_n}$ for any $n \geq 2$, where $D = \sup_{x,y\in U} R(x,y)$.
Proof of Claim 2: By (4.1),

$$|\varphi_n(x) - \varphi_n(y)|^2 \leq \mathcal{E}(\varphi_n, \varphi_n)R(x,y) = \lambda_n R(x,y). \tag{9.7}$$

We have two cases. First if $U \neq X$, then $\varphi_n(y) = 0$ for any $y \in U^c$. Hence (9.7) implies the claim. Secondly, if $U = X$, then (X, R) is compact. It follows that $\lambda_1 = 0$ and φ_1 is constant on X. Hence $\int_X \varphi_n(x)\mu(dx) = 0$ for any $n \geq 2$. For any $x \in X$, we may find $y \in X$ so that $\varphi_n(x)\varphi_n(y) \leq 0$. Since $|\varphi_n(x)|^2 \leq |\varphi_n(x) - \varphi_n(y)|^2$, (9.7) yields the claim.
Claim 3: $\sum_{n\geq 1} e^{-\lambda_n t}\varphi_n(x)\varphi_n(y)$ converges uniformly on $[T, \infty) \times X \times X$ for any $T > 0$.
Proof of Claim 3: Note that $e^{-a} \leq 2/a^2$ for any $a > 0$. This fact with Claim 2 shows that $|e^{-\lambda_n t}\varphi_n(x)\varphi_n(y)| \leq 2/(\lambda_n t^2)$. Using Claim 1, we immediately obtain Claim 3.

Now, let $\tilde{p}_U(t,x,y) = \sum_{n\geq 1} e^{-\lambda_n t}\varphi_n(x)\varphi_n(y)$. By Claim 3, $\tilde{p}$ is continuous on $(0, +\infty) \times X \times X$. Also, $\tilde{p}_U$ is the integral kernel of the strongly continuous semigroup $\{T^U_t\}_{t>0}$ associated with the Dirichlet form $(\mathcal{E}_U, \mathcal{D}_U)$ on $L^2(U, \mu_U)$. Let A be a Borel set. Then

$$(T^U_t\chi_A)(x) = \int_X \tilde{p}_U(t,x,y)\chi_A(y)\mu(dy).$$

In particular $T_t^U \chi_A$ is continuous on X. By [17, Theorem 4.2.3], $p_t^U \chi_A$ is a quasi continuous version of $T_t^U \chi_A$. Since any quasi continuous function is continuous by Proposition 8.12, we have $(T_t^U \chi_A)(x) = (p_t^U \chi_A)(x)$ for any $x \in X$. Letting $p_U(t,x,y) = \tilde{p}_U(t,x,y)$, we have (TD3). The rest of the required properties are straightforward. $\square$

The following facts are well-known in general setting. See [17] for example. In this particular situation, they can be checked by the eigenfunction expansion of the heat kernel above.

Lemma 9.8. *Assume that $\overline{U}$ is compact.*
(1) For any $t > 0$ and any $x, y \in X$,

$$\frac{\partial p_U}{\partial t}(t,x,y) = -\mathcal{E}(p_U^{t/2,x}, p_U^{t/2,y}).$$

(2) For any $t, s > 0$ and any $x \in X$,

$$\mathcal{E}(p_U^{t,x}, p_U^{s,x}) \le \frac{2}{t+s} p_U\left(\frac{t+s}{2}, x, x\right).$$

Lemma 9.9. *If $\overline{U}$ is compact, then* (9.3) *holds for any Borel subset A of X which satisfies $0 < \mu(A) < \infty$.*

Proof. Since $\int_A p_U(t,x,y)\mu(dy) \le \int_X p_U(t,x,y)\mu(dy) \le 1$, there exists $y_* \in A$ such that $p_U(t,x,y_*) \le 1/\mu(A)$. By this fact along with Lemma 9.8-(2),

$$\begin{aligned}\frac{1}{2}p_U(t,x,x)^2 &\le p_U(t,x,y_*)^2 + |p_U(t,x,x) - p_U(t,x,y_*)|^2 \\ &\le \frac{1}{\mu(A)^2} + \overline{R}(x,A)\mathcal{E}(p_U^{t,x}, p_U^{t,x}) \le \frac{1}{\mu(A)^2} + \frac{\overline{R}(x,A)}{t} p_U(t,x,x)\end{aligned}$$

Solving this with respect to $p_U(t,x,x)$, we have

$$p_U(t,x,x) \le \frac{\overline{R}(x,A)}{t} + \left(\frac{2}{\mu(A)^2} + \frac{\overline{R}(x,A)^2}{t^2}\right)^{\frac{1}{2}} \le \frac{2\overline{R}(x,A)}{t} + \frac{\sqrt{2}}{\mu(A)}.$$

$\square$

Remark. To get (9.4), we only need to use

$$p_U(t,x,x)^2 \le (1+\epsilon)p_U(t,x,y)^2 + \left(1 + \frac{1}{\epsilon}\right)|p_U(t,x,x) - p_U(t,x,y)|^2$$

in place of the inequality with $\epsilon = 1$ in the above proof.

Thus we have shown Theorem 9.4 if $\overline{U}$ is compact.

Proof of Theorem 9.4. If $\overline{U}$ is compact, then we have completed the proof. Assume that $\overline{U}$ is not compact. Fix $x_* \in X$ and set $U_n = B_R(x_*, n) \cap U$ for any $n = 1, 2, \ldots$. Note that $\overline{U_n}$ is compact. Write $p_n(t,x,y) = p_{U_n}(t,x,y)$.
Claim 1 $p_n(t,x,y) \le p_{n+1}(t,x,y)$ for any $x, y \in X$ and any $n \ge 1$.
Proof of Claim 1. Let $\sigma_n = \sigma_{X\setminus U_n}$. Then $\sigma_n \le \sigma_{n+1}$ for any n. Hence

$$(p_t^{U_n}u)(x) = E_x(\chi_{t<\sigma_n}u(X_t)) \le E_x(\chi_{t<\sigma_{n+1}}u(X_t)) = (p_t^{U_{n+1}}u)(x)$$

for any non-negative measurable function u and any $x \in X$. By (TD3), we deduce Claim 1.

Let A be a Borel subset of X which satisfies $0 < \mu(A) < \infty$. By (TD4) and (9.3), we have

$$\begin{aligned} p_n(t,x,y) &\le \sqrt{p_n(t,x,x)}\sqrt{p_n(t,y,y)} \\ &\le \left(\frac{2\overline{R}(x,A)}{t} + \frac{\sqrt{2}}{\mu(A)}\right)^{\frac{1}{2}}\left(\frac{2\overline{R}(y,A)}{t} + \frac{\sqrt{2}}{\mu(A)}\right)^{\frac{1}{2}} \end{aligned} \tag{9.8}$$

for any $x \in X$, any $t > 0$ and any n. Hence $p_n(t,x,y)$ is uniformly bounded and monotonically nondecreasing as $n \to \infty$. This shows that $p_n(t,x,y)$ converges as $n \to \infty$. If $p(t,x,y) = \lim_{n\to\infty} p_n(t,x,y)$, then $p(t,x,y)$ satisfies the same inequality as (9.8). In particular (9.3) holds for $p(t,x,x)$. Also, we immediately verity (TD2) and (TD4) for $p(t,x,y)$ from corresponding properties of $p_n(t,x,y)$. About (TD3), let u be a non-negative Borel-measurable function. Then by (TD3) for $p_n(t,x,y)$,

$$(p_t^{U_n}u)(x) = E_x(\chi_{\{t<\sigma_n\}}u(X_t)) = \int_X p_n(t,x,y)u(y)\mu(dy)$$

for any $x \in X$. The monotone convergence theorem shows that

$$E_x(\chi_{t<\sigma_{X\setminus U}}u(X_t)) = \int_X p(t,x,y)u(y)\mu(dy).$$

Since the left-hand side of the about equality equals $(p_t^U u)(x)$, we have (TD3).

Finally we show (TD1). Fix $(t,x,y) \in (0,\infty) \times X \times X$. Define $V = (t-\epsilon, t+\epsilon) \times B_R(x,r) \times B_R(y,r)$, where $r > 0$ and $0 < \epsilon < t$. (9.3) shows that

$$C = \sup_{(s,x',y')\in V_1, n\ge 1}\left(\sqrt{\frac{p_n(s,x',x')}{s}} + \sqrt{\frac{p_n(s,y',y')}{s}}\right) < \infty,$$

where $V_1 = ((t-\epsilon)/2, t+\epsilon) \times B_R(x,r) \times B_R(y,r)$. By Lemma 9.8, for any

$(s, a, b) \in V$ and any $n \geq 1$,

$$\begin{aligned}
&|p_n(t,x,y) - p_n(s,a,b)| \\
\leq &|p_n(t,x,y) - p_n(t,x,b)| + |p_n(t,x,b) - p_n(t,a,b)| + |p_n(t,a,b) - p_n(s,a,b)| \\
\leq &\sqrt{\mathcal{E}(p^{t,x}_{U_n}, p^{t,x}_{U_n})R(y,b)} + \sqrt{\mathcal{E}(p^{t,b}_{U_n}, p^{t,b}_{U_n})R(x,a)} + |t-s||\frac{\partial p_n}{\partial t}(t',a,b)| \\
\leq &\sqrt{\frac{p_n(t,x,x)R(y,b)}{t}} + \sqrt{\frac{p_n(t,b,b)R(x,a)}{t}} + 2|t-s|\frac{\sqrt{p_n(t'/2,a,a)p(t'/2,b,b)}}{t'} \\
\leq &C\sqrt{R(x,a)} + C\sqrt{R(y,b)} + C^2|t-s|
\end{aligned}$$

where t' is a value between t and s. Letting $n \to \infty$, we have

$$|p(t,x,y) - p(s,a,b)| \leq C\sqrt{R(x,a)} + C\sqrt{R(y,b)} + C^2|t-s|.$$

Hence $p(t,x,y)$ is continuous on $(0,\infty) \times X \times X$. By (TD4), $p^{t,x}_U \in L^2(X,\mu)$ for any $t > 0$ and any $x \in X$. Using (TD3) and (TD4), we see

$$p^{t,x}_U = p^U_{t/2}(p^{t/2,x}_U) = T_t(p^{t/2,x}_U)$$

for any $t > 0$ and any $x, y \in X$, where $\{T_t\}_{t>0}$ is the strongly continuous semigroup associated with the Dirichlet form $(\mathcal{E}_U, \mathcal{D}_U)$ on $L^2(X,\mu)$. Hence $p^{t,x}_U \in \mathcal{D}$ for any $t > 0$ and any $x \in X$. □

Part II
Quasisymmetric metrics and volume doubling measures

The main subject of this part is the notion of qausisymmetric maps, which has been introduced in [45] as certain generalization of quasiconformal mappings of the complex plane. The results in this part will play an indispensable role to in the next part, where we will modify the original resistance metric quasisymmetrically to obtain a metric which is suitable for describing the asymptotic behavior of the associated heat kernel.

At the first section, we present several notions, whose combinations are shown to be equivalent to being quasisymmetric in the second section. In other words, we resolve the notion of being quasisymmetric into geometric and analytic components. In the latter two sections, we discuss relations between a metric and a measure. Under the volume doubling property of the measure, we will construct a quasisymmetric metric which satisfies certain desired relations.

Since [45], quasisymmetric maps and related subjects have been studied deeply by many authors. See Heinonen [28] for example. Some of the results in this section may be included in some of the preceeding articles. However, we give all the proofs since it is difficult to find an exact reference from such a huge number of literatures.

10 Semi-quasisymmetric metrics

In this section, we introduce several notions associated with quasisymmetric mappings and clarify their relations.

Notation. Let X be a set and let d be a distance on X. $\overline{B}_d(x, r)$ is the closed ball, i.e. $\overline{B}_d(x, r) = \{y | y \in X, d(x, y) \leq r\}$. For any $A \subseteq X$, $\text{diam}(A, d)$ is the diameter of A with respect to d defined by $\text{diam}(A, d) = \sup_{x,y \in A} d(x, y)$. Moreover, we set $d_*(x) = \sup_{y \in X} d(x, y)$ for any $x \in X$.

In the rest of this section, we assume that d and ρ are distances on a set X.

The following notion "semi-quasisymmetric" is called "weakly quasisymmetric" in [45] and can be traced back to [12] and [29]. See [45] for details.

Definition 10.1. ρ is said to be semi-quasisymmtric with respect to d, or $(\text{SQS})_\text{d}$ for short, if and only if there exist $\epsilon \in (0, 1)$ and $\delta > 0$ such that $\rho(x, z) < \epsilon\rho(x, y)$ whenever $d(x, z) < \delta d(x, y)$.

In the above definition, we may assume $\delta < 1$ without loss of generality.

Proposition 10.2. *If ρ is $(\text{SQS})_\text{d}$, then the identity map from (X, d) to (X, ρ) is continuous.*

This fact has been obtained in [45].

Proof. Assume that $d(x_n, x) \to 0$ as $n \to \infty$ and $\rho(x_n, x) \to a$ as $n \to \infty$, where $a > 0$. Choose $\epsilon_1 \in (\epsilon, 1)$. Then $\rho(x_{n+m}, x) > \epsilon_1\rho(x_n, x)$ and $d(x_{n+m}, x) < \delta d(x_n, x)$ for sufficiently large n and m. By $(\text{SQS})_\text{d}$, it follows that $\rho(x_{n+m}, x) < \epsilon\rho(x_n, x)$. This contradiction implies the desired conclusion. □

Proposition 10.3. *Assume that (X, d) is uniformly perfect. Then ρ is $(\text{SQS})_\text{d}$ if and only if, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\rho(x, z) < \epsilon\rho(x, y)$ whenever $d(x, z) < \delta d(x, y)$.*

Proof. Assume that ρ is $(\text{SQS})_\text{d}$. We will show that $d(x, z) < (c\delta)^n d(x, y)$ implies $\rho(x, z) < \epsilon^n\rho(x, y)$ by induction, where c is the constant appearing in Definition 6.1. The case $n = 1$ is obvious. Suppose this is true for n. Suppose that $d(x, z) < (c\delta)^{n+1}d(x, y)$. Since (X, d) is uniformly perfect, there exits $y' \in X$ such that $c(c\delta)^n d(x, y) \leq d(x, y') < (c\delta)^n d(x, y)$. By induction assumption, $\rho(x, y') < \epsilon^n\rho(x, y)$. Also since $d(x, z) < \delta d(x, y')$. we have $\rho(x, z) < \epsilon\rho(x, y')$. Therefore $\rho(x, z) < \epsilon^{n+1}\rho(x, y)$.

The converse is obvious. □

Next we consider geomtric interpretation of semi-quasisymmetricity. We say that ρ is semi-quasiconformal with respect to d, $(\text{SQC})_\text{d}$ for short, if ρ-balls are equivalent to d-balls with a uniform distortion. $(\text{SQS})_\text{d}$ implies $(\text{SQS})_\text{d}$ but not vise versa. To get an "if and only if" assertion, we need a kind of uniform distortion condition regarding annuli instead of balls, called annulus semi-quasiconformality. To give a precise statement, we introduce the followings.

Definition 10.4. (1) Define $\overline{d}_\rho(x,r) = \sup_{y \in B_\rho(x,r)} d(x,y)$ for $x \in X$ and $r > 0$. d is said to be doubling with respect to ρ if and only if there exist $\alpha > 1$ and $c > 0$ such that $\overline{d}_\rho(x, \alpha r) \leq c\overline{d}_\rho(x,r) < \infty$ for any $r > 0$ and any $x \in X$.
(2) ρ is said to be semi-quasiconformal with respect to d, or $(\mathrm{SQC})_\mathrm{d}$ for short, if and only if $\overline{d}_\rho(x,r) < +\infty$ for any $x \in X$ and any $r > 0$ and there exists $\delta \in (0,1)$ such that $B_d(x, \delta\overline{d}_\rho(x,r)) \subseteq B_\rho(x,r)$ for any $x \in X$ and $r > 0$.
(3) ρ is said to be annulus semi-quasiconformal with respect to d, or $(\mathrm{ASQC})_d$ for short, if and only if $\overline{d}_\rho(x,r) < \infty$ for any $x \in X$ and $r > 0$ and, for any $\epsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that $d(x,y) \geq \delta\overline{d}_\rho(x,r)$ whenever $\rho(x,y) \geq \epsilon r$.
(4) ρ is said to be weak annulus semi-quasiconformal with respect to d, or $(\mathrm{wASQC})_\mathrm{d}$ for short, if and only if $\overline{d}_\rho(x,r) < \infty$ for any $x \in X$ and $r > 0$ and there exist $\epsilon \in (0,1)$ and $\delta \in (0,1)$ such that $d(x,y) \geq \delta\overline{d}_\rho(x,r)$ whenever $\rho(x,y) \geq \epsilon r$.

Remark. (1) If d is doubling with respect to ρ, then

$$\overline{d}_\rho(x, ar) \leq c_0 a^\omega \overline{d}_\rho(x,r)$$

for any $r > 0, a \geq 1$ and $x \in X$, where c_0 and ω are positive constants which are independent of x, a and r. Hence the value of α itself is not essential. An easy choice of α is two, and this is why we call this notion "doubling".
(2) Note that $B_\rho(x,r) \subseteq \overline{B}_d(x, \overline{d}_\rho(x,r))$. Hence ρ is $(\mathrm{SQC})_\mathrm{d}$ if and only if, for any $x \in X$ and $r > 0$, there exist R_1 and R_2 such that $B_d(x, R_1) \subseteq B_\rho(x,r) \subseteq B_d(x, R_2)$ and $R_1 \geq CR_2$, where $C \in (0,1)$ is independent of x and r. Therefore, a ρ-ball is equivlent to a d-ball with a uniformly bounded distortion.
(3) Assume that $\overline{d}_\rho(x,r) < \infty$ for any x and r. Then $(\mathrm{ASQC})_\mathrm{d}$ is equivalent to the following statement: for any $\epsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that $d(x,y) \geq \delta\overline{d}_\rho(x,r)$ whenever $r > \rho(x,y) \geq \epsilon r$. Also $(\mathrm{ASQC})_\mathrm{d}$ implies that a ρ-annulus $B_\rho(x,r)\backslash B_\rho(x, \epsilon r)$ is contained in a d-annulus $B_d(x, (1+\gamma)\overline{d}_\rho(x,r))\backslash B_d(x, \delta\overline{d}_\rho(x,r))$ for any $\gamma > 0$.

Theorem 10.5. *Assume that both (X,d) and (X,ρ) are uniformly perfect and that $\overline{d}_\rho(x,r) < +\infty$ for any $x \in X$ and $r > 0$. Then the following four conditions are equivalent.*
(a) *ρ is* $(\mathrm{SQS})_\mathrm{d}$.
(b) *d is doubling with respect to ρ and ρ is* $(\mathrm{SQC})_\mathrm{d}$.
(c) *ρ is* $(\mathrm{ASQC})_\mathrm{d}$.
(d) *ρ is* $(\mathrm{wASQC})_\mathrm{d}$.

Proof. *(a)* $\Rightarrow$ *(b)*: First we show that d is doubling with respect to ρ. Since (X,ρ) is uniformly perfect, $B_\rho(x,r)\backslash B_\rho(x,cr) \neq \emptyset$ unless $X\backslash B_\rho(x,r) = \emptyset$, where c is independent of x and r. By Proposition 10.3, we may assume $\epsilon < c^2$. Now by $(\mathrm{SQS})_\mathrm{d}$, $\rho(x,z)/\epsilon \geq \rho(x,y)$ implies $d(x,z) \geq \delta d(x,y)$.
Claim: Suppose $r/\sqrt{\epsilon} > \rho(x,y)$. Then there exists $z \in B_\rho(x,r)$ such that $\rho(x,z)/\epsilon > \rho(x,y)$.
Proof of the claim: If $X\backslash B_\rho(x,r) \neq \emptyset$, then there exists $z \in X$ such that $r >$

$\rho(x,z) \geq cr$. Hence $\rho(x,z)/\epsilon \geq cr/\epsilon > r/\sqrt{\epsilon} > \rho(x,y)$. In case $X = B_\rho(x,r)$, let $\rho_*(x) = \sup_{x' \in X} \rho(x,x')$. Then $\rho_*(x)/\epsilon > \rho_*(x) \geq \rho(x,y)$. Hence there exists $z \in B_\rho(x,r) = X$ such that $\rho(x,z)/\epsilon > \rho(x,y)$. Thus we have shown the claim.
If $\rho(x,z)/\epsilon > \rho(x,y)$, $(\mathrm{SQS})_{\mathrm{d}}$ implies $d(x,z) \geq \delta d(x,y)$. By the above claim, we obtain that $\overline{d}_\rho(x,z) \geq \delta \overline{d}_\rho(x, y/\sqrt{\epsilon})$. Hence d is doubling with respect to ρ.

Next we show that ρ is $(\mathrm{SQC})_{\mathrm{d}}$. Suppose that $d(x,z) < \delta \overline{d}_\rho(x,r)$. Then there exists $y \in B_\rho(x,r)$ such that $d(x,z) < \delta d(x,y)$. Hence by $(\mathrm{SQS})_{\mathrm{d}}$, $\rho(x,z) < \epsilon\rho(x,y) < r$ and hence $z \in B_\rho(x,r)$.
$(b) \Rightarrow (c)$: Let $\epsilon \in (0,1)$. By $(\mathrm{SQC})_{\mathrm{d}}$, $B_\rho(x,\epsilon r) \supseteq B_d(x, \delta\overline{d}_\rho(x,\epsilon r))$. Since d is doubling with respect to ρ, $\overline{d}_\rho(x,\epsilon r) > c'\overline{d}_\rho(x,r)$, where c' is independent of x and r. Therefore, $B_\rho(x,\epsilon r) \supseteq B_d(x, \delta c' \overline{d}_\rho(x,r))$. This immediately imply $(\mathrm{ASQC})_{\mathrm{d}}$.
$(c) \Rightarrow (d)$: This is obvious.
$(d) \Rightarrow (a)$: Let $\rho(x,z) \geq \epsilon r$. Then $\epsilon^{-n+1} r \leq \rho(x,z) < \epsilon^{-n} r$ for some $n \geq 0$. By $(\mathrm{ASQC})_{\mathrm{d}}$, $\delta\overline{d}_\rho(x,r) \leq \delta\overline{d}_\rho(x, \epsilon^{-n} r) \leq d(x,z)$. Hence, $d(x,z) < \delta\overline{d}_\rho(x,r)$ implies $\rho(x,z) < \epsilon r$. Now suppose $d(x,z) < \delta d(x,y)$. Since $\delta d(x,y) \leq \delta \overline{d}_\rho(x, \rho(x,y))$, we have $\rho(x,y) < \epsilon\rho(x,y)$. □

Next we present useful implications of $(\mathrm{SQS})_{\mathrm{d}}, (\mathrm{SQC})_{\mathrm{d}}$ and $(\mathrm{ASQC})_{\mathrm{d}}$.

Definition 10.6. ρ is said to decay uniformly with respect to d if and only if
(i) $\mathrm{diam}(X,d) < +\infty$ and there exist $r_* > \mathrm{diam}(X,d)$ and $(a,\lambda) \in (0,1)^2$ such that $\overline{\rho}_d(x,\lambda r) \leq a\overline{\rho}_d(x,r)$ for any $x \in X$ and $r \in (0, r_*]$
or
(ii) $\mathrm{diam}(X,d) = +\infty$ and there exists $(a,\lambda) \in (0,1)^2$ such that $\overline{\rho}_d(x,\lambda r) \leq a\overline{\rho}_d(x,r)$ for any $x \in X$ and $r > 0$.

Proposition 10.7. *Assume that (X,d) is uniformly perfect and ρ is $(\mathrm{SQS})_{\mathrm{d}}$. Then ρ decays unifromly with respect to d. More precisely, if $\mathrm{diam}(X,d) < \infty$, then, for any $r_* > 0$, there exists $(a,\lambda) \in (0,1)^2$ such that $\overline{\rho}_d(x,\lambda r) \leq a\overline{\rho}_d(x,r)$ for any $x \in X$ and $r \in (0, r_*]$.*

Remark. If ρ is $(\mathrm{SQS})_{\mathrm{d}}$, then $B_d(x, \delta d(x,y)) \subseteq B_\rho(x, \epsilon\rho(x,y))$. Hence $\overline{\rho}_d(x,r) < \infty$ if $r < \delta d_*(x)$. Note that $d_*(x) \geq \mathrm{diam}(X,d)/2$.

Proof. Since ρ is $(\mathrm{SQS})_{\mathrm{d}}$, there exist $\epsilon \in (0,1)$ and $\delta \in (0,1)$ such that $\rho(x,z) < \epsilon\rho(x,y)$ whenever $d(x,z) < \delta d(x,y)$. Also there exists $c \in (0,1)$ such that $B_d(x,r) \backslash B_d(x,cr) \neq \emptyset$ unless $X = B_d(x,r)$. Suppose that $\mathrm{diam}(X,d) < \infty$. Choose $n \geq 1$ so that $c^{n-1} r_* < \mathrm{diam}(X,d)/2$. Since $d_*(x) \geq \mathrm{diam}(X,d)/2$, it follows that $X \neq B_d(x, c^{n-1} r)$ for any $r \in (0, r_*]$. Therefore, there exists $y \in X$ such that $c^n r \leq d(x,y) < c^{n-1} r$. If $d(x,z) < c^n \delta r$, we have $d(x,z) < \delta d(x,y)$. Hence, $\rho(x,z) < \epsilon\rho(x,y)$. This shows that $\overline{\rho}_d(x, c^n\delta r) \leq \epsilon\rho(x,y) \leq \epsilon\overline{\rho}_d(x,r)$. The similar arguments suffice as well in the case where $\mathrm{diam}(X,d) = \infty$. □

Proposition 10.8. *Assume that d is doubling with respect to ρ and ρ is $(\mathrm{SQC})_{\mathrm{d}}$. Let μ be a Borel regular measure on X. Then μ is $(\mathrm{VD})_\rho$ if μ is $(\mathrm{VD})_{\mathrm{d}}$.*

Combining this proposition with Thereom 10.5, we see that the volume doubling property is inherited from d to ρ if ρ is $(\mathrm{SQS})_{\mathrm{d}}$ under the uniform perfectness.

Proof. Since d is $(\mathrm{VD})_\rho$,

$$B_\rho(x, 2r) \subseteq B_d(x, 2\overline{d}_\rho(x, 2r)) \subseteq B_d(x, c'\overline{d}_\rho(x, r)),$$

where $c' > 1$. If μ is $(\mathrm{VD})_{\mathrm{d}}$, then

$$\mu(B_d(x, c'\overline{d}_\rho(x, r))) < c\mu(B_d(x, \delta\overline{d}_\rho(x, r))).$$

Moreover, by $(\mathrm{SQC})_{\mathrm{d}}$, $B_d(x, \delta\overline{d}_\rho(x, r)) \subseteq B_\rho(x, r)$. Thus, we have

$$\mu(B_\rho(x, 2r)) \leq c\mu(B_d(x, \delta\overline{d}_\rho(x, r)) \leq c\mu(B_\rho(x, r)).$$

□

The following lemma is quite similar to Theorem 10.5 but is a little stronger since it does not assume that (X, d) is uniformly perfect. We will take advantage of this stronger statement later.

Lemma 10.9. *Assume that (X, ρ) is uniformly perfect. If ρ is $(\mathrm{ASQC})_{\mathrm{d}}$, then d is doubling with respect to ρ.*

Proof. By the assumption, $B_\rho(x, r)\backslash B_\rho(x, cr) \neq \emptyset$ unless $B_\rho(x, cr) = X$ for some $c \in (0, 1)$. Let $\epsilon = c^2$. By $(\mathrm{ASQC})_{\mathrm{d}}$, for some $\delta \in (0, 1)$, $d(x, y) \geq \delta\overline{d}_\rho(x, r)$ when $\rho(x, y) \geq \epsilon r$. If $B_\rho(x, r) = B_\rho(x, cr)$, then $\overline{d}_\rho(x, r) = \overline{d}_\rho(x, cr) \geq \delta\overline{d}_\rho(x, r)$. If $B_\rho(x, r)\backslash B_\rho(x, cr) \neq \emptyset$, then there exists $y \in X$ such that $\epsilon r \leq \rho(x, y) \leq cr$. This also implies that $\overline{d}_\rho(x, cr) \geq \delta\overline{d}_\rho(x, r)$. Hence we have the doubling property of $\overline{d}_\rho$. □

Proposition 10.10. *Assume that (X, ρ) is uniformly perfect and that $\overline{d}_\rho(x, r) < \infty$ for any $x \in X$ and $r > 0$. If ρ is $(\mathrm{ASQC})_{\mathrm{d}}$, then*

$$d(x, y) \asymp \overline{d}_\rho(x, \rho(x, y))$$

for any $x, y \in X$.

Proof. There exist $\epsilon \in (0, 1)$ and $\delta \in (0, 1)$ such that $\epsilon r \leq \rho(x, y) < r$ implies $d(x, y) \geq \delta\overline{d}_\rho(x, r)$. Choose $\alpha > 1$ so that $\alpha\epsilon < 1$. For any $y \in X$, we have $\epsilon\alpha\rho(x, y) \leq \rho(x, y) < \alpha\rho(x, y)$. Hence $d(x, y) \geq \delta\overline{d}_\rho(x, \alpha\rho(x, y)) \geq \delta\overline{d}_\rho(x, \rho(x, y))$. By Lemma 10.9, d is doubling with respect to ρ. Therefore, $c_2\overline{d}_\rho(x, \rho(x, y)) \leq \overline{d}_\rho(x, \alpha\rho(x, y)) \leq d(x, y)$, where c_2 only depends on α. □

11 Quasisymmetric metrics

In this section, we will introduce the notion of being quasisymmetric and relate it to the notions obtained in the last section.

d and ρ are distances on a set X through this section.

Definition 11.1. ρ is said to be quasisymmetric, or QS for short, with respect to d if and only if there exists a homeomorphism h from $[0, \infty)$ to itself such that $h(0) = 0$ and, for any $t > 0$, $\rho(x, z) < h(t)\rho(x, y)$ whenever $d(x, z) < td(x, y)$. We write $\rho \underset{\text{QS}}{\sim} d$ if ρ is quasisymmetric with respect to d.

The followings are basic properties of quasisymmetric distances.

Proposition 11.2. *Assume that ρ is quasisymmetric with respect to d. Then*
(1) *d is quasisymmetric with respect to ρ.*
(2) *The identity map from (X, d) to (X, ρ) is a homeomorphism.*
(3) *(X, d) is uniformly perfect if and only if (X, ρ) is uniformly perfect.*
(4) *(X, d) is bounded if and only if (X, ρ) is bounded.*
(5) *Define $\overline{d}_\rho(x, r) = \sup_{y \in B_\rho(x,r)} d(x, y)$ and $\overline{\rho}_d(x, r) = \sup_{y \in B_d(x,r)} \rho(x, y)$. Then $\overline{d}_\rho(x, r)$ and $\overline{\rho}_d(x, r)$ are finite for any $x \in X$ and any $r > 0$.*

Those statements, in particular (1) and (3), have been obtained in the original paper [45].

Proof. (1) Note that $\rho(x, z) \geq h(t)\rho(x, y)$ implies $d(x, z) \geq td(x, y)$. Hence if $h(t)^{-1}\rho(x, z) > \rho(x, y)$, then $2t^{-1}d(x, z) > d(x, y)$. Set $g(s) = 2/h^{-1}(1/t)$. Then $g(s)$ is a homeomorphism from $[0, \infty)$ to itself and $g(s)d(x, z) > d(x, y)$ whenever $t\rho(x, z) > \rho(x, y)$. Thus d is QS with respect to ρ.
(2) If $\rho \underset{\text{QS}}{\sim} d$, then ρ is $(\text{SQS})_{\text{d}}$ and d is $(\text{SQS})_\rho$. Now, Proposition 10.2 suffices.
(3) There exists $\delta \in (0, 1)$ such that $B_d(x, r/\delta)\backslash B_d(x, r) \neq \emptyset$ if $B_d(x, r) \neq X$ by the uniform perfectness. Choose $t_* \in (0, 1)$ so that $h(t_*) < 1$. Suppose $B_\rho(x, r) \neq X$. There exists $y \in X$ such that $\rho(x, y) > r$. Let $r = \delta t_* d(x, y)$. Since $r < d(x, y)$, $B_d(x, r) \neq \emptyset$. Hence there exists $y_1 \in X$ such that $\delta t_* d(x, y) \leq d(x, y_1) < t_* d(x, y)$. Since $\rho \underset{\text{QS}}{\sim} d$, we have $\lambda_1 \rho(x, y) < \rho(x, y_1) < \lambda_2 \rho(x, y)$, where $0 < \lambda_1 = h(2/(\delta t_*)) < \lambda_2 = h(t_*) < 1$. In the same way, we have y_2 which satisfies $\lambda_1 \rho(x, y_1) < \rho(x, y_2) < \lambda_2 \rho(x, y_1)$. Inductively, we may construct $\{y_n\}_{n \geq 1}$ such that $\lambda_1 \rho(x, y_n) < \rho(x, y_{n+1}) < \lambda_2 \rho(x, y_n)$. Choose m so that $\rho(x, y_{m+1}) < r \leq \rho(x, y_m)$. Then $y_m \in B_\rho(x, r/\lambda_1)\backslash B_\rho(x, r)$. Hence (X, ρ) is uniformly perfect.
(4), (5) Obvious. $\square$

By (1) of the above proposition, $\underset{\text{QS}}{\sim}$ is an equivalence relation.

The following theorem relates the notion of begin semi-quasisymmetric with being quasisymmetric. It has essentially been obtained in [45, Theorem 3.10], where the notion of "unifromly perfect" is called "homogeneously dense".

Theorem 11.3. *Assume that both (X, d) and (X, ρ) are uniformly perfect. Then ρ is QS with respect to d if and only if ρ is* $(\mathrm{SQS})_{\mathrm{d}}$ *and d is* $(\mathrm{SQS})_{\rho}$.

Proof. If ρ is QS with respect to d, then it is straight forward to see that d and ρ are semi-quasisymmetric with respect to each other. Conversely, assume that d and ρ are semi-quasisymmetric with respect to each other. Then by Proposition 10.3, we may construct homeomorphisms $h_1 : [0.\delta_1] \to [0, \epsilon_1]$ and $h_2 : [0, \delta_2] \to [0, \epsilon_2]$ which satisfy
(i) $h_1(0) = 0, h_2(0) = 0$,
(ii) $\rho(x, z) < h_1(\delta)\rho(x, y)$ whenever $d(x, z) < \delta d(x, y)$ for any $\delta \in (0, \delta_1]$ and
(iii) $d(x, z) < h_2(\delta)d(x, y)$ whenever $\rho(x, z) < \delta\rho(x, y)$ for any $\delta \in (0, \delta_2]$.

Define
$$h_3(t) = \begin{cases} 2/(h_2)^{-1}(1/t) & \text{for } t \in [1/\delta_2, \infty), \\ 2/\epsilon_2 & \text{for } t \in [0, 1/\delta_2]. \end{cases}$$
Then $\rho(x, z) < h_3(\delta)\rho(x, y)$ whenever $d(x, z) < \delta d(x, y)$ for any $\delta \in (0, \infty)$. There is no difficulty to find a homeomorphism $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ that satisfies $h(t) \geq h_1(t)$ for any $t \in [0, \delta_1]$ and $h(t) \geq h_3(t)$ for any $t \in [\delta_1, \infty)$. Obviously $d(x, z) < td(x, y)$ implies $\rho(x, z) < h(t)\rho(x, y)$ for any $t > 0$. Therefore, ρ is QS with respect to d. □

Combining this theorem with Theorem 10.5, we can produce several equivalent conditions for quasisymmetricity under uniform perfectness.

The next corollary is a modified version of Proposition 10.8.

Corollary 11.4. *Assume that (X, d) is uniformly perfect and that $\rho \underset{\mathrm{QS}}{\sim} d$. Let μ be a Borel regular measure on (X, d). Then μ is* $(\mathrm{VD})_{\mathrm{d}}$ *if and only if it is* $(\mathrm{VD})_{\rho}$.

Proof. By Proposition 11.2-(3), (X, ρ) is uniformly perfect. Hence by Theorem 11.3, ρ is $(\mathrm{SQS})_{\mathrm{d}}$ and d is $(\mathrm{SQS})_{\rho}$. Theorem 10.5 shows that d is doubling with respect to ρ and ρ is $(\mathrm{SQC})_{\mathrm{d}}$. By Proposition 10.8, if μ is $(\mathrm{VD})_{\mathrm{d}}$, then μ is $(\mathrm{VD})_{\rho}$. The converse follows by exchanging d and ρ. □

12 Relations of measures and metrics

To obtain a heat kernel estimate, one often show a certain kind of relations concerning a measure and a distance. The typical example in the following relation:
$$d(x, y)\mu(B_{\rho}(x, \rho(x, y))) \asymp \rho(x, y)^{\beta}, \tag{12.1}$$
where $d(x, y)$ is the resistance metric (may be written as $R(x, y)$), ρ is a distance used in the heat kernel estimate and β is a positive exponent. The left hand side corresponds the escape time from a ρ-ball. We generalize such a kind of relations and study them in the light of quasisymmetricity in the present section.

Troughout this section, d and ρ are distances on a set X which give the same topology on X. μ is a Borel regular measure on (X, d). We assume that

$0 < \mu(B_d(x,r)) < +\infty$ and $0 < \mu(B_\rho(x,r)) < +\infty$ for any $x \in X$ and any $r > 0$.

Notation. We set $V_d(x,r) = \mu(B_d(x,r))$ and $V_\rho(x,r) = \mu(B_\rho(x,r))$.

Let $H : (0,\infty)^2 \to (0,\infty)$ satisfy the following two conditions:
(H1) if $0 < s_1 \le s_2$ and $0 < t_1 \le t_2$, then $H(s_1,t_1) \le H(s_2,t_2)$,
(H2) for any $(a,b) \in (0,\infty)^2$, define

$$h(a,b) = \sup_{(s,t)\in(0,\infty)^2} \frac{H(as,bt)}{H(s,t)}.$$

Then $h(a,b) < \infty$ for any $(a,b) \in (0,\infty)^2$ and there exists $c_0 > 0$ such that $h(a,b) < 1$ for any $(a,b) \in (0,c_0)^2$.

Also $g : (0,\infty) \to (0,\infty)$ is a monotonically increasing function satisfying $g(t) \downarrow 0$ as $t \downarrow 0$ and the doubling property, i.e. there exists $c > 0$ such that $g(2t) \le cg(t)$ for any $t > 0$.

We will study several relations between conditions concerning d, ρ, μ, H and g.

Definition 12.1. (1) We say that the condition (DM1) holds if and only if there exists $\eta : (0,1] \to (0,\infty)$ such that η is monotonically nondecreasing, $\eta(t) \downarrow 0$ as $t \downarrow 0$ and

$$\eta(\lambda)\frac{g(\rho(x,y))}{H(d(x,y),V_\rho(x,\rho(x,y)))} \ge \frac{g(\lambda\rho(x,y)))}{H(d(x,y),V_\rho(x,\lambda\rho(x,y)))}$$

for any $x,y \in X$ and any $\lambda \in (0,1]$.
(2) We say that the condition (DM2) holds if and only if

$$H(d(x,y),V_\rho(x,\rho(x,y))) \asymp g(\rho(x,y))$$

for any $x,y \in X$.
(3) We say that the condition (DM3) holds if and only if there exist $r_* > \mathrm{diam}(X,\rho)$ such that

$$H(\overline{d}_\rho(x,r),V_\rho(x,r)) \asymp g(r)$$

for any $x \in X$ and any $r \in (0,r_*]$.

The relation (DM2) can be seen as a generalization of the above mentioned relation (12.1), where $H(s,t) = st$ and $g(r) = r^\beta$. The relation (DM1) looks too complicated but it is shown to be necessary if $d \underset{\mathrm{QS}}{\sim} \rho$ and (DM2) is satisfied. See Corollary 12.3.

Remark. If $\mathrm{diam}(X,\rho) = \infty$, then we remove the statement "$r_* > \mathrm{diam}(X,\rho)$" and replace "$r \in (0,r_*]$" by "$r > 0$" in (3) of the above definition.

In the next section, we are going to construct a distance ρ on X which satisfies all three conditions (DM1), (DM2) and (DM3) with $g(r) = r^\beta$ for sufficiently large β under a certain assumptions. See Theorem 13.1 for details.

The next theorem gives the basic relations. Much clearer description from quasisymmetric point of view can be found in the corollary below.

Theorem 12.2. *Assume that* (X, ρ) *is uniformly perfect, that* $\lim_{s\downarrow 0} h(s,1) = \lim_{t\downarrow 0} h(1,t) = 0$ *and that there exists* $c_* > 0$ *such that* $\mu(X) \leq c_* V_\rho(x, \rho_*(x))$ *for any* $x \in X$*, where* $\rho_*(x) = \sup_{y\in X} \rho(x,y)$*.*
(1) μ *is* $(\mathrm{VD})_\rho$ *under* (DM1) *and* (DM2).
(2) (DM1) *and* (DM2) *hold if and only if* (DM3) *holds,* d *decays uniformly with respect to* ρ *and* ρ *is* $(\mathrm{ASQC})_{\mathrm{d}}$.

Remark. If $\mathrm{diam}(X, \rho) = \infty$, then $\rho_*(x) = \infty$ for any $x \in X$. In this case, we define $B_\rho(x, \infty) = X$ and $V_\rho(x, \infty) = \mu(X)$. Hence letting $c_* = 1$, we always have $\mu(X) \leq c_* V_\rho(x, \rho_*(x))$.

On the other hand, if $\mathrm{diam}(X, \rho) < \infty$, then $\mathrm{diam}(X, \rho)/2 \leq \rho_*(x) \leq \mathrm{diam}(X, \rho)$. In this case, $X = \overline{B}(x, \rho_*(x))$.

Corollary 12.3. *In addition to the assumptions in Theorem 12.2, suppose that* (X, d) *is uniformly perfect. Then the following four conditions are equivalent:*
(a) (DM1) *and* (DM2) *hold.*
(b) $\rho \underset{\mathrm{QS}}{\sim} d$ *and* (DM2) *holds.*
(c) $\rho \underset{\mathrm{QS}}{\sim} d$ *and* (DM3) *holds.*
(d) (DM3) *holds,* d *decays uniformly with respect to* ρ *and* ρ *is* $(\mathrm{SQS})_{\mathrm{d}}$.
Moreover, if any of the above conditions is satisfied, then μ *is* $(\mathrm{VD})_{\mathrm{d}}$ *and* $(\mathrm{VD})_\rho$.

The rest of this section is devoted to proving the above theorem and the corollary.

Lemma 12.4. *If* (DM1) *and* (DM2) *are satisfied, than, for any* $\epsilon > 0$*, there exists* $\delta > 0$ *such that* $d(x,z) < \epsilon d(x,y)$ *whenever* $\rho(x,z) < \delta\rho(x,y)$*. In particular,* d *is* $(\mathrm{SQS})_\rho$*.*

Proof. Assume that $d(x,z) \geq \epsilon d(x,y)$ and that $\rho(x,z) \leq \rho(x,y)$. Let $\lambda = \rho(x,z)/\rho(x,y)$. Then by (DM2),

$$\begin{aligned} c_2 g(\lambda g(x,y)) &\geq H(d(x,z), V_\rho(x, \rho(x,z))) \\ &\geq H(\epsilon d(x,y), V_\rho(x, \lambda\rho(x,y))) \\ &\geq h(1/\epsilon, 1)^{-1} H(d(x,y), V_\rho(x, \lambda\rho(x,y))). \end{aligned}$$

Hence

$$c_3 \leq \frac{g(\lambda\rho(x,y))}{H(d(x,y), V_\rho(x, \lambda\rho(x,y)))},$$

where c_3 is a positive constant which depends only on ϵ. This combined with (DM1) and (DM2) implies that $0 < c_4 \leq \eta(\lambda)$, where c_4 depends only on ϵ. Hence, there exists $\delta > 0$ such that $\rho(x,z) \geq \delta\rho(x,y)$. Thus we have the contraposition of the statement. □

Lemma 12.5. *Assume* (DM1) *and that* $\lim_{t\downarrow 0} h(1,t) = 0$*. Then, for any* $\lambda > 0$*, there exists* $a > 0$ *such that* $V_\rho(x, \lambda\rho(x,y)) \geq a V_\rho(x, \rho(x,y))$ *for any* $x, y \in X$*.*

Remark. Note that $h(1, t^n) \leq h(1,t)^n$. Hence $\lim_{t\downarrow 0} h(1,t) = 0$ if and only if there exists $t_* \in (0,1)$ such that $h(1, t_*) < 1$.

Proof. For $\lambda \geq 1$, we may choose $a = 1$. Suppose that $\lambda \in (0,1)$. Then by (DM1) and the doubling property of g,

$$h\Big(1, \frac{V_\rho(x, \lambda\rho(x,y))}{V_\rho(x, \rho(x,y))}\Big) \geq \frac{H(d(x,y), V_\rho(x, \lambda\rho(x,y)))}{H(d(x,y), V_\rho(x, \rho(x,y)))} \geq \eta(\lambda)\frac{g(\lambda\rho(x,y))}{g(\rho(x,y))} \geq c_\lambda > 0,$$

where c_λ depends only on λ. Since $\lim_{t\downarrow 0} h(1,t) = 0$, we have the desired conclusion. $\square$

Lemma 12.6. *Assume that* (X, ρ) *is uniformly perfect and that* $\lim_{t\downarrow 0} h(1,t) = 0$*. If* d *is* $(\mathrm{SQS})_\rho$ *and* (DM2) *holds, then, for any sufficiently small* $\lambda \in (0,1)$*, there exists* $a > 0$ *such that* $V_\rho(x, \lambda\rho(x,y)) \geq aV_\rho(x, \rho(x,y))$ *for any* $x, y \in X$*.*

Proof. Since d is $(\mathrm{SQS})_\rho$, there exists $\epsilon \in (0,1)$ and $\delta_0 \in (0,1)$ such that $d(x,z) < \epsilon d(x,y)$ whenever $\rho(x,z) < \delta_0\rho(x,y)$. If $\delta \leq \delta_0$, then $d(x,z) < \epsilon d(x,y)$ whenever $\rho(x,z) < \delta\rho(x,y)$. Also there exists $c \in (0,1)$ such that $B_\rho(x,r)\backslash B_\rho(x,cr) \neq \emptyset$ if $B_\rho(x,r) \neq X$ because (X,ρ) is uniformly perfect. Let x and $y \in X$. Then we may choose $z \in B_\rho(x,\delta r)\backslash B_\rho(x, c\delta r)$, where $r = \rho(x,y)$. Note that $d(x,z) < \epsilon d(x,y) < d(x,y)$. By the doubling property of g and (DM2),

$$\begin{aligned}
c' g(\rho(x,z)) &\geq c_2 g(\rho(x,z)/(c\delta)) \geq c_2 g(r) \geq H(d(x,y), V_\rho(x,r)) \\
&\geq H(d(x,z), V_\rho(x,r)) \geq h\Big(1, \frac{V_\rho(x,\rho(x,z))}{V_\rho(x,\rho(x,y))}\Big)^{-1} H(d(x,z), V_\rho(x,\rho(x,z))) \\
&\geq c_1 h\Big(1, \frac{V_\rho(x,\delta\rho(x,y))}{V_\rho(x,\rho(x,y))}\Big)^{-1} g(\rho(x,z)).
\end{aligned}$$

Therefore, it follows that

$$h\Big(1, \frac{V_\rho(x,\delta\rho(x,y))}{V_\rho(x,\rho(x,y))}\Big) \geq c_3,$$

where $c_3 > 0$ is independent of x and y. Since $\lim_{t\downarrow 0} h(1,t) = 0$, we have $V_\rho(x,\delta\rho(x,y)) \geq aV_\rho(x,\rho(x,y))$. Letting $\lambda = \delta$, we have the desired statement. $\square$

Lemma 12.7. *Assume that* (X, ρ) *is uniformly perfect, that* $\lim_{t\downarrow 0} h(1,t) = 0$*, and that there exists* $c_* > 0$ *such that* $\mu(X) \leq c_* V_\rho(x, \rho_*(x))$ *for any* $x \in X$*. If either* (DM1) *is satisfied or* d *is* $(\mathrm{SQS})_\rho$ *and* (DM2) *is satisfied, then* μ *is volume doubling with respect to* ρ*.*

Proof. $B_\rho(x,r)\backslash B_\rho(x,cr) \neq \emptyset$ unless $B_\rho(x,cr) = X$ by the uniform perfectness. Suppose $r < \rho_*(x) = \sup_{y\in X}\rho(x,y)$. Choose λ so that $0 < \lambda < c$. Then there exists $y \in X$ such that $cr \leq \rho(x,y) < r$. By Lemmas 12.5 and 12.6, we have $V_\rho(x,\lambda\rho(x,y)) \geq aV_\rho(x,\rho(x,y))$ in either case. This immediately implies

$V_\rho(x, \lambda r) \geq aV_\rho(x, cr)$. Therefore, if $r < c\rho_*(x)$, then $V_\rho(x, \lambda' r) \geq aV_\rho(x, r)$, where $\lambda' = \lambda/c < 1$. If $\mathrm{diam}(X, \rho) = \infty$, then we have finished the proof. Otherwise, $\rho_*(x) < \infty$ for any $x \in X$.

If $r \in [c\rho_*(x), \rho_*(x))$, there exists $y \in X$ such that $r \leq \rho(x, y) \leq \rho_*(x)$. Lemma 12.5 implies that $V_\rho(x, \lambda\rho(x, y)) \geq aV_\rho(x, \rho(x, y)) \geq aV_\rho(x, r)$. Since $r/c \geq \rho_*(x) \geq \rho(x, y)$, we have $V_\rho(x, \lambda' r) \geq V_\rho(x, \lambda\rho(x, y)) \geq aV_\rho(x, r)$.

Finally, let $r \geq \rho_*(x)$. Then there exists $y \in X$ such that $\rho_*(x)/2 < \rho(x, y) \leq \rho_*(x)$ and $V_\rho(x, \rho_*(x))/2 \leq V_\rho(x, \rho(x, y))$. By Lemma 12.5, $V_\rho(x, \lambda'\rho(x, y) \geq a'V_\rho(x, \rho(x, y))$, where a' is independent fo x and y. Hence

$$\begin{aligned}\frac{a'c_*}{2}V_\rho(x, r) = \frac{a'c_*}{2}\mu(X) \leq \frac{a'}{2}V_\rho(x, \rho_*(x)) &\leq a'V_\rho(x, \rho(x, y)) \\ &\leq V_\rho(x, \lambda'\rho(x, y)) \leq V_\rho(x, \lambda' r)\end{aligned}$$

□

Lemma 12.8. *Assume that (X, ρ) is uniformly perfect and that μ is volume doubling with respect to ρ. If* (DM1) *and* (DM2) *are satisfied and* $\lim_{s\downarrow 0} h(s, 1) = 0$, *then ρ is* $(\mathrm{ASQC})_{\mathrm{d}}$.

Proof. First we suppose that $\mathrm{diam}(X, \rho) < \infty$. Lemma 12.4 implies that d is $(\mathrm{SQS})_\rho$. Let $r_* > \mathrm{diam}(X, \rho)$. By Proposition 10.7-(1), there exist $\lambda \in (0, 1)$ and $a \in (0, 1)$ such that $\overline{d}_\rho(x, \lambda r) \leq a\overline{d}_\rho(x, r)$ for any $x \in X$ and any $r \in (0, r_*]$. Let $r \in (0, r_*]$. (Note that $\overline{d}_\rho(x, \lambda^n r) \leq a^n\overline{d}_\rho(x, r)$. Hence λ can be arbitrarily small.) Then

$$\overline{d}_\rho(x, r) = \sup\{d(x, y)|y \in B_\rho(x, r)\backslash B_\rho(x, \lambda r)\}. \tag{12.2}$$

Since μ is $(\mathrm{VD})_\rho$, there exists $\alpha > 0$ such that $\alpha V_\rho(x, \lambda r) \geq V_\rho(x, r)$. Now choose $x, y \in B_\rho(x, r)\backslash B_\rho(x, \lambda r)$. Then $\lambda\rho(x, y) \leq \rho(x, z) \leq \rho(x, y)/\lambda$. Therefore,

$$\begin{aligned}c_1 g(\rho(x, z)) &\leq H(d(x, z), V_\rho(x, \rho(x, z))) \leq H(d(x, z), V_\rho(x, \rho(x, y)/\lambda))) \\ &\leq H(d(x, z), \alpha V_\rho(x, \rho(x, y))) \leq c_2 h(1, \alpha) h\Big(\frac{d(x, z)}{d(x, y)}, 1\Big) g(\rho(x, y)).\end{aligned}$$

This along with the doubling property of g shows that

$$h\Big(\frac{d(x, z)}{d(x, y)}, 1\Big) \geq c_3 > 0,$$

where c_3 is independent of x, y and z. Since $h(s, 1) \downarrow 0$ as $s \downarrow 0$, there exists $\delta > 0$ such that $d(x, z) \geq \delta d(x, y)$. By (12.2), we see that $d(x, z) \geq \delta\overline{d}_\rho(x, r)$. Hence $B_\rho(x, r)\backslash B_\rho(x, \lambda r) \subseteq \overline{B}_\rho(x, \overline{d}_\rho(x, r))\backslash B_\rho(x, \delta\overline{d}_\rho(x, r))$. Next we consider the case where $r > r_*$. Note that $r_*\mathrm{diam}(X, d) \geq \rho_*(x)$. Hence $B_\rho(x, r) = B_\rho(x, r_*) = X$ and $\overline{d}_\rho(x, r) = \overline{d}_\rho(x, r_*) = d_*(x)$. Also, $\rho(x, z) \leq r_*$ for any $z \in X$. Therefore if $\lambda r \leq \rho(x, y) < r$, then $\lambda r_* \leq \rho(x, z) < r_*$ and hence $d(x, z) \geq \delta\overline{d}_\rho(x, r_*) = \delta\overline{d}_\rho(x, r_*)$. This completes the proof when $\mathrm{diam}(X, \rho) < \infty$. Using the similar arguments, we immediately obtain the case where $\mathrm{diam}(X, \rho) = \infty$. □

Lemma 12.9. *Assume that* (X, ρ) *is uniformly perfect, that* μ *is* $(\mathrm{VD})_\rho$ *and that* ρ *is* $(\mathrm{ASQC})_{\mathrm{d}}$. *If* (DM2) *holds, then, for any* $r_* > \mathrm{diam}(X, \rho)$,

$$H(\overline{d}_\rho(x, r), V_\rho(x, r)) \asymp g(r)$$

for any $x \in X$ *and any* $r \in (0, r_*]$. *In particular,* (DM3) *holds.*

Remark. If $\mathrm{diam}(X, \rho) = \infty$, then we remove ", for any $r_* > \mathrm{diam}(X, \rho)$," and replace "$r \in (0, r_*]$" by "$r > 0$" in the statement of the above lemma.

Proof. There exists $c \in (0, 1)$ such that $B_\rho(x, r)\backslash B_\rho(x, cr) \neq \emptyset$ if $X \neq B_\rho(x, cr)$ by the uniform perfectness. Recall that $\rho_*(x) \geq \mathrm{diam}(X, \rho)/2$ for any $x \in X$. Hence if $c^{n-1}r_* < \mathrm{diam}(X, \rho)/2$, then $c^{n-1}r < \rho_*(x)$ for any $r \in (0, r_*]$. Therefore $X \neq B_\rho(x, c^{n-1}r)$. So, we have $y \in X$ satisfying $c^n r \leq \rho(x, y) < c^{n-1}r$. By Proposition 10.10 and (DM2),

$$H(\overline{d}_\rho(x, \rho(x, y)), V_\rho(x, \rho(x, y)) \asymp g(\rho(x, y)).$$

On the other hand, Lemma 10.9 implies that d is doubling with respect to ρ. This and the doubling property of μ shows that

$$H(\overline{d}_\rho(x, \rho(x, y)), V_\rho(x, \rho(x, y))) \asymp H(\overline{d}_\rho(x, r), V_\rho(x, r)).$$

Moreover, by the doubling property of g, $c_6 g(r) \leq g(\rho(x, y)) \leq g(r)$. Combining the last three inequalities, we immediately obtain the desired statement. □

Lemma 12.10. *Assume that* (X, ρ) *is uniformly perfect and that* ρ *is* $(\mathrm{ASQC})_{\mathrm{d}}$. *Then* (DM3) *implies* (DM2).

Proof. By Proposition 10.10,

$$\overline{d}_\rho(x, \rho(x, y)) \asymp d(x, y).$$

Hence, letting $r = \rho(x, y)$, we obtain

$$H(\overline{d}_\rho(x, r), V_\rho(x, r)) \asymp H(d(x, y), V_\rho(x, r))$$

This immediately imply (DM2). □

Lemma 12.11. *Assume that* (X, ρ) *is uniformly perfect and* $\lim_{s \downarrow 0} h(s, 1) = 0$. *If* ρ *is* $(\mathrm{ASQC})_{\mathrm{d}}$, d *decays uniformly with respect to* ρ *and* (DM3) *holds, then* (DM1) *holds.*

Proof. Since d decays uniformly with respect to ρ, there exists $c_0 > 0$ and $\tau > 0$ such that $\overline{d}_\rho(x, \lambda r) \leq c_0 \lambda^\tau \overline{d}_\rho(x, r)$ for any $x \in X$ and $r \in (0, r_*]$. (If $\mathrm{diam}(X, \rho) = \infty$, we always replace $(0, r_*]$ by $(0, \infty)$ in this proof.) Let $r = \rho(x, y)$. By Proposition 10.10 and (DM3),

$$\begin{aligned} c_1 g(\lambda r) \leq H(\overline{d}_\rho(x, \lambda r), V_\rho(x, \lambda r)) &\leq H(c_0 \lambda^\tau \overline{d}_\rho(x, r), V_\rho(x, \lambda r)) \\ &\leq h(c_3 \lambda^\tau, 1) H(d(x, y), V_\rho(x, \lambda r)), \end{aligned}$$

where $c_3 > 0$ is independent of x and y. Moreover, by Lemma 12.10, we have (DM2). Hence $H(d(x,y), V_\rho(x,r))/g(r)$ is uniformly bounded. So, there exists $c_4 > 0$ such that

$$c_4 h(c_0\lambda^\tau, 1) \geq \frac{H(d(x,y), V_\rho(x, \rho(x,y)))}{g(\rho(x,y))} \frac{g(\lambda\rho(x,y))}{H(d(x,y), V_\rho(x, \lambda\rho(x,y)))}$$

for any x and y. Since $\lim_{\lambda\downarrow 0} h(c_0\lambda^\tau, 1) = 0$, we have (DM1) with $\eta(\lambda) = c_4 h(c_0\lambda^\tau, 1)$. □

Proof of Theorem 12.2. (1) This is immediate by Lemma 12.7.
(2) Assume (DM1) and (DM2). Then, Lemma 12.8 shows that ρ is $(\mathrm{ASQC})_\mathrm{d}$. By Lemma 12.4, d is $(\mathrm{SQS})_\rho$. This along with Proposition 10.7 implies that d decays uniformly with respect to ρ. Now (DM3) follows by using Lemma 12.9.

Lemmas 12.10 and 12.11 suffice for the converse direction. □

Proof of Corollary 12.3. $(a) \Rightarrow (b)$ Using Lemma 12.4 and 12.8 and applying Theorem 10.5, we see that d and ρ are semi-quasisymmetric with respect to each other. Hence by Theorem 11.3, $\rho \underset{\mathrm{QS}}{\sim} d$.
$(b) \Rightarrow (c)$ Since $\rho \underset{\mathrm{QS}}{\sim} d$, Theorems 10.5 and 11.3 shows that d is $(\mathrm{SQS})_\rho$ and that ρ is $(\mathrm{ASQC})_\mathrm{d}$. By Lemma 12.7, μ is $(\mathrm{VD})_\rho$. Therefore Lemma 12.9 yields (DM3).
$(c) \Rightarrow (d)$ Since $\rho \underset{\mathrm{QS}}{\sim} d$, Theorems 10.5 and 11.3 shows that d is $(\mathrm{SQS})_\rho$ and that ρ is $(\mathrm{ASQC})_\mathrm{d}$. Then by Proposition 10.7, d decays uniformly with respect to ρ.
$(d) \Rightarrow (a)$ This immediately follows by Theorem 12.2. □

13 Construction of quasisymmetric metrics

The main purpose of this section is to construct a distance ρ which satisfy the conditions (DM1) and (DM2) in Section 12 in the case where $g(r) = r^\beta$.

In this section, (X,d) is a metric space and μ is a Borel regular measure on (X,d) which is volume doubling with respect to d. We also assume that $0 < \mu(B_d(x,r)) < +\infty$ for any $x \in X$ and $r > 0$. Let $H : (0,\infty)^2 \to (0,\infty)$ satisfy (H1) and (H2) in Section 12.

Theorem 13.1. *Assume that (X,d) is uniformly perfect and that μ is* $(\mathrm{VD})_\mathrm{d}$. *For sufficiently large $\beta > 0$, there exists a distance ρ on X such that $\rho \underset{\mathrm{QS}}{\sim} d$ and* (DM3) *holds with $g(r) = r^\beta$.*

Remark. If $\rho \underset{\mathrm{QS}}{\sim} d$ and (X,d) is uniformly perfect, then Proposition 11.2 and Corollary 11.4 impliy that (X,ρ) is uniformly perfect and that μ is $(\mathrm{VD})_\rho$.

Our distance ρ satisfies the condition (c) of Corollary 12.3. If $\lim_{s\downarrow 0} h(s,1) = \lim_{t\downarrow 0} h(1,t) = 0$, then we have all the assumption of the corollary and hence obtain the statements (a) through (d). In particular, d, μ and ρ satisfy (DM1) and (DM2) with $g(r) = r^\beta$. In particular, letting $H(s,t) = st$, we establish the existence of a distance which is quasisymmetric to the resistance metric and satisfies (12.1) if μ is volume doubing with respect to the resistance meric. This fact play an important role in the next part.

Example 13.2. (1) If $H(s,t) = t$, (DM3) is

$$\mu(B_\rho(x,r)) \asymp r^\beta.$$

Hence in this case, Theorem 13.1 implies that following well-known theorem: if (X,d) is unifromly perfect and μ is $(\mathrm{VD})_{\mathrm{d}}$, then there exists a metric ρ such that the metric measure space (X,ρ,μ) is Ahlfors regular. See Heinonen [28, Chapter 14] and Semmes [43, Section 4.2] for details.
(2) Let $F : (0,\infty) \to (0,\infty)$ be monotonically nondecreasing. Suppose that there exist positive constants c_1, τ_1 and τ_2 such that

$$F(xy) \leq c_1 \max\{x^{\tau_1}, x^{\tau_2}\}F(y)$$

for any $x, y \in (0,\infty)$. Define $H(s,t) = F(s^p t^q)$. If $p \geq 0$, $q \geq 0$ and $(p,q) \neq (0,0)$, then H satisfies (H1) and (H2). In fact, $H(as,bt) = F(a^p b^q s^p t^q) \leq c_1 \max\{(a^p b^q)^{\tau_1}, (a^p b^q)^{\tau_2}\}H(s,t)$. Hence $h(a,b) \leq c_1 \max\{(a^p b^q)^{\tau_1}, (a^p b^q)^{\tau_2}\}$.

To prove Theorem 13.1, we need several preparations.

Notation. Define $v(x,y) = V_d(x, d(x,y)) + V_d(y, d(x,y))$. Also define

$$\varphi(x,y) = \begin{cases} H(d(x,y), v(x,y)) & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\varphi(x,y) = \varphi(y,x) \geq 0$ and that $\varphi(x,y) = 0$ implies $x = y$.

Hereafter in this section, we always assume that (X,d) is uniformly perfect and that μ is $(\mathrm{VD})_{\mathrm{d}}$. By the volume doubling property, we have the following lemma.

Lemma 13.3. *For any $x, y \in X$,*

$$v(x,y) \asymp V_d(x, d(x,y)).$$

Lemma 13.4. *Define*

$$f_{\tau_1,\tau_2}(t) = \begin{cases} t^{\tau_1} & \text{if } t \in (0,1), \\ t^{\tau_2} & \text{if } t \geq 1. \end{cases}$$

Then there exist positive constants c_1, τ_1 and τ_2 such that $V_d(x, \delta d(x,y)) \leq c_1 f_{\tau_1,\tau_2}(\delta) V_d(x, d(x,y))$ for any $x, y \in X$ and any $\delta > 0$.

Proof. If $\delta \geq 1$, this is immediate from the volume doubling property. Since (X,d) is uniformly perfect, there exists $c \in (0,1)$ such that $B_d(x,r) \neq X$ implies $B_d(x,r) \backslash B_d(x,cr) \neq \emptyset$. Let $r = d(x,y)$. Choose $z \in B_d(x,r/2) \backslash B_d(x,cr/2)$. It follows that $V_d(x,cr/4) + V_d(z,cr/4) \leq V_d(x,r)$. Now by the volume doubling property, $V_d(z,cr/4) \geq aV_d(x,cr/4)$, where a is independent of x, z and r. Hence $V_d(x,cr/4) \leq (1+a)^{-1} V_d(x,r)$. This shows the desired inequality when $\delta \in (0,1)$. □

Lemma 13.5. *There exists a homeomorphism $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and $\varphi(x,z) < g(t)\varphi(x,y)$ whenever $d(x,z) < td(x,y)$.*

Proof. Assume that $d(x,z) < td(x,y)$. Write $f = f_{\tau_1,\tau_2}$. Then by (H1), (H2) and the above lemmas,

$$\begin{aligned}\varphi(x,z) &= H(d(x,z), v(x,z)) \le H(td(x,y), Mc_1 f(t)V_d(x, d(x,y)))\\ &\le H(td(x,y), M^2 c_1 f(t)v(x,y)) \le h(1, M^2c_1)h(t, f(t))H(d(x,y), v(x,y)).\end{aligned}$$

By the definition of $h(a,b)$, it follows that $h(t, f(t))$ is monotonically nondecreasing. Also if $t < c_0$, then $h(t, t^{\tau_1}) < 1$. Since $h(t^n, t^{n\tau_1}) \le h(t, t^{\tau_1})^n$ for $n \ge 0$, we see that $h(t, f(t)) \to 0$ as $t \downarrow 0$. Therefore, there exists a homeomorphism $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and $g(t) \ge h(1, M^2c_1)h(t, f(t))$ for any $t > 0$. □

Definition 13.6. $f : X \times X \to [0,\infty)$ is called a quasidistance on X if and only if f satisfies the following three conditions:
(QD1) $f(x,y) \ge 0$ for any $x, y \in X$. $f(x,y) = 0$ if and only if $x = y$.
(QD2) $f(x,y) = f(y,x)$ for any $x, y \in X$.
(QD3) There exists $K > 0$ such that $f(x,y) \le K(f(x,z) + f(z,y))$ for any $x, y, z \in X$.

Lemma 13.7. *$\varphi(x,y)$ is a quasidistance.*

Proof. Since $d(x,y) \le d(x,z) + d(z,y)$, either $d(x,y) \le d(x,z)/2$ or $d(x,y) \le d(z,y)$. Assume that $d(x,y) \le d(x,z)/2$. Then Lemma 13.5 implies that $\varphi(x,y) \le g(1/2)\varphi(x,z) \le g(1/2)(\varphi(x,z) + \varphi(z,y))$. □

Lemma 13.8. *If $f : X \times X \to [0,\infty)$ is a quasidistance on X, then there exists $\epsilon_0 > 0$ such that f^ϵ is equivalent to a distance for any $\epsilon \in (0, \epsilon_0]$, i.e.*

$$f(x,y)^\epsilon \asymp \rho(x,y)$$

for any $x, y \in X$, where ρ is a distance on X.

See Heinonen [28, Proposition 14.5] for the proof of this lemma.

Lemma 13.9. *For sufficiently large $\beta > 0$, there exists a distance ρ on X such that $\rho \underset{\mathrm{QS}}{\sim} d$ and*

$$\varphi(x,y) \asymp \rho(x,y)^\beta \tag{13.1}$$

for any $x, y \in X$.

Proof. By Lemmas 13.7 and 13.8, if β is large enough, then there exists a distance ρ which satisfies (13.1). By Lemma 13.5, $d(x,z) < td(x,y)$ implies $\rho(x,z) < cg(t)^{1/\beta}\rho(x,y)$ for some $c > 0$. Hence $\rho \underset{\mathrm{QS}}{\sim} d$. □

Since $\rho \underset{\text{QS}}{\sim} d$, d and ρ define the same topology on X. Also since (X, d) is uniformly perfect, so is (X, ρ). Then, Theorem 11.3 shows that d and ρ are semi-quasisymmetric with respect to each other. So we may enjoy the results in Theorem 10.5 in the rest of discussions.

Lemma 13.10. *For any $x \in X$ and any $r > 0$,*

$$V_\rho(x, r) \asymp V_d(x, \overline{d}_\rho(x, r)). \tag{13.2}$$

Proof. Since ρ is $(\mathrm{SQC})_{\mathrm{d}}$,

$$B_d(x, c\overline{d}_\rho(x, r)) \subseteq B_\rho(x, r) \subseteq B_d(x, c'\overline{d}_\rho(x, r)).$$

This and the volume doubling property of μ imply (13.2). □

Proof of Theorem 13.1. The rest is to show (DM3). Since (X, ρ) is uniformly perfect, there exists $c \in (0, 1)$ such that $B_\rho(x, r)\backslash B_\rho(x, cr) \neq \emptyset$ unless $B_\rho(x, r) = X$. We will consider the case when $\mathrm{diam}(X, \rho) < \infty$. Let $r_* > \mathrm{diam}(X, \rho)$. Choose $n \geq 1$ so that $c^n r < \mathrm{diam}(X, \rho)/2$. Note that $\mathrm{diam}(X, \rho)/2 \leq \rho_*(x)$. Hence if $r \in (0, r_*]$, then $c^n r < \rho_*(x)$. Hence there exists $y \in X$ such that $c^{n+1} r \leq \rho(x, y) < c^n r$. By $(\mathrm{ASQC})_{\mathrm{d}}$, there exists $\delta > 0$ such that $d(x, z) \geq \delta\overline{d}_\rho(x, r)$ for any $r > 0$ and any $z \in B_\rho(x, r)\backslash B_\rho(x, cr)$. This along with the doubling property of $\overline{d}_\rho(x, r)$ implies that $\overline{d}_\rho(x, r) \geq d(x, y) \geq \delta\overline{d}_\rho(x, c^n r) \geq c'\overline{d}_\rho(x, r)$. Then by Lemma 13.3. the volume doubling property of μ, (13.1) and (13.2),

$$\begin{aligned}(c^n r)^\beta \geq \rho(x, y)^\beta &\geq c_3 H(d(x, y), c_4 V_d(x, d(x, y))) \\ &\geq c_3 H(c'\overline{d}_\rho(x, r), c_5 V_d(x, \overline{d}_\rho(x, r))) \geq c_6 H(\overline{d}_\rho(x, r), V_\rho(x, r))\end{aligned}$$

Similarly,

$$\begin{aligned}(c^{n+1} r)^\beta \leq \rho(x, y)^\beta &\leq c_7 H(d(x, y), c_8 V_d(x, d(x, y))) \\ &\leq c_7 H(\overline{d}_\rho(x, r), V_d(x, \overline{d}_\rho(x, r))) \leq c_8 H(\overline{d}_\rho(x, r), V_\rho(x, r))\end{aligned}$$

Thus we obtain (DM3) if $\mathrm{diam}(X, \rho) < \infty$. The other case follows by almost the same argument. □

Part III
Volume doubling measures and heat kernel estimates

In this part, we will show results on heat kernel estimates, which answer the questions in the introduction, under the foundation laid by the previous two parts. The first question is how and when we can find a metric which is suitable for describing an asymptotic behavoir of a heat kernel. We will give an answer to this question in Theorem 14.11, which says that if the measure is volume doubling with respect to the resistance metric, then we can get a good (on-diagonal estimate, at least) heat kernel estimate by quasisymmetric modification of the resistance metric. The second question concerns jumps. Namely, what kind of jump can we allow to get a good heat kernel estimate? Theorem 14.11 also gives an answer to this question, saying that the annulus comparable condition (with the volume doubling property) is sufficient and necessary for a good heat kernel estimate.

14 Main results on heat kernel estimates

In this section, we present the main results on heat kernel estimates. There will be three main theorems, 14.6, 14.10 and 14.11. The first one gives a good (on-diagonal and lower near diagonal) heat kernel estimate if the measure is $(\mathrm{VD})_{\mathrm{R}}$ and the distance is quasisymmetric with respect to R, where R is the resistance metric. The second one provides geometrical and analytical equivalent conditions for having a good heat kernel estimate. Finally in the third theorem, $(\mathrm{VD})_{\mathrm{R}}$ and (ACC) ensures the existence of a distance d which is quasisymmetric to R and under which a good heat kernel estimate holds.

Let $(\mathcal{E}, \mathcal{F})$ be a regular resistance form on a set X and let R be the associated resistance metric on X. We assume that (X, R) is separable, complete, uniformly perfect and locally compact. Let μ be a Borel regular measure on (X, R) which satisfies $0 < \mu(B_R(x,r)) < \infty$ for any $x \in X$ and any $r > 0$. Under those assumptions, if $\mathcal{D}$ is the closure of $\mathcal{F} \cap C_0(X)$ with respect to the $\mathcal{E}_1$-norm, then $(\mathcal{E}, \mathcal{D})$ is a regular Dirichlet form $L^2(X, \mu)$. Let $(\{X_t\}_{t>0}, \{P_x\}_{x \in X})$ be the Hunt process associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(X, \mu)$.

As we have shown in Section 6, if (X, R) is complete and μ is $(\mathrm{VD})_{\mathrm{R}}$, then $\overline{B_R(x,r)}$ is compact for any $x \in X$ and any $r > 0$. Hence under $(\mathrm{VD})_{\mathrm{R}}$, Theorem 9.4 implies the existence of a jointly continuous heat kernel (i.e. transition density) $p(t,x,y)$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(X, \mu)$.

Definition 14.1. Let d be a distance on X giving the same topology as R. Define $\overline{R}_d(x,r) = \sup_{y \in B_d(x,r)} R(x,y)$, $V_d(x,r) = \mu(B_d(x,r))$ and $h_d(x,r) = \overline{R}_d(x,r)V_d(x,r)$ for any $r > 0$ and any $x \in X$.

Lemma 14.2. *For each $x \in X$, $\overline{R}_d(x,r)$ and $V_d(x,r)$ are monotonically nondecreasing left-continuous function on $(0, \infty)$. Moreover $\lim_{r \downarrow 0} \overline{R}_d(x,r) = 0$ and $\lim_{r \downarrow 0} V_d(x,r) = \mu(\{x\})$.*

By the above lemma, $h_d(x,r)$ is monotonically nondecreasing left-continuous function on $(0, \infty)$ and $\lim_{r \downarrow 0} h_d(x,r) = 0$.

Definition 14.3. (1) For a Borel set $B \subseteq X$, define the exit time from B, τ_B by $\tau_B = \inf\{t > 0 | X_t \notin B\}$. Note that $\tau_B = \sigma_{B^c}$, where σ_{B^c} is the hitting time of B^c.

(2) Let d be a distance on X which gives the same topology as R. We say that the Einstein relation with respect d, $(\mathrm{EIN})_\mathrm{d}$ for short, holds if and only if

$$E_x(\tau_{B_d(x,r)}) \asymp h_d(x,r), \tag{14.1}$$

for any $x \in X$ and $r > 0$ with $X \neq B_d(x,r)$.

The name "Einstein relation" have been use by several authors . See [22] and [44] for example.

We have two important equivalences between the resistance estimate, the annulus comparable condition and the Einstein relation.

Proposition 14.4. *Assume that d is a distance on X and $d \underset{\mathrm{QS}}{\sim} R$. Then* (RES) *is equivalent to*

$$R(x, B_d(x,r)^c) \asymp \overline{R}_d(x,r) \tag{14.2}$$

for any $x \in X$ and any $r > 0$ with $B_d(x,r) \neq X$.

Proposition 14.5. *Assume that μ is $(\mathrm{VD})_\mathrm{R}$, that d is a distance on X and that $d \underset{\mathrm{QS}}{\sim} R$. Then* (RES), (ACC) *and* $(\mathrm{EIN})_\mathrm{d}$ *are equivalent to one another.*

The proofs of the above propositions are in Section 17.

Now we have the first result on heat kernel estimate.

Theorem 14.6. *Assume* (ACC). *Suppose μ has volume doubling property with respect to R. Then, there exists a jointly continuous heat kernel $p(t,x,y)$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(X,\mu)$. Moreover, if a distance d on X is quasisymmetric with respect to R, then* $(\mathrm{EIN})_\mathrm{d}$ *holds and*

$$\frac{c_1}{V_d(x,r)} \leq p(h_d(x,r), x, y) \tag{14.3}$$

and

$$p(h_d(x,r), x, x) \leq \frac{c_2}{V_d(x,r)} \tag{14.4}$$

for any $x \in X$, any $r > 0$ and any $y \in X$ with $d(x,y) \leq c_3 \min\{r, \mathrm{diam}(X,d)\}$.

(14.3) is called the lower near diagonal estimate. If the distance is not geodesic, the lower near diagonal estimate is known as a substitute of the lower off-diagonal sub-Gaussian estimate for diffusion case.

Note that $R \underset{\mathrm{QS}}{\sim} R$ and $\overline{R}_d(x,r) \asymp r$ if (X,R) is uniformly perfect. Hence, $h_d(x,r) = rV_R(x,r)$ and the above theorem shows

$$p(rV_R(x,r), x, x) \asymp \frac{1}{V_R(x,r)}.$$

This have essentially been obtained in [37].

To state the next theorem, we need several notions and results on monotonically non-decreasing functions on $(0,\infty)$ and their inverse.

Definition 14.7. Let $f : (0, \infty) \to (0, \infty)$.
(1) f is said to be doubling if there exists $c > 0$ such that $f(2t) \leq cf(t)$ for any $t \in (0, \infty)$.
(2) f is said to be to decay uniformly if and only if there exists $(\delta, \lambda) \in (0, 1)^2$ such that $f(\delta t) \leq \lambda f(t)$ for any $t \in (0, \infty)$.
(3) f is said to be a monotone function with full range if and only if f is monotonically non-decreasing, $\lim_{t\downarrow 0} f(t) = 0$ and $\lim_{t\to\infty} f(t) = +\infty$. For a monotone function with full range on $(0, \infty)$, we define $f^{-1}(y) = \sup\{x|f(x) \leq y\}$ and call f^{-1} the right-cotinuous inverse of f.

Lemma 14.8. *Let $f : (0, \infty) \to (0, \infty)$ be a monotone function with full range.*
(1) If f is doubling, then f^{-1} decays uniformly and $f(f^{-1}(y)) \asymp y$ for any $y \in (0, \infty)$.
(2) If f decays uniformly, then f^{-1} is doubling and $f^{-1}(f(x)) \asymp x$ for any $x \in (0, \infty)$.

This lemma is rather elementary and we omit its proof.

The following definition is a list of important relations or properties between a heat kernel, a measure and a distance.

Definition 14.9. Let d be a distance of X giving the same topology as R and let $g : (0, \infty) \to (0, \infty)$ be a monotone function with full range.
(1) A heat kernel $p(t, x, y)$ is said to satisfy on-diagonal heat kernel estimate of order g with respect to d, $(\mathrm{DHK})_{g,d}$ for short, if and only if

$$p(t, x, x) \asymp \frac{1}{V_d(x, g^{-1}(t))}$$

for any $x \in X$ and any $t > 0$, where g^{-1} is the right-continuous inverse of g.
(2) A heat kernel $p(t, x, y)$ is said to have the doubling property. (KD) for short, if and only if there exists $c_1 > 0$ such that

$$p(t, x, x) \leq c_1 p(2t, x, x).$$

for any $x \in X$ and any $t > 0$.
(3) We say that $(\mathrm{DM1})_{\mathrm{g,d}}$ holds if and only if there exists $\eta : (0, 1] \to [0, \infty)$ such that η is monotonically nondecreasing, $\lim_{t\downarrow 0} \eta(t) = 0$ and

$$\frac{g(\lambda d(x, y))}{V_d(x, \lambda d(x, y))} \leq \frac{g(d(x, y))}{V_d(x, d(x, y))}\eta(\lambda)$$

for any $x, y \in X$ and any $\lambda \in (0, 1]$.
(4) We say that $(\mathrm{DM2})_{\mathrm{g,d}}$ holds if and only if

$$R(x, y)V_d(x, d(x, y)) \asymp g(d(x, y))$$

for any $x, y \in X$.

The conditions $(\mathrm{DM1})_{\mathrm{g,d}}$ and $(\mathrm{DM2})_{\mathrm{g,d}}$ corresponds to (DM1) and (DM2) with $H(s, t) = st$ respectively. $(\mathrm{DM2})_{\mathrm{g,d}}$ is the counterpart of $\mathrm{R}(\beta)$ defined in [6] and it will relate the exit time with $g(d(x, y))$ through $(\mathrm{EIN})_{\mathrm{d}}$.

Remark. If $\mathrm{diam}(X, d)$ is bounded, it is enough for g to be only defined on $(0, \mathrm{diam}(X, d))$, for example, to describe $(\mathrm{DM2})_{\mathrm{g,d}}$. In such a case, the value of g for $[\mathrm{diam}(X, d), \infty)$ does not make any essential differences. One can freely extend $g : (0, \mathrm{diam}(X, d)) \to (0, \infty)$ to $g : (0, \infty) \to (0, \infty)$ so that g satisfy required conditions as being doubling, decaying uniformly or being strictly monotone.

Here is our second theorem giving equivalent conditions for a good heat kernel.

Theorem 14.10. *Assume* (ACC). *Let d be a distance on X giving the same topology as R and let $g : (0, \infty) \to (0, \infty)$ be a monotone function with full range and doubling. Then the following statements* (a), (b), (c) *and* $(\mathrm{HK})_{\mathrm{g,d}}$ *are equivalent.*

(a) *(X, d) is uniformly perfect,* $(\mathrm{DM1})_{\mathrm{g,d}}$ *and* $(\mathrm{DM2})_{\mathrm{g,d}}$ *hold.*

(b) $d \underset{\mathrm{QS}}{\sim} R$ *and* $(\mathrm{DM2})_{\mathrm{g,d}}$ *holds.*

(c) $d \underset{\mathrm{QS}}{\sim} R$ *and, for any $x \in X$ and any $r \leq \mathrm{diam}(X, d)$,*

$$h_d(x, r) \asymp g(r) \tag{14.5}$$

$(\mathrm{HK})_{\mathrm{g,d}}$ $d \underset{\mathrm{QS}}{\sim} R$*, g decays uniformly, a jointly continuous heat kernel $p(t, x, y)$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(X, \mu)$ exists and satisfies* (KD) *and* $(\mathrm{DHK})_{\mathrm{g,d}}$.

Moreover, if any of the above conditions holds, then there exist positive constants c and c' such that

$$\frac{c'}{V_d(x, g^{-1}(t))} \leq p(t, x, y) \tag{14.6}$$

for any $y \in B_d(x, cg^{-1}(t))$. Furthermore, assume that $\Phi(r) = g(r)/r$ is a monotone function with full range and decays uniformly. We have the following off-diagonal estimates:

Case 1: If $(\mathcal{E}, \mathcal{F})$ has the local property, then

$$p(t, x, y) \leq \frac{c_1}{V_d(x, g^{-1}(t))} \exp\left(- c_2 \left(\frac{d(x, y)}{\Phi^{-1}(t/d(x, y))} \right) \right) \tag{14.7}$$

for any $x, y \in X$ and any $t > 0$, where $c_1, c_2 > 0$ are independent of x, y and t.

Case 2: Assume that $d(x, y)$ has the chain condition, i.e. for any $x, y \in X$ and any $n \in \mathbb{N}$, there exist $x_0, \ldots, x_n$ such that $x_0 = x, x_n = y$ and $d(x_i, x_{i+1}) \leq Cd(x, y)/n$ for any $i = 0, \ldots, n - 1$, where $C > 0$ is independent of x, y and n. Then,

$$\frac{c_4}{V_d(x, g^{-1}(t))} \exp\left(- c_5 \left(\frac{d(x, y)}{\Phi^{-1}(t/d(x, y))} \right) \right) \leq p(t, x, y) \tag{14.8}$$

for any $x, y \in X$ and any $t > 0$, where $c_3, c_4 > 0$ are independent of x, y and t.

Note that if $d = R$, then the above theorem says $(\mathrm{DM2})_{\mathrm{g,R}}$ implies $(\mathrm{DM1})_{\mathrm{g,R}}$. Moreover, in this case, $(\mathrm{DM2})_{\mathrm{g,R}}$ show the uniform volume doubling property given by Kumagai in [37]. In fact, he have shown the above theorem in this special case including the off-diagonal estimates when $(\mathcal{E}, \mathcal{F})$ satisfy the local property.

The above theorem is useful to show a heat kernel estimate for a specific example. In the next section, we will apply this theorem to (traces of) α-stable process on $\mathbb{R}$ for $\alpha \in (1, 2]$. Also, in Section 23, we will apply (14.7) and (14.8) to homogeneous random Sierpinski gaskets and recover the off-diagonal heat kernel estimate obtained by Barlow and Hambly in [7].

The next theorem assures the existence of a distance d which satisfies the conditions in Theorem 14.10 for certain g if μ is $(\mathrm{VD})_{\mathrm{R}}$ and (ACC).

Theorem 14.11. *Assume that (X, R) is uniformly perfect. Then the following conditions* (C1), (C2), ..., (C6) *are equivalent.*
(C1) (ACC) *holds and μ is* $(\mathrm{VD})_{\mathrm{R}}$.
(C2) *μ is* $(\mathrm{VD})_{\mathrm{R}}$ *and* $(\mathrm{EIN})_{\mathrm{R}}$ *holds.*
(C3) (ACC) *holds and there exist a distance d on X and $\beta > 1$ such that* $(\mathrm{HK})_{\mathrm{g,d}}$ *with $g(r) = r^\beta$ is satisfied.*
(C4) *There exist a distance d on X and $\beta > 1$ such that $d \underset{\mathrm{QS}}{\sim} R$,*

$$E_x(\tau_{B_d(x,r)}) \asymp r^\beta \asymp h_d(x, r) \tag{14.9}$$

for any $x \in X$ and any $r > 0$ with $B_d(x, r) \neq X$.
(C5) *μ is* $(\mathrm{VD})_{\mathrm{R}}$. *If d is a distance on X and $d \underset{\mathrm{QS}}{\sim} R$, then* $(\mathrm{EIN})_{\mathrm{d}}$ *holds.*
(C6) *μ is* $(\mathrm{VD})_{\mathrm{R}}$. *There exists a distance d on X such that $d \underset{\mathrm{QS}}{\sim} R$ and* $(\mathrm{EIN})_{\mathrm{d}}$ *holds.*

Moreover, if any of the above conditions holds, then we can choose the distance d in (C3) *and* (C4) *so that*

$$d(x, y)^\beta \asymp R(x, y)(V_R(x, R(x, y)) + V_R(y, R(x, y))) \tag{14.10}$$

for any $x, y \in X$.

Both the voulme doubing property and the Einstein relation are known to be necessary to obtain a good heat kernel estimate. Hence the implication (C2) $\Rightarrow$ (C3) shows that (ACC) is also necessary to get a reasonable both side heat kernel estimate.

Remark. By Theorem 6.10, we may replace (ACC) by (RES) in (C1) and (C3).

We have simpler statement in the local case. Recall that (X, R) is assumed to be unifromly perfect. Using Corollary 6.11, we have the next corollary.

Corollary 14.12. *Assume that $(\mathcal{E}, \mathcal{F})$ has the local property. Then the following conditions* (C1)' *and* (C3)' *are equivalent:*
(C1)' *μ is* $(\mathrm{VD})_{\mathrm{R}}$.

(C3)' *There exist a distance d on X and $\beta > 1$ such that* $(\mathrm{HK})_{\mathrm{g,d}}$ *with* $g(r) = r^{\beta}$ *holds.*

Moreover, if any of the above conditions is satisfied, then we have the near diagonal lower estimate (14.6) *and off-diagonal sub-Gaussian upper estimate* (14.7)*.*

Next we apply the above theorems to the Dirichlet form associated with a trace of a resistance form $(\mathcal{E}, \mathcal{F})$ on X. Let Y be a closed subset of X which is uniformly perfect. Assume that $(\mathcal{E}, \mathcal{F})$ satisfy (RES). By Theorem 7.8, $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ satisfy (RES) as well. Let ν be a Borel regular measure on $(Y, R|_Y)$ which satisfy $0 < \mu(B_R(x, r) \cap Y) < +\infty$ for any $x \in Y$ and $r > 0$. If ν is $(\mathrm{VD})_{\mathrm{R}|_{\mathrm{Y}}}$, then (ACC) for $(Y, R|_Y)$ follows by Theorem 6.10. Therefore, the counterpart of Theorems 14.6, 14.10 and 14.11 hold for $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ with (ACC) granted. (Note that the conditions (a), (b), (c) and $(\mathrm{HK})_{\mathrm{g,d}}$ imply the volume doubling property.) In particular, we have the following result.

Theorem 14.13. *Let μ be a Borel regular measure on (X, R) with satisfies $0 < \mu(B_R(x, r)) < +\infty$ for any $x \in X$ and any $r > 0$. Assume that* (ACC) *holds for $(\mathcal{E}, \mathcal{F})$ and that there exists a distance d on X such that $d \underset{\mathrm{QS}}{\sim} R$ and* $(\mathrm{HK})_{\mathrm{g,d}}$ *with $g(r) = r^{\beta}$ is satisfied. Let Y be a non-empty closed subset of X. If $(Y, R|_Y)$ is uniformly perfect and there exist $\gamma > 0$ and a Borel regular measure ν on $(Y, R|_Y)$ such that*

$$\mu(B_d(x, r)) \asymp r^{\gamma} \nu(B_d(x, r) \cap Y) \tag{14.11}$$

for any $x \in Y$ and any $r > 0$ with $B_d(x, r) \neq X$, then it follows that $\beta > \gamma$, that there exists a jointly continuous heat kernel $p_{\nu}^{Y}(t, x, y)$ associated with the regular Dirichlet form $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ on $L^2(Y, \nu)$ and that

$$p_{\nu}^{Y}(t, x, x) \asymp \frac{1}{\nu(B_d(x, t^{1/(\beta-\gamma)}) \cap Y)} \tag{14.12}$$

for any $x \in X$ and any $t > 0$. In particular, if $\mu(B_d(x, r)) \asymp r^{\alpha}$ for any $x \in X$ and any $r > 0$ with $B_d(x, r) \neq X$, then

$$p_{\nu}^{Y}(t, x, x) \asymp t^{-\frac{\alpha-\gamma}{\beta-\gamma}} \tag{14.13}$$

for any $x \in Y$ and any $t > 0$ with $B_d(x, t^{1/(\beta-\gamma)}) \neq X$.

If $\mu(B_r(x, d)) \asymp r^{\alpha}$, then the Hausdorff dimension of (X, d) is α and $\mu \asymp \mathcal{H}^{\alpha}$, where $\mathcal{H}^{\alpha}$ is the α-dimensional Hausdorff measure of (X, d). In other word, (X, d) is Alfors α-regular set. In this case, (14.11) implies that $(Y, d|_Y)$ is Alfors $(\alpha - \gamma)$-regular set.

We will apply the above theorem for the traces of the standard resistance form on the Sierpinski gasket in Example 19.8.

15 Example: the α-stable process on $\mathbb{R}$

In this section, we will apply the results in the last section to the resistance forms associated with the α-stable process on $\mathbb{R}$ for $\alpha \in (1, 2]$. For $\alpha = 2$, the α-stable process is the Brownian motion on $\mathbb{R}$. We denote the Euclidean distance on $\mathbb{R}$ by d_E.

Definition 15.1. (1) For $\alpha \in (0, 2)$, define

$$\mathcal{F}^{(\alpha)} = \left\{ u \middle| u \in C(\mathbb{R}), \int_{\mathbb{R}^2} \frac{(u(x) - u(y))^2}{|x - y|^{1+\alpha}} dxdy < \infty \right\} \tag{15.1}$$

and

$$\mathcal{E}^{(\alpha)}(u, v) = \int_{\mathbb{R}^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{1+\alpha}} dxdy \tag{15.2}$$

for any $u, v \in \mathcal{F}^{\alpha}$. Moreover, define $\mathcal{D}^{(\alpha)} = \mathcal{F}^{(\alpha)} \cap L^2(\mathbb{R}, dx)$.
(2) For $\alpha = 2$, define

$$\mathcal{F}_0^{(2)} = \{u | u \in C^1(\mathbb{R}), \int_{\mathbb{R}} (u'(x))^2 dx < +\infty\}$$

and

$$\mathcal{E}^{(2)}(u, v) = \int_{\mathbb{R}} u'(x) v'(x) dx$$

for any $u, v \in \mathcal{F}_0^{(2)}$.

For $\alpha = 2$, $(\mathcal{E}^{(2)}, \mathcal{F}_0^{(2)})$ does not satisfy (RF2). To make a resistance form, we need to take a kind of closure of $(\mathcal{E}^{(2)}, \mathcal{F}_0^{(2)})$.

Proposition 15.2. *If $\{u_n\}_{n \geq 1} \subseteq \mathcal{F}_0^{(2)}$ satisfies $\mathcal{E}^{(2)}(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to \infty$ and $u_n(0) \to a$ as $n \to \infty$ for some $a \in \mathbb{R}$, then $\{u_n\}_{n \geq 1}$ converges compact uniformly to $u \in C(\mathbb{R})$ as $n \to \infty$.*

Definition 15.3. We use $\mathcal{F}^{(2)}$ to denote the collection of all such limits u in the sense of Proposition 15.2. Define

$$\mathcal{E}^{(2)}(u, v) = \lim_{n \to \infty} \mathcal{E}^{(2)}(u_n, v_n)$$

for any $u, v \in \mathcal{F}^{(2)}$, where $\{u_n\}_{n \geq 1}$ and $\{v_n\}_{n \geq 1}$ are the sequences convergent to u and v respectively in the sense of Proposition 15.2. Also set $\mathcal{D}^{(2)} = \mathcal{F}^{(2)} \cap L^2(\mathbb{R}, dx)$.

It is well-known that, for $\alpha \in (0, 2]$, $(\mathcal{E}^{(\alpha)}, \mathcal{D}^{(\alpha)})$ is a regular Dirichlet form on $L^2(\mathbb{R}, dx)$ and the associated non-negative self-adjoint operator on $L^2(\mathbb{R}, dx)$ is $(-\Delta)^{\alpha/2}$, where $\Delta = d^2/dx^2$ is the Laplacian. The corresponding Hunt process is called the α-stable process on $\mathbb{R}$. See [36, 13] for example. Note that

$(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ has a natural scaling property. More precisely, for $u \in \mathcal{F}^{(\alpha)}$, define $u_t(x) = u(tx)$ for any $t > 0$. Then,

$$\mathcal{E}^{(\alpha)}(u_t, u_t) = t^{\alpha-1}\mathcal{E}^{(\alpha)}(u, u)$$

for any $t > 0$. Combining this scaling property with [20, Theorem 8.1], we have the following.

Proposition 15.4. *For $\alpha \in (1, 2]$, $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ is a regular resistance form on $\mathbb{R}$. The corresponding resistance metric $R^{(\alpha)}(x, y) = \gamma_\alpha|x - y|^{\alpha-1}$ for any $x, y \in \mathbb{R}$, where γ_α is independent of x and y.*

By this proposition, for $\alpha \in (1, 2]$, if $\mathcal{D}_\mu^{(\alpha)} = L^2(\mathbb{R}, \mu) \cap \mathcal{F}^{(\alpha)}$, then $(\mathcal{E}^{(\alpha)}, \mathcal{D}_\mu^{(\alpha)})$ is a regular Dirichlet form on $L^2(\mathbb{R}, \mu)$ for any Radon measure μ on $\mathbb{R}$.

Theorem 15.5. *$(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(a)})$ satisfies the annulus comparable condition* (ACC) *for $\alpha \in (1, 2]$.*

Proof. By the scaling property with the invariance under parallel translations, there exist $c_1, c_2 > 0$ such that

$$R^{(\alpha)}(x, B(x, r)^c) = c_1 r$$
$$R^{(\alpha)}(x, B(x, r)^c \cap \overline{B(x, 2r)}) = c_2 r,$$

where $B(x, r) = B_{R^{(\alpha)}}(x, r)$. Now, it is obvious that (ACC) holds. $\square$

Due to this theorem, we can apply Theorems 14.6 and 14.11 to get an estimate of the heat kernel associated with the Dirichlet form $(\mathcal{E}^{(\alpha)}, \mathcal{D}_\mu^{(\alpha)})$ on $L^2(\mathbb{R}, \mu)$ if μ has the volume doubling property with respect to the Euclidean distance. (Note that $R^{(\alpha)}$ is a power of the Euclidean distance.) As a special case, we have the following proposition.

Proposition 15.6. *Let $p_\delta^{(\alpha)}(t, x, y)$ be the heat kernel associated with the Dirichlet form $(\mathcal{E}^{(\alpha)}, \mathcal{D}^{(\alpha)})$ on $L^2(\mathbb{R}, x^\delta dx)$ for $\delta > -1$. For $\alpha \in (1, 2]$,*

$$p_\delta^{(\alpha)}(t, 0, 0) \asymp t^{-\frac{\delta+1}{\delta+\alpha}}$$

for any $t > 0$.

Recall that $(-\Delta)^{\alpha/2}$ is the associated self-adjoint operator for $\delta = 0$. Hence $p_0^{(\alpha)}(t, x, y) = P^\alpha(t, |x - y|)$, where $P^\alpha(t, \cdot)$ is the inverse Fourier transform of $e^{-ct|x|^\alpha}$ for some $c > 0$. This immediately imply that $p_0^{(\alpha)}(t, x, x) = a/t^{1/\alpha}$ for some $a > 0$.

Next we consider the trace of $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ on the Cantor set. Let K be the Cantor set, i.e.

$$K = \left\{ \sum_{m=1}^{\infty} \frac{i_m}{3^m} \middle| i_1, i_2, \ldots \in \{0, 2\} \right\}.$$

The Hausdorff dimension d_H of (K, d_E) is $\log 2/\log 3$, where d_E is the Euclidean distance. Let ν be the d_H-dimensional normalized Hausdorff measure. Define

$$K_{i_1\ldots i_m} = \left\{ \sum_{k=1}^{m} \frac{i_k}{3^k} + \frac{1}{3^m}\sum_{n=1}^{\infty}\frac{j_n}{3^n} \middle| j_1, j_2, \ldots \in \{0,2\} \right\}$$

for any $i_1, \ldots, i_m \in \{0,2\}$. Then $\nu(K_{i_1\ldots i_m}) = 2^{-m}$. Hence $\nu(B(x,r)) \asymp r^{d_H}$ for any $r \in [0,1]$ and any $x \in K$. It is easy to see that ν has the volume doubling property with respect to d_E. Also $(K, d_E|_K)$ is uniformly perfect. Recall that $R^{(\alpha)} = \gamma_\alpha (d_E)^{\alpha-1}$. Also we have

$$\mu(B_{d_E}(x,r)) \asymp r^{1-d_H}\nu(B_{d_E}(x,r)\cap K)$$

Using Theorem 14.13, we have the following result.

Theorem 15.7. *Let $\alpha \in (1,2]$. There exists a jointly continuous heat kernel $p_K^{(\alpha)}(t,x,y)$ on $(0,\infty)\times K^2$ associated with the Dirichlet form $(\mathcal{E}^{(\alpha)}|_K, \mathcal{F}^{(\alpha)}|_K)$ on $L^2(K,\nu)$. Moreover,*

$$p_K^{(\alpha)}(t,x,x) \asymp t^{-\eta} \tag{15.3}$$

for any $t \in (0,1]$ and any $x \in K$, where $\eta = \frac{\log 2}{(\alpha-1)\log 3 + \log 2}$.

If $\alpha = 2$, the process associated with $(\mathcal{E}^{(2)}|_K, \mathcal{F}^{(2)}|_K)$ on $L^2(K,\nu)$ is called the generalized diffusion on the Cantor set. Fujita has studied the heat kernel associated with the generalized diffusion on the Cantor set extensively in [16]. He has obtained (15.3) for this case by a different method.

16 Basic tools in heat kernel estimates

The rest of this part is devoted to proving the theorems in Section 14. In this section, we review the general methods of estimates of a heat kernel and make necessary modifications to them. The results in this section have been developed by several authors, for example, [1], [35] and [18].

In this section, (X,d) is a metric space and $(\mathcal{E},\mathcal{D})$ is a regular Dirichlet form on $L^2(X,\mu)$, where μ is a Radon measure on X. (We do not assume that $(\mathcal{E},\mathcal{D})$ is derived from a resistance form.) We assume that there exists a jointly continuous heat kernel (i.e. transition density) $p(t,x,y)$ associated with this Dirichlet form.

First we introduce a result on diagonal-lower estimate of a heat kernel. The Chapman-Kolmogorov equation imply the following fact.

Lemma 16.1. *For any Borel set $A \subseteq X$, any $t > 0$ and any $x \in X$,*

$$\frac{P_x(X_t \in A)^2}{\mu(A)} \leq p(2t,x,x).$$

The next lemma can be extracted from [18, Proof of Theorem 9.3].

Lemma 16.2. *Let $h : X \times (0, \infty) \to [0, \infty)$ satisfy the following conditions* (A)*,* (B) *and* (C)*:*
(A) *For any $x \in X$, $h(x, r)$ is a monotonically non decreasing function of r and $\lim_{r \downarrow 0} h(x, r) = 0$.*
(B) *There exists $a_1 > 0$ such that $h(x, 2r) \leq a_1 h(x, r)$ for any $x \in X$ and any $r > 0$.*
(C) *There exists $a_2 > 0$ such that $h(x, r) \leq a_2 h(y, r)$ for any $x, y \in X$ with $d(x, y) \leq r$.*

Assume that there exist positive constants c_1, c_2 and $r_* \in (0, \infty) \cup \{\infty\}$ such that

$$c_1 h(x, r) \leq E_x(\tau_{B_d(x,r)}) \leq c_2 h(x, r) \tag{16.1}$$

for any $x \in X$ and any $r \in (0, r_*]$. Then,
(1) *There exist $\epsilon \in (0, 1)$ and $c > 0$ such that*

$$E_x(e^{-\lambda \tau_{B_d(x,r)}}) \leq \epsilon \tag{16.2}$$

whenever $\lambda h(x, r) \leq c$ and $r \in (0, r_*/2]$.
(2) *For any $r \in (0, r_*/2]$ and any $t > 0$,*

$$P_x(\tau_{B_d(x,r)} \leq t) \leq \epsilon e^{\frac{ct}{h(x,r)}}, \tag{16.3}$$

where ϵ and c are the same as in (1).

Combining the above two lemmas, we immediately obtain the following theorem.

Theorem 16.3. *Under the same assumptions of Lemma 16.1, there exist positive constants α and δ such that*

$$\frac{\alpha}{\mu(B_d(x, r))} \leq p(\delta h(x, r), x, x)$$

for any $x \in X$ and any $r \in (0, r_*/2]$. Moreover, if μ has the volume doubling property with respect to d and $h(x, \lambda r) \leq \eta h(x, r)$ for any $x \in X$ and any $r \in (0, r_*]$, where λ and η belong to $(0, 1)$ and are independent of x and r, then there exist $\alpha' > 0$ and $c_* \in (0, 1)$ such that

$$\frac{\alpha'}{\mu(B_d(x, r))} \leq p(h(x, r), x, x)$$

for any $x \in X$ and $r \in (0, c_* r_*]$.

Next we give a result on off-diagonal upper estimate.

Hereafter, $h(x, r)$ is assumed to be independent on $x \in X$. We write $h(r) = h(x, r)$.

The following line of reasoning has essentially been developed in the series of papers by Barlow and Bass [2, 3, 4]. It has presented in [1] in a concise and organized manner. Here we follow a sophisticated version in [18]. Generalizing the discussion in [18, Proof of Theorem 9.1], we have the following lemma.

Lemma 16.4. *Let $(\mathcal{E},\mathcal{D})$ be strongly local, i.e. the Hunt process associated with the Dirichlet form $(\mathcal{E},\mathcal{D})$ on $L^2(X,\mu)$ is a diffusion process. Also let $h : (0,\infty) \to (0,\infty)$ be a monotone function with full range, continuous, strictly increasing and doubling.*
(1) *If there exist $\epsilon \in (0,1)$ and $c > 0$ such that (16.2) holds for any $r \in (0, r_*]$ and any $x \in X$ with $\lambda h(r) \geq c$, then, for any $q > 0$,*

$$E_x(e^{-\lambda \tau_{B_d(x,r)}}) \leq c_1 \exp\left(- c_2 \frac{r}{h^{-1}(c/\lambda)}\right) \tag{16.4}$$

for any $\lambda > 0$ and any $r \in (0, qr_*]$, where $c_1 = \epsilon^{-2\max\{1,q\}}$ and $c_2 = -\log \epsilon$.
(2) *Moreover, assume that $\Psi(r) = h(r)/r$ is a monotone function with full range, strictly increasing. If (16.4) holds for any $\lambda > 0$ and any $r \in (0, R]$, then, for any $\delta \in (0,1)$, any $t > 0$ and any $r \in (0,R]$,*

$$P_x(\tau_{B_d(x,r)} \leq t) \leq c_1 \exp\left(- \frac{c_3 r}{\Psi^{-1}(c_4 t/r)}\right), \tag{16.5}$$

where $c_3 = c_2(1-\delta)$ and $c_4 = c/(c_2\delta)$.

Proof. (1) First assume that $r/h^{-1}(c/\lambda) \geq 2$ and $r_*/h^{-1}(c/\lambda) \geq 2$. Then there exists $n \in \mathbb{N}$ such that

$$\frac{r}{h^{-1}(c/\lambda)} \geq n \geq \frac{r}{2h^{-1}(c/\lambda)} \geq \frac{r}{r_*}. \tag{16.6}$$

If n is the maximum natural number satisfying (16.6), then $n \geq r/h^{-1}(c/\lambda) - 1$. Since $\lambda h(r/n) \geq c$ and $r/n \in (0, r_*]$, the arguments in [18, Proof of Theorem 9.1] work and imply

$$E_x(e^{-\lambda \tau_{B_d(x,r)}}) \leq \epsilon^n \leq \frac{1}{\epsilon} \exp\left(- \frac{c_2 r}{h^{-1}(c/\lambda)}\right). \tag{16.7}$$

If $r/h^{-1}(x/\lambda) \leq 2$, then

$$E_x(e^{-\lambda \tau_{B_d(x,r)}}) \leq 1 \leq \frac{1}{\epsilon^2} \exp\left(- \frac{c_2 r}{h^{-1}(c/\lambda)}\right). \tag{16.8}$$

Finally if $r_*/h^{-1}(c/\lambda) \leq 2$, then $r/h^{-1}(c/\lambda) \leq qr_*/h^{-1}(x/\lambda) \leq 2q$ for any $r \in (0, qr_*]$. Hence

$$E_x(e^{-\lambda \tau_{B_d(x,r)}}) \leq 1 \leq \frac{1}{\epsilon^{2q}} \exp\left(- \frac{c_2 r}{h^{-1}(c/\lambda)}\right). \tag{16.9}$$

Combining (16.7), (16.8) and (16.9), we obtain the desired inequality.
(2) By [18, Proof of Theorem 9.1],

$$P_x(\tau_{B_d(x,r)} \leq t) \leq e^{\lambda t} E_x(e^{-\lambda \tau_{B_d(x,r)}}) \leq c_1 \exp\left(\lambda t - \frac{c_2 r}{h^{-1}(c/\lambda)}\right) \tag{16.10}$$

for any $t > 0$, any $\lambda > 0$ and any $r \in (0, R]$. Let $\lambda = \frac{\delta c_2 r}{t\Psi^{-1}(ct/(\delta c_2 r))}$. Then $\lambda t = \delta \frac{c_2 r}{h^{-1}(c/\lambda)} = \frac{\delta c_2 r}{\Psi^{-1}(ct/(\delta c_2 r))}$. Hence we have (16.5). $\square$

Theorem 16.5. *Let $(\mathcal{E}, \mathcal{D})$ be strongly local. Also let $h : (0, \infty) \to (0, \infty)$ be a monotone function with full range, continuous, strictly increasing, doubling and decays unifromly. Assume that if $\Psi(r) = h(r)/r$ is a monotone function with full range and strictly increasing. Assume that μ is* $(\mathrm{VD})_{\mathrm{d}}$. *If there exist $c_1, c_3, c_4 > 0$ such that (16.5) holds for any $t > 0$ and any $r \in (0, R]$ and*

$$p(t, x, x) \leq \frac{c_5}{\mu(B_d(x, h^{-1}(t)))}$$

for any $t \in (0, t_*]$ and any $x \in X$, then there exist c_6 and c_7 such that

$$p(t, x, y) \leq \frac{c_6}{\mu(B_d(x, h^{-1}(t)))} \exp\left(-c_7 \frac{d(x, y)}{\Psi^{-1}(2c_4 t/d(x, y))}\right) \tag{16.11}$$

for any $t \in (0, t_*]$ and any $x, y \in X$ with $d(x, y) \leq R$.

The next two lemmas are technically the keys in proving the above theorem. The first one is well-known. See [18, Lemma 11.1].

Lemma 16.6. *Assume that μ is* $(\mathrm{VD})_{\mathrm{d}}$. *There exist $c_0 > 0$ and $\alpha > 0$ such that*

$$\mu(B_d(x, r_1)) \leq c_0 (r_1/r_2)^\alpha \mu(B_d(x, r_2))$$

for any $r_1 \geq r_2 > 0$ and

$$\mu(B_d(x, r)) \leq c_0 \Big(1 + \frac{d(x, y)}{r}\Big)^\alpha \mu(B_d(y, r))$$

for any $x, y \in X$ and $r > 0$. In particular, there exists $M > 0$ such that

$$\mu(B_d(x, r)) \leq M \mu(B_d(y, r))$$

if $d(x, y) \leq r$.

Lemma 16.7. *Let Ψ be a monotone function with full range, strictly increasing and continuous. Set $h(r) = r\Psi(r)$. For any $\gamma > 0$, any $\epsilon > 0$, any $s > 0$ and any $r > 0$,*

$$1 + \frac{r}{h^{-1}(s)} \leq \max\{\epsilon^{-1}, 1 + \gamma\} \exp\left(\frac{\epsilon r}{\Psi^{-1}(\gamma s/r)}\right). \tag{16.12}$$

Proof. Set $x = r/h^{-1}(s)$. If $0 \leq x \leq \gamma$, then (16.12) holds. Assume that $x \geq \gamma$. Then

$$\psi^{-1}\left(\frac{\gamma s}{r}\right) = \Psi^{-1}\left(\frac{\gamma}{x}\Psi\left(\frac{r}{x}\right)\right) \leq \frac{r}{x}.$$

This implies

$$\exp\left(\frac{\epsilon r}{\Psi^{-1}(\gamma s/r)}\right) \geq \exp \epsilon x \geq 1 + \epsilon x.$$

Hence we have (16.12). $\square$

Proof of Theorem 16.5. One can modify the discussions in [18, Section 12.3]. In particular, the counterpart of [18, (12.20)] is obtained by Lemma 16.7. □

Next we give an off-diagonal lower estimate. For our theorem, the local property of the Dirichlet form is not required but the estimate should not be best possible without the local property, i.e. if the Hunt process associated with the Dirichlet form has jumps. One can find the original form on this theorem in [1].

Theorem 16.8. *Let $\Psi : (0,\infty) \to (0,\infty)$ be a monotone function with full range, strictly increasing and continuous. Set $h(r) = r\Psi(r)$. Assume that μ is $(\mathrm{VD})_{\mathrm{d}}$ and that $d(x,y)$ satisfies the chain condition, i.e. for any $x, y \in X$ and any $n \in \mathbb{N}$, there exist $x_0, \ldots, x_n$ such that $x_0 = x, x_n = y$ and $d(x_i, x_{i+1}) \le Cd(x,y)/n$ for any $i = 0, \ldots, n-1$, where $C \ge 1$ is independent of x, y and n. Also assume that there exist c_1 and c_2 such that*

$$\frac{c_1}{V_d(x, h^{-1}(t))} \le p(t,x,y) \tag{16.13}$$

for any $t \in (0, t_*]$ and any $x, y \in X$ with $d(x,y) \le c_2 h^{-1}(t)$. Then

$$\frac{c_3}{V_d(x, h^{-1}(t))} \exp\left(- c_4 \frac{d(x,y)}{\Psi^{-1}(c_5 t/d(x,y))} \right) \le p(t,x,y) \tag{16.14}$$

for any $t \in (0, t_*]$ and any $x, y \in X$.

Lemma 16.9. *Let $C, D, T \in (0,\infty)$. Then $D \le Ch^{-1}(T)$ if and only if $D/C \le \Psi^{-1}(TC/D)$. Also $D \ge Ch^{-1}(T)$ if and only if $D/C \ge \Psi^{-1}(TC/D)$.*

The ideas of the following proof is essentially found in [1]. We modify a version in [35].

Proof of Theorem 16.8. If $d(x,y) \le c_2 h^{-1}(t)$, then (16.13) implies (16.14). So we may assume that $d(x,y) \ge ch^{-1}(t)$, where $c = \min\{c_2/(6C), 1/(2C)\}$, without loss of generality. By Lemma 16.9, we have

$$\frac{d(x,y)}{c\Psi^{-1}(ct/d(x,y))} \ge 1.$$

Therefore, there exists $n \in \mathbb{N}$ such that

$$\frac{d(x,y)}{2c\Psi^{-1}(2ct/d(x,y))} \le n \le \frac{d(x,y)}{c\Psi^{-1}(ct/d(x,y))}. \tag{16.15}$$

Note that (16.15) is equivalent to

$$ch^{-1}\Big(\frac{t}{n}\Big) \le \frac{d(x,y)}{n} \le 2ch^{-1}\Big(\frac{t}{n}\Big). \tag{16.16}$$

Now we use the classical chaining argument. (See [1] for example.) Note that

$$p(t,x,y) = \int_{X^{n-1}} p\Big(\frac{t}{n}, x, z_1\Big) p\Big(\frac{t}{n}, z_1, z_2\Big) \cdots p\Big(\frac{t}{n}, z_{n-1}, y\Big) \mu(dz_1) \cdots \mu(dz_{n-1}).$$

By the chain condition, we may choose a sequence $\{x_i\}_{i=0}^n$ such that $x_0 = x, x_n = y$ and $d(x_i, x_{i+1}) \le D/n$ for any $i = 0, 1, \ldots, n-1$, where $D = Cd(x, y)$. Define $B_i = B_{D/n}(x_i)$ for $i = 1, \ldots, n-1$. If $z_i \in B_i$ and $z_{i+1} \in B_{i+1}$, then $d(z_i, z_{i+1}) \le 3D/n$. By (16.16), $3D/n \le c_2 h^{-1}(t/n)$ and $D/n \le h^{-1}(t/n)$, (16.13) and Lemma 16.6 yield

$$p(t, z_i, z_{i+1}) \ge \frac{c_1}{V_d(z_i, h^{-1}(t/n))} \ge \frac{c_1}{MV_d(x_i, h^{-1}(t/n))},$$

Hence

$$\begin{aligned} &p(t, x, y) \\ &\ge \int_{B_1 \times \ldots \times B_{n-1}} p\Big(\frac{t}{n}, x, z_1\Big) p\Big(\frac{t}{n}, z_1, z_2\Big) \cdots p\Big(\frac{t}{n}, z_{n-1}, y\Big) \mu(dz_1) \cdots \mu(dz_{n-1}) \\ &\ge (c_1/M)^n \frac{1}{V_d(x, h^{-1}(t))} \prod_{i=1}^{n-1} \frac{V_d(x_i, D/n)}{V_d(x_i, h^{-1}(t/n))} \\ &\ge (c_1/M)^n \frac{1}{V_d(x, h^{-1}(t))} \prod_{i=1}^{n-1} \frac{V_d(x_i, D/n)}{V_d(x_i, h^{-1}(t/n))} \end{aligned}$$

By Lemma 16.6 and (16.16),

$$\frac{V_d(x_i, D/n)}{V_d(x_i, h^{-1}(t/n))} \ge (c_0)^{-1} \Big(\frac{D}{nh^{-1}(t/n)}\Big)^{\lambda} \ge {c_0}^{-1} (cC)^{\lambda}.$$

Therefore there exists $L > 1$ such that

$$p(t, x, y) \ge \frac{L^{-n}}{V_d(x, h^{-1}(t))}.$$

Now the desired estimate follows immediately from (16.15). □

17 Proof of Theorem 14.6

We assume the same prerequisites on a resistance form $(\mathcal{E}, \mathcal{F})$ and the associated resistance metric R as Section 14.

Lemma 17.1. *Let A be an open set containing $x \in X$. Assume that $A \neq X$, $\mu(A) < \infty$ and $\overline{R}(x, A) < \infty$. Then, for any $\gamma \in (0, 1)$,*

$$(1 - \gamma) R(x, A^c) V_R(x, \gamma R(x, A^c)) \le E_x(\tau_A) \le R(x, A^c) \mu(A)$$

Proof. Set $B = A^c$. Note that $E_x(\tau_A) = \int_A g_B(x, y) \mu(dy)$. Since $g_B(x, y) \le g_B(x, x) = R(x, B)$, the upper estimate is obvious. If $y \in B_R(x, \gamma R(x, B))$, then (GF4) implies that $g_B^x(y) \ge (1 - \gamma) g_B^x(x)$. Therefore,

$$E_x(\tau_A) \ge \int_{B_R(x, \gamma R(x, B))} g_B^x(y) \mu(dy) \ge (1 - \gamma) R(x, B) V_R(x, \gamma R(x, B)).$$

□

Proposition 17.2. *Assume that* $d \underset{\mathrm{QS}}{\sim} R$.

(1) *There exists* $\delta > 0$ *such that* $B_d(x, \delta \overline{d}_R(x,r)) \subseteq B_R(x,r) \subseteq B_d(x, \overline{d}_R(x,r))$ *and* $B_R(x, \delta \overline{R}_d(x,r)) \subseteq B_d(x,r)$ *for any* $x \in X$ *and any* $r > 0$, *where* $\overline{d}_R(x,r) = \sup_{y \in B_R(x,r)} d(x,y)$.

(2) *There exists* $c > 0$ *such that* $\overline{R}_d(x, 2r) \leq c\overline{R}_d(x,r)$ *for any* $x \in X$ *and any* $r > 0$.

(3) *If* $\mathrm{diam}(X,d) < \infty$, *then, for any* $r_* > 0$, *there exist* $\lambda \in (0,1)$ *and* $\delta \in (0,1)$ *such that* $\overline{R}_d(x, \lambda r) \leq \delta \overline{R}_d(x,r)$ *for any* $x \in X$ *and any* $r \in (0, r_*]$. *If* $\mathrm{diam}(X,d) = \infty$, *then we have the same statement with* $r_* = \infty$.

(4) *If* μ *is* $(\mathrm{VD})_{\mathrm{R}}$, *then it is* $(\mathrm{VD})_{\mathrm{d}}$.

Proof. If $d \underset{\mathrm{QS}}{\sim} R$, then by Theorem 11.3, d is $(\mathrm{SQS})_{\mathrm{R}}$ and R is $(\mathrm{SQS})_{\mathrm{d}}$. Since (X,R) is assumed to be uniformly perfect, Proposition 11.2-(3) implies that (X,d) is uniformly perfect. Hence we may apply Theorem 10.5. Note that the statement (a) of Theorem 10.5 holds.

(1) By the statement (b) of Theorem 10.5, d is $(\mathrm{SQC})_{\mathrm{R}}$ and R is $(\mathrm{SQC})_{\mathrm{d}}$.

(2) By the statement (b) of Theorem 10.5, R is doubling with respect to d.

(3) Proposition 10.7 suffices to deduce the desired result. □

Proof of Proposition 14.4. Define $\widetilde{R}(x,r) = R(x, B_R(x,r)^c)$.

Assume (RES). By Proposition 17.2-(1),

$$B_R(x, \delta \overline{R}_d(x,r)) \subseteq B_d(x,r) \subseteq B_R(x, \overline{R}_d(x,r)). \tag{17.1}$$

Hence by (RES),

$$\widetilde{R}(x,r) \geq \widetilde{R}(x, \delta \overline{R}_d(x,r)) \geq c\delta \overline{R}_d(x,r).$$

If $B_R(x, \overline{R}_d(x,r)) \neq X$, then (RES) also shows

$$\widetilde{R}(x,r) \leq \widetilde{R}(x, \overline{R}_d(x,r)) \leq c\overline{R}_d(x,r).$$

In case $X = B_R(x, \overline{R}_d(x,r))$, we have $\mathrm{diam}(X,R)/2 \leq \overline{R}_d(x,r)$. Hence

$$\widetilde{R}(x,r) \leq \mathrm{diam}(X,R) \leq 2\overline{R}_d(x,r).$$

Conversely assume that $\widetilde{R}(x,r) \asymp \overline{R}_d(x,r)$ for any $x \in X$ and any $r > 0$ with $B_d(x,r) \neq X$. By (17.1),

$$c_1 \overline{R}_d(x,r) \leq \widetilde{R}(x, \overline{R}_d(x,r)) \quad \text{and} \quad \widetilde{R}(x, \delta \overline{R}_d(x,r)) \leq c_2 \overline{R}_d(x,r) \tag{17.2}$$

On the other hand, by Proposition 17.2-(1) and (2), there exists $\eta \in (0,1)$ such that

$$\eta\theta(r) \leq \delta \overline{R}_d(x, \delta \overline{d}_R(x,r)) \leq r \leq \theta(r),$$

for any $x \in X$ and $r > 0$, where $\theta(r) = \overline{R}_d(x, \overline{d}_R(x,r))$. Hence by (17.2),

$$r \leq \theta(r) \leq \widetilde{R}(x, \theta(r)) \leq \widetilde{R}(x, r/\eta)$$
$$\widetilde{R}(x, \delta r) \leq \widetilde{R}(x, \delta\theta(r)) \leq c_2\theta(r) \leq c_2 r/\eta$$

This suffices for (RES). □

Proof of Proposition 14.5. Assume (RES). Let $\gamma \in (0, 1)$. By Proposition 14.4 and the volume doubling property of μ, we obtain

$$\mu(B_R(x, \gamma R(x, B_d(x, r)^c)) \geq \mu(B_R(x, \gamma c_3 \overline{R}_d(x, r)))$$
$$\geq c' \mu(B_R(x, \overline{R}_d(x, r))) \geq c' \mu(B_d(x, r)).$$

By Lemmas 17.1 and 14.4 , $c'(1 - \gamma)c_3 h_d(x, r) \leq E_x(\tau_{B_d(x,r)}) \leq c_4 h_d(x, r)$.
Conversely, assume $(\mathrm{EIN})_{\mathrm{d}}$. By Lemma17.1,

$$c_1 \overline{R}_d(x, r) \leq R(x, B_d(x, r)^c). \tag{17.3}$$

Also Lemma 17.1 and the volume doubling property of μ yield

$$c_2 R(x, B_d(x, r)^c) \mu(B_R(x, R(x, B_d(x, r)^c)) \leq \overline{R}_d(x, r) V_d(x, r). \tag{17.4}$$

By (17.3), it follows that $V_d(x, r) \leq V_R(x, \overline{R}_d(x, r)) \leq c_3 V_R(x, c_1 \overline{R}_d(x, r)) \leq V_R(x, B_d(x, r)^c)$. This and (17.4) show that $c_4 R(x, B_d(x, r)^c) \leq \overline{R}_d(x, r)$. Thus we obtain (14.2). Now Proposition 14.4 implies (RES). □

Proof of Theorem 14.6. By Lemma 6.7, $B_R(x, r)$ is totally bounded for any $x \in X$ and any $r > 0$. Hence $\overline{B_R(x, r)}$ is compact. By Theorem 9.4, there exists a jointly continuous heat kernel $p(t, x, y)$. Since $d \underset{\mathrm{QS}}{\sim} R$, $B_d(x, r)$ is compact for any $x \in X$ and any $r > 0$. Using (9.3) with $A = B_d(x, r)$ and letting $t = h_d(x, r)$, we have

$$p(h_d(x, r), x, x) \leq \frac{2 + \sqrt{2}}{V_d(x, r)}.$$

The rest is to show $(\mathrm{EIN})_{\mathrm{d}}$ and the lower estimate of the heat kernel. Since μ is $(\mathrm{VD})_{\mathrm{R}}$, (X, R) has the doubling property. Hence by Theorem 6.10, (ACC) implies (RES). Proposition 14.5 shows $(\mathrm{EIN})_{\mathrm{d}}$. Next we show that $h_d(x, r)$ satisfies the conditions (A), (B) and (C) in Lemma 16.2. (A) is immediate. (B) follows from Proposition 17.2-(2) and (4). Note that $\mathrm{diam}(B_d(x, r), R) \geq \overline{R}_d(x, r) \geq \mathrm{diam}(B_d(x, r), R)/2$. If $d(x, y) \leq r$, then $B_d(y, r) \leq B_d(x, 2r)$. Hence,

$$\overline{R}(y, r) \leq \mathrm{diam}(B_d(y, r), R) \leq \mathrm{diam}(B_d(x, 2r)) \leq 2\overline{R}(x, 2r).$$

By Proposition 17.2-(2), we have (C). Now assume that $\mathrm{diam}(X, R) = \infty$. Then we have $(\mathrm{EIN})_{\mathrm{d}}$. By Proposition 17.2-(3), Theorem 16.3 shows that, for some $c > 0$,

$$\frac{c}{V_d(x, r)} \leq p(h_d(x, r), x, x) \tag{17.5}$$

for any $x \in X$ and any $r > 0$. Next we consider the case where $\mathrm{diam}(X, R) < \infty$. If $B_d(x, r) = X$, then $r \geq \mathrm{diam}(X, d)/2$. Therefore, assumptions of Lemma 16.2 hold with $r_* = \mathrm{diam}(X, d)/3$. Hence by Theorem 16.3, (17.5) is satisfied for any $x \in X$ and any $r \in (0, \alpha \mathrm{diam}(X, d)]$, where α is independent of x. Next we show

$$\frac{1}{\mu(X)} \leq \inf_{x \in X, r \geq \alpha \mathrm{diam}(X, d)} p(h_d(x, r), x, x) \tag{17.6}$$

and

$$\sup_{x \in X, r \geq \alpha \mathrm{diam}(X,d)} \frac{1}{V_d(x,r)} < \infty. \tag{17.7}$$

Letting $A = X$ in Lemma 16.1, we have $\mu(X)^{-1} \leq p(t,x,x)$ for any $x \in X$ and any $t > 0$. This yields (17.6). Let $r_1 = \alpha \mathrm{diam}(X,d)$. Since $X \subseteq B_R(x,r)$ for some $r > 0$, it follows that X is compact. Hence we may choose $N > 0$ so that, for any $x, y \in X$, there exists $\{x_i\}_{i=1,\ldots,N+1} \subseteq X$ such that $x_1 = x, x_{N+1} = y$ and $d(x_i, x_{i+1}) \leq r_1$ for any $i = 1, \ldots, N-1$. Since μ has the volume doubling property with respect to d, there exists $a_1 > 0$ such that $V_d(y_1, r_1) \leq a_1 V_d(y_2, r_1)$ for any $y_1, y_2 \in X$ with $d(x,y) \leq r_1$. Hence, for any $x, y \in X$, $V_d(x, r_1) \leq (a_1)^N V_d(y, r_1)$. This shows (17.7). Thus we have obtained (17.6) and (17.7). Therefore, there exists $C > 0$ such that

$$\frac{C}{V_d(x,r)} \leq p(h_d(x,r), x, x)$$

for any $x \in X$ and any $r \geq \alpha \mathrm{diam}(X,d)$. Hence changing c, we have (17.5) for any $x \in X$ and any $r > 0$ in this case as well.

Now, by Proposition 17.2-(3), there exists $\lambda \in (0,1)$ such that $\overline{R}_d(x, \lambda r) \leq (c/4)\overline{R}_d(x,r)$ for any $x \in X$ and any $r \leq \mathrm{diam}(X,d)$, where c is the constant appearing in (17.5). Since $\overline{R}_d(x,r) = \mathrm{diam}(X,d)$ for any $r \geq \mathrm{diam}(X,d)$, we see that $R(x,y) \leq (c/4)\overline{R}_d(x,r)$ if $d(x,y) \leq \lambda \min\{r, \mathrm{diam}(X,d)\}$. Let $T = h_d(x,r)$. Then, this and (17.5) imply

$$\begin{aligned} |p(T,x,x) - p(T,x,y)|^2 &\leq R(x,y)\mathcal{E}(p^{T,x}, p^{T,x}) \leq \frac{R(x,y)p(T,x,x)}{T} \\ &\leq \frac{c\overline{R}_d(x,r)p(T,x,x)}{4\overline{R}_d(x,r)V_d(x,r)} = \frac{1}{4}\frac{c}{V_d(x,r)}p(T,x,x) \leq \frac{1}{4}p(T,x,x)^2. \end{aligned}$$

Hence,

$$p(h_d(x,r),x,y) \geq \frac{p(h_d(x,r),x,x)}{2} \geq \frac{1}{2}\frac{c}{V_d(x,r)}.$$

Thus we have shown Theorem 14.6. $\square$

18 Proof of Theorems 14.10, 14.11 and 14.13

The proofs of Theorems 14.10 and 14.11 depend on the results in Sections 12 and 13. We use those results by letting $H(s,t) = st$. Note that all the assumptions on H in Sections 12 and 13 are satisfied for this particular H.

Proof of Theorem 14.10. The equivalence between (a), (b) and (c) is immediate form the corresponding part of Corollary 12.3. Next assume that (a), (b) and (c) hold. By Corollary 12.3, μ has the volume doubling property with respect to d and R. Also by Lemma 13.4 and Proposition 17.2-(2), there exists $\lambda, \delta \in (0,1)$ such that $h_d(x,r) \leq \lambda h_d(x, \delta r)$ for any $x \in X$ and $r > 0$. Hence by

(14.5), g decays uniformly. Lemma 14.8 implies that g^{-1} is doubling and decays uniformly. Now apply Theorem 14.6. There exists a jointly continuous heat kernel $p(t,x,y)$. Furthermore, combining (14.3), (14.4) and (14.5) along with the volume doubling property and the above mentioned property of g and g^{-1}, we obtain

$$\frac{1}{V_d(x, g^{-1}(t))} \asymp p(t,x,x)$$

for any $t \le cg(\mathrm{diam}(X,d))$ and any $x \in X$, where c is independent of x and r. The same arguments as in the proof of Theorem 14.6, in particular, (17.6) and (17.7) show $(\mathrm{DHK})_{\mathrm{g,d}}$ for $t \ge cg(\mathrm{diam}(X,d))$. Now (KD) is straight forward by the volume doubling property. Thus we have obtained $(\mathrm{HK})_{\mathrm{g,d}}$.

Conversely, assume $(\mathrm{HK})_{\mathrm{g,d}}$. (KD) and $(\mathrm{DHK})_{\mathrm{g,d}}$ imply that μ has the volume doubling property with respect to d. Since $d \underset{\mathrm{QS}}{\sim} R$, we have the volume doubling property of μ with respect to R. Also (X,d) is uniformly perfect. By Theorem 14.6, we have (14.3) and (14.4). Comparing those with $(\mathrm{DHK})_{\mathrm{g,d}}$, we see that

$$V_d(x, g^{-1}(h_d(x,r))) \asymp V_d(x,r) \tag{18.1}$$

for any $x \in X$ and any $r > 0$. Set $r_* = \mathrm{diam}(X,d)$. Note that $B_d(x,r) \ne X$ for any $r < r_*/2$. By Lemma 13.4, for any $\delta > 1$, there exists $\lambda \in (0,1)$ such that $V_d(x,\lambda r) \le \delta^{-1}V_d(x,r)$ for any $r < r_*/2$. This along with (18.1) shows that $r \asymp g^{-1}(h_d(x,r))$ for any $r < \lambda r_*/2$. This and Lemma 14.8 show (14.5) for $r < \lambda r_*/2$. Let us think about $r \in [\lambda r_*/2, r_*]$. If $r_* < \infty$, then (X,d) is compact and so is (X,R). Therefore, $\overline{R}_d(x,r) \le \mathrm{diam}(X,R)$ and $V_d(x,r) \le \mu(X) < \infty$. Let $r \in [\lambda r_*/2, r_*]$. By the volume doubling property, $V_d(x,r) \ge V_d(x,\lambda r_*/2) \ge cV_d(x,r_*) = c\mu(X)$. Also, by Proposition 6.4-(2), we have $\overline{R}_d(x,r) \ge \overline{R}_d(x,\lambda r_*/2) \ge cV_d(x,r_*) \ge c'\mathrm{diam}(X,R)/2$. Hence $cc'\mathrm{diam}(X,R)\mu(X)/2 \le h_d(x,r) \le \mathrm{diam}(X,R)\mu(X)$ for any $r \in [\lambda r_*/2, r_*]$. Also $g(\lambda r_*/2) \le g(r) \le g(r_*)$ for any $r \in [\lambda r_*/2, r_*]$. Therefore, adjusting constants, we obtain (14.5) for $r \in (0, r_*]$. Thus the condition (c) have been verified.

(14.6) follows from its counterpart (14.3).

The rest is off-diagonal estimates. Note that both g and Φ are doubling and decay uniformly. Then by Lemma18.1 below, we may replace g and Φ by h and Ψ which are continuous and strictly increasing. For the upper off-diagonal estimate, since $d \underset{\mathrm{QS}}{\sim} R$, Theorem 14.6 implies (16.1) with $r_* = \mathrm{diam}(X,d)/2$. Then by Lemmas 16.2 and 16.4, we obtain (16.5) for any $t > 0$ and any $r \in (0, \mathrm{diam}(X,d))$. Applying Theorem 16.5, replacing h and Ψ by g and Φ, and using the doubling properties, we obtain (14.7) for any $x, y \in X$ and any $t > 0$. Finally, since we have (14.6), Theorem 16.8 shows an off-diagonal lower estimate, which easily implies (14.8) by the similar arguments as in the case of upper off-diagonal estimate. □

Lemma 18.1. *Suppose that $g : (0,\infty) \to (0,\infty)$ is a monotone function with full range and doubling. Then there exists $h : (0,\infty) \to (0,\infty)$ such that h is continuous and strictly monotonically increasing on $(0,\infty)$ and $g(r) \asymp h(r)$ for* *any $r \in (0, \infty)$. Moreover, if g decays uniformly, then $g^{-1}(t) \asymp h^{-1}(t)$ for any $t \in (0, \infty)$.*

Proof. Assume that $g(2r) \leq cg(r)$ for any r. Set $\theta(r) = 1 + (1 + e^{-r})^{-1}$. Note that θ is strictly monotonically increasing and $1 < \theta(r) < 2$ for any r. Let $G(r) = \theta(r)g(r)$. Then H is strictly monotonically increasing. There exists a continuous function $F : (0, \infty) \to (0, \infty)$ such that $F(G(r)) = r$ for any $r > 0$. Define $f(x) = \theta(x)F(x)$. Then f is strictly monotonically increasing and continuous and so is the inverse of f, which is denoted by h. Since $f(G(r)) = \theta(G(r))F(G(r)) = \theta(G(r))r$, we have $\theta(r)g(r) = h(\theta(G(r))r)$. This implies $h(r)/2 \leq g(r) \leq ch(r)$.

Now assume that g decays uniformly. Then so does h. By Lemma 14.8, h^{-1} is doubling. Since $h(r)/2 \leq g(r) \leq ch(r)$, we have $h^{-1}(t/c) \leq g^{-1}(t) \leq h^{-1}(2t)$ for any $t \in (0, \infty)$. Hence the doubling property of h^{-1} shows $h^{-1}(t) \asymp g^{-1}(t)$ for any $t \in (0, \infty)$. $\square$

Proof of Theorem 14.11. By Theorem 14.6, (C1) implies (C5). Note that $R \underset{\mathrm{QS}}{\sim} R$. Since (X, R) is uniformly perfect, it follows that $\overline{R}_R(x, r) \asymp r$. Hence (C5) implies (C2). Obviously (C2) implies (C6). By Proposition 14.5, (C6) implies (C1).

(C1) $\Rightarrow$ (C3) and (C4): Assume (C1). Then by Theorem 13.1, there exists a metric d which satisfies the condition (c) of Theorem 14.10 with some $\beta > 1$. Therefore, we have (C3) by Theorem 14.10. Also, (C4) follows by Theorem 14.6.

(C3) $\Rightarrow$ (C1): Assume (C3). Then, $(\mathrm{DHK})_{\mathrm{g,d}}$ and (KD) imply the volume doubling property of μ with respect to d. Since $d \underset{\mathrm{QS}}{\sim} R$, μ has the volume doubling property with respect to R as well. Hence we have (C1).

(C4) $\Rightarrow$ (C1): Assume (C4). Then $(2r)^\beta \asymp \overline{R}_d(x, 2r)V_d(x, 2r)$. By Proposition 17.2-(2), $\overline{R}_d(x, 2r) \asymp \overline{R}_d(x, r)$. Hence μ is $(\mathrm{VD})_{\mathrm{d}}$. Since $d \underset{\mathrm{QS}}{\sim} R$, μ is $(\mathrm{VD})_{\mathrm{R}}$. Also we have $(\mathrm{EIN})_{\mathrm{d}}$. Hence Proposition 14.5 shows (ACC). Thus (C1) is verified.

Finally, (14.10) follows by the process of construction of d in Section 13, in particular, by (13.1). $\square$

Proof of Theorem 14.13. Let $g(r) = r^\beta$ and let $h(r) = r^{\beta-\gamma}$. By Theorem 14.10, $(\mathrm{HK})_{\mathrm{g,d}}$ shows $(\mathrm{DM2})_{\mathrm{g,d}}$ and $(\mathrm{DM1})_{\mathrm{g,d}}$. Using (14.11), we obtain $(\mathrm{DM2})_{\mathrm{h,d|_Y}}$ and $(\mathrm{DM1})_{\mathrm{h,d|_Y}}$, where we replace μ by ν. Since we have (ACC) for $(\mathcal{E}|_Y, \mathcal{F}_Y)$, Theorem 14.10 implies the counterpart of $(\mathrm{HK})_{\mathrm{h,d|_Y}}$. Thus we have (14.12). $\square$

Part IV

Random Sierpinski gaskets

The main purpose of this part is to apply theorems in the last part to resistance forms on random Sierpinski gaskets. The notion of random (recursive) self-similar set has introduced in [39], where basic properties, Hausdorff dimension for example, have been studied. Analysis on random Sierpinski gaskets has been developed in a series of papers by Hambly[23, 24, 25]. He has defined "Brownian motion" on a random Sierpinski gasket associated with a natural resistance form and studied an asymptotic behavior of associated heat kernel and eigenvalue counting function. He has found possible fluctuations in those assymptotics, which have later confirmed in [26].

In this part, we will first establish a sufficient and necessary condition for a measure to be volume doubling with respect to the resistance metric in Theorem 22.2. This result is a generalization of the counterpart in [30] on self-similar sets. Using this result, we show that a certain class of random self-similar measure always has the volume doubling property with respect to the resistance metric, so that we may apply theorems on heat kernel estimates in the last part. Note that Hambly has used the Hausdorff measure associated with the resistance metric, which is not a random self-similar measure in general. In fact, in Section 24, we show that the Hausdorff measure is not volume doubling with respect to the resistance metric for almost sure cases. On the contrary, in the homogeneous case, the Hausdorff measure is a random self-similar measure and is shown to satisfy the volume doubling condition in Section 23. Applying Theorem 14.10, we will recover the both-side off-diagonal heat kernel estimate in [7]. See Theorem 23.7.

Troughout this part, we fix $p_1 = \sqrt{-1}, p_2 = -\sqrt{3}/2 - \sqrt{-1}/2$ and $p_3 = \sqrt{3}/2 - \sqrt{-1}/2$ and set $V_0 = \{p_1, p_2, p_3\}$. Note that $p_1 + p_2 + p_3 = 0$ and that V_0 is the set of vertices of a regular triangle. Let T be the convex hull of V_0. We will always identify $\mathbb{R}^2$ with $\mathbb{C}$ if no confusion may occur.

19 Generalized Sierpinski gasket

In this section, as a basic component of random Sierpinski gasket, we define a family of self-similar sets in $\mathbb{R}^2$ which can be considered as a modification of the original Sierpinski gasket. Then according to the theory in [33], we briefly review the construction of resistance forms on those sets. Also, in Example 19.8, we apply Theorem 14.13 to the subsets of the Original Sierpinski gasket and obtain heat kernel estimates for the traces onto those sets.

The following is a standard set of definitions for self-similar sets.

Definition 19.1. Let S be a finite set.
(1) We define $W_m(S) = S^m = \{w_1w_2\cdots w_m | w_j \in S \text{ for } j = 1, \ldots, m\}$ for $m \geq 1$ and $W_0(S) = \{\emptyset\}$. Also $W_*(S) = \cup_{m\geq 0} W_m(S)$. For any $w \in W_*(S)$, the length of w, $|w|$, is defined to be m where $w \in W_m(S)$. For any $w = w_1w_2\cdots w_m \in W_m(S)$, define

$$[w]_n = \begin{cases} w_1w_2\cdots w_n & \text{if } 0 \leq n < m, \\ w & \text{if } n \geq m. \end{cases}$$

(2) $\Sigma(S)$ is defined by $\Sigma(S) = S^{\mathbb{N}} = \{\omega_1\omega_2\ldots | \omega_j \in S \text{ for any } j \in \mathbb{N}\}$. For any

$\omega = \omega_1\omega_2\ldots \in \Sigma(S)$, define $[\omega]_n = \omega_1\omega_2\cdots\omega_n$ for any $n \geq 0$. For $w \in W_*(S)$, define $\Sigma_w(S) = \{\omega|\omega \in \Sigma(S), [\omega]_{|w|} = w\}$ and define $\sigma_w : \Sigma(S) \to \Sigma(S)$ by $\sigma_w(\omega) = w\omega$.

A generalized Sierpinski gasket is defined as a self-similar set which preserves some of the good properties possessed by the original Sierpinski gasket.

Definition 19.2. Let K be a non-empty compact subset of $\mathbb{R}^2$ and let $S = \{1, \ldots, N\}$ for some integer $N \geq 3$. Also let $F_i(x) = \alpha_i A_i x + q_i$ for any $i \in S$, where $\alpha_i \in (0,1)$, $A_i \in O(2)$, where $O(2)$ is the 2-dimensional orthogonal matrices, and $q_i \in \mathbb{R}^2$. Then $(K, S, \{F_i\}_{i\in S})$ is called a generalized Sierpinski gasket, GSG for short, if and only if the following four conditions are satisfied:
(GSG1) $K = \cup_{i\in S} F_i(K)$,
(GSG2) $F_i(p_i) = p_i$ for $i = 1, 2, 3$,
(GSG3) $F_i(T) \subseteq T$ for any $i \in S$ and $F_i(T) \cap F_j(T) \subseteq F_i(V_0) \cap F_j(V_0)$ for any $i, j \in S$ with $i \neq j$,
(GSG4) For any $i, j \in \{1, 2, 3\}$, there exist $i_1, \ldots, i_m$ such that $i_1 = i$, $i_m = j$ and $F_{i_k}(V_0) \cap F_{i_{k+1}}(V_0) \neq \emptyset$ for all $k = 1, \ldots, m-1$.
Write $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$. For any $w = w_1w_2\cdots w_m \in W_*(S)\backslash W_0(S)$, we define $F_w = F_{w_1} \circ \ldots \circ F_{w_m}$ and $K_w = F_w(K)$. Also $V_m(\mathcal{L}) = \cup_{w\in W_m(S)} F_w(V_0)$.

By (GSG1), (GSG2) and (GSG3), the results in [33, Sections 1.2 and 1.3] show that a GSG $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is a post critically finite self-similar structure whose post critical set is V_0. Also by (GSG4), K is connected.

Next we give a brief survey on how to construct a resistance form on a self-similar set. See [33] for details.

Definition 19.3. Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a GSG. For a pair $(D, \mathbf{r}) \in \mathcal{LA}(V) \times (0,\infty)^S$, we define a symmetric bilinear form $\mathcal{E}_m$ on $\ell(V_m(\mathcal{L}))$ by

$$\mathcal{E}_m(u, v) = \sum_{w\in W_m(S)} \frac{1}{r_w} \mathcal{E}_D(u \circ F_w, v \circ F_w),$$

where $\mathbf{r} = (r_i)_{i\in S}$ and $r_w = r_{w_1}\cdots r_{w_m}$ for $w = w_1w_2\cdots w_m \in W_m(S)$. $(D, \mathbf{r})$ is called a regular harmonic structure if and only if $\mathbf{r} \in (0,1)^S$ and $\mathcal{E}_D(u, u) = \min\{\mathcal{E}_1(v, v)|v \in \ell(V_1), v|_{V_0} = u\}$ for any $u \in \ell(V_0)$.

By the results in [33, Chapter 3], we may construct a resistance form on K from a regular harmonic structure $(D, \mathbf{r})$ as a limit of the resistance forms $\{\mathcal{E}_m\}_{m\geq 0}$ on V_m.

Proposition 19.4. *Let $C(K)$ be the collection of continuous functions on K with respect to the restriction of the Euclidean metric. For any $u \in C(K)$, $\mathcal{E}_m(u_m, u_m)$ is monotonically non-decreasing with respect to m, where $u_m = u|_{V_m(\mathcal{L})}$. Define*

$$\mathcal{F} = \{u|u \in C(K), \lim_{m\to\infty} \mathcal{E}_m(u_m, u_m) < \infty\}$$

and

$$\mathcal{E}(u,v) = \lim_{m\to\infty} \mathcal{E}_m(u_m, v_m)$$

for any $u, v \in C(K)$. Then $(\mathcal{E}, \mathcal{F})$ is a resistance form on K and the associated resistance metric R gives the same topology as the restriction of the Euclidean metric. In particular, (K, R) is compact and $(\mathcal{E}, \mathcal{F})$ is a regular resistance form.

Recall that the chain condition of a distance is required to get a (lower) off-diagonal estimate of a heat kernel. In [32], we have obtained a condition for the existence of a shortest path metric, which posseses the chain condition by definition. The next definitions and the following theorems are essentially included in [32].

Definition 19.5. Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a GSG.
(1) For $p, q \in V_n(\mathcal{L})$, $(p_1, \ldots, p_m)$ is called an n-path between p and q if $p_1 = p, p_m = q$ and for any $i = 1, \ldots, m-1$, there exists $w \in W_n(S)$ such that $p_i, p_{i+1} \in F_w(V_0)$.
(2) $\mathcal{L}$ is said to admit a symmetric self-similar geodesic metric if and only if there exists $\gamma \in (0,1)$ such that

$$\gamma^{-1} = \min\{m-1 | (p_1, \ldots, p_m) \text{ is a 1-path between } p \text{ and } q\}$$

for any $p, q \in V_0$ with $p \neq q$. γ is called the symmetric geodesic ratio of $\mathcal{L}$.

Definition 19.6. Let (X, d) be a metric space. For $x, y \in X$, a continuous curve $g : [0, d(x,y)] \to X$ is called a geodesic between x and y if and only if $d(g(s), g(t)) = |s-t|$ for any $s, t \in [0,1]$. If there exists a geodesic between x and y for any $x, y \in X$, then d is called a geodesic metric on X.

Obviously, a geodesic metric satisfies the chain condition. The following theorem shows the existence of geodesic metric.

Theorem 19.7. *Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a GSG. Assume that $\mathcal{L}$ admits a symmetric self-similar geodesic metric. Then there exists a geodesic distance d on K which gives the same topology as the Euclidian metric on K and*

$$d(F_i(x), F_i(y)) = \gamma d(x,y)$$

for any $x, y \in K$ and any $i \in S$. Moreover, for any $p, q \in V_n(S)$,

$$d(p,q) = \gamma^n \min\{m-1 | (p_1, \ldots, p_m) \text{ is an } n\text{-path between } p \text{ and } q\},$$

where γ is the symmetric geodesic ratio of $\mathcal{L}$.

Proof. We can verify all the conditions in [32, Theorem 4.3] and obtain this theorem. □

We present two examples which will used as a typical component of random Siepinski gaskets in the following sections.

Example 19.8 (the (original) Sierpinski gasket)**.** For $i = 1, 2, 3$, define $f_i(z) = (z - p_i)/2 + p_i$ for any $z \in \mathbb{C}$. Then there exists a unique non-empty compact subset K of $\mathbb{C}$ such that $K = f_1(K) \cup f_2(K) \cup f_3(K)$. K is called the Sierpinski gasket. To distinguish this K from other generalized Sierpinski gaskets, we call K the original Sierpinski gasket, the OSG for short. Let $S = \{1, 2, 3\}$. Then $(K, S, \{f_i\}_{i\in S})$ is a generalized Sierpinski gasket. We write $\mathcal{L}_{SG} = (K, S, \{f_i\}_{i\in S})$. Define

$$D_h = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -(1+h) & h \\ 1 & h & -(1+h) \end{pmatrix}$$

for $h > 0$. By [33, Exercise 3.1], there exists a unique (r_1, r_2, r_3) such that $(D_h, (r_1, r_2, r_3))$ is a harmonic structure for each $h > 0$. Also the unique (r_1, r_2, r_3) satisfies $r_2 = r_3$ and $(D_h, r_1, r_2, r_3))$ is regular. We write $r_i = r_i^{SG}$ for $i = 1, 2, 3$. Hereafter in this example, we set $h = 1$. Then $r_1^{SG} = r_2^{SG} = r_3^{SG} = 3/5$. Set $\mathbf{r} = (3/5, 3/5, 3/5)$. Let μ be the self-similar measure on K with weight $(1/3, 1/3, 1/3)$. Let $(\mathcal{E}, \mathcal{F})$ be the regular resistance form on K associated with $(D_1, \mathbf{r})$ and let R be the associated resistance metric on K. Then by Barlow-Perkins [9], it has been known that the heat kernel $p(t, x, y)$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ satisfies

$$\begin{aligned} c_1 t^{-\frac{d_S}{2}} \exp\left(- c_2 \left(\frac{|x-y|^{d_w}}{t} \right)^{\frac{1}{d_w - 1}} \right) &\le p(t, x, y) \\ &\le c_3 t^{-\frac{d_S}{2}} \exp\left(- c_4 \left(\frac{|x-y|^{d_w}}{t} \right)^{\frac{1}{d_w - 1}} \right) \end{aligned} \tag{19.1}$$

for any $t \in (0, 1]$ and any $x, y \in K$, where $d_s = \log 9/\log 5$ and $d_w = \log 5/\log 2$, $c_1, \ldots, c_4$ are constants independent of x, y and t. The exponents d_s and d_w are called the spectral dimension and the walk dimension of the Sierpinski gasket respectively. In this case, $\mathcal{L}_{SG}$ admits a symmetric self-similar geodesic metric with the geodesic ratio $1/2$. The resulting geodesic metric on K is equivalent to the Euclidean metric.

Next we consider the traces of $(\mathcal{E}, \mathcal{F})$ on Alfors regular subset of K. It is known that

$$R(x, y) \asymp d_E(x, y)^{(\log 5 - \log 3)/\log 2}$$

for any $x, y \in K$, where $d_E(x, y) = |x - y|$. Hence $d_E \underset{\text{QS}}{\sim} R$. Also,

$$\mu(B_{d_E}(x, r)) \asymp r^{d_H} \tag{19.2}$$

for any $x \in K$ and $r \in (0, 1]$, where $d_H = \log 3/\log 2$ is the Hausdorff dimension of (K, d_E). Let Y be a closed Alfors δ-regular subset of K, i.e. there exists a Borel regular measure ν on Y such that $\nu(B_d(x, r) \cap Y) \asymp r^\delta$ for any $x \in Y$ and any $r \in (0, 1]$. Then by (19.1) and (19.2), we may verify all the assumptions

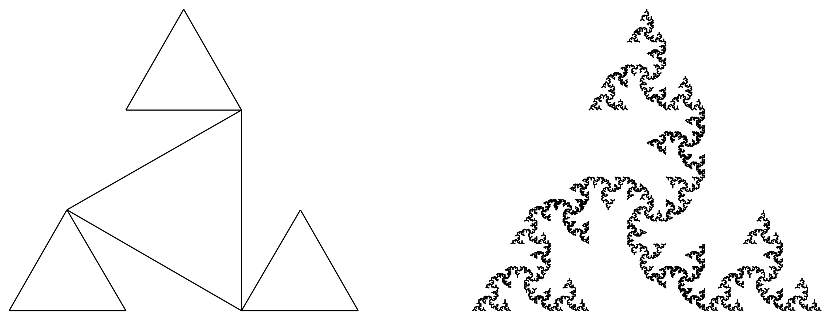

Figure 1: the Sierpinski spiral

of Theorem 14.13. Thus there exists a jointly continuous heat kernel $p_\nu^Y(t,x,y)$ associated with the regular Dirichlet form $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ on $L^2(Y,\nu)$ and

$$p_\nu^Y(t,x,x) \asymp t^{-\eta}$$

for any $x \in K$ and any $t \in (0,1]$, where $\eta = \frac{\delta \log 2}{\log 5 - \log 3 + \delta \log 2}$. In particular, if Y is equal to the line segment $\overline{p_2p_3}$, then $\delta = 1$ and $\eta = \log 2/\log(10/3)$.

Example 19.9 (the Sierpinski spiral)**.** For $i = 1,2,3$, define $h_i(z) = (z - p_i)/3 + p_i$ for any $z \in \mathbb{C}$. Also define $h_4(z) = -z/\sqrt{-3}$. The unique non-empty compact subset K of $\mathbb{C}$ satisfying $K = \cup_{i=1,2,3,4} h_i(K)$ is called the Sierpinski spiral, the S-spiral for short. See Figure 1. Let $S = \{1,2,3,4\}$. Then $(K, S, \{h_i\}_{i\in S})$ is a generalized Sierpinski gasket. We use $\mathcal{L}_{SP}$ to denote this generalize Sierpinski gasket associated with the S-spiral. Let D_h be the same as in Example 19.8 for $h > 0$. Define $r_1^{SP} = (h-\gamma)/(h+1)$, $r_2^{SP} = (1-\gamma h)/(h+1)$, $r_3^{SP} = (1-\gamma)/2$ and $r_4^{SP} = \gamma$. Then $(D_h, (r_i^{SP})_{i\in S})$ is a regular harmonic structure for $\gamma \in (0, \min\{h, 1/h\})$. Let $(\mathcal{E}, \mathcal{F})$ be the regular resistance form on K associated with $(D_h, (r_i^{SP})_{i\in S})$ and let R be the resistance distance induced by $(\mathcal{E}, \mathcal{F})$. Note that K is a dendrite, i.e. for any two points $x, y \in K$, there is a unique path between x and y. It follows that R is a geodesic metric. The Hausdorff dimension d_H of (K, R) is given by the unique d which satisfies

$$\sum_{i=1}^{4} (r_i^{SP})^d = 1.$$

By Theorem 22.8, any self-similar measure on K has the volume doubling property with respect to R. (Note that a generalize Sierpinski gasket itself is a special random Sierpinski gasket. Also for the Spiral SG, all the adjoining pair are trivial, i.e. $((j_1,i_1),(j_2,i_2))$ is an adjoining pair if and only if $j_1 = j_2$ and

$i_1 = i_2$. See Definition 22.7 for the definition of adjoining pair.) In particular, letting ν be the self-similar measure with weight $((r_i^{SP})^{d_H})_{i \in S}$, we have

$$R(x,y)V_R(x, R(x,y)) \asymp R(x,y)^{d_H+1}.$$

By Theorem 14.10, the heat kernel $p(t,x,y)$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K,\nu)$ satisfies

$$\begin{aligned} c_1 t^{-\frac{d_H}{d_H+1}} \exp\left(- c_2 \left(\frac{R(x,y)^{d_H+1}}{t} \right)^{\frac{1}{d_H}} \right) &\le p(t,x,y) \\ &\le c_3 t^{-\frac{d_H}{d_H+1}} \exp\left(- c_4 \left(\frac{R(x,y)^{d_H+1}}{t} \right)^{\frac{1}{d_H}} \right) \end{aligned}$$

for any $t \in (0,1]$ and any $x, y \in K$, where $c_1, \ldots, c_4$ are constants independent of x, y and t. Note that the spiral SG admits a symmetric self-similar geodesic metric with the ration $1/3$ and this geodesic metric coincides with the resistance metric R when $h = 1$ and $\gamma = 1/3$.

20 Random Sierpinski gasket

In this section, we will give basic definitions and notations for random (recursive) Sierpinski gaskets. Essentially the definition is the same as in [39, 23, 24, 25]. However, we will not introduce the randomness until Section 24.

Let $\mathcal{L}_j = (K(j), S_j, \{F_i^j\}_{i \in S_j})$ be a generalized Sierpinski gasket for $j = 1, \ldots, M$, where $S_j = \{1, \ldots, N_j\}$. Set $N = \max_{j=1,\ldots,M} N_j$ and define $S = \{1, \ldots, N\}$. Those generalized Sierpinski gaskets $\mathcal{L}_1, \ldots, \mathcal{L}_M$ are the basic components of our random Sierpinski gasket.

Definition 20.1. Let $W_* \subseteq W_*(S)$ and let $\Gamma : W_* \to \{1, \ldots, M\}$. (W_*, Γ) is called a random Sierpinski gasket generated by $\{L_1, \ldots, \mathcal{L}_M\}$ if and only if the following properties are satisfied:
(RSG) $\emptyset \in W_*$ and, for $m \ge 1$, $w = w_1 w_2 \cdots w_m \in W_m(S)$ belongs to W_* if and only if $[w]_{m-1} \in W_*$ and $w_m \in S_{\Gamma([w]_{m-1})}$.

Strictly speaking, to call (W_*, Γ) a "random" Sierpinski gasket, one need to introduce a randomness in the choice of $\Gamma(w)$ for every w, i.e. a probability measure on the collections of (W_*, Γ). We will do so in the final section, Section 24. Until then, we study each (W_*, Γ) respectively without randomness.

Note that (W_*, Γ) is not a geometrical object. The set $K(W_*, \Gamma) \subseteq \mathbb{R}^2$ defined in Proposition 20.3-(2) is the real geometrical object considered as the random self-similar "set" generated by (W_*, Γ).

Definition 20.2. Let (W_*, Γ) be a random Sierpinski gasket generated by $\{L_1, \ldots, \mathcal{L}_M\}$. Define $W_m = W_* \cap W_m(S)$.

(1) Define $F_\emptyset = I$, where I is the identity map from $\mathbb{R}^2$ to itself. For any $m \geq 1$ and $w = w_1 w_2 \cdots w_m \in W_m$, define $F_w : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F_w = F_{w_1}^{\Gamma([w]_0)} \circ F_{w_2}^{\Gamma([w]_1)} \circ \ldots \circ F_{w_m}^{\Gamma([w]_{m-1})}.$$

(2) $\Sigma(W_*, \Gamma) = \{w_1 w_2 \ldots | w_1 w_2 \ldots \in \Sigma(S), w_1 \ldots w_m \in W_m$ for any $m \geq 1\}$.
(3) Define $T_m(W_*, \Gamma) = \cup_{w \in W_m} F_w(T)$ and $V_m(W_*, \Gamma) = \cup_{w \in W_m} F_w(V_0)$ for any $m \geq 0$.

The followings are basic properties of random Sierpinski gaskets which are analogous to the self-similar sets.

Proposition 20.3. *Let (W_*, Γ) be a random Sierpinski gasket generated by $\{L_1, \ldots, \mathcal{L}_M\}$.*
(1) $\cap_{m \geq 0} T_m(W_*, \Gamma)$ equals to the closure of $\cup_{m \geq 0} V_m(W_*, \Gamma)$ with respect to the Euclidean metric.
(2) Define $K(W_*, \Gamma) = \cap_{m \geq 0} T_m$ and $K_w(W_*, \Gamma) = K(W_*, \Gamma) \cap F_w(T)$ for any $w \in W_*$. Then, $K_w(W_*, \Gamma) \cap K_v(W_*, \Gamma) = F_w(V_0) \cap F_v(V_0)$ for any $w, v \in W_*$ with $\Sigma_w(S) \cap \Sigma_v(S) = \emptyset$.
(3) Let $\omega = \omega_1 \omega_2 \ldots \in \Sigma(W_*, \Gamma)$. Then

$$K_{[\omega]_m}(W_*, \Gamma) \supseteq K_{[\omega]_{m+1}}(W_*, \Gamma)$$

for any $m \geq 0$ and $\cap_{m \geq 1} K_{[\omega]_m}(W_*, \Gamma)$ is a single point. If we denote this single point by $\pi_{W_*, \Gamma}(\omega)$, then the map $\pi_{W_*, \Gamma} : \Sigma(W_*, \Gamma) \to K(W_*, \Gamma)$ is continuous and onto. For any $k = 1, 2, 3$, $(\pi_{W_*, \Gamma})^{-1}(p_k) = \{(k)^\infty\}$, where $(k)^\infty = kkk \ldots \in \Sigma(S)$.
(4) For any $x \in K(W_*, \Gamma)$, set $n(x) = \#(\pi_{W_*, \Gamma}^{-1}(x))$. Then $n(x) \leq 5$ and $n(x) \geq 2$ if and only if there exist $w \in W_*$, $i_1, \ldots, i_{n(x)} \in S_{\Gamma(w)}$ with $i_m \neq i_n$ for any $m \neq n$ and $k_1, \ldots, k_{n(x)} \in \{1, 2, 3\}$ such that

$$\pi_{W_*, \Gamma}^{-1}(x) = \{w i_m (k_m)^\infty | m = 1, \ldots, n(x)\}.$$

Next we try to describe the self-similarity of random Sierpinski gasket.

Definition 20.4. Let (W_*, Γ) be a random Sierpinski gasket generated by $\{\mathcal{L}_1, \ldots, \mathcal{L}_M\}$.
(1) For any $w \in W_*$, define $W_*^w = \{v | wv \in W_*\}$ and $\Gamma^w : W_*^w \to \{1, \ldots, M\}$ by $\Gamma^w(v) = \Gamma(wv)$ for any $v \in W_*^w$.
(2) A subset $\Lambda \subseteq W_*$ is called a partition of W_* if and only if $\Sigma(W_*, \Gamma) \subseteq \cup_{w \in \Lambda} \Sigma_w(S)$ and $\Sigma_{w(1)}(S) \cap \Sigma_{w(2)}(S) = \emptyset$ for any $w(1), w(2) \in \Lambda$ with $w(1) \neq w(2)$.

The following theorem gives the self-similarity of random Sierpinski gasket. (20.1) is the counterpart of the ordinary self-similarity $K = \cup_{i=1}^N F_i(K)$.

Proposition 20.5. *Let (W_*, Γ) be a random Sierpinski gasket generated by $\{\mathcal{L}_1, \ldots, \mathcal{L}_M\}$. For any $w \in W_*$, (W^w, Γ^w) is a random Sierpinski gasket*

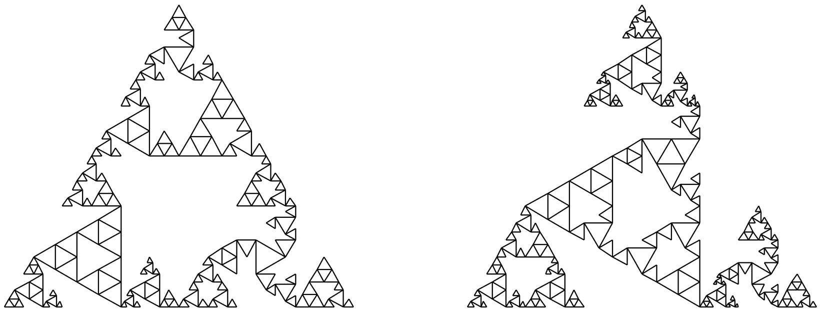

Figure 2: Random Sierpinski gaskets

generated by $\{\mathcal{L}_1, \ldots, \mathcal{L}_M\}$, $K_w(W_*, \Gamma) = F_w(K(W_*^w, \Gamma^w))$ *and* $F_w \circ \pi_{W_*^w, \Gamma^w} = \pi_{W_*, \Gamma} \circ \sigma_w$. *Moreover, if* Λ *is a partition of* W_**, then*

$$K(W_*, \Gamma) = \bigcup_{w \in \Lambda} K_w(W_*, \Gamma^w) = \bigcup_{w \in \Lambda} F_w(K(W_*^w, \Gamma^w)). \tag{20.1}$$

The following proposition describes the topological structure of a random Sierpinski gasket.

Proposition 20.6. *Let* $K_{m,x}(W_*, \Gamma) = \cup_{w \in W_m, x \in K_w(W_*, \Gamma)} K_w(W_*, \Gamma)$*. Then* $K_{m,x}(W_*, \Gamma)$ *is a neighborhood of* x *and* $\sup_{x \in K(W_*, \Gamma)} \mathrm{diam}(K_{m,x}, d_E) \to 0$ *as* $m \to \infty$*, where* d_E *is the Euclidean distance.*

Proof. Write $K_{m,x} = K_{m,x}(W_*, \Gamma)$. Set $A_{m,x} = \cup_{w \in W_m, x \notin K_w(W_*, \Gamma)} K_w(W_*, \Gamma)$. Then $A_{m,x}$ is compact and $x \notin A_{m,x}$. Hence $\alpha = \min_{y \in A_{m,x}} |x - y| > 0$. For any $s \in (0, \alpha)$, $B_{d_E}(x, s) \cap K(W_*, \Gamma) \subseteq K_{m,x}$. Hence $K_{m,x}$ is a neighborhood of x. Let $\overline{L}$ be the maximum of the Lipschitz constants of F_i^j for $j \in \{1, \ldots, M\}$ and $i \in S_j\}$. Then $\mathrm{diam}(K_w(W_*, \Gamma), d_E) \leq \overline{L}^m \mathrm{diam}(T, d_E)$. Thus $\sup_{x \in K(W_*, \Gamma)} \mathrm{diam}(K_w(W_*, \Gamma)) \leq \overline{L}^m \mathrm{diam}(T, d_E) \to 0$ as $m \to \infty$. $\square$

Figure 2 shows two random Sierpinski gaskets generated by $\{\mathcal{L}_{SG}, \mathcal{L}_{SP}\}$.

21 Resistance forms on Random Sierpinski gaskets

The main purpose of this section is to introduce the construction of a (random self-similar) resistance form on a random Sierpinski gasket. We follow the method of construction given in [24]. Furthermore, we are going to study the resistance metric associated with the constructed resistance form.

In this section, we fix a random Sierpinski gasket (W_*, Γ) generated by $\{\mathcal{L}_1, \ldots, \mathcal{L}_M\}$, where $\mathcal{L}_j = (K(j), S_j, \{F_i^j\}_{i \in S_j})$ and $S_j = \{1, \ldots, N_j\}$. We write T_m, V_m, K, K_w and π in place of $T_m(W_*, \Gamma), V_m(W_*, \Gamma)$ and so on.

Let $(D, \mathbf{r}^{(j)})$ be a regular harmonic structure for each $j \in \{1, \ldots, M\}$. Set $\mathbf{r}^{(j)} = (r_i^{(j)})_{i \in S_j}$. (Note that D is independet of j.) Define $\overline{r} = \max\{r_i^{(j)} | j \in \{1, \ldots, M\}, i \in S_j\}$ and $\underline{r} = \min\{r_i^{(j)} | j \in \{1, \ldots, M\}, i \in S_j\}$.

We first construct a series of a resistance from on $\{V_m\}_{m \geq 0}$ as in the case of p. c. f. self-similar sets.

Definition 21.1. Let (W_*, Γ) be a random Sierpinski gasket generated by $\{\mathcal{L}_1, \ldots, \mathcal{L}_M\}$. For any $w \in W_m$, define $r_w = r_{w_1}^{\Gamma([w]_0)} r_{w_2}^{\Gamma([w]_1)} \cdots r_{w_m}^{\Gamma([w]_{m-1})}$. (We set $r_\emptyset = 1$.) Define a symmetric bilinear form $\mathcal{E}_m$ on $\ell(V_m)$ by

$$\mathcal{E}_m(u, v) = \sum_{w \in W_m} \frac{1}{r_w} \mathcal{E}_D(u \circ F_w, v \circ F_w)$$

for any $u, v \in \ell(V_m)$. We use H_m to denote the symmetric linear operator from $\ell(V_m)$ to itself satisfying $\mathcal{E}_m(u, v) = -(u, H_m v)_{V_m}$ for any $u, v \in \ell(V_m)$.

Since each $(D, \mathbf{r}^{(j)})$ is a harmonic structure, we have the following fact immediately.

Proposition 21.2. *$\mathcal{E}_m$ is a resistance form on V_m for any $m \geq 1$ and H_m is a Laplacian on V_m. Moreover, $\{(V_m, H_m)\}_{m \geq 0}$ is a compatible sequence.*

Since $\{(V_m, H_m)\}_{m \geq 0}$ is a compatible sequence, $\mathcal{E}_m(u|_{V_m}, u|_{V_m})$ is monotonically non-decreasing for any $u \in \ell(V_*)$. Define

$$\mathcal{F} = \{u | u \in \ell(V), \lim_{m \to \infty} \mathcal{E}_m(u|_{V_m}, u|_{V_m}) < \infty\}$$

and

$$\mathcal{E}(u, v) = \lim_{m \to \infty} \mathcal{E}_m(u|_{V_m}, v|_{V_m})$$

for any $u, v \in \mathcal{F}$. Then by [33, Chapter 2], $(\mathcal{E}, \mathcal{F})$ is a resistance form. We use $R(\cdot, \cdot)$ to denote the associated resistance distance on V_*. Note that if $x, y \in V_m$, then $R(x, y)$ is equal to the effective resistance with respect to the resistance form $(\mathcal{E}_m, \ell(V_m))$ on V_m. We use this fact in the followings.

At this point, $(\mathcal{E}, \mathcal{F})$ is merely a resistance form on the countable set V_*. We need to show that $(\mathcal{E}, \mathcal{F})$ is naturally extended to a resistance form on K and that the associated resistance distance R gives the same topology as the Euclidean metric. Such a result will be obtained in Theorem 21.7 after rather lengthy but necessary steps. The following definition is an analogue of the notion of scales in [30].

Definition 21.3. For $s \in (0, 1)$ define

$$\Lambda_s = \{w | w \in W_* \backslash W_0, r_{[w]_{|w|-1}} > s \geq r_w\}$$

and $\Lambda_1 = \{\emptyset\}$. For any $x \in X$ and any $s \in (0,1]$,

$$\Lambda_{s,x} = \{w | w \in \Lambda_s, x \in K_w\}, \qquad K_s(x) = \cup_{w \in \Lambda_{s,x}} K_w,$$
$$\Lambda^1_{s,x} = \{w | w \in \Lambda_s, K_w \cap K_s(x) \neq \emptyset\} \text{ and } \qquad U_s(x) = \cup_{w \in \Lambda^1_{s,x}} K_w.$$

Also $Q_s(x) = \cup_{w \in \Lambda_s \setminus \Lambda^1_{s,x}} K_w$, $C_s(x) = U_s(x) \cap Q_s(X)$.

We think of K_w's for $w \in \Lambda_s$ a "ball" of radius s with respect to the resistance metric. Also, $U_s(x)$ is regarded as a s-neighborhood of x. Such a viewpoint will be justified in Corollary 21.8. First we show that $\{U_s(x)\}_{s>0}$ is a fundamental system of neighborhood with respect to the Euclidean metric.

Lemma 21.4. *Let d_E be the restriction of Euclidean metric on K. In this lemma, we use the topology of K induced by d_E.*
(1) *$K_s(x), U_s(x)$ and $V_s(x)$ are compact.*
(2) *$U_s(x)$ is a neighborhood of x with respect to d_E. Moreover, as $s \downarrow 0$, $\sup_{x \in K} \mathrm{diam}(U_s(x), d_E) \to 0$.*
(3) *$C_s(x) \subseteq \cup_{w \in \Lambda^1_{s,x}} F_w(V_0)$ and $C_s(x)$ is the topological boundary of $U_s(x)$.*

Proof. (1) and (3) are immediate. About (2), for $w = w_1 w_2 \cdots w_m \in \Lambda_s$, since $r_{w_1 w_2 \cdots w_{m-1}} > s \geq r_w$, it follows that $\overline{r}^{m-1} \geq s \geq \underline{r}^m$. Hence

$$\frac{\log s}{\log \underline{r}} \leq m \leq \frac{\log s}{\log \overline{r}} + 1 \tag{21.1}$$

Let $\underline{m}(s)$ be the integral part of $\log s / \log \underline{r}$ and let $\overline{m}(s)$ be the integral part of $\log s / \log \overline{r} + 2$. Then (21.1) implies $U_s(x) \supseteq K_{\overline{m}(s),x}$. By Proposition 20.6, $U_s(x)$ is a neighborhood of x. Also by (21.1),

$$\mathrm{diam}(U_s(x), d_E) \leq 4 \sup_{w \in \Lambda_s} \mathrm{diam}(K_w, d_E) \leq 4 \sup_{x \in K} \mathrm{diam}(K_{\overline{m}(s),x}).$$

Now Proposition 20.6 yields the desired result. □

In the next lemma, we show that the diameter of K_w for $w \in \Lambda_s$ is roughly s.

Lemma 21.5. (1) *There exists $c_0 > 0$ such that $\sup_{x,y \in K_w \cap V_*} R(x,y) \leq c_0 r_w$ for any $w \in W_*$.*
(2) *There exists $c_1 > 0$ such that $R(x,y) \leq c_1 s$ for any $s \in (0,1]$, any $x \in V_*$ and any $y \in U_s(x) \cap V_*$.*

Proof. (1) First we enumerate two basic facts.
Fact 1: $r_w \leq (\overline{r})^{|w|}$.
Fact 2: Define $R_* = \max\{R(x,y) | x, y \in V_0\}$. Then $R(x,y) \leq r_w R_*$ for any $x, y \in F_w(V_0)$.

Assume that $x \in F_w(V_0)$ and $y \in F_w(V_1(\mathcal{L}_{\Gamma(w)}))$ for some $w \in W_*$. Note that $F_w(V_1(\mathcal{L}_{\Gamma(w)})) = \cup_{i \in S_{\Gamma(w)}} F_{wi}(V_0)$. Since $\#(V_1(\mathcal{L}_{\Gamma(w)})) \leq 3N$, we may find $m \leq 3N$, $i_1, \ldots, i_m \in S_{\Gamma(w)}$ and $x_0, x_1, \ldots, x_m \in F_w(V_1(\mathcal{L}_{\Gamma(w)}))$ satisfying

$x_0 = x$, $x_m = y$ and $x_{k-1}, x_k \in F_{wi_k}(V_0)$ for any $k = 1, \ldots, m$. By the above facts,

$$R(x, y) \le \sum_{k=1}^{m} R(x_{k-1}, x_k) \le 3NR_*\overline{r}r_w.$$

Now, let $x \in F_w(V_0)$ and let $y \in K_w \cap V_*$. Then $y \in F_{wv}(V_0)$ for some $wv \in W_*$. Choose $y_i \in F_{[wv]_{|w|+i}}$ for $i = 1, \ldots, |v| - 1$. Set $y_0 = x$ and $y_{|v|} = y$. By the above arguments,

$$\begin{aligned} R(x, y) \le \sum_{i=0}^{|v|-1} R(y_i, y_{i+1}) \le \sum_{i=0}^{|v|-1} 3NR_*\overline{r}r_{[wv]_{|w|+i}} \\ \le \sum_{i=0}^{\infty} 3NR_*(\overline{r})^{i+1} r_w = \frac{3NR_*\overline{r}r_w}{1 - \overline{r}}. \end{aligned}$$

This shows that $\sup_{x,y \in K_w \cap V_*} R(x, y) \le 6NR_*\overline{r}(1 - \overline{r})^{-1} r_w$.
(2) Let $y \in U_s(x) \cap V_*$. There exist $w(1), w(2) \in \Lambda^1_{s,x}$ and $z \in F_{w(1)}(V_0) \cap F_{w(2)}(V_0)$ such that $x \in K_{w(1)} \cap V_*$ and $y \in K_{w(2)} \cap V_*$. By (1), $R(x, y) \le R(x, z) + R(z, y) \le c_0(r_{w(1)} + r_{w(2)}) \le 2c_0 s$. □

Next lemma is the heart of the series of discussions. It shows that $U_s(x)$ contains a resistance ball of radius cs, where c is independent of s.

Lemma 21.6. *There exists $c_2 > 0$ such that $R(x, y) \ge c_2 s$ for any $s \in (0, 1]$, any $x \in V_*$ and any $y \in Q_s(x) \cap V_*$.*

Proof. Set $K_* = K_s(x) \cap V_*$ and $Q_* = Q_s(x) \cap V_*$.
Claim 1 Let $z \in (K_s(x) \cup Q_s(x))^c \cap V_*$. For any $a, b, c \in \mathbb{R}$, there exists $u \in \mathcal{F}$ such that $u|_{K_*} = a, u|_{Q_*} = b$ and $u(z) = c$.
Proof of Claim 1 Set $m_* = \max_{w \in \Lambda^1_{s,x}} |w|$. We may choose $m \ge m_*$ so that $z \in F_w(V_0)$, $K_w \cap Q_s(x) = \emptyset$ and $K_w \cap K_s(x) = \emptyset$ for some $w \in W_m$. Considering the resistance form $(\mathcal{E}_m, \ell(V_m))$, we find $\tilde{u} \in \ell(V_m)$ such that $\tilde{u}|_{K_s(x) \cap V_m} = a, \tilde{u}|_{Q_s(x) \cap V_m} = b$ and $\tilde{u}(z) = c$. Since $(\mathcal{E}, \mathcal{F})$ is the limit of the compatible sequence (V_m, H_m), the harmonic extension of $\tilde{u}$ possesses the desired properties.
Claim 2 Let $\mathcal{F}_{s,x} = \{u | u \in \mathcal{F},\ u|_{K_*}$ and $u|_{Q_*}$ are constants$\}$. Then $(\mathcal{E}, \mathcal{F}_{s,x})$ is a resistance form on $(V_* \backslash (K_s(x) \cap Q_s(X))) \cup \{K_*\} \cup \{Q_*\}$.
Proof of Claim 2 By Claim 1, we see that $(K_*)^{\mathcal{F}} = K_*$. Theorem 4.11 implies that $(\mathcal{E}, \mathcal{F}^{K_*})$ is a resistance form on $(V_* \backslash K_s(x)) \cup \{K_*\}$. Again by Claim 1, $(Q_*)^{\mathcal{F}^{K_*}} = Q_*$. Using Theorem 4.11, we verify Claim 2.
Claim 3 Let $R_*(\cdot, \cdot)$ be the resistance metric associated with $(\mathcal{E}, \mathcal{F}_{s,x})$. Then $R_*(K_*, Q_*) \ge c_2 s$ for any $x \in V_*$ and any $s \in (0, 1]$, where c_2 is independent of x and s.
Proof of Claim 3 Let $\widetilde{V} = \cup_{w \in \Lambda^1_{s,x} \backslash \Lambda_{s,x}} F_w(V_0)$. Define $V = (\widetilde{V} \backslash (K_s(x) \cup Q_s(x))) \cup \{K_0\} \cup \{Q_0\}$. Note that V is naturally regarded as a subset of

$(V_*\backslash(K_s(x)\cap Q_s(X)))\cup\{K_*\}\cup\{Q_*\}$. Also, $\Phi:\widetilde{V}\to V$ is defined by

$$\Phi(x)=\begin{cases}x & \text{if } x\notin K_s(x)\cup Q_s(x),\\ K_0 & \text{if } x\in K_s(x),\\ Q_0 & \text{if } x\in Q_s(X).\end{cases}$$

Let

$$\mathcal{E}_V(u,v)=\sum_{w\in\Lambda^1_{s,x}\backslash\Lambda_{s,x}}\frac{1}{r_w}\mathcal{E}_D(u\circ\Phi,v\circ\Phi)$$

for any $u,v\in\ell(V)$. Then $(\mathcal{E}_V,\ell(V))$ is a resistance form on V. If $R_V(\cdot,\cdot)$ is the resistance metric associated with $(\mathcal{E}_V,\ell(V))$, then $R_V(K_0,Q_0)=R_*(K_*,Q_*)$. Let us consider $R_V(K_0,Q_0)$. Any path of resistors between K_0 and Q_0 corresponds to $(r_w)^{-1}\mathcal{E}_D(u\circ\Phi,v\circ\Phi)$ for some $w\in\Lambda^1_{s,x}\backslash\Lambda_{s,x}$. Let $w\in\Lambda^1_{s,x}\backslash\Lambda_{s,x}$. If $F_w(V_0)\cap K_s(x)$ or $F_w(V_0)\cap Q_s(x)$ is empty, then this part does not contribute to the effective resistance between K_0 and Q_0. So assume that both $p_w=K_s(x)\cap F_w(V_0)$ and $q_w=Q_s(x)\cap F_w(V_0)$ are non-empty. Let $r(w)$ be the effective resistance between p_w and q_w with respect to the resistance form derived from the resistance form $(r_w)^{-1}\mathcal{E}_D(\cdot,\cdot)$ on $F_w(V_0)$. Since the choice of p_w and q_w in $F_w(V_0)$ is finite, it follows that $\alpha_1 r_w\le r(w)\le\alpha_2 r_w$, where α_1 and α_2 are independent of x,s and w. Since $r_w\ge\underline{r}s$, we have $\alpha_3 s\le r(w)\le\alpha_2 s$, where $\alpha_3=\alpha_1\underline{r}$. Now $R_V(K_0,Q_0)$ is the resistance of the parallel circuit with the resistors of resistances $r(w)$. Since $\#(\Lambda_1^{s,x})$ is uniformly bounded with respect to x and s, in fact 45 is a sufficient upper bound, we have

$$\alpha_4 s\le R_V(K_0,Q_0)\le\alpha_5 s,$$

where α_4 and α_5 are independent of x and s. This completes the proof of Claim 3.

Since $R_*(K_*,Q_*)\le R(x,y)$ for any $y\in Q_s(x)$, Claim 3 suffices for the proof of this lemma. $\square$

Combining all the lemmas, we finally obtain the desired result.

Theorem 21.7. *The resistance distance R and the Euclidean distance d_E give the same topology on V_*. Moreover, the identity map on V_* is extended to a homeomorphism between the completions of (V_*,R) and (V_*,d_E).*

Proof. If $\{x_n\}_{n\ge1}\subseteq V_*$ and $x\in V_*$, then the following three conditions (A), (B) and (C) are equivalent.
(A) $\lim_{n\to\infty}R(x_n,x)=0$
(B) For any $s>0$, there exists $N>0$ such that $x_n\in U_s(x)$ for any $n\ge N$.
(C) $\lim_{n\to\infty}|x_n-x|\to0$.

In fact, by Lemmas 21.5-(2) and 21.6, (A) is equivalent to (B). Lemma 21.4-(2) shows that (B) is equivalent to (C).

Hence, the identity map between (V_*,R) and (V_*,d_E) is homeomorphism. Next assume that $\{x_n\}_{n\ge1}$ is a d_E-Cauchy sequence. Let $x\in K$ be the limit of

$\{x_n\}_{n\geq 1}$ with respect to d_E. Since $U_s(x)$ is a neighborhood of x with respect to d_E by Lemma 21.4-(2), $x_n \in U_s(x)$ for sufficiently large n. Lemma 21.5-(2) shows that $\{x_n\}_{n\geq 1}$ is an R-Cauchy sequence. Conversely assume that $\{x_n\}_{n\geq 1}$ is not a d_E-Cauchy sequence. There exist $\delta > 0$ and subsequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$ such that $|x_{n_i} - x_{m_i}| \geq \delta$ for any $i \geq 1$. By Lemma 21.5-(2), we may choose $s \in (0, 1]$ so that $\mathrm{diam}(U_s(x), d_E) < \delta$. This shows that $x_{n_i} \notin U_s(x_{m_i})$. By Lemma 21.6, it follows that $R(x_{n_i}, x_{m_i}) \geq c_2 s$. Hence $\{x_n\}_{n\geq 1}$ is not an R-Cauchy sequence. Thus we have shown that the completions of (V_*, R) and (V_*, d_E) are naturally homeomorphic. □

By this theorem, we are going to identify the completion of (V_*, R) with K. In other words, the resistance distance R is naturally extended to K. Using [33, Theorem 2.3.10], we think of $(\mathcal{E}, \mathcal{F})$ as a resistance form on K and R as the associated resistance metric from now on. Note that (K, R) is compact and hence $(\mathcal{E}, \mathcal{F})$ is regular.

By the identification described above, Lemmas 21.4, 21.5 and 21.6 imply that $U_s(x)$ is comparable with the resistance ball of radius s.

Corollary 21.8. *There exist $\alpha_1, \alpha_2 > 0$ such that*

$$B_R(x, \alpha_1 s) \subseteq U_s(x) \subseteq B_R(x, \alpha_2 s)$$

for any $x \in X$ and any $s \in (0, 1]$.

Since $(\mathcal{E}, \mathcal{F})$ is a resistance form on K and (K, R) is compact, we immediately obtain the following result.

Corollary 21.9. *$(\mathcal{E}, \mathcal{F})$ is a local regular resistance form on K.*

Definition 21.10. $(\mathcal{E}, \mathcal{F})$ and R constructed in this section are called the resistance form and the resistance metric on K associated with $((D, \mathbf{r}^{(j)}))_{j=1,\ldots,M}$ respectively.

22 Volume doubling property

In this section, we will give a criterion for the volume doubling property of a measure with respect to the resistance metric in Theorem 22.2. For random self-similar measures, we will obain a simpler condition in Theorem 22.8.

As in the last section, (W_*, Γ) is a random Sierpinski gasket generated by $\{\mathcal{L}_1, \ldots, \mathcal{L}_M\}$, $(D, \mathbf{r}^{(j)})$ is a regular harmonic structure on $\mathcal{L}_j$ for any j and $(\mathcal{E}, \mathcal{F})$ is the resistance form on K associated with $\{(D, \mathbf{r}^{(j)})\}_{j=1,\ldots,M}$. We continue to use the same notations as in the previous section.

The first theorem is immediate from Theorem 8.4 and Corollary 21.9.

Theorem 22.1. *Let μ be a finite Borel regular measure on K. Then $(\mathcal{E}, \mathcal{F})$ is a local regular Dirichlet form on $L^2(K, \mu)$.*

The following theorem gives a necessary and sufficient condition for the volume doubling property with respect to the resistance metric. It is a generalization of [30, Theorem 1.3.5]. The conditions (EL) and (GE) correspond to (ELm) and (GE) in [30] respectively.

Theorem 22.2. *Let μ be a finite Borel regular measure on K. μ has the volume doubling property with respect to the resistance distance R if and only if the following two conditions* (GE) *and* (EL) *are satisfied:*
(GE) *There exists $c_1 > 0$ such that $\mu(K_w) \leq c_1\mu(K_v)$ for any $w, v \in \Lambda_s$ with $K_w \cap K_v \neq \emptyset$ and any $s \in (0, 1]$.*
(EL) *There exists $c_2 > 0$ such that $\mu(K_{wi}) \geq c_2\mu(K_w)$ for any $w \in W_*$ and any $i \in S_{\Gamma(w)}$.*

We need two lemmas to prove this theorem.

Lemma 22.3. *Let μ be a finite Borel regular measure on K. μ has the volume doubling property with respect to the resistance distance R if and only if there exist $\alpha \in (0, 1)$ and $c > 0$ such that $\mu(U_s(x)) \leq c\mu(U_{\alpha s}(x))$ for any $x \in X$ and any $s \in (0, 1]$.*

Proof. By Corollary 21.8, $B_R(x, \alpha_1 s) \subseteq U_s(x) \subseteq B_R(x, \alpha_2 s)$. Assume that $\mu(U_s(x)) \leq c\mu(U_{\alpha s}(x))$. Choose n so that $\alpha^n\alpha_2 < \alpha_1$. Then

$$\mu(B_R(x, \alpha_1 s)) \leq \mu(U_s(x)) \leq c^n\mu(U_{\alpha^n s}(x)) \leq \mu(B_R(x, \alpha^n\alpha_2 s)).$$

Hence μ has the volume doubling property with respect to R.

Conversely, assume that $\mu(B_R(x, s)) \leq c_*\mu(B_R(x, \delta s))$ for some $c_* > 0$ and $\delta \in (0, 1)$. Choose n so that $\delta^n\alpha_2 < \alpha_1$. Then

$$\mu(U_s(x)) \leq \mu(B_R(x, \alpha_2 s)) \leq (c_*)^n\mu(B_R(x, \delta^n\alpha_2 s)) \leq (c_*)^n\mu(U_{\delta^n\alpha_2(\alpha_1)^{-1}s}(x)).$$

Letting $\alpha = \delta^n\alpha_2(\alpha_1)^{-1}$, we have the desired statement. □

Lemma 22.4. *Let $s \in (0, 1]$ and let $w \in \Lambda_s$. If $\alpha \leq \underline{r}^2$, then there exists $x \in K_w$ such that $U_s(x) \subseteq K_w$.*

Proof. Set $w = w_1w_2\cdots w_m$, where $m = |w|$. Choose k and l so that $k, l \in \{1, 2, 3\}$, $k \neq w_m$ and $l \neq k$. Note that $K_{wkl} \cap F_w(V_0) = \emptyset$. Since $r_{w_1w_2\cdots w_{m-1}} > s$, it follows that $r_{wk} > \underline{r}^2 s$. If $\alpha \leq \underline{r}^2$, then $r_{wklv} \in \Lambda_{\alpha s}$ for some $v \in W_*(S)$. Set $w_* = wklv$. Choose $x \in K_{w_*}\backslash F_{w_*}(V_0)$. By Proposition 20.3-(2) and (4), $\Lambda_{\alpha s,x} = \{w_*\}$ and $[w']_m = w$ for any $w' \in \Lambda^1_{\alpha s,x}$. Hence $U_{\alpha s}(x) \subseteq K_w$. □

Proof of Theorem 22.2. Assume (GE) and (EL). Fix $\alpha \in (0, 1)$. Let $w \in \Lambda_{s,x}$ and let $wv \in \Lambda_{\alpha s,x}$. For any $w' \in \Lambda^1_{s,x}$, there exists $w'' \in \Lambda_{s,x}$ such that $K_{w''} \cap K_{w'} \neq \emptyset$ and $K_{w''} \cap K_w \neq \emptyset$. Hence by (GE), $\mu(K_{w''}) \leq (c_1)^2\mu(K_w)$. Since $\#(\Lambda^1_{s,x}) \leq 45$,

$$\mu(U_s(x)) \leq 45(c_1)^2\mu(K_w). \tag{22.1}$$

Now, since $wv \in \Lambda_{\alpha s}$ and $w \in \Lambda_s$,

$$\alpha s < r_w r_{v_*} \leq s r_{v_*} \leq s\overline{r}^{|v|-1},$$

where $v_* = [v]_{|v|-1}$. Letting m_* be the integral part of $\frac{\log \alpha}{\log \overline{r}} + 2$, we have $|v| \leq m_*$. Note that m_* only depends on α. By (EL), $\mu(K_{wv}) \geq (c_2)^{m_*}\mu(K_w)$. Hence (22.1) shows that

$$\mu(U_{\alpha s}) \geq (c_2)^{m_*}\mu(K_w) \geq (c_2)^{m_*}(c_1)^{-2}\frac{1}{45}\mu(U_s(x)).$$

By Lemma 22.3, μ has the volume doubling property with respect to R.

Next assume that (GE) do not hold. For any $C > 0$, there exist $s \in (0, 1]$ and $w, v \in \Lambda_s$ with $K_w \cap K_v \neq \emptyset$ such that $\mu(K_v) \geq C\mu(K_w)$. Let $\alpha \in (0, \underline{r}^2]$. By Lemma 22.4, $U_{\alpha s}(x) \subseteq K_w$ for some $x \in K_w$. Since $v \in \Lambda^1_{s,x}$,

$$\mu(U_s(x)) \geq (1 + C)\mu(K_w) \geq (1 + C)\mu(U_{\alpha s}(x)).$$

Lemma 22.3 shows that μ does not have the volume doubling property with respect to R.

Finally, if (EL) do not hold, then for any $\epsilon > 0$ there exist $w \in W_*$ and $i \in S_{\Gamma(w)}$ such that $\mu(K_{wi}) \leq \epsilon\mu(K_w)$. Set $s = r_w$. Let $\alpha \in (0, \underline{r}^3]$. Then $\alpha s \leq \underline{r}^3 s \leq \underline{r}^2 r_{wi}$. By Lemma 22.4, there exists $x \in K_{wi}$ such that $U_{\alpha s}(x) \subseteq K_{wi}$. Now,

$$\mu(U_{\alpha s}(x)) \leq \mu(K_{wi}) \leq \epsilon\mu(K_w) \leq \epsilon\mu(U_s(x)).$$

Using Lemma 22.3, we see that μ does not have the volume doubling property with respect to R. $\square$

Next we introduce the notion of random self-similar measures, which is a natural generalization of self-similar measures on ordinary self-similar sets.

Proposition 22.5. *Let $\mu^{(j)} = (\mu^{(j)}_i)_{i \in S_j} \in (0, 1)^{\mathbb{S}_j}$ satisfy $\sum_{i \in S_j} \mu^{(j)}_i = 1$ for each $j = 1, \ldots, M$. Define $\mu_w = \mu^{\Gamma([w]_0)}_{w_1}\mu^{\Gamma([w]_1)}_{w_2} \cdots \mu^{\Gamma([w]_{m-1})}_{w_m}$ for any $w = w_1w_2 \cdots w_m \in W_*$. Then there exists a unique Borel regular probability measure μ on K such that $\mu(K_w) = \mu_w$ for any $w \in W_*$. Moreover, μ satisfies the condition* (EL) *in Theorem 22.2*

Note that the Hausdorff measure associated with the resistance metric, which has been studied in [24, 25, 26] is not a random self-similar measure in general except for a homogeneous case.

Definition 22.6. The Borel regular probability measure μ in Proposition 22.5 is called the random self-similar measure on (W_*, Γ) generated by $(\mu^{(1)}, \ldots, \mu^{(M)})$.

In the next definition, we introduce a notion describing relations of neighboring K_w's for $w \in \Lambda_s$ in order to apply Theorem 22.2.

Definition 22.7. A pair $((j_1, i_1), (j_2, i_2)) \in \{(j, i)|j = 1, \ldots, M, i \in \{1, 2, 3\}\}^2$ is called an adjoining pair for (W_*, Γ) if and only if there exist $w, v \in W_*$ such that $wi_1, vi_2 \in \Lambda_s$ for some $s \in (0, 1]$, $w \neq v$, $j_1 = \Gamma(w), j_2 = \Gamma(v)$ and $\pi(w(i_1)^\infty) = \pi(v(i_2)^\infty)$.

Theorem 22.8. *Let μ be a random self-siimilar measure on (W_*, Γ) generated by $(\mu^{(1)}, \dots, \mu^{(M)})$. μ has the volume doubling property with respect to the resistance distance R if*

$$\frac{\log \mu_{i_1}^{(j_1)}}{\log r_{i_1}^{(j_1)}} = \frac{\log \mu_{i_2}^{(j_2)}}{\log r_{i_2}^{(j_2)}} \tag{22.2}$$

for any adjoining pair $((j_1, i_1), (j_2, i_2))$ for (W_*, Γ).

Before proving this theorem, we give an example where the condition of the above theorem is realized.

Example 22.9. Let $\mathcal{L}_1 = \mathcal{L}_{SG}$ and let $\mathcal{L}_2 = \mathcal{L}_{SP}$, where $\mathcal{L}_{SG}$ and $\mathcal{L}_{SP}$ are the original Sierpinski gasket and the Sierpinski spiral respectively introduced in Section 19. Set $S_1 = \{1, 2, 3\}$ and $S_2 = \{1, 2, 3, 4\}$. Define $H = \{(h, \gamma)|0 < h, \gamma \in (0, \min\{h, 1/h\})\}$. Fix $(h, \gamma) \in H$ and set $r_i^{(1)} = r_i^{SG}$ for $i \in S_1$ and $r_i^{(2)} = r_i^{SP}$ for $i \in S_2$. (Recall that r_i^{SG} only depends on h and r_i^{SP} depend on h and γ. See Examples 19.8 and 19.2.) Denote $\mathbf{r}^{(j)} = (r_i^{(j)})_{j \in S_j}$ for $j = 1, 2$. Define α_* by the unique α satisfying $\sum_{i \in S_1} (r_i^{(1)})^\alpha = 1$. Note that α_* depends only on h. When $h = 1$, then $r_i^{(1)} = 3/5$ for any $i \in S_1$ and hence $\alpha_* = \log 3/(\log 5 - \log 3)$. Let $\mu_i^{(1)} = (r_i^{(1)})^{\alpha_*}$ for $i \in S_1$. Define

$$H_0 = \{(h, \gamma)|(h, \gamma) \in H, \sum_{i=1,2,3} (r_i^{(2)})^{\alpha_*} < 1\}.$$

If $h = 1$, $r_i^{(2)} = (1 - \gamma)/2$ for any $i \in \{1, 2, 3\}$. This implies $(1, \gamma) \in H_0$ for any $\gamma \in (0, 1)$. Hence H_0 is a non-empty open subset of $\mathbb{R}^2$. Let $\mu_i^{(2)} = (r_i^{(2)})^{\alpha_*}$ for any $i \in \{1, 2, 3\}$ and let $\mu_4^{(2)} = 1 - \sum_{i=1}^{3} \mu_i^{(2)}$. Applying Theorem 22.8, we have the following proposition:

Proposition Assume that $(h, \gamma) \in H_0$. Let (W_*, Γ) be any random Sierpinski gasket generated by generated by $\{\mathcal{L}_1, \mathcal{L}_2\}$. Let μ_* be the random self-similar measure generated by $((\mu_i^{(j)})_{i \in S_j})_{j=1,2}$ and let R be the resistance distance on $K = K(W_*, \Gamma)$ associated with $((D_h, \mathbf{r}^{(j)}))_{j=1,2}$. Then μ_* has the volume doubling property with respect to R.

The rest of this section is devoted to the proof of Theorem 22.8.

Proof of Theorem 22.8. Let $\omega(1) = \omega(1)_1\omega(1)_2\dots, \omega(2) = \omega(2)_1\omega(2)_2\dots \in \Sigma(W_*, \Gamma)$. Assume that $\omega(1) = wi_1(k)^\infty, \omega(2) = wi_2(l)^\infty \in \Sigma(W_*, \Gamma)$, where $w \in W_* \backslash W_0$, $i_1 \neq i_2 \in S_{\Gamma(w)}$, $k, l \in \{1, 2, 3\}$ and $\pi(\omega(1)) = \pi(\omega(2))$. Set $r_{i,n} = r_{\omega(i)_n}^{(\Gamma([\omega(i)]_{n-1}))}$ and $\mu_{i,n} = \mu_{\omega(i)_n}^{(\Gamma([\omega(i)]_{n-1}))}$ for $i = 1, 2$ and $n \geq 1$. Define $\{m_n\}_{n \geq 0}$ and $\{M_n\}_{n \geq 0}$ inductively by

$$\begin{cases} m_0 = M_0 & = |w| \\ m_{n+1} & = \inf\{m | m > m_n, r_{[\omega(1)]_m} = r_{[\omega(2)]_{m'}} \text{ for some } m'\} \\ M_{n+1} & = \inf\{m | m > M_n, r_{[\omega(1)]_{m'}} = r_{[\omega(2)]_m} \text{ for some } m'\} \end{cases}$$

(If $\inf\{m | m > m_n, r_{[\omega(1)]_m} = r_{[\omega(2)]_{m'}}$ for some $m'\} = \emptyset$, then we define $m_N = M_N = \infty$ for all $N \geq n+1$.) Also define $s_n = r_{[\omega(1)]_{m_n}}$ for $n \geq 0$. (If $m_n = \infty$, then define $s_n = 0$.) Note that $s_n = r_{[\omega(1)]_{m_n}} = r_{[\omega(2)]_{M_n}}$.

Claim 1 Let $n \geq 1$. Then there exists α_n such that $\mu_{1,m} = (r_{1,m})^{\alpha_n}$ for any $m = m_n + 1, \ldots m_{n+1}$ and $\mu_{2,m} = (r_{2,m})^{\alpha_n}$ for any $m = M_n + 1, \ldots, M_{n+1}$.

Proof of Claim 1 For sufficiently small $\epsilon > 0$, $[w(1)]_{m_n+1}, [w(2)]_{M_n+1} \in \Lambda_{s_n+\epsilon}$. Hence $(\Gamma([w(1)]_{m_n}), k), (\Gamma([w(2)]_{M_n}), l)$ is an adjoint pair. By (22.2),

$$\frac{\log \mu_{1,m_n+1}}{\log r_{1,m_n+1}} = \frac{\log \mu_{2,M_n+1}}{\log r_{2,M_n+1}} \tag{22.3}$$

Set $\alpha_n = \log \mu_{1,m_n+1} / \log r_{1,m_n+1}$. Let $m_n + 1 \leq m < m_{n+1}$. Then there exists $m' \in [M_n, M_{n+1} - 1]$ such that $r_{[w(2)]_{m'}} < r_{[w(1)]_m} < r_{[w(2)]_{m'+1}}$. Set $s_* = r_{[w(1)]_m}$. Then $[w(1)]_m, [w(2)]_{m'+1} \in \Lambda_{s_*}$ and $[w(1)]_{m+1}, [w(2)]_{m'+1} \in \Lambda_{s_*+\epsilon}$ for sufficiently small $\epsilon > 0$. Hence $((\Gamma([w(1)]_{m-1}), k), (\Gamma([w(2)]_{m'}), l))$ and $((\Gamma([w(1)]_m, k), (\Gamma([w(2)]_{m'}, l))$ are adjoint pairs. Using (22.2), we see that

$$\frac{\log \mu_{1,m}}{\log r_{1,m}} = \frac{\log \mu_{2,m'+1}}{\log r_{2,m'+1}} = \frac{\log \mu_{1,m+1}}{\log r_{1,m+1}}. \tag{22.4}$$

By the similar arguments,

$$\frac{\log \mu_{2,m}}{\log r_{2,m}} = \frac{\log \mu_{2,m+1}}{\log r_{2,m+1}} \tag{22.5}$$

for any $m = M_n + 1, \ldots, M_{n+1} - 1$. The equations (22.3), (22.4) and (22.5) immediately imply the claim. (End of Proof of Claim 1)

Claim 2 Set $s_* = \min\{r_{wi_1}, r_{wi_2}\}$. Define $m_* = \min\{m | s_* > r_{[\omega(1)]_m} \geq s_1\}$ and $M_* = \min\{m' | s_* > r_{[\omega(2)]_{m'}} \geq s_1\}$. There exists $\alpha_0 > 0$ such that $\mu_{1,m} = (r_{1,m})^{\alpha_0}$ and $\mu_{2,m'} = (r_{2,m'})^{\alpha_0}$ for any $m = m_*, \ldots, m_1$ and any $m' = M_*, \ldots, M_1$.

Proof of Claim 2 If $m_1 = m_0 + 1$, then $s_* = s_1$. Hence we have Claim 2. Similarly, if $M_1 = M_0 + 1$, then we have Claim 2. Thus we may assume that $m_1 \geq m_0 + 2$ and $M_1 \geq M_0 + 2$. Then $[\omega(1)]_{m_1}, [\omega(2)]_{M_1} \in \Lambda_{s_1}$, and so $(\Gamma([\omega(1)]_{m_1-1}), k), (\Gamma([\omega(2)]_{M_1-1}), l))$ is an adjoining pair. By (22.2),

$$\frac{\log \mu_{1,m_1}}{\log r_{1,m_1}} = \frac{\log \mu_{2,M_1}}{\log r_{2,M_1}} \tag{22.6}$$

Let $m \in \{m_*, \ldots, m_1 - 1\}$. Then there exists $m' \in [M_0 + 1, M_1 - 1]$ such that $r_{[w(2)]_{m'}} < r_{[w(1)]_m} < r_{[w(2)]_{m'+1}}$. Using the similar arguments as in the proof of Claim 1, we obtain counterparts of (22.4) and (22.5). These equalities along with (22.6) yield the claim. (End of Proof of Claim 2)

Claim 3 Define $L = \min\{n | n \in \mathbb{N}, \overline{r}^n < \underline{r}\}$. If $[\omega(1)]_m, [\omega(2)]_{m'} \in \Lambda_s$ for some $s \in [s_1, s_0)$, then

$$(\underline{\mu})^L (\underline{r})^{\alpha_0(L+1)} \mu_{[\omega(1)]_m} \leq \mu_{[\omega(2)]_{m'}} \leq (\underline{\mu})^{-L} (\underline{r})^{-\alpha_0(L+1)} \mu_{[\omega(1)]_m}, \tag{22.7}$$

where $\underline{\mu} = \min\{\mu_i^{(j)} | j \in \{1, \dots, M\}, i \in \{1, 2, 3\}\}$.
Proof of Claim 3 Assume that $r_{wi_1} = s_*$. Note that $m_* = m_0 + 2$. First we consider the case where $s \in [s_*, s_0)$. It follows that $m = m_0 + 1$ and $wi_1 \in \Lambda_s$. Since $r_{[\omega(2)]_{m'-1}} \geq s_* = r_{wi_1}$, we have $r_{2,M_0+1} \cdots r_{2,m'-1} \geq r_{1,m_0+1}$. This shows that $\overline{r}^{m'-M_0-1} \geq \underline{r}$. Therefore, $m' - M_0 \leq L$. Now $\mu_w \leq \mu_{wi_1} \leq \mu_w \underline{\mu}$ and $\mu_w \leq \mu_{[\omega(2)]_{m'}} \leq \mu_w (\underline{\mu})^{m'-M_0} \leq \mu_w (\underline{\mu})^L$. This immediately imply Claim3 in this case.
Next suppose $s \in [s_1, s_*)$. By Claim 2, $\mu_{1,m_0+2} \cdots \mu_{1,m} = (r_{1,m_0+2} \cdots r_{1,m})^{\alpha_0}$ and $\mu_{2,M_*} \cdots \mu_{2,m'} = (r_{2,M_*} \cdots r_{2,m'})^{\alpha_0}$. On the other hand, $r_{[\omega(1)]_{m-1}} > s \geq r_{[\omega(1)]_m}$. Hence

$$\frac{s}{r_{wi_1}} \geq r_{1,m_0+2} \cdots r_{1,m} \geq \frac{\underline{r}s}{r_{wi_1}}.$$

This implies

$$\frac{\mu_w s^{\alpha_0}}{(r_w)^{\alpha_0}} \frac{1}{\underline{r}} \geq \mu_{[\omega(1)]_m} \geq \frac{\mu_w s^{\alpha_0}}{(r_w)^{\alpha_0}} \underline{r}^{\alpha_0} \underline{\mu}. \tag{22.8}$$

Similarly, we have

$$\frac{s}{r_{[\omega(2)]_{M_*-1}}} \geq r_{2,M_*} \cdots r_{2,m'} \geq \frac{\underline{r}s}{r_{[\omega(2)]_{M_*-1}}}$$

and hence

$$\frac{\mu_w s^{\alpha_0}}{(r_w)^{\alpha_0}} \frac{1}{\underline{r}^{\alpha_0 L}} \geq \mu_{[\omega(2)]_{m'}} \geq \frac{\mu_w s^{\alpha_0}}{(r_w)^{\alpha_0}} \underline{\mu}^L \underline{r}^{\alpha_0} \tag{22.9}$$

Combining (22.8) and (22.9), we have the claim. If $r_{wi_2} = s_*$, then an analogous discussion yields the claim as well. (End of Proof of Claim 3)
Claim 4 Define $\alpha_* = \max\{\log \mu_i^{(j)} / \log r_i^{(j)} | j \in \{1, \dots, M\}, i \in \{1, 2, 3\}\}$. If $[\omega(1)]_m, [\omega(2)]_{m'} \in \Lambda_s$ for some $s \in (0, s_0)$, then

$$(\underline{\mu})^L (\underline{r})^{\alpha_*(L+2)} \mu_{[\omega(1)]_m} \leq \mu_{[\omega(2)]_{m'}} \leq (\underline{\mu})^{-L} (\underline{r})^{-\alpha_*(L+2)} \mu_{[\omega(1)]_m} \tag{22.10}$$

Proof of Claim 4 By Claim 1, $\mu_{1,m_n+1} \cdots \mu_{2,m_{n+1}} = (r_{1,m_n+1} \cdots r_{1,m_{n+1}})^{\alpha_n} = (r_{2,M_n+1} \cdots r_{2,M_{n+1}})^{\alpha_n} = \mu_{2,M_n+1} \cdots \mu_{2,M_{n+1}}$. for any $n \geq 1$. Suppose $s \in [s_{p+1}, s_p)$ for some $p \geq 1$. Then

$$\frac{\mu_{[\omega(1)]_p}}{\mu_{[\omega(2)]_p}} = \frac{\mu_{1,m_0+1} \cdots \mu_{1,m_1}}{\mu_{2,M_0+1} \cdots \mu_{2,M_1}} \times \frac{\mu_{1,m_p+1} \cdots \mu_{1,m}}{\mu_{2,M_p+1} \cdots \mu_{2,m'}}.$$

By Claim 3, we have an estimate of the first part of the right-hand side of the above equality. For the second part, $\mu_{1,m_p+1} \cdots \mu_{1,m} = (r_{1,m_p+1} \cdots r_{1,m'})^{\alpha_p}$. On the other hand, $s/s_p \geq r_{1,m_p+1} \cdots r_{1,m} \geq \underline{r}s/s_p$. Hence we have

$$\left(\frac{s}{s_p}\right)^{\alpha_p} \geq \mu_{1,m_p+1} \cdots \mu_{1,m} \geq (\underline{r})^{\alpha_p} \left(\frac{s}{s_p}\right)^{\alpha_p}.$$

Similarly,

$$\left(\frac{s}{s_p}\right)^{\alpha_p} \geq \mu_{2,M_p+1} \cdots \mu_{2,m'} \geq (\underline{r})^{\alpha_p} \left(\frac{s}{s_p}\right)^{\alpha_p}.$$

Hence $(\mu_{1,m_p+1}\cdots\mu_{1,m})/(\mu_{2,M_p+1}\cdots\mu_{2,m'}) \le (\underline{r})^{-\alpha_p} \le (\underline{r})^{-\alpha_*}$. Combining this with Claim 3, we obtain Clam 4. (End of Proof of Claim 4)

Finally, we give a proof of the theorem. If $w(1), w(2) \in \Lambda_s$ for some $s \in (0,1]$, $w(1) \neq w(2)$ and $K_{w(1)} \cap K_{w(2)} \neq \emptyset$, then $w(1) = [\omega(1)]_m$ and $w(2) = [\omega(2)]_{m'}$ for some $\omega(1) = wi_1(k)^\infty, \omega(2) = wi_2(l)^\infty \in \Sigma(W_*, \Gamma)$, where $w \in W_*\backslash W_0$, $i_1 \neq i_2 \in S_{\Gamma(w)}$, $k, l \in \{1,2,3\}$ and $\pi(\omega(1)) = \pi(\omega(2))$. By Claim 4, $\mu(K_{w(2)}) \le (\underline{\mu})^{-L}(\underline{r})^{-\alpha_*(L+2)}\mu(K_{w(1)})$. Hence we have (GE). Proposition 22.5 shows that μ satisfies (EL). Using Theorem 22.2, we see that μ has the volume doubling property with respect to R. □

23 Homogeneous case

In this section, we treat a special class of random Sierpinski gasket called homogeneous random Sierpinski gaskets. In this case, the Hausdorff measure is a random self-similar measure and it is always volume doubling with respect to the resistance metric. The associated diffusion process has been extensively studied in [23, 25, 7]. Most of the results in this section are the reproduction of their works from the our view point.

As in the previous sections, $\mathcal{L}_j = (K(j), S_j, \{F_j^i\}_{i\in S_j})$ is a generalized Sierpinski gasket for $j = 1, \ldots, M$ and $S_j = \{1, \ldots, N_j\}$.

Definition 23.1. Let (W_*, Γ) be a random Sierpinski gasket generated by $\{\mathcal{L}_1, \ldots, \mathcal{L}_M\}$.
(1) (W_*, Γ) is called homogeneous if and only if $\Gamma(w) = \Gamma(v)$ for any $w, v \in W_m$ and for any $m \ge 0$.
(2) Let (W_*, Γ) be homogeneous. For $m \ge 1$, we define $\Gamma_m = \Gamma(w)$ for $w \in W_{m-1}$. Set $\nu_i^{(j)} = (N_j)^{-1}$ for any $j = 1, \ldots, M$ and any $i \in S_j$. The random self-similar measure ν on (W_*, Γ) generated by $\{(\nu_i^{(1)})_{i\in S_1}, \ldots, (\nu_i^{(M)})_{i\in S_M}\}$ is called the canonical measure on (W_*, Γ).

This canonical measure coincides with the measure used in [23, 24, 25]. We will show in Theorem 23.5 that the canonical measure is equivalent to the Hausdorff measure associated with the resistance metric.

Throughout this section, (W_*, Γ) is a homogeneous random Sierpinski gasket. Let $(D, \mathbf{r}^{(j)})$ be a regular harmonic structure on $\mathcal{L}_j$ for each $j = 1, \ldots, M$. We will also require homogeneity for the resistance scaling ratio $\mathbf{r}^{(j)}$. Namely, the following condition (HG):
(HG) $r_{i_1}^{(j)} = r_{i_2}^{(j)}$ for any j and any $i_1, i_2 \in \mathcal{S}_j$.
is assumed hereafter in this section. Under (HG), we write $r_i^{(j)} = r^{(j)}$.

Proposition 23.2. *Assume* (HG).
(1) *Define* $r(m) = r^{(\Gamma_1)}\cdots r^{(\Gamma_m)}$. *Then* $\Lambda_s = W_m$ *for* $s \in (r(m-1), r(m)]$.
(2) *Let* ν *be the canonical measure on* (W_*, Γ). *Then* $\nu(K_w) = \#(W_m)^{-1} = (N_{\Gamma_1}\cdots N_{\Gamma_m})^{-1}$ *for any* $w \in W_m$.

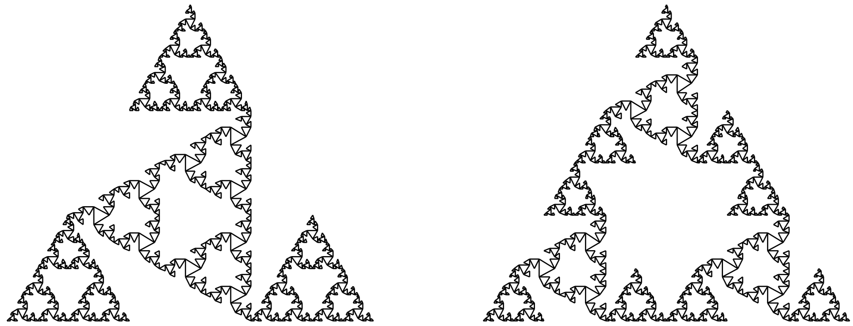

Figure 3: Homogeneous random Sierpinski gaskets

Note that $j_1 = j_2$ for any adjoining pair $((j_1, i_1), (j_2, i_2))$ in the case of a homogeneous random Sierpinski gasket. Hence by Theorem 22.8, we immediately obtain the following result.

Theorem 23.3. *Assume* (HG). *The canonical measure ν has the volume doubling property with respect to the resistance metric R on $K = K(W_*, \Gamma)$ associated with $((D, \mathbf{r}^{(j)}))_{j=1,\ldots,M}$.*

We can describe more detailed structure of the canonical measure ν in terms of the resistance metric.

Definition 23.4. Define $\psi(s) = \#(\Lambda_s)^{-1}$ for any $s \in (0, 1]$. For $s \geq 1$, we define $\psi(s) = \psi(1)$. For any $\delta > 0$ and any $A \subseteq K$, we define

$$\mathcal{H}^{\psi}_{\delta}(A) = \inf\big\{\sum_{i\geq 1}\psi(\mathrm{diam}(U_i, R))|A \subseteq \cup_{i\geq 1}U_i, \mathrm{diam}(U_i, R) \leq \delta \text{ for any } i \geq 1\big\}$$

and $\mathcal{H}^{\psi}(A) = \lim_{\delta\downarrow 0}\mathcal{H}^{\psi}_{\delta}(A)$. $\mathcal{H}^{\psi}$ is called the ψ-Hausdorff measure on (K, R).

It is known that $\mathcal{H}^{\psi}$ is a Borel regular measure. See [41]. The next theorem shows that ν is equivalent to the ψ-Hausdorff measure.

Theorem 23.5. *Assume* (HG). *The canonical measure ν is equivalent to the ψ-dimensional Hausdorff measure $\mathcal{H}^{\psi}$ on (K, R). More precisely, there exist $c_1, c_2 > 0$ such that*

$$c_1\nu(A) \leq \mathcal{H}^{\psi}(A) \leq c_2\nu(A) \tag{23.1}$$

for any Borel set $A \subseteq K$ and

$$c_1\psi(r) \leq \nu(B_R(x, r)) \leq c_2\psi(r) \tag{23.2}$$

for any $x \in X$ and any $r \in (0, 1]$.

Proof. Note that ψ has the doubling property, i.e. $\psi(2s) \leq c\psi(s)$ for any s, where c is independent of s. Now, if $w \in \Lambda_s$, then $\nu(K_w) = \psi(s)$. Since $1 \leq \#(\Lambda^1_{s,x}) \leq 45$,

$$\psi(s) \leq \nu(U_s(x)) \leq 45\psi(s) \tag{23.3}$$

By Corollary 21.8, $\nu(B_R(x, \alpha_1 s)) \leq \nu(U_s(x)) \leq \nu(B_R(x, \alpha_2 s))$. This along with (23.3) and the doubling property of ψ implies (23.2).

Next we show (23.1). By Lemma 21.5-(1), $\alpha_3 r_w \leq \mathrm{diam}(K_w, R) \leq \alpha_4 r_w$ for any $w \in W_*$, where α_3 and α_4 are independent of w. Since $\psi(r_w) = \nu(K_w)$, we have $\alpha_6\nu(K_w) \leq \psi(\mathrm{diam}(K_w, R)) \leq \alpha_7\nu(K_w)$. Let A be a compact subset of K. Define $\Lambda_s(A) = \{w | w \in \Lambda_s, K_w \cap A \neq \emptyset\}$ and $K_s(A) = \cup_{w \in \Lambda_s(A)} K_w$. Then $\cap_{n \geq 1} K_{1/n}(A) = A$. Note that $\max_{w \in \Lambda_s} \mathrm{diam}(K_w, R) \leq \alpha_8 s$. Hence,

$$\mathcal{H}^{\psi}_{\alpha_8 s}(A) \leq \sum_{w \in \Lambda_s(A)} \psi(\mathrm{diam}(K_w, R)) \leq \sum_{w \in \Lambda_s(A)} \alpha_7 \nu(K_w) \leq \alpha_7 \nu(K_s(A)).$$

Letting $s \downarrow 0$, we obtain $\mathcal{H}^{\psi}(A) \leq \alpha_7\nu(A)$ for any compact set A. Since both $\mathcal{H}^{\psi}$ and ν are Borel regular, $\mathcal{H}^{\psi}(A) \leq \alpha_7\nu(A)$ for any Borel set A. Finally, let A be a Borel set and let $A \subseteq \cup_{i \geq 1} U_i$. Choose $x_i \in U_i$. Then by (23.2)

$$\nu(A) \leq \sum_{i \geq 1} \nu(U_i) \leq \sum_{i \geq 1} \mu(B_R(x, \mathrm{diam}(U_i, R))) \leq c_2 \sum_{i \geq 1} \psi(\mathrm{diam}(U_i, R))$$

This shows $\nu(A) \leq c_2\mathcal{H}^{\psi}(A)$. Thus we have (23.1). □

By (23.3), we have the uniform volume doubling property, which has been defined in [37]. By the above theorem, we have

$$R(x,y)V_R(x, R(x,y)) \asymp R(x,y)\psi(R(x,y)),$$

Hence Theorem 14.10 implies the following theorem. (In fact, since we have the uniform volume doubling property, [37, Theorem 3.1] suffices to have (23.4) and (23.5).)

Theorem 23.6. *Let $(\mathcal{E}, D)$ be the regular local Dirichlet form on $L^2(K, \nu)$ associated with $((D, \mathbf{r}^{(j)}))_{j=1,\ldots,M}$. Assume* (HG). *There exists a jointly continuous heat kernel $p(t,x,y)$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(K, \nu)$. Define $g(r) = r\psi(r)$. (Note that $\psi(g^{-1}(t)) \asymp t/g^{-1}(t)$.) Then*

$$p(t,x,x) \asymp \frac{g^{-1}(t)}{t} \tag{23.4}$$

for any $t > 0$ and any $x \in K$ and

$$p(t,x,y) \leq \frac{c_1 g^{-1}(t)}{t} \exp\left(- c_2\left(\frac{R(x,y)}{\psi^{-1}(t/R(x,y))} \right)\right) \tag{23.5}$$

for any $t > 0$ and any $x, y \in K$.

By (23.4), the fluctuation from a power law in the on-diagonal behavior of heat kernels given in [23, 25] is now understood as the fluctuation of $\psi(r)$ versus r^α, where α is the Hausdorff dimension of (K, R).

Unfortunately, the resistance metric is not (equivalent to a power of) a geodesic metric in general, and hence (23.5) may not be best possible. To construct a geodesic metric, we define the notion of n-paths for homogeneous random Sierpinski gaslets in the similar way as in Definition 19.5.

Theorem 23.7. *Assume that $\mathcal{L}_j$ admits a symmetric self-similar geodesic metric with the ration γ_j. Set $\gamma(m) = \gamma_{\Gamma_1} \dots \gamma_{\Gamma_j}$. Then there exists a geodesic metric d on K which satisfies*

$$d(p,q) = \gamma(n) \min\{m-1 | (p_1, \dots, p_m) \text{ is an } n\text{-path between } p \text{ and } q\}$$

for any $p, q \in V_n$. Moreover, assume (HG) *and that $r^{(j)}/N_j < \gamma_j$ for any $j = 1, \dots, M$. Set $\nu(m) = \#(W_m)^{-1}$ and $T_m = \nu(m) r(m)$ for any $m \geq 0$. Define $\beta_m = \log T_m / \log \gamma(m)$ and*

$$h(s) = \begin{cases} s^{\beta_m} & \text{if } T_m \leq s < T_{m-1}, \\ s^2 & \text{if } t \geq 1. \end{cases}$$

Then

$$\frac{c_1}{V_d(x, h^{-1}(t))} \exp\left(- c_2 \left(\frac{d(x,y)}{\Phi^{-1}(t/d(x,y))} \right) \right) \leq p(t,x,y) \leq \frac{c_3}{V_d(x, h^{-1}(t))} \exp\left(- c_4 \left(\frac{d(x,y)}{\Phi^{-1}(t/d(x,y))} \right) \right), \quad (23.6)$$

where $\Phi(s) = h(s)/s$.

The above both-side off-diagonal estimate is essentially same as that obtained by Barlow and Hambly in [7]. More precisely, they have shown (23.7) and (23.8) given below.

We will prove this theorem at the end of this section.

Remark. (23.6) has equivalent expressions. Set $\alpha_m = \log \nu(m) / \log \gamma(m)$. Then (23.6) is equivalent to

$$\frac{c_5}{t^{\alpha_m/\beta_m}} \exp\left(- c_6 \left(\frac{d(x,y)^{\beta_n}}{t} \right)^{\frac{1}{\beta_n - 1}} \right) \leq p(t,x,y) \leq \frac{c_7}{t^{\alpha_m/\beta_m}} \exp\left(- c_8 \left(\frac{d(x,y)^{\beta_n}}{t} \right)^{\frac{1}{\beta_n - 1}} \right). \quad (23.7)$$

if $T_m \leq t < T_{m-1}$ and $T_n/\gamma(n) \leq t/d(x,y) < T_{n-1}/\gamma(n-1)$.

Also (23.6) is equivalent to

$$\frac{c_9}{\nu(m)} \exp\left(- c_{10}\frac{T_m}{T_n}\right) \le p(t,x,y) \le \frac{c_{11}}{\nu(m)} \exp\left(- c_{12}\frac{T_m}{T_n}\right). \tag{23.8}$$

if $T_m \le t < T_{m-1}$ and $T_n/\gamma(n) \le t/d(x,y) < T_{n-1}/\gamma(n-1)$.

Example 23.8. As in Example 22.9, let $\mathcal{L}_1 = \mathcal{L}_{SG}$ and let $\mathcal{L}_2 = \mathcal{L}_{SP}$. We consider homogeneous random Sierpinski gasket generated by $\{\mathcal{L}_1, \mathcal{L}_2\}$. See Figure 3. Fix $h = 1$ and set $\mathbf{r}^{(1)} = (3/5, 3/5, 3/5)$ and $\mathbf{r}^{(2)} = (1/3, 1/3, 1/3, 1/3)$. Then $(D_1, \mathbf{r}^{(j)})$ is a regular harmonic structure on $\mathcal{L}_j$ for $j = 1, 2$. Note that $((D_1, \mathbf{r}^{(j)}))_{j=1,2}$ satisfies the assumption (HG). In this case, $\nu_i^{(1)} = 1/3$ for $i \in S_1$ and $\nu_i^{(2)} = 1/4$ for $i \in S_2$. Also in this case, by Examples 19.8 and 19.9, both $\mathcal{L}_1$ and $\mathcal{L}_2$ admit symmetric self-similar geodesic metrics with geodesic ratio $\gamma_1 = 1/2$ and $\gamma_2 = 1/3$ respectively. Since $r^{(1)}/N_1 = 1/5 < \gamma_1$ and $r^{(2)}/N_2 = 1/12 < \gamma_2$, we have (23.6), (23.7) and (23.8).

The rest of this section is devoted to proving Theorem 23.7. The existence of a geodesic distance d is shown by the similar arguments in the proof of [32, Theorem 4.3]. Using the same arguments, we obtain the following lemma at the same time.

Lemma 23.9. *Define* $\overline{\gamma}(m) = \max_{w \in W_m} \max_{x,y \in K_w} d(x,y)$. *Then* $\overline{\gamma}(m) \asymp \gamma(m)$.

Lemma 23.10. *For* $x, y \in K$, *define* $M(x,y) = \inf\{m | y \notin U_{r(m)}(x)\}$. *Then*
(1) *For some* $m_* \in \mathbb{N}$,

$$1 \le \inf\{n | (p_0, \ldots, p_n) \textit{ is an } M(x,y)\textit{-path and there exist } w(1), w(2) \in W_{M(x,y)} \textit{ such that } x, p_0 \in K_{w(1)} \textit{ and } p_n, y \in K_{w(2)}\} \le m_*$$

for any $x, y \in K$,
(2) $R(x,y) \asymp r(M(x,y))$,
(3) $d(x,y) \asymp \gamma(M(x,y))$

Proof. Note that $\Lambda_{r(k)} = W_k$ for any k. Since $y \in U_{r(m-1)}(x) \backslash U_{r(m)}(x)$ for $m = M(x,y)$, we have (1). Combining (1) and Corollary 21.8, we obtain (2). (3) follows from Lemma 23.9 and (1). □

Lemma 23.11. $d \underset{\mathrm{QS}}{\sim} R$.

Proof. By Lemma 23.10-(2), for any $n \ge 1$, there exists $\delta_n > 0$ such that $R(x,z) \le \delta_n R(x,y)$ implies $M(x,z) \ge M(x,y) + n$. Fix $\epsilon \in (0,1)$. By Lemma 23.10-(2), $M(x,z) \ge M(x,y) + n$ implies $d(x,z) \le \epsilon d(x,y)$ for sufficiently large n. Hence d is $(\mathrm{SQS})_{\mathrm{R}}$. The similar discussion shows that R is $(\mathrm{SQS})_{\mathrm{d}}$. Hence Theorem 11.3 shows that $d \underset{\mathrm{QS}}{\sim} R$. □

Lemma 23.12. $V_d(x, d(x,y)) \asymp \nu(M(x,y))$.

Proof. Since ν is $(\mathrm{VD})_{\mathrm{R}}$ and $d \underset{\mathrm{QS}}{\sim} R$, ν is $(\mathrm{VD})_{\mathrm{d}}$. Lemma 23.10-(3) implies $V_d(x, d(x,y)) \asymp V_d(x, \gamma(M(x,y))$. Set $m = M(x,y)$. Note that $B_d(x, \gamma(m)) \subseteq U_{r(m)}(x)$. Hence

$$V_d(x, \gamma(m)) \leq \nu(U_{r(m)}(x)) \leq \#(\Lambda^1_{r(m),x})\nu(m) \leq C\nu(m),$$

where C is independent of x and m. On the other hand, if $w \in W_m$ and $x \in K_w$, then $K_w \subseteq B_d(x, \overline{\gamma}(m))$. Combining this with Lemma 23.9, we obtain $\nu(m) = \nu(K_w) \leq V_d(x, \overline{\gamma}(m)) \leq cV_d(x, \gamma(m))$. Thus, we have shown the desired result. □

Proof of Theorem 23.7. By Lemmas 23.10 and 23.12, we obtain

$$R(x,y)V_d(x, d(x,y)) \asymp T_{M(x,y)} \asymp h(d(x,y)).$$

Hence we obtain $(\mathrm{DM2})_{\mathrm{h,g}}$. Also, h is a monotone function with full range and doubling. Now we have the condition (b) of Theorem 14.10. Moreover, since $r^{(j)}/N_j < \gamma_j$ for any j, Φ is a monotone function with full range and decays uniformly. Hence Theorem 14.10 implies (23.6). □

24 Introducing randomness

Finally in this section, we introduce randomness in the random Sierpinski gaskets. As is mentioned before, the Hausdorff measure associated with the resistance metric is almost surely not (equivalent to) a random self-similar measure.

As in the previous section, we fix a family of generalized Sierpinski gaskets $\{\mathcal{L}_1, \ldots, \mathcal{L}_M\}$. Let $\mathcal{L}_j = (K(j), S_j, \{F^j_i\}_{i \in S_j})$, where $S_j = \{1, \ldots, N_j\}$. Set $N = \max_{j=1,\ldots,M} N_j$ and $S = \{1, \ldots, N\}$ as before.

Definition 24.1. Let $\Omega =$

$$\{(W_*, \Gamma) | (W_*, \Gamma) \text{ is a random Sierpinski gasket generated by } \{\mathcal{L}_1, \ldots, \mathcal{L}_M\}\}.$$

Define $\Omega_{w,j} = \{(W_*, \Gamma) | (W_*, \Gamma) \in \Omega, w \in W_*, \Gamma(i) = j\}$ for $i \in W_*(S)$ and $j \in \{1, \ldots, M\}$. Let $\mathcal{B}_m$ be the σ-algebra generated by $\{\Omega_{w,j} | w \in \cup_{n=0}^{m-1} W_n(S), j \in \{1, \ldots, M\}\}$ and define $\mathcal{B} = \cup_{m \geq 1} \mathcal{B}_m$.

For $\omega = (W_*, \Gamma) \in \Omega$, we write $W_*(\omega) = W_*, \Gamma(\omega) = \Gamma$, $K^\omega = K(W_*, \Gamma)$, $W^\omega_m = W_m(W_*, \Gamma)$ and so on.

According to [24, 25], we have the following fact.

Proposition 24.2. *Let $(p_j)_{j=1,\ldots,M} \in (0,1)^M$ satisfy $\sum_{j=1}^M p_j = 1$. Then there exists a probability measure P on $(\Omega, \mathcal{B})$ such that $\{\Omega_{w,j} | w \in W_*(S), j \in \{1, \ldots, M\}\}$ is independent and $P(\Gamma(w) = j | w \in W_*) = p_j$ for any $w \in W_*(S)$ and any $j \in \{1, \ldots, M\}$.*

We fix such a probability measure P on $(\Omega, \mathcal{B})$ as in Proposition 24.2.

Now let $(D, \mathbf{r}^{(j)})$ be a regular harmonic structure on $\mathcal{L}_j$ for $j = 1, \ldots, M$. We use $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ to denote the resistance form on K^ω associated with $\{(D, \mathbf{r}^{(j)})$ for $\omega \in \Omega$. In [24], Hambly has introduced a probability measure μ on K^ω which is natural from the view point of the resistance metric in the following way.

Definition 24.3. Let $\omega = (W_*, \Gamma) \in \Omega$. Choose $x_w \in K_w^\omega$ for $w \in W_*$. For $n \geq 1$, define

$$\mu_n = \sum_{w \in W_m^\omega} \frac{(r_w)^{-1}}{\sum_{v \in W_m^\omega} (r_v)^{-1}} \delta_{x_w},$$

where δ_x is the Dirac's point mass. Let $\mu = \mu^\omega$ be one of the accumulating points of $\{\mu_n\}$ in the weak sense.

Note that since K^ω is compact, $\{\mu_n\}$ has accumulating points. This measure μ^ω is known to be equivalent to the proper dimensional Hausdorff measure and it is not a random self-similar measure for P-a. s. $\omega \in \Omega$. See [24, 25] for details. In [24, 26], Hambly and Kumagai have shown some fluctuations in the asymptotic behavior of heat kerenels associated with the Dirichlet form $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ on $L^2(K^\omega, \mu^\omega)$ for P-a. s. $\omega \in \Omega$. In particular, by [26, Theorem 5.5], we have the following theorem.

Theorem 24.4. *μ^ω is not* $(\mathrm{VD})_{\mathrm{R}}$ *for P-a. s. ω.*

As in the homogeneous case, a fluctuation of the diagonal behavior of heat kerenels from a power law has been shown in [26] as well. By the above theorem, however, the fluctuation in this case may be caused by the lack of volume doubling property. (Recall that the volume doubling property always holds in the homogeneous case.) Hence those two fluctuations in homogeneous and non-homogeneous cases are completely different in nature.

Proof. Using [26, Theorem 5.5], we see that (GE) do not hold for P-a. s. ω. Hence by Theorem 22.2, μ^ω is not $(\mathrm{VD})_{\mathrm{R}}$ for P-a. s. ω. □

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