Resistance forms, quasisymmetric maps and heat kernel estimates

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Abstract

Assume that there is some analytic structure, a differential equation or a stochastic process for example, on a metric space. To describe asymptotic behaviors of analytic objects, the original metric of the space may not be the best one. Every now and then one can construct a better metric which is somehow "intrinsic" with respect to the analytic structure and under which asymptotic behaviors of the analytic objects have nice expressions. The problem is when and how one can find such a metric.

In this paper, we consider the above problem in the case of stochastic processes associated with Dirichlet forms derived from resistance forms. Our main concerns are following two problems:

(I) When and how can we find a metric which is suitable for describing asymptotic behaviors of the heat kernels associated with such processes?

(II) What kind of requirement for jumps of a process is necessary to ensure good asymptotic behaviors of the heat kernels associated with such processes?

Note that in general stochastic processes associated with Dirichlet forms have jumps, i. e. paths of such processes are not continuous.

The answer to (I) is for measures to have volume doubling property with respect to the resistance metric associated with a resistance form. Under volume doubling property, a new metric which is quasisymmetric with respect to the resistance metric is constructed and the Li-Yau type diagonal sub-Gaussian estimate of the heat kernel associated with the process using the new metric is shown.

About the question (II), we will propose a condition called annulus comparable condition, (ACC) for short. This condition is shown to be equivalent to the existence of a good diagonal heat kernel estimate.

As an application, asymptotic behaviors of the traces of 1-dimensional α-stable processes are obtained.

In the course of discussion, considerable numbers of pages are spent on the theory of resistance forms and quasisymmetric maps.

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1. Introduction

Originally, the main purpose of this paper is to give answers to the following two questions on heat kernels associated with Dirichlet forms derived from resistance forms. Such Dirichlet forms roughly correspond to Hunt processes for which every point has positive capacity.

(I) When and how can we find metrics which are suitable for describing asymptotic behaviors of heat kernels?

(II) What kind of requirement for jumps of processes and/or Dirichlet forms is necessary to ensure good asymptotic behaviors of associated heat kernels?

Eventually we are going to make these questions more precise. For the moment, let us explain what a heat kernel is. Assume that we have a regular Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(X, \mu)$, where X is a metric space, μ is a Borel regular measure on X, $\mathcal{E}$ is a nonnegative closed symmetric form on $L^2(X, \mu)$ and $\mathcal{D}$ is the domain of $\mathcal{E}$. Let L be the "Laplacian" associated with this Dirichlet form, i.e. Lv is characterized by the unique element in $L^2(X, \mu)$ which satisfies

$$\mathcal{E}(u, v) = -\int_X u(Lv)d\mu$$

for any $u \in \mathcal{F}$. A nonnegative measurable function $p(t, x, y)$ on $(0, +\infty) \times X^2$ is called a heat kernel associated with the Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(X, \mu)$ if

$$u(t, x) = \int_X p(t, x, y)u(y)\mu(dy)$$

for any $(t, x, y) \in (0, +\infty) \times X^2$ and any initial value $u \in L^2(X, \mu)$, where $u(t, x)$ is the solution of the heat equation associated with the Laplacian L:

$$\frac{\partial u}{\partial t} = Lu.$$

The heat kernel may not exist in general. However, it is know to exist in many cases like the Brownian motions on Euclidean spaces, Riemannian manifolds and certain classes of fractals.

If the Dirichlet form $(\mathcal{E}, \mathcal{D})$ has the local property, in other words, the corresponding stochastic process is a diffusion, then one of the preferable goals on an asymptotic estimate of a heat kernel is to show the so-called Li-Yau type (sub-)Gaussian estimate, which is

$$p(t, x, y) \asymp \frac{c_1}{V_d(x, t^{1/\beta})} \exp\left(- c_2\left(\frac{d(x, y)^\beta}{t}\right)^{1/(\beta-1)}\right), \tag{1.1}$$

where d is a metric on X, $V_d(x, r)$ is the volume of a ball $B_d(x, r) = \{y|d(x, y) < r\}$ and $\beta \geq 2$ is a constant. It is well-known that the heat kernel of the Brownian motion on $\mathbb{R}^n$ is Gaussian which is a special case of (1.1) with $d(x, y) = |x-y|$, $\beta = 2$ and $V_d(x, r) = r^n$. Li and Yau have shown in [**42**] that, for a complete Riemannian manifold with non-negative Ricci curvature, (1.1) holds with $\beta = 2$, where d is the geodesic metric and $V_d(x, r)$ is the Riemannian volume. In this case, (1.1) is called the Li-Yau type Gaussian estimate. Note that $V_d(x, t^{1/\beta})$ may have inhomogeneity with respect to x in this case. For fractals, Barlow and Perkins have shown in [**9**] that the Brownian motion on the Sierpinski gasket satisfies sub-Gaussian estimate, that is, (1.1) with $d(x, y) = |x - y|$, $\beta = \log 5/\log 2$ and $V_d(x, r) = r^\alpha$, where

$\alpha = \log 3/\log 2$ is the Hausdorff dimension of the Sierpinski gasket. Note that $V_d(x,r)$ is homogeneous in this particular case. Full generality of (1.1) is realized, for example, by a certain time change of the Brownian motion on $[0,1]$, whose heat kernel satisfies (1.1) with $\beta > 2$ and inhomogeneous $V_d(x,r)$. See [**38**] for details.

There have been extensive studies on the conditions which are equivalent to (1.1). For Riemannian manifolds, Grigor'yan [**23**] and Saloff-Coste [**47**] have independently shown that the Li-Yau type Gaussian estimate is equivalent to the Poincaré inequality and the volume doubling property. For random walks on weighted graphs, Grigor'yan and Telcs have obtained several equivalent conditions for general Li-Yau type sub-Gaussian estimate, for example, the combination of the volume doubling property, the elliptic Harnack inequality and the Poincaré inequality in [**25, 26**]. Similar results have been obtained for diffusions. See [**31**] and [**10**] for example.

The importance of the Li-Yau type (sub-)Gaussian estimate (1.1) is that it describes asymptotic behaviors of an analytical object, namely, $p(t,x,y)$ in terms of geometrical objects like the metric d and the volume of a ball $V_d(x,r)$. Such an interplay of analysis and geometry makes the study of heat kernels interesting. In this paper, we have resistance forms on the side of analysis and quasisymmetric maps on the side of geometry. To establish a foundation in studying heat kernel estimates, we first need to do considerable works on both sides, i.e. resistance forms and quasisymmetric maps. Those two subjects come to the other main parts of this paper as a consequence.

The theory of resistance forms has been developed to study analysis on "low-dimensional" fractals including the Sierpinski gasket, the 2-dimensional Sierpinski carpet, random Sierpinski gaskets and so on. Roughly, a symmetric non-negative definite quadratic form $\mathcal{E}$ on a subspace $\mathcal{F}$ of real-valued functions on a set X is called a resistance form on X if it has the Markov property and

$$\min\{\mathcal{E}(u,u)|u \in \mathcal{F}, u(x) = 1 \text{ and } u(y) = 0\}$$

exists and is positive for any $x \neq y \in X$. The reciprocal of the above minimum, denoted by $R(x,y)$, is known to be a metric (distance) and is called the resistance metric associated with $(\mathcal{E},\mathcal{F})$. See [**36**] for details. Note that unlike the Dirichlet forms, a resistance form is defined without referring to any measure on the space and hence it is not necessarily a Dirichlet form as it is. In Part 1, we are going to establish fundamental notions on resistance forms, for instance, the existence and properties of the Green function with an infinite set as a boundary, regularity of a resistance form, traces, construction of a Dirichlet form from a resistance form, the existence and continuity of heat kernels. Assume that (X,R) is compact for simplicity and let μ be a Borel regular measure on (X,R). In Section 9, a regular resistance form $(\mathcal{E},\mathcal{F})$ is shown to be a regular Dirichlet form on $L^2(X,\mu)$. We also prove that the associated heat kernel $p(t,x,y)$ exists and is continuous on $(0,+\infty) \times X \times X$ in Section 10. Even if (X,R) is not compact, we are going to obtain a modified versions of those statements under mild assumptions. See Part I for details.

The notion of quasisymmetric maps has been introduced by Tukia and Väisälä in [**51**] as a generalization of quasiconformal mappings in the complex plane. Soon its importance has been recognized in wide areas of analysis and geometry. There have been many works on quasisymmetric maps since then. See Heinonen [**32**] and Semmes [**48**] for references. In this paper, we are going to modify the resistance metric R quasisymmetrically to obtain a new metric which is more suitable for describing asymptotic behaviors of a heat kernel. The key of modification is to realize the following relation:

$$\text{Resistance} \times \text{Volume} \asymp (\text{Distance})^{\beta}, \tag{1.2}$$

where "Volume" is the volume of a ball and "Distance" is the distance with respect to the new metric. With (1.5), we are going to show that the mean exit time from a ball is comparable with r^{β} and this fact will lead us to Li-Yau type on-diagonal estimate of the heat kernel described in (1.6). Quasisymmetric modification of a metric has many advantages. For example, it preserves the volume doubling property of a measure. In Part 2, we will study quasisymmetric homeomorphisms on a metric space. In particular, we are going to establish relations between properties such as (1.2) concerning the original metric D, the quasisymmetrically modified metric d and the volume of a ball $V_d(x,r) = \mu(B_d(x,r))$ and show how to construct a metric d which is quasisymmetric to the original metric D and satisfy a desired property like (1.2).

Let us return to question (I). We will confine ourselves to the case of diffusion processes for simplicity. The lower part of the Li-Yau type (sub-)Gaussian estimate (1.1) is known to hold only when the distance is geodesic, i.e. any two points are connected by a geodesic curve. This is not the case for most of metric spaces. So, we use an adequate substitute called near diagonal lower estimate, $(\text{NDL})_{\beta,d}$ for short. We say that $(\text{NDL})_{\beta,d}$ holds if and only if

$$\frac{c_3}{V_d(x,t^{1/\beta})} \le p(t,x,y) \tag{1.3}$$

for any x, y which satisfy $d(x,y)^{\beta} \le c_4 t$. For upper estimates, the Li-Yau type (sub-)Gaussian upper estimate of order β, $(\text{LYU})_{\beta,d}$ for short, is said to hold if and only if

$$p(t,x,y) \le \frac{c_5}{V_d(x,t^{1/\beta})} \exp\left(- c_6 \left(\frac{d(x,y)^{\beta}}{t}\right)^{1/(\beta-1)}\right). \tag{1.4}$$

Another important property is the doubling property of a heat kernel, (KD) for short, that is,

$$p(t,x,x) \le c_7 p(2t,x,x). \tag{1.5}$$

Note that $p(t,x,x)$ is monotonically decreasing with respect to t. It is easy to see that the Li-Yau type (sub-)Gaussian heat kernel estimate together with the volume doubling property implies (KD). Let $p(t,x,y)$ be the heat kernel associated with a diffusion process. Now, the question (I) can be rephrased as follows:

Question When and how can we find a metric d under which $p(t,x,y)$ satisfies $(\text{LYU})_{\beta,d}$, $(\text{NDL})_{\beta,d}$ and (KD)?

In Corollary 15.12, we are going to answer this if the Dirichlet form associated with the diffusion process is derived from a resistance form. Roughly speaking, we obtain the following statement.

Answer The underlying measure μ has the volume doubling property with respect to the resistance metric R if and only if there exist $\beta > 1$ and a metric d which is quasisymmetric with respect to R such that $(\text{LYU})_{\beta,d}$, $(\text{NDL})_{\beta,d}$ and (KD) hold.

Of course, one can ask the same question for general diffusion process with a heat kernel. Such a problem is very interesting. In this paper, however, we only consider the case where the process is associated with a Dirichlet form induced by a resistance form.

Next, we are going to explain the second problem, the question (II). Recently, there have been many results on asymptotic behaviors of a heat kernel associated with a jump process. See [**12, 15, 16, 5**] for example. They have dealt with a specific class of jump processes and studied a set of conditions which is equivalent to certain kind of (off-diagonal) heat kernel estimate. For example, in [**15**], they have shown the existence of jointly continuous heat kernel for a generalization of α-stable process on an Ahlfors regular set and given a condition for best possible off-diagonal heat kernel estimate. In this paper, we will only consider the following Li-Yau type on-diagonal estimate, $(\mathrm{LYD})_{\beta,d}$ for short,

$$p(t,x,x) \asymp \frac{1}{V_d(x,t^{1/\beta})} \tag{1.6}$$

which is the diagonal part of (1.1). Our question is

Question When and how can we find a metric d with $(\mathrm{LYD})_{\beta,d}$ for a given (jump) process which possesses a heat kernel?

In this case, the "when" part of the question includes the study of the requirement on jumps. In this paper, again we confine ourselves to the case where Dirichlet forms are derived from resistance forms. Our proposal for a condition on jumps is the annulus comparable condition, (ACC) for short, which says that the resistance between a point and the complement of a ball is comparable with the resistance between a point and an annulus. More exactly, (ACC) is formulated as

$$R(x, B_R(x,r)^c) \asymp R(x, A_R(x,r,(1+\epsilon)r)) \tag{1.7}$$

on $(x,r) \in X \times (0,+\infty)$ for some $\epsilon > 0$, where R is a resistance metric, $B_R(x,r)$ is a resistance ball and $A_R(x,r,(1+\epsilon)r) = \overline{B_R(x,(1+\epsilon)r)} \backslash B_R(x,r)$ is an annulus. ($\overline{B_R(x,s)}$ is the closure of a ball $B_R(x,r)$ with respect to the resistance metric.) If the process in question has no jump, i.e. is a diffusion process, then the quantities in the both sides of (1.7) coincide and hence (ACC) holds. As our answer to the above question, we obtain the following statement in Theorem 15.11:

THEOREM 1.1. *The following three conditions are equivalent:*
(C1) *The underlying measure μ has the volume doubling property with respect to R and* (ACC) *holds.*
(C2) *The underlying measure μ has the volume doubling property with respect to R and the so-called "Einstein relation":*

$$\text{Resistance} \times \text{Volume} \asymp \textit{Mean exit time}$$

holds for the resistance metric.
(C3) (ACC) *and* (KD) *is satisfied and there exist $\beta > 1$ and a metric d which is quasisymmetric with respect to R such that* $(\mathrm{LYD})_{\beta,d}$ *holds.*

See [**26, 50**] on the Einstein relation, which is known to be implied by the Li-Yau type (sub-)Gaussian heat kernel estimate.

Our work on heat kernel estimates is largely inspired by the previous two papers [**6**] and [**41**]. In [**6**], the strongly recurrent random walk on an infinite graph has been studied by using two different metrics, one is the shortest path metric d on the graph and the other is the resistance metric R. In [**6**], the condition R(β), that is,

$$\mathrm{R}(\beta) \qquad\qquad R(x,y)V_d(x,d(x,y)) \asymp d(x,y)^\beta$$

has been shown to be essentially equivalent to the random walk version of (1.1). Note the resemblance between (1.2) and R(β). The metric d is however fixed in their case. In [**41**], Kumagai has studied the (strongly recurrent) diffusion process associated with a resistance form using the resistance metric R. He has shown that the uniform volume doubling property with respect to R is equivalent to the combination of natural extensions of $(\mathrm{LYU})_{\beta,d}$ and $(\mathrm{NDL})_{\beta,d}$ with respect to R. See the remark after Theorem 15.10 for details. Examining those results carefully from geometrical view point, we have realized that quasisymmetric change of metrics (implicitly) plays an important role. In this respect, this paper can be though of an extension of those works.

There is another closely related work. In [**39**], a problem which is very similar to our question (I) has been investigated for a heat kernel associated with a self-similar Dirichlet form on a self-similar set. The result in [**39**] is also quite similar to ours. It has been shown that the volume doubling property of the underlying measure is equivalent to the existence of a metric with $(\mathrm{LYD})_{\beta,d}$. Note that the results in [**39**] include higher dimensional Sierpinski carpets where the self-similar Dirichlet forms are not induced by resistance forms. The processes studied in [**39**], however, have been all diffusions

Finally, we present an application of our results to an α-stable process on $\mathbb{R}$ for $\alpha \in (1,2]$. Define

$$\mathcal{E}^{(\alpha)}(u,u) = \int_{\mathbb{R}^2} \frac{(u(x)-u(y))^2}{|x-y|^{1+\alpha}} dxdy$$

and $\mathcal{F}^{(\alpha)} = \{u|\mathcal{E}^{(\alpha)}(u,u) < +\infty\}$ for $\alpha \in (1,2)$ and $(\mathcal{E}^{(2)},\mathcal{F}^{(2)})$ is the ordinary Dirichlet form associated with the Brownian motion on $\mathbb{R}$. Then $(\mathcal{E}^{(\alpha)},\mathcal{F}^{(\alpha)})$ is a resistance form for $\alpha \in (1,2]$ and the associated resistance metric is $c|x-y|^{\alpha-1}$. If $\alpha \neq 2$, then the corresponding process is not a diffusion but has jumps. Let $p^{(\alpha)}(t,x,y)$ be the associated heat kernel. (We will show the existence of the heat kernel $p^{(\alpha)}(t,x,y)$ in Section 16.) It is well known that $p^{(\alpha)}(t,x,x) = ct^{1/\alpha}$. Let $(\mathcal{E}^{(\alpha)}|_K,\mathcal{F}^{(\alpha)}|_K)$ be the trace of $(\mathcal{E}^{(\alpha)},\mathcal{F}^{(\alpha)})$ onto the ternary Cantor set K. Let $p_K^{(\alpha)}(t,x,y)$ be the heat kernel associated with the Dirichlet form on $L^2(K,\nu)$ induced by $(\mathcal{E}^{(\alpha)}|_K,\mathcal{F}^{(\alpha)}|_K)$, where ν is the normalized Hausdorff measure of K. By Theorem 15.13, we confirm that (ACC) holds and obtain

$$p_K^{(\alpha)}(t,x,x) \asymp t^{-\eta},$$

where $\eta = \frac{\log 2}{(\alpha-1)\log 3 + \log 2}$. See Section 16 for details.

This paper consists of four parts. In Part 1, we will develop basic theory of resistance forms regarding the Green function, trace of a form, regularity and heat kernels. This part is the foundation of the discussion in Part 3. Part 2 is devoted to studying quasisymmetric homeomorphisms. This is another foundation of the discussion in Part 3. After preparing those basics, we will consider heat kernel estimates in Part 3. Finally in Part 4, we consider estimates of heat kernels on random Sierpinski gaskets as an application of the theorems in Part 3.

The followings are conventions in notations in this paper.
(1) Let f and g be functions with variables $x_1, \ldots, x_n$. We use "$f \asymp g$ for any $(x_1, \ldots, x_n) \in A$" if and only if there exist positive constants c_1 and c_2 such that

$$c_1 f(x_1, \ldots, x_n) \leq g(x_1, \ldots, x_n) \leq c_2 f(x_1, \ldots, x_n)$$

for any $(x_1, \ldots, x_n) \in A$.
(2) The lower case c (with or without a subscript) represents a constant which is independent of the variables in question and may have different values from place to place (even in the same line).

Part 1

Resistance forms and heat kernels

In this part, we will establish basics of resistance forms such as the Green function, harmonic functions, traces and heat kernels. In the previous papers [**36, 34, 37**], we have established the notions of the Green function, harmonic functions and traces if a boundary is a finite set. One of the main subjects is to extend those results to the case where a boundary is an infinite set. In fact, we should determine what kind of an infinite set can be regarded as a proper boundary in the first place. To do so, we introduce a new topology determined by the domain of a resistance form and show that the closed set with respect to this new topology can be regarded as a boundary. Moreover, we will establish the existence of jointly continuous heat kernel associated with the Dirichlet form derived from a resistance form under several mild assumptions, which do not include the ultracontractivity. Recall that if the transition semigroup associated with a Dirichlet form is ultracontractive, then there exists a (pointwise) integral kernel of the transition semigroup, which will give a heat kernel after necessary modifications. See [**18**, Lemma 2.1.2], [**22**, Lemma 3.2] and [**5**, Theorem 2.1] for details.

The followings are basic notations used in this paper.

NOTATION. (1) For a set V, we define $\ell(V) = \{f|f : V \to \mathbb{R}\}$. If V is a finite set, $\ell(V)$ is considered to be equipped with the standard inner product $(\cdot,\cdot)_V$ defined by $(u,v)_V = \sum_{p\in V} u(p)v(p)$ for any $u, v \in \ell(V)$. Also $|u|_V = \sqrt{(u,u)_V}$ for any $u \in \ell(V)$.
(2) Let V be a finite set. The characteristic function χ_U^V of a subset $U \subseteq V$ is defined by

$$\chi_U^V(q) = \begin{cases} 1 & \text{if } q \in U, \\ 0 & \text{otherwise.} \end{cases}$$

If no confusion can occur, we write χ_U instead of χ_U^V. If $U = \{p\}$ for a point $p \in V$, we write χ_p instead of $\chi_{\{p\}}$. If $H : \ell(V) \to \ell(V)$ is a linear map, then we set $H_{pq} = (H\chi_q)(p)$ for $p, q \in V$. Then $(Hf)(p) = \sum_{q\in V} H_{pq}f(q)$ for any $f \in \ell(V)$.
(3) Let (X, d) be a metric space. Then

$$B_d(x,r) = \{y|y \in X, d(x,y) < r\}$$

for $x \in X$ and $r > 0$.

2. Topology associated with a subspace of functions

In this section, we will introduce an operation $B \to B^{\mathcal{F}}$ from subsets of a space X to itself associated with a linear subspace $\mathcal{F}$ of real valued functions $\ell(X)$. This operation will turn out to be essential in describing whether a set can be treated as a boundary or not. More precisely, we will show in Theorems 4.1 and 4.3 that a set B can be a proper boundary if and only if $B^{\mathcal{F}} = B$. Also the importance of the condition that $B^{\mathcal{F}} = B$ is revealed in Theorem 6.3 where we have equivalent conditions for regularity of resistance forms.

DEFINITION 2.1. Let $\mathcal{F}$ be a linear subspace of $\ell(X)$ for a set X. For a subset $B \subseteq X$, define

$$\mathcal{F}(B) = \{u|u \in \mathcal{F}, u(x) = 0 \text{ for any } x \in B\}.$$

and

$$B^{\mathcal{F}} = \bigcap_{u\in\mathcal{F}(B)} u^{-1}(0)$$

The following lemma is immediate from the definition.

LEMMA 2.2. *Let $\mathcal{F}$ be a linear subspace of $\ell(X)$ for a set X.*
(1) *For any $B \subseteq X$, $B \subseteq B^{\mathcal{F}}$, $\mathcal{F}(B) = \mathcal{F}(B^{\mathcal{F}})$ and $(B^{\mathcal{F}})^{\mathcal{F}} = B^{\mathcal{F}}$.*
(2) *$X^{\mathcal{F}} = X$.*
(3) *$\emptyset^{\mathcal{F}} = \emptyset$ if and only if $\{u(x)|u \in \mathcal{F}\} = \mathbb{R}$ for any $x \in X$.*

The above lemma suggests that the operation $B \to B^{\mathcal{F}}$ satisfies the axiom of closure and hence it defines a topology on X. Indeed, this is the case if $\mathcal{F}$ is stable under the unit contraction.

DEFINITION 2.3. (1) For $u : X \to \mathbb{R}$, define $\overline{u} : X \to [0, 1]$ by

$$\overline{u}(p) = \begin{cases} 1 & \text{if } u(p) \geq 1, \\ u(p) & \text{if } 0 < u(p) < 1, \\ 0 & \text{if } u(p) \leq 0. \end{cases}$$

$\overline{u}$ is called the unit contraction of u.
(2) Let $\mathcal{F}$ be a linear subspace of $\ell(X)$ for a set X. $\mathcal{F}$ is said to be stable under the unit contraction if and only if $\overline{u} \in \mathcal{F}$ for any $u \in \mathcal{F}$.

In the case of Dirichlet forms, the condition that the domain is stable under the unit contraction is one of the equivalent conditions of the Markov property. See [**21**, Section 1.1] for details. (In [**21**], their terminology is that "the unit contraction operates on $\mathcal{F}$" in place of that "$\mathcal{F}$ is stable under the unit contraction".)

THEOREM 2.4. *Let $\mathcal{F}$ be a linear subspace of $\ell(X)$ for a set X. Assume that $\{u(x)|u \in \mathcal{F}\} = \mathbb{R}$ for any $x \in X$ and that $\mathcal{F}$ is stable under the unit contraction. Define*

$$\mathcal{C}_{\mathcal{F}} = \{B|B \subseteq X, B^{\mathcal{F}} = B\}.$$

Then $\mathcal{C}_{\mathcal{F}}$ satisfies the axiom of closed sets and it defines a topology of X. Moreover, the T_1-axiom of separation holds for this topology, i.e. $\{x\}$ is a closed set for any $x \in X$, if and only if, for any $x, y \in X$ with $x \neq y$, there exists $u \in \mathcal{F}$ such that $u(x) \neq u(y)$.

As a topology given by a family of real-valued function, the notion of "fine" topology has been introduced in classical axiomatic potential theory by Brelot [**13**]. In the case of resistance forms, we will see that the topology given by $\mathcal{C}_{\mathcal{F}}$ coincides with the fine topology associated with the cone of nonnegative functions in the domain of resistance form in Theorem 5.7. Our proof of Theorem 5.7, however, depends essentially on Theorem 4.3, where the condition $B^F = B$ has already played a crucial role. So the coincidence of $\mathcal{C}_{\mathcal{R}}$ and the fine topology does give us small help in studying resistance forms. See the comments after Theorem 5.7 for details.

The rest of this section is devoted to proving the above theorem.

LEMMA 2.5. *Under the assumptions of Theorem 2.4, if $B \in \mathcal{C}_{\mathcal{F}}$ and $x \in X \backslash B$, then there exists $u \in \mathcal{F}$ such that $u \in \mathcal{F}(B)$, $u(x) = 1$ and $0 \leq u(y) \leq 1$ for any $y \in X$.*

PROOF. Since $B^{\mathcal{F}} = B$, there exists $v \in \mathcal{F}(B)$ such that $v(x) \neq 0$. Let $u = \overline{v/v(x)}$. Then u satisfies the required properties. $\square$

Proof of Theorem 2.4. First we show that $\mathcal{C}_{\mathcal{F}}$ satisfies the axiom of closed sets. Since $\mathcal{F}(X) = \{0\}$, $X^{\mathcal{F}} = X$. Also we have $\emptyset^{\mathcal{F}} = \emptyset$ by Lemma 2.2-(3). Let $B_i \in \mathcal{C}_{\mathcal{F}}$ for $i = 1, 2$ and let $x \in (B_1 \cup B_2)^c$, where A^c is the complement of A in X, i.e. $A^c = X \backslash A$. By Lemma 2.5, there exists $u_i \in \mathcal{F}(B_i)$ such that $\overline{u_i} = u_i$ and $u_i(x) = 1$. Let $v = u_1 + u_2 - 1$. Then $v(x) = 1$ and $v(y) \leq 0$ for any $y \in B_1 \cup B_2$. If $u = \overline{v}$, then $u \in \mathcal{F}(B_1 \cup B_2)$ and $u(x) = 1$. Hence $B_1 \cup B_2 \in \mathcal{C}_{\mathcal{F}}$. Let $B_\lambda \in \mathcal{C}_{\mathcal{F}}$ for any $\lambda \in \Lambda$. Set $B = \cap_{\lambda\in\Lambda} B_\lambda$. If $x \notin B$, then there exists $\lambda_* \in \Lambda$ such that $x \notin B_{\lambda_*}$. We have $u \in \mathcal{F}(B_\lambda) \subseteq \mathcal{F}(B)$ satisfying $u(x) \neq 0$. Hence $x \notin B^{\mathcal{F}}$. This shows $B \in \mathcal{C}_{\mathcal{F}}$. Thus $\mathcal{C}_{\mathcal{F}}$ satisfies the axiom of closed sets.

Next define $U_{x,y} = \left\{\left(\begin{smallmatrix} f(x) \\ f(y)\end{smallmatrix}\right)\Big| f \in \mathcal{F}\right\}$. We will show that $U_{x,y} = \mathbb{R}^2$ if there exists $u \in \mathcal{F}$ such that $u(x) \neq u(y)$. Suppose that $u(x) \neq 0$. Considering $u/u(x)$, we see that $\left(\begin{smallmatrix} 1 \\ a\end{smallmatrix}\right) \in U_{x,y}$, where $a \neq 1$. Since there exists $v \in \mathcal{F}$ with $v(y) \neq 0$, it follows that $\left(\begin{smallmatrix} b \\ 1\end{smallmatrix}\right) \in U_{x,y}$ for some $b \in \mathbb{R}$. Now we have five cases.
Case 1: Assume that $a \leq 0$. Considering the operation $u \to \overline{u}$, we have $\left(\begin{smallmatrix} 1 \\ 0\end{smallmatrix}\right) \in U_{x,y}$. Also $\left(\begin{smallmatrix} b \\ 1\end{smallmatrix}\right) \in U_{x,y}$. Since $U_{x,y}$ is a linear subspace of $\mathbb{R}^2$, $U_{x,y} = \mathbb{R}^2$.
Case 2: Assume that $b \leq 0$. By similar arguments as Case 1, we have $U_{x,y} = \mathbb{R}^2$.
Case 3: Assume that $b \geq 1$. The $\overline{u}$-operation shows that $\left(\begin{smallmatrix} 1 \\ 1\end{smallmatrix}\right) \in U_{x,y}$. Since $\left(\left(\begin{smallmatrix} 1 \\ 1\end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ a\end{smallmatrix}\right)\right)$ is independent, $U_{x,y} = \mathbb{R}^2$.
Case 4: Assume that $a \in (0, 1)$ and $b \in (0, 1)$. Then $\left(\left(\begin{smallmatrix} 1 \\ a\end{smallmatrix}\right), \left(\begin{smallmatrix} b \\ 1\end{smallmatrix}\right)\right)$ is independent. Hence $U_{x,y} = \mathbb{R}^2$.
Case 5: Assume that $a > 1$ and $b \in (0, 1)$. The $\overline{u}$-operation shows $\left(\begin{smallmatrix} 1 \\ 1\end{smallmatrix}\right) \in U_{x,y}$. Then $\left(\left(\begin{smallmatrix} 1 \\ 1\end{smallmatrix}\right), \left(\begin{smallmatrix} b \\ 1\end{smallmatrix}\right)\right)$ is independent and hence $U_{x,y} = \mathbb{R}^2$.
Thus $U_{x,y} = \mathbb{R}^2$ in all the cases. Exchanging x and y, we also deduce the same conclusion even if $u(x) = 0$. In particular, the fact that $U_{x,y} = \mathbb{R}^2$ implies that $y \notin \{x\}^{\mathcal{F}}$. Hence if there exists $u \in \mathcal{F}$ such that $u(x) \neq u(y)$ for any $x, y \in X$ with $x \neq y$, then $\{x\} \in \mathcal{C}_{\mathcal{F}}$ for any $x \in X$. The converse direction is immediate. $\square$

3. Basics on resistance forms

In this section, we first introduce definition and basics on resistance forms.

Definition 3.1 (Resistance form). Let X be a set. A pair $(\mathcal{E}, \mathcal{F})$ is called a resistance form on X if it satisfies the following conditions (RF1) through (RF5).
(RF1) $\mathcal{F}$ is a linear subspace of $\ell(X)$ containing constants and $\mathcal{E}$ is a non-negative symmetric quadratic form on $\mathcal{F}$. $\mathcal{E}(u, u) = 0$ if and only if u is constant on X.
(RF2) Let $\sim$ be an equivalent relation on $\mathcal{F}$ defined by $u \sim v$ if and only if $u - v$ is constant on X. Then $(\mathcal{F}/\!\sim, \mathcal{E})$ is a Hilbert space.
(RF3) If $x \neq y$, then there exists $u \in \mathcal{F}$ such that $u(x) \neq u(y)$.
(RF4) For any $p, q \in X$,

$$\sup\left\{\frac{|u(p) - u(q)|^2}{\mathcal{E}(u, u)}\Big| u \in \mathcal{F}, \mathcal{E}(u, u) > 0\right\}$$

is finite. The above supremum is denoted by $R_{(\mathcal{E},\mathcal{F})}(p, q)$.
(RF5) $\overline{u} \in \mathcal{F}$ and $\mathcal{E}(\overline{u}, \overline{u}) \leq \mathcal{E}(u, u)$ for any $u \in \mathcal{F}$, where $\overline{u}$ is defined in Definition 2.3.

Note that the definition of resistance forms does not require any measure on the space X at all. Being combined with a measure μ, a resistance form may induce a Dirichlet form on $L^2(X, \mu)$ and the associated process, semigroup and Laplacian. See Section 9.

By (RF3) and (RF5) along with Theorem 2.4, the axiom of closed sets holds for $\mathcal{C}_{\mathcal{F}}$ and the associated topology satisfies the T_1-separation axiom.

PROPOSITION 3.2. *Assume that $\overline{u} \in \mathcal{F}$ for any $u \in \mathcal{F}$. Then* (RF3) *in the above definition is equivalent to the following conditions:*
(RF3-1) $F^{\mathcal{F}} = F$ *for any finite subset* $F \subseteq X$.
(RF3-2) *For any finite subset $F \subset X$ and any $v \in \ell(F)$, there exists $u \in \mathcal{F}$ such that $u|_F = v$.*

PROOF. (RF3) $\Rightarrow$ (RF3-1) By Theorem 2.4, (RF3) implies that $\{x\}^{\mathcal{F}} = \{x\}$ for any $x \in X$. Let F be a finite subset of X. Again by Theorem 2.4, $F^{\mathcal{F}} = (\cup_{x \in F}\{x\})^{\mathcal{F}} = \cup_{x \in F}\{x\}^{\mathcal{F}} = F$.
(RF3-1) $\Rightarrow$ (RF3-2) Let F be a finite subset of X. Set $F_x = F \backslash \{x\}$ for $x \in F$. Since $(F_x)^{\mathcal{F}} = F_x$, there exists $u_x \in \mathcal{F}$ such that $u_x|_{F_x} \equiv 0$ and $u_x(x) = 1$. For any $v \in \ell(F)$, define $u = \sum_{x \in F} v(x)u_x$. Then $u|_F = v$ and $u \in \mathcal{F}$.
(RF3-2) $\Rightarrow$ (RF3) This is obvious. $\square$

REMARK. In the previous literatures [**36, 34, 37**], (RF3-2) was employed as a part of the definition of resistance forms in place of the current (RF3).

By the results in [**36**, Chapter 2], we have the following fact.

PROPOSITION 3.3. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X. The supremum in* (RF4) *is the maximum and $R_{(\mathcal{E},\mathcal{F})}$ is a metric on X.*

DEFINITION 3.4. Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X. $R_{(\mathcal{E},\mathcal{F})}$ is called the resistance metric on X associated with the resistance form $(\mathcal{E}, \mathcal{F})$.

One of the most simple examples of resistance forms is weighted graphs.

EXAMPLE 3.5 (Weighted graph). Let V be an (infinite) countable set and let $H = \{H_{xy}\}_{x,y \in X}$ satisfy the following three conditions (WG1), (WG2) and (WG3).

(WG1) $H_{xy} = H_{yx} \geq 0$ and $H_{xx} = 0$ for any $x, y \in X$.
(WG2) $N(x) = \{y | H_{xy} > 0\}$ is a finite set for any $x \in X$.
(WG3) For any $x, y \in X$, there exist $x_1, \ldots, x_n \in X$ such that $x_1 = x$, $x_n = y$ and $H_{x_i x_{i+1}} > 0$ for any $i = 1, \ldots, n-1$.
Then (V, H) is called a (locally finite irreducible) weighted graph. (The condition (WG3) is called the locally finiteness and the condition (WG4) is called the irreducibility.) Define

$$\mathcal{F}_{(V,H)} = \Big\{u \Big| u \in \ell(V), \sum_{x,y \in V} H_{xy}(u(x) - u(y))^2 < +\infty\Big\}$$

and, for $u, v \in \mathcal{F}_{(V,H)}$,

$$\mathcal{E}_{(V,H)}(u, v) = \frac{1}{2} \sum_{x,y \in V} H_{xy}(u(x) - u(y))(v(x) - v(y)).$$

Then $(\mathcal{E}_{(V,H)}, \mathcal{F}_{(V,H)})$ is a resistance form on V. There exists a random walk on V associated with the weight graph (V, H). Namely define $\mu_x = \sum_{y \in N(x)} H_{xy}$ and $P(x, y) = H_{xy}/\mu_x$. We give the transition probability from x to y in the unit time by $P(x, y)$. This random walk is called the random walk associated with (V, H).

Relations between $(\mathcal{E}_{(V,H)}, \mathcal{F}_{(V,H)})$ and the random walk associated with (V, H) have been one of the main subjects in the theory of random walks and discrete potential theory. See [**52, 53**], [**43**] and [**49**] for example. In particular, as we mentioned in the introduction, asymptotic behaviors of the heat kernel associated with the random walk has been studied in [**6**].

Note that in the above example, the set V is countable. This is not the case for general resistance forms. For example, we have the resistance forms on $\mathbb{R}$ associated with α-stable process defined in the introduction and the resistance forms on the (random) Sierpinski gaskets in Part 4.

If no confusion can occur, we use R to denote $R_{(\mathcal{E},\mathcal{F})}$. By (RF4), we immediately obtain the following fact.

Proposition 3.6. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X and let R be the associated resistance metric. For any $x, y \in X$ and any $u \in \mathcal{F}$,*

$$|u(x) - u(y)|^2 \leq R(x, y)\mathcal{E}(u, u). \tag{3.1}$$

In particular, $u \in \mathcal{F}$ is continuous with respect to the resistance metric.

Next we introduce the notion of Laplacians on a finite set and harmonic functions with a finite set as a boundary. See [**36**, Section 2.1] for details, in particular, the proofs of Proposition 3.8 and 3.10.

Definition 3.7. Let V be a non-empty finite set. Recall that $\ell(V)$ is equipped with the standard inner-product $(\cdot, \cdot)_V$. A symmetric linear operator $L : \ell(V) \to \ell(V)$ is called a Laplacian on V if it satisfies the following three conditions:

(L1) L is non-positive definite,

(L2) $Lu = 0$ if and only if u is a constant on V,

(L3) $L_{pq} \geq 0$ for all $p \neq q \in V$.

We use $\mathcal{LA}(V)$ to denote the collection of Laplacians on V.

By [**36**, Proposition 2.1.3], we have the next proposition, which says that a resistance form on a finite set corresponds to a Laplacian.

Proposition 3.8. *Let V be a non-empty finite set and let L be a symmetric linear operator form $\ell(V)$ to itself. Define a symmetric bilinear form $\mathcal{E}_L$ on $\ell(V)$ by $\mathcal{E}_L(u, v) = -(u, Lv)_V$ for any $u, v \in \ell(V)$. Then, $\mathcal{E}_L$ is a resistance form on V if and only if $L \in \mathcal{LA}(V)$.*

Using the standard inner-product $(\cdot, \cdot)_V$, we implicitly choose the uniform distribution, i.e. the sum of all the Dirac masses on the space, as our measure on V. This is why we may relate Laplacians with resistance forms on a finite set without mentioning any measure explicitly. (Recall the comment after Definition 3.1.)

Definition 3.9. Let V be a finite set and let $L \in \mathcal{LA}(V)$. The resistance form $(\mathcal{E}_L, \ell(V))$ on V is called the resistance form associated with L.

The harmonic function with a finite set as a boundary is defined as the energy minimizing function.

Proposition 3.10. [**36**, Lemma 2.3.5] *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X and let V be a finite subset of X. Let $\rho \in \ell(V)$. Then there exists a unique $u \in \mathcal{F}$ such that $u|_V = \rho$ and u attains the following minimum:*

$$\min\{\mathcal{E}(v, v) | v \in \mathcal{F}, v|_V = \rho\}.$$

Moreover, the map from ρ to u is a linear map from $\ell(V)$ to $\mathcal{F}$. Denote this map by h_V. Then there exists a Laplacian $L \in \mathcal{LA}(V)$ such that

$$\mathcal{E}_L(\rho, \rho) = \mathcal{E}(h_V(\rho), h_V(\rho)). \tag{3.2}$$

In Lemma 8.2, we are going to extend this proposition when V is an infinite set.

DEFINITION 3.11. $h_V(\rho)$ defined in Proposition 3.10 is called the V-harmonic function with the boundary value ρ. Also we denote the above $L \in \mathcal{LA}(V)$ in (3.2) by $L_{(\mathcal{E},\mathcal{F}),V}$.

To construct concrete examples of resistance forms, we often start from a sequence of resistance forms on finite sets which has certain compatibility as follows.

DEFINITION 3.12. Let $\{V_m\}_{m\geq 0}$ be a sequence of finite sets and let $L_m \in \mathcal{LA}(V_m)$ for $m \geq 0$. $\{(V_m, L_m)\}_{m\geq 0}$ is called a compatible sequence if and only if $L_{(\mathcal{E}_{L_{m+1}},\ell(V_{m+1})),V_m} = L_m$ for any $m \geq 0$, i.e.

$$\mathcal{E}_{L_m}(u, u) = \min\{\mathcal{E}_{L_{m+1}}(v, v) | v \in \ell(V_{m+1}), u = v|_{V_m}\}$$

for any $u \in \ell(V_m)$.

Combining the results in [**36**, Sections 2.2 and 2.3], in particular [**36**, Theorems 2.2.6 and 2.3.10], we obtain the following theorem on construction of a resistance form from a compatible sequence.

THEOREM 3.13. *Let $\{V_m\}_{m\geq 0}$ be a sequence of finite sets and let $L_m \in \mathcal{LA}(V_m)$ for $m \geq 0$. Assume that $\mathcal{S} = \{(V_m, L_m)\}_{m\geq 0}$ is a compatible sequence. Let $V_* = \cup_{m\geq 0} V_m$. Define*

$$\mathcal{F}_{\mathcal{S}} = \{u | u \in \ell(V_*), \lim_{m\to+\infty} \mathcal{E}_{L_m}(u|_{V_m}, u|_{V_m}) < +\infty\}$$

and

$$\mathcal{E}_{\mathcal{S}}(u, v) = \lim_{m\to+\infty} \mathcal{E}_{L_m}(u|_{V_m}, v|_{V_m})$$

for any $u, v \in \mathcal{F}_{\mathcal{S}}$. Then $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$ is a resistance form on V_* and $L_{(\mathcal{E}_{\mathcal{S}},\mathcal{F}_{\mathcal{S}}),V_m} = L_m$ for any $m \geq 0$. Moreover, let $R_{\mathcal{S}}$ be the resistance metric associated with $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$ and let (X, R) be the completion of $(V_*, R_{\mathcal{S}})$. Then there exists a unique resistance form $(\mathcal{E}, \mathcal{F})$ on X such that, for any $u \in \mathcal{F}$, u is a continuous function on X, $u|_{V_*} \in \mathcal{F}_{\mathcal{S}}$ and $\mathcal{E}(u, u) = \mathcal{E}_{\mathcal{S}}(u|_{V_*}, u|_{V_*})$. In particular, R coincides with the resistance metric associated with $(\mathcal{E}, \mathcal{F})$.

We are going to use this theorem to construct resistance forms in Example 5.5 and Section 22.

In contrast with the above theorem, if (X, R) is separable, the resistance form $(\mathcal{E}, \mathcal{F})$ is always expressed as a limit of a compatible sequence. More precisely, the following fact has been shown in [**36**, Section 2.3].

THEOREM 3.14. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a set X and let R be the associated resistance metric. Let $\{V_m\}_{m\geq 1}$ be an increasing sequence of finite subsets of X. Assume that $V_* = \cup_{m\geq 1} V_m$ is dense in X. Set $L_m = L_{(\mathcal{E},\mathcal{F}),V_m}$ where $L_{(\mathcal{E},\mathcal{F}),V_m}$ is defined in Definition 3.11. Then $\{(V_m, L_m)\}_{m\geq 1}$ is a compatible sequence. Moreover,*

$$\mathcal{F} = \{u | u \in C(X, R), \lim_{m\to+\infty} \mathcal{E}_{L_m}(u|_{V_m}, u|_{V_m}) < +\infty\},$$

where $C(X,R)$ is the collection of real-valued continuous functions with respect to the resistance metric, and

$$\mathcal{E}(u,v) = \lim_{m\to+\infty} \mathcal{E}_{L_m}(u|_{V_m}, v|_{V_m})$$

for any $u, v \in \mathcal{F}$.

By using this theorem, we have the following fact, which is used in Section 5.

Proposition 3.15. *Let $(\mathcal{E},\mathcal{F})$ be a resistance form on a set X and let R be the resistance metric. Assume that (X,R) is separable. For $u : X \to \mathbb{R}$, define $u_+ : \mathbb{R} \to [0,+\infty)$ by $u_+(x) = \max\{u(x), 0\}$. Then $u_+ \in \mathcal{F}$ and $\mathcal{E}(u_+,u_+) \le \mathcal{E}(u,u)$ for any $u \in \mathcal{F}$.*

Proof. Since (X,R) is separable, we may choose a sequence $\{V_m\}_{m\ge0}$ of subsets of X so that $V_m \subseteq V_{m+1}$ for any $m \ge 0$ and $V_* = \cup_{m\ge0} V_m$ is dense in (X,R). Let L_m as in Theorem 3.14. Then $\mathcal{E}_{L_m}(u,u) = \sum_{x,y\in V_m}(L_m)_{xy}(u(x)-u(y))^2/2$ for any $u \in \ell(V_m)$. Since $(u_+(x)-u_+(y))^2 \le (u(x)-u(y))^2$, it follows that $\mathcal{E}_{L_m}(u_+|_{V_m}, u_+|_{V_m}) \le \mathcal{E}_{L_m}(u|_{V_m}.u|_{V_m})$. Hence by Theorem 3.14, we see that $u_+ \in \mathcal{F}$ and $\mathcal{E}(u_+,u_+) \le \mathcal{E}(u,u)$. $\square$

4. the Green function

In this section, we study the Green function associated with an infinite set as a boundary. In the course of discussion, we will show that a set B is a suitable boundary if and only if $B^{\mathcal{F}} = B$. Conditions ensuring $B^{\mathcal{F}} = B$ are given in the next section. For example, if B is compact with respect to the resistance metric, then $B^{\mathcal{F}}$ will be shown to coincide with B.

Throughout this section, $(\mathcal{E},\mathcal{F})$ is a resistance form on a set X and R is the associated resistance metric. The next theorem establishes the existence and basic properties of the Green function with an infinite set as a boundary.

Theorem 4.1. *Let $B \subseteq X$ be non-empty. Then $(\mathcal{E},\mathcal{F}(B))$ is a Hilbert space and there exists a unique $g_B : X \times X \to \mathbb{R}$ that satisfies the following condition* (GF1)*:*

(GF1) *Define $g_B^x(y) = g_B(x,y)$. For any $x \in X$, $g_B^x \in \mathcal{F}(B)$ and $\mathcal{E}(g_B^x, u) = u(x)$ for any $u \in \mathcal{F}(B)$.*

Moreover, g_B satisfies the following properties (GF2), (GF3) *and* (GF4)*:*

(GF2) *$g_B(x,x) \ge g_B(x,y) = g_B(y,x) \ge 0$ for any $x,y \in X$. $g_B(x,x) > 0$ if and only if $x \notin B^{\mathcal{F}}$.*

(GF3) *Define $R(x,B) = g_B(x,x)$ for any $x \in X$. If $x \notin B^{\mathcal{F}}$, then*

$$\begin{aligned} R(x,B) &= \big(\min\{\mathcal{E}(u,u) | u \in \mathcal{F}(B), u(x) = 1\}\big)^{-1} \\ &= \sup\left\{\frac{|u(x)|^2}{\mathcal{E}(u,u)}\middle| u \in \mathcal{F}(B), u(x) \neq 0\right\}. \end{aligned}$$

(GF4) *For any $x,y,z \in X$, $|g_B(x,y) - g_B(x,z)| \le R(y,z)$.*

By (GF2), if $B \neq B^{\mathcal{F}}$, then $g_B^x \equiv 0$ for any $x \in B^{\mathcal{F}}\backslash B$. Such a set B is not a good boundary.

We will prove this and the next theorem at the same time.

Definition 4.2. The function $g_B(\cdot,\cdot)$ given in the above theorem is called the Green function associated with the boundary B or the B-Green function.

As we have remarked after Definition 3.1, no measure is required to define a resistance form. Thus the definition of the Green function is also independent of a choice of measures. After introducing a measure μ and constructing the Dirichlet form on $L^2(X,\mu)$ induced by a resistance form in Section 8, we may observe that the Green function $g_B(x,y)$ defined above coincides with the Green function associated with the Dirichlet form. See Theorem 10.10 and Corollary 10.11 for details. For example, $g_B(x,y)$ is shown to be the integral kernel of the nonnegative self-adjoint operator associated with the Dirichlet form.

The next theorem assures another advantage of being $B = B^{\mathcal{F}}$. Namely, if $B = B^{\mathcal{F}}$, we may reduce B to a one point, consider the "shorted" resistance form $(\mathcal{E}, \mathcal{F}^B)$ and obtain a expression of the Green function (4.1) by the "shorted" resistance metric $R_B(\cdot,\cdot)$. In the case of weighted graphs, such an identification of a subset of domain is called Rayleigh's shorting method. See [**19**] for details.

THEOREM 4.3. *Let $(\mathcal{E},\mathcal{F})$ be a resistance form and let $B \subseteq X$ be non-empty. Suppose that $B^{\mathcal{F}} = B$. Set*

$$\mathcal{F}^B = \{u | u \in \mathcal{F}, u \text{ is a constant on } B\}$$

and $X_B = \{B\} \cup (X \backslash B)$. Then $(\mathcal{E}, \mathcal{F}^B)$ is a resistance form on X_B. Furthermore, if $R_B(\cdot,\cdot)$ is the resistance metric associated with $(\mathcal{E},\mathcal{F}^B)$, then

$$g_B(x,y) = \frac{R_B(x,B) + R_B(y,B) - R_B(x,y)}{2} \tag{4.1}$$

for any $x, y \in X$. In particular, $R(x,B) = R_B(x,B)$ for any $x \in X\backslash B$.

REMARK. In [**45**, Section 3], V. Metz has shown (4.1) in the case where B is a one point.

The proofs of the those two theorems are divided into several parts.

Note that by Proposition 5.1 B is closed with respect to R if $B^{\mathcal{F}} = B$.

PROOF OF THE FIRST HALF OF THEOREM 4.1. Let $x \in B$ and let $\mathcal{F}(x) = \mathcal{F}(\{x\})$. By (RF2), $(\mathcal{E},\mathcal{F}(x))$ is a Hilbert space. Note that $\mathcal{F}(B) \subseteq \mathcal{F}(x)$. If $\{u_m\}_{m\geq 1}$ is a Cauchy sequence in $\mathcal{F}(B)$, there exists the limit $u \in \mathcal{F}(x)$. For $y \in B$,

$$|u_m(y) - u(y)|^2 \leq R(x,y)\mathcal{E}(u_m - u, u_m - u).$$

Letting $m \to +\infty$, we see that $u(y) = 0$. Hence $u \in \mathcal{F}(B)$. This shows that $(\mathcal{E},\mathcal{F}(B))$ is a Hilbert space. For any $z \in X$ and any $u \in \mathcal{F}(B)$, $|u(z)|^2 \leq R(x,y)\mathcal{E}(u,u)$. The map $u \to u(z)$ is continuous linear functional and hence there exists a unique $\varphi_z \in \mathcal{F}(B)$ such that $\mathcal{E}(\varphi_z, u) = u(z)$ for any $u \in \mathcal{F}(B)$. Define $g_B(z,w) = \varphi_z(w)$. Since $\mathcal{E}(\varphi_z,\varphi_w) = \varphi_z(w) = \varphi_w(z)$, we have (GF1) and $g_B(z,w) = g_B(w,z)$. If $z \in B^{\mathcal{F}}$, then $u(z) = 0$ for any $u \in \mathcal{F}(B)$. Hence $g_B(z,z) = g_B^z(z) = 0$. Conversely, assume $g_B(z,z) = 0$. Since $g_B(z,z) = \mathcal{E}(g_B^z, g_B^z)$, (RF1) implies that g_B^z is constant on X. On the other hand, $g_B^z(y) = 0$ for any $y \in B$. Hence $g_B^z \equiv 0$. For any $u \in \mathcal{F}(B)$, $u(z) = \mathcal{E}(g_B^z, u) = 0$. Therefore, $z \in B^{\mathcal{F}}$. $\square$

DEFINITION 4.4. Let $B \subseteq X$ be non-empty. If $x \notin B^{\mathcal{F}}$, we define $\psi_B^x = g_B^x / g_B(x,x)$.

Note that $g_B(x,x) > 0$ if and only if $x \notin B^{\mathcal{F}}$ by the above proof. Hence ψ_B^x is well-defined.

Lemma 4.5. *Let $B \subseteq X$ be non-empty and let $x \notin \mathcal{B}^{\mathcal{F}}$. Then ψ_B^x is the unique element which attains the following minimum:*

$$\min\{\mathcal{E}(u,u)|u \in \mathcal{F}(B), u(x) = 1\}.$$

In particular, (GF3) *holds.*

Proof. Let $u \in \mathcal{F}(B)$ with $u(x) = 1$. Since

$$\mathcal{E}(u - \psi_B^x, \psi_B^x) = \frac{\mathcal{E}(u - \psi_B^x, g_B^x)}{g_B(x,x)} = \frac{(u(x) - 1)}{g_B(x,x)} = 0,$$

we have

$$\mathcal{E}(u,u) = \mathcal{E}(u - \psi_B^x, u - \psi_B^x) + \mathcal{E}(\psi_B^x, \psi_B^x)$$

Hence $\mathcal{E}(u,u) \geq \mathcal{E}(\psi_B^x, \psi_B^x)$ and if the equality holds, then $u = \psi_B^x$. Now,

$$\mathcal{E}(\psi_B^x, \psi_B^x) = \frac{\mathcal{E}(g_B^x, g_B^x)}{g_B(x,x)^2} = \frac{1}{g_B(x,x)}.$$

This suffices for (GB3). □

Lemma 4.6. *Let $B \subseteq X$ be non-empty. Then $g_B(x,x) \geq g_B(x,y) \geq 0$ for any $x, y \in X$.*

Proof. If $x \in B^{\mathcal{F}}$, then $g_B^x \equiv 0$. Otherwise, define $v = \overline{\psi_B^x}$. Then by (RF5), $\mathcal{E}(\psi_B^x, \psi_B^x) \geq \mathcal{E}(v,v)$. The above lemma shows that $\psi_x^B = v$. Hence $0 \leq \psi_B^x \leq 1$. □

Combining Lemma 4.6 and the results from "Proof of the first half of Theorem 4.1", we have (GF2).

So far, we have obtained (GF1), (GF2) and (GF3). Before showing (GF4), we prove Theorem 4.3.

Proof of Theorem 4.3. (RF1), (RF2) and (RF5) are immediate by the definition of $\mathcal{F}^B$. To show (RF3), let x and $y \in X$ with $x \neq y$. We may assume $y \neq B$ without loss of generality. Set $B_x = B \cup \{x\}$. Since $(B_x)^{\mathcal{F}} = B_x$, there exists $u \in \mathcal{F}(B_x)$ such that $u(y) \neq 0$. Hence we obtain (RF3). To see (RF4), note that

$$\sup\left\{\frac{|u(x) - u(y)|^2}{\mathcal{E}(u,u)}\middle| u \in \mathcal{F}^B, \mathcal{E}(u,u) > 0\right\} \leq R_{(\mathcal{E},\mathcal{F})}(x,y)$$

because $\mathcal{F}^B \subseteq \mathcal{F}$. Hence we have (RF4). To prove (4.1), it is enough to show the case where B is a one point. Namely we will show that

$$g_{\{z\}}(x,y) = \frac{R(x,z) + R(y,z) - R(x,y)}{2} \tag{4.2}$$

for any $x, y, z \in X$. We write $g(x,y) = g_{\{z\}}(x,y)$. The definition of $R(\cdot,\cdot)$ along with Lemma 4.5 shows that $g(x,x) = R(x,z)$. Also by Lemma 4.5, if $u_*(y) = g(x,y)/g(x,x)$, then u_* is the $\{x,z\}$-harmonic function whose boundary values are $u_*(z) = 0$ and $u_*(x) = 1$. Let $V = \{x,y,z\}$. Then by Proposition 3.10, there exists a Laplacian $L \in \mathcal{LA}(V)$ with (3.2). Note that

$$\mathcal{E}_L(u_*|_V, u_*|_V) = \min\{\mathcal{E}_L(v,v)|v \in \ell(V), v(x) = 1, v(z) = 0\}.$$

Therefore, $(Lu_*)(y) = 0$. Set $L = (L_{pq})_{p,q \in V}$. Hereafter we assume that $L_{pq} > 0$ for any $p, q \in V$ with $p \neq q$. (If this condition fails, the proof is easier.) Let $R_{pq} = (L_{pq})^{-1}$. Solving $Lu_*(y) = 0$, we have

$$u_*(y) = \frac{L_{xy}}{L_{xy} + L_{yz}} = \frac{R_{yz}}{R_{xy} + R_{yz}}. \tag{4.3}$$

Define $R_x = R_{xy}R_{xz}/R_*, R_y = R_{yx}R_{yz}/R_*$ and $R_z = R_{zx}R_{zy}/R_*$, where $R_* = R_{xy} + R_{yz} + R_{zx}$. By the Δ-Y transform, $R(p,q) = R_p + R_q$ for any p and q with $p \neq q$. (See [**36**, Lemma 2.1.15] for "Δ-Y transform".) Hence

$$\frac{R(x,z) + R(y,z) - R(x,y)}{2} = R_z \tag{4.4}$$

Since $g(x,x) = R(x,z)$, (4.3) implies

$$g(x,y) = g(x,x)u_*(y) = R(x,z)u_*(y) = \frac{R_{xz}(R_{xy} + R_{yz})}{R_*}u_*(y) = R_z.$$

By (4.4), we have (4.2). □

Proof of (GF4) of Theorem 4.1. Let $K = B^{\mathcal{F}}$. Note that $g_B(x,y) = g_K(x,y)$. By (4.1),

$$\begin{aligned}|g_B(x,y) - g_B(x,z)| &\leq \frac{|R(y,K) - R(z,K)| + |R_K(x,y) - R_K(x,z)|}{2} \\ &\leq R_K(y,z) \leq R(y,z).\end{aligned}$$

□

5. Topologies associated with resistance forms

We continue to assume that $(\mathcal{E}, \mathcal{F})$ is a resistance form on a set X and R is the associated resistance metric. In the previous section, we have seen that the condition for a subset $B \subset X$ being a "good" boundary is that $B^{\mathcal{F}} = B$. Note that by Theorem 2.4, $\mathcal{C}_{\mathcal{F}}$ gives a topology on X which satisfies T_1-axiom of separation. Lemma 2.2 implies that $B^{\mathcal{F}} = B$ if and only if B is a closed set with respect to the $\mathcal{C}_{\mathcal{F}}$-topology. Therefore, to see when $B^{\mathcal{F}} = B$ occurs is to consider the relation between the topologies induced by the resistance metric R and $\mathcal{C}_{\mathcal{F}}$. Furthermore, there exists a classical notion of "fine topology" associated with a cone of non-negative functions introduced by Brelot in [**13**]. In this section, we will study relations of those topologies on X. The topologies induced by the resistance metric and $\mathcal{C}_{\mathcal{F}}$ are called the R-topology and the $\mathcal{C}_{\mathcal{F}}$-topology respectively.

First we show that the $\mathcal{C}_{\mathcal{F}}$-topology is coarser than the R-topology in general.

Proposition 5.1. *$B^{\mathcal{F}}$ is a closed set with respect to the resistance metric R. In other word, the $\mathcal{C}_{\mathcal{F}}$-topology is coarser (i.e. weaker) than that given by the R-topology.*

Proof. Let $\{x_n\}_{n\geq 1} \subset B^{\mathcal{F}}$. Assume $\lim_{n\to+\infty} R(x, x_n) = 0$. If $u \in \mathcal{F}(B)$, then $u(x) = \lim_{n\to+\infty} u(x_n) = 0$ for any $u \in \mathcal{F}(B)$. Hence $x \in B^{\mathcal{F}}$. □

Later in Example 5.5, we have a resistance form on $\{0, 1, \ldots\}$ where there exists an R-closed set which is not $\mathcal{C}_{\mathcal{F}}$-closed.

Next, we study a sufficient condition ensuring that $B^{\mathcal{F}} = B$.

Definition 5.2. (1) Let (X, d) be a metric space. For a non-empty subset of $B \subseteq X$, define

$$N_d(B, r) = \min\{\#(A) | A \subseteq B \subseteq \cup_{y\in A} B_d(y, r)\}$$

for any $r > 0$, where $\#(A)$ is the number of the elements of A.
(2) Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X and let R be the associated resistance metric. For any subsets $U, V \subset X$, define

$$\underline{R}(U, V) = \inf\{R(x, y) | x \in U, y \in V\}.$$

Hereafter in this section, we use $N(B, r)$ to denote $N_R(B, r)$.

The following theorem plays an important role in proving heat kernel estimates in Part III.

THEOREM 5.3. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X. Let B be a non-empty subset of X and let $x \in X \backslash B$. If $N(B, \underline{R}(x, B)/2) < +\infty$, then $x \notin B^{\mathcal{F}}$ and*

$$\frac{\underline{R}(x, B)}{4N(B, \underline{R}(x, B)/2)} \leq R(x, B) \leq \underline{R}(x, B).$$

The key idea of the following proof has been extracted from [**6**, Lemma 2.4] and [**41**, Lemma 4.1].

PROOF. Write $u_y = \psi^x_{\{y\}}$ for any $x, y \in X$. Then, by (GF4),

$$u_y(z) = u_y(z) - u_y(y) \leq \frac{R(y, z)}{R(y, x)}.$$

If $y \in B, x \in X \backslash B$ and $z \in B_R(x, \underline{R}(x, B)/2)$, then $u_y(z) \leq 1/2$. Suppose that $n = N(B, \underline{R}(x, B)/2)$ is finite. We may choose $y_1, \ldots, y_n \in B$ so that $B \subseteq \cup_{i=1}^n B_R(y_i, \underline{R}(x, B)/2)$. Define $v(z) = \min_{i=1,\ldots,n} u_{y_i}(z)$ for any $z \in X$. Then $v \in \mathcal{F}, v(x) = 1$ and $v(z) \leq 1/2$ for any $z \in B$. Letting $h = 2\overline{(v - 1/2)}$, we see that $0 \leq h(z) \leq 1$ for any $z \in X$, $h(x) = 1$ and $h \in \mathcal{F}(B)$. Hence $x \notin B^{\mathcal{F}}$. Moreover,

$$\mathcal{E}(h, h) \leq 4\mathcal{E}(v, v) \leq 4\sum_{i=1}^n \mathcal{E}(u_{y_i}, u_{y_i}) \leq 4\sum_{i=1}^n \frac{1}{R(x, y_i)} \leq \frac{4n}{\underline{R}(x, B)}.$$

Therefore,

$$R(x, B) = \big(\min\{\mathcal{E}(u, u) | u \in \mathcal{F}(B), u(x) = 1\}\big)^{-1} \geq \frac{\underline{R}(x, B)}{4n}.$$

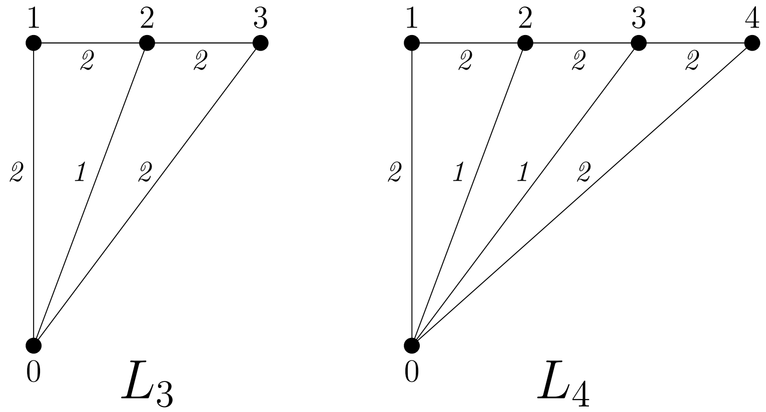
$\square$

COROLLARY 5.4. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X. If B is compact with respect to the resistance metric associated with $(\mathcal{E}, \mathcal{F})$, then $B^{\mathcal{F}} = B$.*

In general, even if (X, R) is locally compact, $B^{\mathcal{F}} \neq B$ may happen for an R-closed set B as you can see in the next example.

EXAMPLE 5.5. Let $X = \mathbb{N} \cup \{0\}$ and let $V_m = \{1, \ldots, m\} \cup \{0\}$. Define a linear operator $L_m : \ell(V_m) \to \ell(V_m)$ by

$$(L_m)_{ij} = \begin{cases} 2 & \text{if } |i - j| = 1 \text{ or } |i - j| = m, \\ 1 & \text{if } \{i, j\} = \{0, k\} \text{ for some } k \in \{1, \ldots, m\} \backslash \{1, m\}, \\ -4 & \text{if } i = j \text{ and } i \in \{1, m\}, \\ -5 & \text{if } i = j \text{ and } i \in \{1, \ldots, m\} \backslash \{1, m\}, \\ -(m+2) & \text{if } i = j = 0, \\ 0 & \text{otherwise.} \end{cases}$$



The italic number between vertices i and j is the value of $(L_m)_{i,j}$.

FIGURE 1. L_3 and L_4

See Figure 1 for L_3 and L_4. L_m is a Laplacian on V_m and $\{(V_m, L_m)\}_{m\geq 1}$ is a compatible sequence. Let $\mathcal{S} = \{(V_m, L_m)\}_{m\geq 1}$. Then by Theorem 3.13, we have a resistance form $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$ on $V_* = \cup_{m\geq 1} V_m$. Note that $V_* = X$. We use $\mathcal{E}$ and $\mathcal{F}$ instead of $\mathcal{E}_{\mathcal{S}}$ and $\mathcal{F}_{\mathcal{S}}$ respectively for ease of notation. Let R be the resistance metric associated with $(\mathcal{E}, \mathcal{F})$. Hereafter in this example, we only consider the topology induced by the resistance metric R. Using the fact that $R(i, j) = R_m(i, j)$ for $i, j \in V_m$, where R_m is the resistance metric with respect to $\mathcal{E}_{L_m}$, we may calculate $R(i, j)$ for any $i, j \in X$. As a result,

$$\begin{cases} R(0, j) & = \frac{1}{3} \quad \text{for any } j \geq 1, \\ R(i, j) & = \frac{2}{3}(1 - 2^{-|i-j|}) \quad \text{if } i, j \geq 1. \end{cases}$$

Since $1/3 \leq R(i, j) \leq 2/3$ for any $i, j \in X$ with $i \neq j$, any one point set $\{x\}$ is closed and open. In particular, (X, R) is locally compact. Let $B = \mathbb{N}$. Since B is the complement of a open set $\{0\}$, B is closed. Define $\psi \in \ell(X)$ by $\psi(0) = 1$ and $\psi(x) = 0$ for any $x \in B$. Since $\mathcal{E}_m(\psi|_{V_m}, \psi|_{V_m}) = m+2 \to +\infty$ as $m \to +\infty$, we see that $\psi \notin \mathcal{F}$. Therefore if $u \in \mathcal{F}(B)$, then $u(0) = 0$. This shows that $B^{\mathcal{F}} = B \cup \{0\}$.

Now following Brelot [**13**], we introduce the notion of fine topology associated with a cone of extended nonnegative valued functions.

DEFINITION 5.6. (1) Define $\mathbb{R}^+_{+\infty} = [0, +\infty) \cup \{+\infty\}$ and $f : \mathbb{R}^+_{+\infty} \to [0, 1]$ by $f(x) = 1 - (x + 1)^{-1}$. (Note that $f(+\infty) = 1$.) We give a metric d on $\mathbb{R}^+_{+\infty}$ by $d(a, b) = |f(x) - f(y)|$.
(2) Let X be a set and let $\Phi \subset \{f | f : X \to \mathbb{R}^+_{+\infty}\}$. Assume that Φ is a cone, i.e. $a_1 f_1 + a_2 f_2 \in \Phi$ if $a_1, a_2 \in \mathbb{R}$ and $f_1, f_2 \in \Phi$ and that $+\infty \in \Phi$, where $+\infty$ means a constant function whose value is $+\infty$. (We use a convention that $0 \cdot +\infty = 0$.) Let $\mathcal{O}$ be a topology on X, i.e. $\mathcal{O}$ is a family of sets satisfying the axiom of open sets. The coarsest topology which is finer than $\mathcal{O}$ and for which all the functions in Φ are continuous is called the fine topology associated with Φ and $\mathcal{O}$. We use $\mathcal{O}^F(\mathcal{O}, \Phi)$

to denote the family of open sets with respect to the fine topology associated with $\mathcal{O}$ and Φ. In particular, if $\mathcal{O} = \{\emptyset, X\}$, then we write $\mathcal{O}^F(\Phi) = \mathcal{O}^F(\mathcal{O}, \Phi)$.

Considering the resistance form $(\mathcal{E}, \mathcal{F})$, the adequate candidate of Φ is $\mathcal{F}^+_{+\infty}$ defined by

$$\mathcal{F}^+_{+\infty} = \{u | u \in \mathcal{F}, u(x) \geq 0 \text{ for any } x \in X\} \cup \{+\infty\}$$

Define $\mathcal{O}_{\mathcal{F}} = \{U | X \backslash U \in \mathcal{C}_{\mathcal{F}}\}$. Also define $\mathcal{O}_R$ as the collection of open sets with respect to the R-topology. By Proposition 5.1, we have $\mathcal{O}_{\mathcal{F}} \subseteq \mathcal{O}_R$.

Theorem 5.7. *Assume that (X, R) is separable.*
(1) $\mathcal{O}^F(\mathcal{O}_R, \mathcal{F}^+_{+\infty}) = \mathcal{O}_R$.
(2) $\mathcal{O}^F(\mathcal{F}^+_{+\infty}) = \mathcal{O}_{\mathcal{F}}$.

By Theorem 5.7, we realize an odd situation where the "fine" topology is coarser than the R-topology. Because of this and the fact that we need Theorem 4.3, where the condition $B^F = B$ has already been used, to prove Theorem 5.7, we can take small advantage of the axiomatic potential theory developed in [**13**].

To prove the above theorem, we need the following lemma.

Lemma 5.8. *Assume that (X, R) is separable. Then for any $x \in X$, there exists $u \in \mathcal{F}$ such that $u(x) = 0$ and $u(y) > 0$ for any $y \in X \backslash \{x\}$.*

Proof. First note that by (RF2), $(\mathcal{F}_x, \mathcal{E})$ is a Hilbert space, where $\mathcal{F}_x = \{u | u(x) = 0, u \in \mathcal{F}\}$.

For any $y \in X$, define $u_y(z) = g^y_{\{x\}}(z)$. Then by (GF2) and (GF4), if $R(y, z) < R(y, x)$, then

$$u_y(y) - u_y(z) \leq R(y, z) < R(x, y) = u_y(y).$$

Hence $u_y(z) > 0$ for any $z \in B_R(y, R(y, x))$.

Now fix $\epsilon > 0$ and let $\{x_n\}_{n \geq 1}$ be a dense subset of $X \backslash B_R(x, \epsilon)$. Choose $\alpha_n > 0$ so that $\sum_{n \geq 1} \sqrt{\alpha_n R(x_n, x)} < +\infty$. Define $v_n = \sum_{m=1}^n \alpha_n u_{x_m}$. Then $\sum_{n \geq 1} \sqrt{\mathcal{E}(v_n - v_{n+1}, v_n - v_{n+1})} = \sum_{n \geq 1} \sqrt{\alpha_{n+1} R(x_{n+1}, x)} < +\infty$. This implies that $\{v_n\}_{n \geq 1}$ is a Cauchy sequence in the Hilbert space $(\mathcal{F}_x, \mathcal{E})$. Hence $v = \sum_{n \geq 1} \alpha_n u_{x_n} \in \mathcal{F}_x$. By the above argument, for any $z \in X \backslash B_R(x, \epsilon)$, there exists x_n such that $u_{x_n}(z) > 0$. Therefore, $v(z) > 0$ for any $z \in X \backslash B_R(x, \epsilon)$. We use v_ϵ to denote v. Let $u_n = v_{1/n}$. Choose β_n so that $\sum_{n \geq 1} \sqrt{\beta_n \mathcal{E}(u_n, u_n)} < +\infty$. Then similar argument as $\sum_{n \geq 1} \alpha_n u_{x_n}$, we see that $\sum_{n \geq 1} \beta_n u_n$ belongs to $\mathcal{F}_x$. Let $u = \sum_{n \geq 1} \beta_n u_n$. Then $u(z) > 0$ for any $z \in X \backslash \{x\}$. $\square$

Proof of Theorem 5.7. First we show that

$$\mathcal{O}^F(\mathcal{F}^+_{+\infty}) \subseteq \mathcal{O}_{\mathcal{F}} \subseteq \mathcal{O}_R. \tag{5.1}$$

Note that $\mathcal{O}^F(\mathcal{F}^+_{+\infty})$ is generated by $u^{-1}((-\infty, c))$ and $u^{-1}((c, +\infty))$ for any $c \in \mathbb{R}$ and any $u \in \mathcal{F}^+_{+\infty}$. Let $u \in \mathcal{F}$ and let $c \in \mathbb{R}$. Set $B = \{x | u(x) \geq c\}$. Define $v(x) = \max\{u(x) - c, 0\}$. Then Proposition 3.15 shows that $v \in \mathcal{F}$. Furthermore, $v \in \mathcal{F}(B)$ and $v(x) > 0$ for any $x \notin B$. Hence $B^{\mathcal{F}} = B$ and so $B \in \mathcal{C}_{\mathcal{F}}$. Therefore $u^{-1}((-\infty, c)) \in \mathcal{O}_{\mathcal{F}}$. Using similar argument, we also obtain that $u^{-1}((c, +\infty)) \in \mathcal{O}_{\mathcal{F}}$. Thus we have shown (5.1)

Since $\mathcal{O}^F(\mathcal{O}_R, \mathcal{F}^+_{+\infty})$ is finer than $\mathcal{O}_R$, (5.1) implies $\mathcal{O}^F(\mathcal{O}_R, \mathcal{F}^+_{+\infty}) = \mathcal{O}_R$. Hence we have shown (1).

To prove (2), let $B \in \mathcal{C}_{\mathcal{F}}$. By Theorem 4.3, $(\mathcal{E}, \mathcal{F}^B)$ is a resistance form on X_B. Let R_B be the resistance metric associated with $(\mathcal{E}, \mathcal{F}^B)$ on X_B. Then $R_B(x, y) \leq$

$R(x,y)$ for any $x, y \in X \backslash B$ and $R_B(x,B) \leq \inf_{z \in B} R(x,z)$. Therefore, since (X,R) is separable, (X_B, R_B) is separable. Hence applying Lemma 5.8 to $(\mathcal{E}, \mathcal{F}^B)$ with $x = B$, we obtain $u \in \mathcal{F}^B$ which satisfies $u(y) > 0$ for any $y \in X \backslash B$. Since $u \in \mathcal{F}_{+\infty}^+$, $u^{-1}(0) = B$ is closed under $\mathcal{O}^F(\mathcal{F}_{+\infty}^+)$. This yields that $\mathcal{O}_{\mathcal{F}} \subseteq \mathcal{O}^F(\mathcal{F}_{+\infty}^+)$. □

6. Regularity of resistance forms

Does a domain $\mathcal{F}$ of a resistance form $\mathcal{E}$ contain enough many functions? The notion of regularity of a resistance form will provide an answer to such a question. As you will see in Definition 6.2, a resistance form is regular if and only if the domain of the resistance form is large enough to approximate any continuous function with a compact support. It is notable that the operation $B \to B^{\mathcal{F}}$ plays an important role again in this section.

Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a set X and let R be the associated resistance metric on X. We assume that (X,R) is separable in this section.

Definition 6.1. (1) Let $u : X \to \mathbb{R}$. The support of u, $\mathrm{supp}(u)$ is defined by $\mathrm{supp}(u) = \overline{\{x | u(x) \neq 0\}}$, where $\overline{U}$ is the closure of $U \subseteq X$ with respect to the resistance metric. We use $C_0(X)$ to denote the collection of continuous functions on X whose support are compact.
(2) Let K be a subset of X and let $u : X \to \mathbb{R}$. We define the supremum norm of u on K, $||u||_{\infty,K}$ by

$$||u||_{\infty,K} = \sup_{x \in K} |u(x)|.$$

We write $|| \cdot ||_\infty = || \cdot ||_{\infty,X}$ if no confusion can occur.

Definition 6.2. The resistance form $(\mathcal{E}, \mathcal{F})$ on X is called regular if and only if $\mathcal{F} \cap C_0(X)$ is dense in $C_0(X)$ in the sense of the supremum norm $|| \cdot ||_\infty$.

The regularity of a resistance form is naturally associated with that of a Dirichlet form. See Section 9 for details. The following theorem gives a simple criteria for regularity.

Theorem 6.3. *Assume that (X,R) is locally compact. The following conditions are equivalent:*
(R1) *$(\mathcal{E}, \mathcal{F})$ is regular.*
(R2) *$B^{\mathcal{F}} = B$ for any R-closed subset B. In other words, the R-topology coincides with the $\mathcal{C}_{\mathcal{F}}$-topology.*
(R3) *If B is R-closed and $\overline{B^c}$ is R-compact, then $B^{\mathcal{F}} = B$.*
(R4) *If K is a compact subset of X, U is an R-open subset of X, $K \subseteq U$ and $\overline{U}$ is R-compact, then there exists $\varphi \in \mathcal{F}$ such that $\mathrm{supp}(\varphi) \subseteq \overline{U}$, $0 \leq \varphi(y) \leq 1$ for any $y \in X$ and $\varphi|_K \equiv 1$.*

Combining the above theorem with Corollary 5.4, we obtain the following result.

Corollary 6.4. *If (X,R) is R-compact, then $(\mathcal{E}, \mathcal{F})$ is regular.*

In general, even if (X,R) is locally compact, $(\mathcal{E}, \mathcal{F})$ is not always regular. Recall Example 5.5.

Hereafter, the R-topology will be always used when we will consider a resistance form unless we say otherwise. For example, an open set means an R-open set.

To prove Theorem 6.3, we need the following lemma, which can be proven by direct calculation.

LEMMA 6.5. *If $u, v \in \mathcal{F} \cap C_0(X)$, then $uv \in \mathcal{F} \cap C_0(X)$ and*

$$\mathcal{E}(uv, uv) \leq 2||u||_\infty^2 \mathcal{E}(v, v) + 2||v||_\infty^2 \mathcal{E}(u, u).$$

PROOF OF THEOREM 6.3. (R1) $\Rightarrow$ (R2) Let $x \notin B$. Choose $r > 0$ so that $\overline{B(x,r)}$ is compact and $B \cap \overline{B(x,r)} = \emptyset$. Then there exists $f \in C_0(X)$ such that $0 \leq f(y) \leq 1$ for any $y \in X$, $f(x) = 1$ and $\text{supp}(f) \subseteq \overline{B(x,r)}$. Since $(\mathcal{E}, \mathcal{F})$ is regular, we may find $v \in \mathcal{F} \cap C_0(X)$ such that $||v - f||_\infty \leq 1/3$. Define $u = \overline{3v - 1}$. Then $u(x) = 1$ and $u|_B \equiv 0$. Hence $x \notin B^{\mathcal{F}}$.
(R2) $\Rightarrow$ (R3) This is obvious.
(R3) $\Rightarrow$ (R4) By (R3), $(U^c)^{\mathcal{F}} = U^c$. Hence, for any $x \in K$, we may choose r_x so that $B(x, r_x) \subseteq U$ and $\psi^x_{U^c}(y) \geq 1/2$ for any $y \in B(x, r_x)$. Since K is compact, $K \subseteq \cup_{i=1}^n B(x_i, r_{x_i})$ for some $x_1, \ldots, x_n \in K$. Let $v = \sum_{i=1}^n \psi^{x_i}_{U^c}$. Then $v(y) \geq 1/2$ for any $y \in K$ and $\text{supp}(v) \subseteq \overline{U}$. If $\varphi = 2\overline{v}$, then u satisfies the desired properties.
(R4) $\Rightarrow$ (R1) Let $u \in C_0(X)$. Set $K = \text{supp}(u)$. Define $\mathcal{U}_K = \{u|_K : u \in \mathcal{F} \cap C_0(X)\}$. Then by (R4) and Lemma 6.5, we can verify the assumptions of the Stone Weierstrass theorem for $\mathcal{U}_K$ with respect to $|| \cdot ||_{\infty,K}$. (See, for example, [**54**] on the Stone Weierstrass theorem.) Hence, $\mathcal{U}_K$ is dense in $C(K)$ with respect to the supremum norm on K. For any $\epsilon > 0$, there exists $u_\epsilon \in \mathcal{F} \cap C_0(X)$ such that $||u - u_\epsilon||_{\infty,K} < \epsilon$. Let $V = K \cup \{x : |u_\epsilon(x)| < \epsilon\}$. Suppose that $x \in K$ and that there exists $\{x_n\}_{n=1,2,\ldots} \subseteq V^c$ such that $R(x_n, x) \to 0$ as $x \to +\infty$. Then $|u_\epsilon(x_n)| \geq \epsilon$ for any n and hence $|u_\epsilon(x)| \geq \epsilon$. On the other hand, since $x_n \in K^c$, $u(x_n) = 0$ for any n and hence $u(x) = 0$. Since $x \in K$, this contradict to the fact that $||u - u_\epsilon||_{\infty,K} < \epsilon$. Therefore, V is open. We may choose a open set U so that $K \subseteq U$, $\overline{U}$ is compact and $U \subseteq V$. Let φ be the function obtained in (R4). Define $v_\epsilon = \varphi u_\epsilon$. Then by Lemma 6.5, $v_\epsilon \in \mathcal{F} \cap C_0(X)$. Also it follows that $||u - v_\epsilon||_\infty \leq \epsilon$. This shows that $\mathcal{F} \cap C_0(X)$ is dense in $C_0(X)$ with respect to the norm $|| \cdot ||_\infty$. $\square$

7. Annulus comparable condition and local property

A resistance form may have a long-distance connection and/or a jump in general. For instance, let us modify a given resistance form by adding a new resistor between two separate points. The modified resistance form has a “jump” induced by the added resistor. Such a “jump” naturally appears in the associated probabilistic process. In this section, we introduce the annulus comparable condition, (ACC) for short, which ensures certain control of such jumps, or direct connections between two separate points. For example, Theorems in Section 15 will show that (ACC) is necessary to get the Li-Yau type on-diagonal heat kernel estimate.

We need the following topological notion to state (ACC).

DEFINITION 7.1. Let (X, d) be a metric space. (X, d) is said to be uniformly perfect if and only if there exists $\epsilon > 0$ such that $B_d(x, (1 + \epsilon)r) \backslash B_d(x, r) \neq \emptyset$ for any $x \in X$ and $r > 0$ with $X \backslash B_d(x, r) \neq \emptyset$.

In [**51**], the notion of “uniformly perfect” is called “homogeneously dense”.

Note that if (X, d) is connected, then it is uniformly perfect. Uniformly perfectness ensures that widths of gaps in the space are asymptotically of geometric progression. For example, the ternary Cantor set is uniformly perfect. In general, any self-similar set where the contractions are similitudes is uniformly perfect as follows.

EXAMPLE 7.2. Let (X, d) be a complete metric space and let $N \geq 2$. For any $i = 1, \ldots, N$, $F_i : X \to X$ is assumed to be a contraction, i.e.

$$\sup_{x \neq y \in \mathbb{R}^n} \frac{d(F_i(x), F_i(y))}{d(x, y)} < 1.$$

Then there exists a unique non-empty compact set $K \subseteq X$ which satisfies $K = \cup_{i=1}^{N} F_i(K)$. See [**36**] for details. K is called the self-similar set associated with $\{F_i\}_{i=1,\ldots,N}$. If (X, d) is $\mathbb{R}^n$ with the Euclidean metric and every F_i is a similitude, i.e. $F_i(x) = r_i A_i x + a_i$, where $r_i \in (0, 1)$, A_i is an orthogonal matrix and $a_i \in \mathbb{R}^n$, then K is uniformly perfect.

Next, we have an example which is complete and perfect but not uniformly perfect.

EXAMPLE 7.3. Let $X = [0, 1]$. If $F_1(x) = x^2/3$ and $F_2(x) = x/3 + 2/3$, the self-similar set associated with $\{F_1, F_2\}$ is a (topological) Cantor set, i.e. complete, perfect and compact. We denote the n-th iteration of F_1 by $(F_1)^n$, i.e. $(F_1)^0(x) = x$ for any $x \in \mathbb{R}$ and $(F_1)^{n+1} = (F_1)^n \circ F_1$. Define $a_n = (F_1)^n(1/3)$ and $b_n = (F_1)^n(2/3)$. Then by inductive argument, we see that $(a_n, b_n) \cap K = \emptyset$ for any $n \geq 0$. Now, $a_n = 3e^{-2^{n+1}\log 3}$ and $b_n = 3e^{-2^{n+1}(\log 3 - (\log 2)/2)}$. Hence $b_n/a_n = (3/\sqrt{2})^{2^{n+1}}$. This shows that K is not uniformly perfect.

In this section, $(\mathcal{E}, \mathcal{F})$ is a regular resistance form on X and R is the associated resistance metric. We assume that (X, R) is separable and complete.

DEFINITION 7.4. A resistance form $(\mathcal{E}, \mathcal{F})$ on X is said to satisfy the annulus comparable condition, (ACC) for short, if and only if (X, R) is uniformly perfect and there exists $\epsilon > 0$ such that

$$\text{(ACC)} \qquad R(x, B_R(x, r)^c) \asymp R\big(x, \overline{B_R(x, (1+\epsilon)r)} \cap B_R(x, r)^c\big)$$

for any $x \in X$ and any $r > 0$ with $B_R(x, r) \neq X$.

REMARK. It is obvious that

$$R(x, B_R(x, r)^c) \leq R\big(x, \overline{B_R(x, (1+\epsilon)r)} \cap B_R(x, r)^c\big).$$

So the essential requirement of (ACC) is the opposite inequality up to a constant multiplication.

The annulus comparable condition holds if (X, R) is uniformly perfect and $(\mathcal{E}, \mathcal{F})$ has the local property defined below.

DEFINITION 7.5. $(\mathcal{E}, \mathcal{F})$ is said to have the local property if and only if $\mathcal{E}(u, v) = 0$ for any $u, v \in \mathcal{F}$ with $\underline{R}(\mathrm{supp}(u), \mathrm{supp}(v)) > 0$.

PROPOSITION 7.6. *Assume that $(\mathcal{E}, \mathcal{F})$ has the local property and that $\overline{B_R(x, r)}$ is compact for any $x \in X$ and any $r > 0$. If $\overline{B_R(x, (1+\epsilon)r)} \cap B_R(x, r)^c \neq \emptyset$, then*

$$R(x, B_R(x, r)^c) = R\big(x, \overline{B_R(x, (1+\epsilon)r)} \cap B_R(x, r)^c\big).$$

In particular, we have (ACC) *if* (X, R) *is uniformly perfect.*

PROOF. Let $K = \overline{B_R(x, (1+\epsilon)r)} \cap B_R(x, r)^c$. Recall that $\psi_K^x(y) = \frac{g_K(x,y)}{g_K(x,x)}$ and that $\mathcal{E}(\psi_K^x, \psi_K^x) = R(x, K)^{-1}$. By Theorem 6.3, there exists $\varphi \in \mathcal{F}$ such that $\mathrm{supp}(\varphi) \subseteq \overline{B_R(x, (1+\epsilon/2)r)}$, $0 \leq \varphi(y) \leq 1$ for any $y \in X$ and $\varphi(y) = 1$ for any

$y \in \overline{B_R(x,r)}$. By Lemma 6.5, if $\psi_1 = \psi_K^x \varphi$ and $\psi_2 = \psi_K^x(1-\varphi)$, then ψ_1 and ψ_2 belong to $\mathcal{F}$. Since $\mathrm{supp}(\psi_2) \subseteq B_r(x,(1+\epsilon)r)^c$, the local property implies

$$\mathcal{E}(\psi_K^x, \psi_K^x) = \mathcal{E}(\psi_1, \psi_1) + \mathcal{E}(\psi_2, \psi_2) \geq \mathcal{E}(\psi_1, \psi_1).$$

Note that $\psi_1(y) = 0$ for any $y \in B_R(x,r)^c$ and that $\psi_1(x) = 1$. Hence, $\mathcal{E}(\psi_1, \psi_1) \geq \mathcal{E}(\psi_B^x, \psi_B^x)$, where $B = B(x,r)^c$. On the other hand, since $K \subseteq B$, $\mathcal{E}(\psi_B^x, \psi_B^x) \geq \mathcal{E}(\psi_K^x, \psi_K^x)$. Therefore, we have

$$R(x,B)^{-1} = \mathcal{E}(\psi_B^x, \psi_B^x) = \mathcal{E}(\psi_K^x, \psi_K^x) = R(x,K)^{-1}.$$

□

There are non-local resistance forms which satisfy (ACC), for example, the α-stable process on $\mathbb{R}$ and their traces on the Cantor set. See Sections 16. In the next section, we will show that if the original resistance form has (ACC), then so do its traces, which are non-local in general.

To study non-local cases, we need the doubling property of the space.

Definition 7.7. Let (X,d) be a metric space.
(1) (X,d) is said to have the doubling property or be a doubling space if and only if

$$\sup_{x\in X, r>0} N_d(B_d(x,r), \delta r) < +\infty \tag{7.1}$$

for any $\delta \in (0,1)$, where $N_d(B,r)$ is defined in Definition 5.2. (2) Let μ be a Borel regular measure on (X,d) which satisfies $0 < \mu(B_d(x,r)) < +\infty$ for any $x \in X$ and any $r > 0$. μ is said to have the volume doubling property with respect to d or be volume doubling with respect to d, $(\mathrm{VD})_d$ for short, if and only if there exists $c > 0$ such that

$$\mu(B_d(x,2r)) \leq c\mu(B_d(x,r)) \tag*{$(\mathrm{VD})_d$}$$

for any $x \in X$ and any $r > 0$.

Remark. (1) It is easy to see that (7.1) holds for all $\delta \in (0,1)$ if it holds for some $\delta \in (0,1)$. Hence (X,d) is a doubling space if (7.1) holds for some $\delta \in (0,1)$.
(2) If μ is $(\mathrm{VD})_d$, then, for any $\alpha > 1$, $\mu(B_d(x,\alpha r)) \asymp \mu(B_d(x,r))$ for any $x \in X$ and any $r > 0$.

One of the sufficient condition for the doubling property of a space is the existence of a measure which has the volume doubling property. The following theorem is well-known. See [**32**] for example.

Proposition 7.8. *Let (X,d) be a metric space and let μ be a Borel regular measure on (X,R) with $0 < \mu(B_R(x,r)) < +\infty$ for any $x \in X$ and any $r > 0$. If μ is $(\mathrm{VD})_d$, then (X,d) has the doubling property.*

The next proposition is straightforward from the definitions.

Proposition 7.9. *If a metric space (X,d) has the doubling property, then any bounded subset of (X,d) is totally bounded.*

By the above proposition, if the space is doubling and complete, then every bounded closed set is compact.

Now we return to (ACC). The following key lemma is a direct consequence of Theorem 5.3.

LEMMA 7.10. *Assume that* (X, R) *has the doubling property and is uniformly perfect. Then, for some* $\epsilon > 0$,

$$R\big(x, B_R(x,r)^c \cap \overline{B_R(x,(1+\epsilon)r)}\big) \asymp r \tag{7.2}$$

for any $x \in X$ *and any* $r > 0$ *with* $B_R(x,r) \neq X$.

PROOF. Set $B = \overline{B_R(x,(1+\epsilon)r)} \cap B_R(x,r)^c$. Choose ϵ so that $B \neq \emptyset$ for any $x \in X$ and any $r > 0$ with $B_R(x,r) \neq X$. Then, $r \leq \underline{R}(x,B) \leq (1+\epsilon)r$. This and the doubling property of (X, R) imply

$$N(B, \underline{R}(x,B)/2) \leq N(B, r/2) \leq N(B_R(x,(1+2\epsilon)r), r/2) \leq c_*,$$

where c_* is independent of x and r. Using Theorem 5.3, we see

$$\frac{r}{8c_*} \leq R(x,B) \leq (1+\epsilon)r.$$

□

By the above lemma, (ACC) turns out to be equivalent to (RES) defined below if (X, R) is a doubling space.

DEFINITION 7.11. A resistance form $(\mathcal{E}, \mathcal{F})$ on X is said to satisfy the resistance estimate, (RES) for short, if and only if

$$R(x, B_R(x,r)^c) \asymp r \tag{RES}$$

for any $x \in X$ and any $r > 0$ with $B_R(x,r) \neq X$.

THEOREM 7.12. *Assume that* (X, R) *has the doubling property. Then* (X, R) *is uniformly perfect and* (RES) *holds if and only if* (ACC) *holds.*

PROOF OF THEOREM 7.12. If (ACC) holds, then (7.2) and (ACC) immediately imply (RES). Conversely, (RES) along with (7.2) shows (ACC). □

COROLLARY 7.13. *If* $(\mathcal{E}, \mathcal{F})$ *has the local property,* (X, R) *has the doubling property and is uniformly perfect, then* (RES) *holds.*

8. Trace of resistance form

In this section, we introduce the notion of the trace of a resistance form on a subset of the original domain. This notion is a counterpart of the notion of traces in the theory of Dirichlet forms, which has been extensively studied in [**21**, Section 6.2], for example.

Throughout this section, $(\mathcal{E}, \mathcal{F})$ is a resistance form on X and R is the associated resistance distance. We assume that (X, R) is separable and complete.

DEFINITION 8.1. For a non-empty subset $Y \subseteq X$, define $\mathcal{F}|_Y = \{u|_Y : u \in \mathcal{F}\}$.

LEMMA 8.2. *Let* Y *be a non-empty subset of* (X, R). *For any* $u \in \mathcal{F}|_Y$, *there exists a unique* $u_* \in \mathcal{F}$ *such that* $u_*|_Y = u$ *and*

$$\mathcal{E}(u_*, u_*) = \min\{\mathcal{E}(v,v) | v \in \mathcal{F}, v|_Y = u\}.$$

This lemma is an extension of Proposition 3.10. In fact, if Y is a finite set, then we have $\mathcal{F}|_Y = \ell(Y)$ by (RF3-2). Hence Lemma 8.2 holds in this case by Proposition 3.10. The unique u_* is thought of as the harmonic function with the boundary value u on Y.

Proof of Lemma 8.2. Let $p \in Y$. Replacing u by $u - u(p)$, we may assume that $u(p) = 0$ without loss of generality. Choose a sequence $\{v_n\}_{n \geq 1} \subseteq \mathcal{F}$ so that $v_n|_Y = u$ and $\lim_{n \to +\infty} \mathcal{E}(v_n, v_n) = \inf\{\mathcal{E}(v,v) | v \in \mathcal{F}, v|_Y = u\}$. Let $C = \sup_n \mathcal{E}(v_n, v_n)$. By Proposition 3.2, if $v = v_n$, then

$$|v(x) - v(y)|^2 \leq CR(x,y) \tag{8.1}$$

and

$$|v(x)|^2 \leq CR(x,p) \tag{8.2}$$

Let $\{V_m\}_{m \geq 1}$ be an increasing sequence of finite subsets of X. Assume that $V_* = \cup_{m \geq 1} V_m$ is dense in X. (Since (X, R) is separable, such $\{V_m\}_{m \geq 1}$ does exists.) By (8.1) and (8.2), the standard diagonal construction gives a subsequence $\{v_{n_i}\}_{i \geq 1}$ which satisfies $\{v_{n_i}(x)\}_{i \geq 1}$ is convergent as $i \to +\infty$ for any $x \in V_* = \cup_{m \geq 1} V_m$. Define $u_*(x) = \lim_{i \to +\infty} v_{n_i}(x)$ for any $x \in V_*$. Since u_* satisfies (8.1) and (8.2) on V_* with $v = u_*$, u_* is extended to a continuous function on X. Note that the extended function also satisfies (8.1) and (8.2) on X with $v = u_*$. Set $\mathcal{E}_m = \mathcal{E}_{L_{(\mathcal{E},\mathcal{F}),V_m}}$. Then, by Theorem 3.14,

$$\mathcal{E}_m(v_n, v_n) \leq \mathcal{E}(v_n, v_n) \leq C \tag{8.3}$$

for any $m \geq 1$ and any $n \geq 1$. Define $M = \inf\{\mathcal{E}(v,v) | v \in \mathcal{F}, v|_Y = u\}$. For any $\epsilon > 0$, if n is large enough, then (8.3) shows $\mathcal{E}_m(v_n, v_n) \leq M + \epsilon$ for any $m \geq 1$. Since $v_n|_{V_m} \to u_*|_{V_m}$ as $n \to +\infty$, it follows that $\mathcal{E}_m(u_*, u_*) \leq M + \epsilon$ for any $m \geq 1$. Theorem 3.14 implies that $u_* \in \mathcal{F}$ and $\mathcal{E}(u_*, u_*) \leq M$.

Next assume that $u_i \in \mathcal{F}$, $u_i|_Y = u$ and $\mathcal{E}(u_i, u_i) = M$ for $i = 1, 2$. Since $\mathcal{E}((u_1 + u_2)/2, (u_1 + u_2)/2) \geq \mathcal{E}(u_1, u_1)$, we have $\mathcal{E}(u_1, u_2 - u_1) \geq 0$. Similarly, $\mathcal{E}(u_2, u_1 - u_2) \geq 0$. Combining those two inequalities, we obtain $\mathcal{E}(u_1 - u_2, u_1 - u_2) = 0$. Since $u_1 = u_2$ on Y, we have $u_1 = u_2$ on X. □

The next definition is an extension of the notion of h_V defined in Definition 3.11.

Definition 8.3. Define $h_Y : \mathcal{F}|_Y \to \mathcal{F}$ by $h_Y(u) = u_*$, where u and u_* are the same as in Lemma 8.2. $h_V(u)$ is called the Y-harmonic function with the boundary value u. For any $u, v \in \mathcal{F}|_Y$, define $\mathcal{E}|_Y(u,v) = \mathcal{E}(h_Y(u), h_Y(v))$. $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ is called the trace of the resistance form $(\mathcal{E}, \mathcal{F})$ on Y.

Making use of the harmonic functions, we construct a resistance form on a subspace Y of X, which is called the trace.

Theorem 8.4. *Let Y be a non-empty subset of X. Then $h_Y : \mathcal{F}|_Y \to \mathcal{F}$ is linear and $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ is a resistance form on Y. The associated resistance metric equals to the restriction of R on $Y \times$. If Y is closed and $(\mathcal{E}, \mathcal{F})$ is regular, then $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ is regular.*

We denote the restriction of R on $Y \times Y$ by $R|_Y$.

The following lemma is essential to prove the above theorem.

Lemma 8.5. *Let Y be a non-empty subset of X. Define*

$$\mathcal{H}_Y = \{u | u \in \mathcal{F}, \mathcal{E}(u,v) = 0 \text{ for any } v \in \mathcal{F}(Y)\},$$

where $\mathcal{F}(Y)$ is defined in Definition 2.1. Then, for any $f \in \mathcal{F}|_Y$, $u = h_Y(f)$ if and only if $u \in \mathcal{H}_Y$ and $u|_Y = f$.

By this lemma, $\mathcal{H}_Y = \mathrm{Im}(h_Y)$ is the space of Y-harmonic functions and $\mathcal{F} = \mathcal{H}_Y + \mathcal{F}(Y)$, i.e. $\mathcal{F}$ is the direct sum of $\mathcal{H}_Y$ and $\mathcal{F}(Y)$. Moreover, $\mathcal{E}(u,v) = 0$ for any $u \in \mathcal{H}_Y$ and any $v \in \mathcal{F}(Y)$. The counterpart of this fact has been know for Dirichlet forms. See [**21**, Section 6.2] for details. In the case of weighted graphs (Example 3.5) such a decomposition is known as the Royden's decomposition. See [**49**, Theorem (6.3)] for details.

PROOF. Let $f_* = h_Y(f)$. If $v \in \mathcal{F}$ and $v|_Y = f$, then

$$\mathcal{E}(t(v - f_*) + f_*, t(v - f_*) + f_*) \geq \mathcal{E}(f_*, f_*)$$

for any $t \in \mathbb{R}$. Hence $\mathcal{E}(v - f_*, f_*) = 0$. This implies that $f_* \in \mathcal{H}_Y$. Conversely assume that $u \in \mathcal{H}_Y$ and $u|_Y = f$. Then, for any $v \in \mathcal{F}$ with $v|_Y = f$,

$$\mathcal{E}(v,v) = \mathcal{E}((v-u)+u, (v-u)+u) = \mathcal{E}(v-u, v-u) + \mathcal{E}(u,u) \geq \mathcal{E}(u,u).$$

Hence by Lemma 8.2, $u = h_Y(f)$. □

PROOF OF THEOREM 8.4. By Lemma 8.5, if $r_Y : \mathcal{H}_Y \to \mathcal{F}|_Y$ is the restriction on Y, then r_Y is the inverse of h_Y. Hence h_Y is linear. The conditions (RF1) through (RF4) for $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ follow immediately from the counterparts for $(\mathcal{E}, \mathcal{F})$. About (RF5),

$$\begin{aligned}\mathcal{E}|_Y(\overline{u}, \overline{u}) = \mathcal{E}(h_Y(\overline{u}), h_Y(\overline{u})) &\leq \mathcal{E}(\overline{h_Y(u)}, \overline{h_Y(u)}) \\ &\leq \mathcal{E}(h_Y(u), h_Y(u)) = \mathcal{E}|_Y(u,u).\end{aligned}$$

The rest of the statement is straightforward. □

In the rest of this section, the conditions (ACC) and (RES) are shown to be preserved by the traces under reasonable assumptions.

THEOREM 8.6. *Let $(\mathcal{E}, \mathcal{F})$ be a regular resistance form on X and let R be the associated resistance metric. Assume that $(\mathcal{E}, \mathcal{F})$ satisfies* (RES). *If Y is a closed subset of X and $(Y, R|_Y)$ is uniformly perfect, then* (RES) *holds for the trace $(\mathcal{E}|_Y, \mathcal{F}|_Y)$.*

By Theorem 7.12, we immediately have the following corollary.

COROLLARY 8.7. *Let $(\mathcal{E}, \mathcal{F})$ be a regular resistance form on X and let R be the associated resistance metric. Assume that (X, R) has the doubling property. Let Y be a closed subset of X and assume that $(Y, R|_Y)$ is uniformly perfect. If* (ACC) *holds for $(\mathcal{E}, \mathcal{F})$, then so does for the trace $(\mathcal{E}|_Y, \mathcal{F}|_Y)$.*

NOTATION. Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X and let R be the associated resistance metric. For a non-empty subset Y of X, we use R^Y to denote the resistance metric associated with the trace $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ on Y. Also we write $B_R^Y(x,r) = B_R(x,r) \cap Y$ for any $x \in Y$ and $r > 0$.

Note that $R^Y(x,y) = R(x,y)$ for any $x, y \in Y$ by Theorem 8.4. Hence $R^Y = R|_Y$.

PROOF OF THEOREM 8.6. Note that $R^Y(x, Y \backslash B_R^Y(x,r)) = R(x, B_R(x,r)^c \cap Y)$. Hence if (RES) holds for $(\mathcal{E}, \mathcal{F})$ then,

$$R^Y(x, Y \backslash B_R^Y(x,r)) \geq R(x, B_R(x,r)^c) \geq c_1 r. \tag{8.4}$$

On the other hand, since (Y, R^Y) is uniformly perfect, there exists $\epsilon > 0$ such that $B^Y_R(x, (1+\epsilon)r)\backslash B^Y_R(x, r) \neq \emptyset$ for any $x \in Y$ and $r > 0$ with $B^Y_R(x, r) \neq Y$. Let $y \in B^Y_R(x, (1+\epsilon)r)\backslash B^Y_R(x, r)$. Then

$$(1+\epsilon)r \geq R^Y(x, y) \geq R^Y(x, Y\backslash B^Y_R(x, r)). \tag{8.5}$$

Combining (8.4) and (8.5), we obtain (RES) for $(\mathcal{E}|_Y, \mathcal{F}|_Y)$. □

9. Resistance forms as Dirichlet forms

In this section, we will present how to obtain a regular Dirichlet form from a regular resistance form and show that every single point has a positive capacity. As in the previous sections, $(\mathcal{E}, \mathcal{F})$ is a resistance form on X and R is the associated resistance metric on X. We continue to assume that (X, R) is separable, complete and locally compact.

Let μ be a Borel regular measure on (X, R) which satisfies $0 < \mu(B_R(x, r)) < +\infty$ for any $x \in X$ and $r > 0$. Note that $C_0(X)$ is a dense subset of $L^2(X, \mu)$ by those assumptions on μ.

Definition 9.1. For any $u, v \in \mathcal{F} \cap L^2(X, \mu)$, define $\mathcal{E}_1(u, v)$ by

$$\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + \int_X uvd\mu.$$

By [**36**, Theorem 2.4.1], we have the following fact.

Lemma 9.2. *$(\mathcal{F} \cap L^2(X, \mu), \mathcal{E}_1)$ is a Hilbert space.*

Since $\mathcal{F} \cap C_0(X) \subseteq \mathcal{F} \cap L^2(X, \mu)$, the closure of $\mathcal{F} \cap C_0(X)$ is a subset of $\mathcal{F} \cap L^2(X, \mu)$.

Definition 9.3. We use $\mathcal{D}$ to denote the closure of $\mathcal{F} \cap C_0(X)$ with respect to the inner product $\mathcal{E}_1$.

Note that if (X, R) is compact, then $\mathcal{D} = \mathcal{F}$.

Theorem 9.4. *If $(\mathcal{E}, \mathcal{F})$ is regular, then $(\mathcal{E}|_{\mathcal{D}\times\mathcal{D}}, \mathcal{D})$ is a regular Dirichlet form on $L^2(X, \mu)$.*

See [**21**] for the definition of a regular Dirichlet form.
For ease of notation, we write $\mathcal{E}$ instead of $\mathcal{E}_{\mathcal{D}\times\mathcal{D}}$.

Definition 9.5. The Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(X, \mu)$ is called the Dirichlet form induced by a resistance form $(\mathcal{E}, \mathcal{F})$.

Proof. $(\mathcal{E}, \mathcal{D})$ is closed form on $L^2(X, \mu)$. Also, since $C_0(X)$ is dense in $L^2(X, \mu)$, the assumption that $\mathcal{F} \cap C_0(X)$ is dense in $C_0(X)$ shows that $\mathcal{D}$ is dense in $L^2(K, \mu)$. Hence $(\mathcal{E}, \mathcal{D})$ is a regular Dirichlet form on $L^2(X, \mu)$ with a core $\mathcal{F} \cap C_0(X)$. □

Hereafter in this section, $(\mathcal{E}, \mathcal{F})$ is always assumed to be regular. Next we study capacity of points associated with the Dirichlet form constructed above.

Lemma 9.6. *Let $x \in X$. Then there exists $c_x > 0$ such that*

$$|u(x)| \leq c_x\sqrt{\mathcal{E}_1(u, u)}$$

for any $u \in \mathcal{D}$. In other words, the map $u \to u(x)$ from $\mathcal{D}$ to $\mathbb{R}$ is bounded.

PROOF. Assume that there exists a sequence $\{u_n\}_{n\geq 1} \subset \mathcal{F}$ such that $u_n(x) = 1$ and $\mathcal{E}_1(u_n, u_n) \leq 1/n$ for any $n \geq 1$. By (3.1),

$$|u_n(x) - u_n(y)| \leq \frac{\sqrt{R(x,y)}}{\sqrt{n}} \leq \sqrt{R(x,y)}$$

Hence $u_n(y) \geq 1/2$ for any $y \in B_R(x, 1/4)$. This implies that

$$||u_n||_2^2 \geq \int_{B(x,1/4)} u(y)^2 d\mu \geq \mu(B_R(x, 1/4))/4 > 0$$

This contradicts the fact that $\mathcal{E}_1(u_n, u_n) \to 0$ as $n \to +\infty$. □

LEMMA 9.7. *If K is a compact subset of X, then the restriction map ι_K : $\mathcal{D} \to C(K)$ defined by $\iota_K(u) = u|_K$ is a compact operator, where $\mathcal{D}$ and $C(K)$ are equipped with the norms $\sqrt{\mathcal{E}_1(\cdot,\cdot)}$ and $||\cdot||_{\infty,K}$ respectively.*

PROOF. Set $D = \sup_{x,y\in K} R(x,y)$. Let $\mathcal{U}$ be a bounded subset of $\mathcal{D}$, i.e. there exists $M > 0$ such that $\mathcal{E}_1(u,u) \leq M$ for any $u \in \mathcal{U}$. Then by (3.1),

$$|u(x) - u(y)|^2 \leq R(x,y)M$$

for any $x, y \in X$ and any $u \in \mathcal{U}$. Hence $\mathcal{U}$ is equicontinuous. Choose $x_* \in K$. By Lemma 9.6 along with (3.1),

$$u(x)^2 \leq 2|u(x) - u(x_*)|^2 + 2|u(x_*)|^2 \leq 2DM + 2c_{x_*}^2 M$$

for any $u \in \mathcal{U}$ and any $x \in K$. This shows that $\mathcal{U}$ is uniformly bounded on K. By the Ascoli-Arzelà theorem, $\{u|_K\}_{u\in\mathcal{U}}$ is relatively compact with respect to the supremum norm. Hence ι_K is a compact operator. □

DEFINITION 9.8. For an open set $U \subseteq X$, define the $\mathcal{E}_1$-capacity of U, $\mathrm{Cap}(U)$, by

$$\mathrm{Cap}(U) = \inf\{\mathcal{E}_1(u,u) | u \in \mathcal{D}, u(x) \geq 1 \text{ for any } x \in U\}.$$

If $\{u|u \in \mathcal{D}, u(x) \geq 1 \text{ for any } x \in U\} = \emptyset$, we define $\mathrm{Cap}(U) = +\infty$. For any $A \subseteq X$, $\mathrm{Cap}(A)$ is defined by

$$\mathrm{Cap}(A) = \inf\{\mathrm{Cap}(U) | U \text{ is an open subset of } X \text{ and } A \subseteq U\}.$$

THEOREM 9.9. *For any $x \in X$, $0 < \mathrm{Cap}(\{x\}) < +\infty$. Moreover, if K is a compact subset of X, then $0 < \inf_{x\in K} \mathrm{Cap}(\{x\})$.*

The proof of Theorem 9.9 requires the following lemma.

LEMMA 9.10. *For any $x \in X$, there exists a unique $g \in \mathcal{D}$ such that*

$$\mathcal{E}_1(g,u) = u(x)$$

for any $u \in \mathcal{D}$. Moreover, let $\varphi = g/g(x)$. Then, φ is the unique element in $\{u|u \in \mathcal{D}, u(x) \geq 1\}$ which attains the following minimum

$$\min\{\mathcal{E}_1(u,u) | u \in \mathcal{D}, u(x) \geq 1\}.$$

PROOF. The existence of g follows by Lemma 9.6. Assume that $\mathcal{E}_1(f,u) = u(x)$ for any $u \in \mathcal{D}$. Since $\mathcal{E}_1(f-g,u) = 0$ for any $u \in \mathcal{D}$, we have $f = g$. Now, if $u \in \mathcal{D}$ and $u(x) = a > 1$, then $\mathcal{E}_1(u - a\varphi, \varphi) = u(x)/g(x) - 1/g(x) = 0$. Hence,

$$\mathcal{E}_1(u,u) = \mathcal{E}_1(u - a\varphi, u - a\varphi) + \mathcal{E}_1(a\varphi, a\varphi) \geq \mathcal{E}_1(\varphi,\varphi).$$

This immediately shows the rest of the statement. □

Definition 9.11. We denote the function g and φ in Lemma 9.10 by g_1^x and φ_1^x respectively.

Proof of Theorem 9.9. Fix $x \in X$. By the above lemma, for any open set U with $x \in U$,

$$\begin{aligned}\mathrm{Cap}(U) &= \min\{\mathcal{E}_1(u,u) | u \in \mathcal{D}, u(y) \geq 1 \text{ for any } y \in U\} \\ &\geq \min\{\mathcal{E}_1(u,u) | u \in \mathcal{D}, u(x) \geq 1\} \geq \mathcal{E}_1(\varphi_1^x, \varphi_1^x) = \frac{1}{g_1^x(x)}.\end{aligned}$$

Hence $0 < 1/g_1^x(x) < \mathrm{Cap}(\{x\}) < \mathrm{Cap}(U) < +\infty$.

Let K be a compact subset of X. By Lemma 9.7, there exists $c_K > 0$ such that $||u||_{\infty,K} \leq c_K\sqrt{\mathcal{E}_1(u,u)}$ for any $u \in \mathcal{D}$. Now, for $x \in K$,

$$g_1^x(x) = \mathcal{E}_1(g_1^x, g_1^x) = \sup_{u \in \mathcal{D}, u \neq 0} \frac{\mathcal{E}_1(g_1^x, u)^2}{\mathcal{E}_1(u,u)} = \sup_{u \in \mathcal{D}, u \neq 0} \frac{u(x)^2}{\mathcal{E}_1(u,u)} \leq (c_K)^2$$

Hence $\mathrm{Cap}(\{x\}) \geq 1/g_1^x(x) \geq (c_K)^{-2}$. □

Following [**21**, Section 2.1], we introduce the notion of quasi continuous functions.

Definition 9.12. A function $u : X \to \mathbb{R}$ is called quasi continuous if and only if, for any $\epsilon > 0$, there exists $V \subseteq X$ such that $\mathrm{Cap}(V) < \epsilon$ and $u|_{X\setminus V}$ is continuous.

Theorem 9.9 implies that every quasi continuous function is continuous.

Proposition 9.13. *Any quasi continuous function is continuous on X.*

Proof. Let u be a quasi continuous function. Let $x \in X$. Since (X, R) is locally compact, $\overline{B_R(x,r)}$ is compact for some $r > 0$. By Theorem 9.9, we may choose $\epsilon > 0$ so that $\inf_{y \in \overline{B_R(x,r)}} \mathrm{Cap}(\{y\}) > \epsilon$. There exists $V \subseteq X$ such that $\mathrm{Cap}(V) < \epsilon$ and $u|_{X\setminus V}$ is continuous. Since $V \cap \overline{B_R(x,r)} = \emptyset$, u is continuous at x. Hence u is continuous on X. □

10. Transition density

In this section, without ultracontractivity, we establish the existence of jointly continuous transition density (i.e. heat kernel) associated with the regular Dirichlet form derived from a resistance form.

As in the last section, $(\mathcal{E}, \mathcal{F})$ is a resistance form on X and R is the associated resistance metric. We assume that (X, R) is separable, complete and locally compact. μ is a Borel regular measure on X which satisfies $0 < \mu(B_R(x,r)) < +\infty$ for any $x \in X$ and any $r > 0$. We continue to assume that $(\mathcal{E}, \mathcal{F})$ is regular. By Theorem 9.4, $(\mathcal{E}, \mathcal{D})$ is a regular Dirichlet form on $L^2(X, \mu)$, where $\mathcal{D}$ is the closure of $\mathcal{F} \cap C_0(X)$ with respect to the $\mathcal{E}_1$-inner product.

Let H be the nonnegative self-adjoint operator associated with the Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(X, \mu)$ and let T_t be the corresponding strongly continuous semigroup. Since $T_t u \in \mathcal{D}$ for any $u \in L^2(X, \mu)$, we always take the continuous version of $T_t u$. In other words, we may naturally assume that $T_t u$ is continuous for any $t > 0$.

Let $\mathbf{M} = (\Theta, \{X_t\}_{t>0}, \{P_x\}_{x \in X})$ be the Hunt process associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(X, \mu)$. By [**21**, Chapter 7], such a Hunt process is determined outside a set of capacity zero in general. Note that $\mathrm{Cap}(A) = 0$ if and only if $A = \emptyset$ by Theorem 9.9. Hence, the Hunt process $\mathbf{M}$ is determined for every $x \in X$ in our case. Moreover, by [**21**, Theorem 4.2.1], every exceptional set is empty. (See [**21**, Section 4.1] for the definition of exceptional sets and its relation to other notions like "polar sets", "semi-polar sets" and "negligible sets".) Let p_t be the transition semigroup associated with the Hunt process $\mathbf{M}$. For non-negative μ-measurable function u,

$$(p_t u)(x) = E_x(u(X_t))$$

for any $x \in X$. Let $\mathcal{B}$ be the Borel σ-algebra of (X, R). We say that u is Borel measurable, if and only if $u^{-1}((a, b]) \in \mathcal{B}$ for any $a, b \in \mathbb{R}$. Combining Proposition 9.13 and [**21**, Theorem 4.2.3], we have the following statement.

Proposition 10.1. *For any nonnegative $u \in L^2(X, \mu)$, $(p_t u)(x) = (T_t u)(x)$ for any $t > 0$ and any $x \in X$.*

Definition 10.2. Let U be an open subset of X. Define $\mathcal{D}_U = \{u | u \in \mathcal{D}, u|_{U^c} \equiv 0\}$. Also we define $\mathcal{E}_U = \mathcal{E}|_{\mathcal{D}_U \times \mathcal{D}_U}$.

Note that if $\overline{U}$ is compact, then $\mathcal{D}_U = \mathcal{F}(U^c)$.

Combining the results in [**21**, Section 4.4], we have the following facts.

Theorem 10.3. *Let μ_U be the restriction of μ on U, i.e. $\mu_U(A) = \mu(A \cap U)$ for any Borel set U. Then $(\mathcal{E}_U, \mathcal{D}_U)$ is a regular Dirichlet form on $L^2(U, \mu_U)$. Moreover, if $\mathbf{M}_U = (\Theta_U, X_t^U, P_x^U)$ is the associated Hunt process, then*

$$P_x^U(X_t^U \in A) = P(X_t \in A, t < \tau_U)$$

for any Borel set A and any $x \in U$, where τ_U is the exit time of U defined by

$$\tau_U(\omega) = \inf\{t > 0 | X_t(\omega) \in X \backslash U\}.$$

Moreover, if p_t^U is the transition semigroup associated with $\mathbf{M}_U$, hen

$$(p_t^U u)(x) = E_x^U(u(X_t^U)) = E_x(\chi_{\{t < \tau_U\}} u(X_t)) \tag{10.1}$$

for any non-negative measurable function u and any $x \in X$.

Remark. For a function $u : U \to \mathbb{R}$, we define $\epsilon_U(u) : X \to \mathbb{R}$ by $\epsilon_U(u)|_U = u$ and $\epsilon_U(u)|_{U^c} \equiv 0$. Through this extension map, $L^2(U, \mu_U)$ is regarded as a subspace of $L^2(X, \mu)$. Also, if $u \in \mathcal{D}_U$, then $\epsilon_U(u|_U) = u$ and hence we may think of $\mathcal{D}_U$ as a subset of $C(X)$ through ϵ_U. Hereafter, we always use these conventions.

Remark. By the same reason as in the case of $\mathbf{M}$, the process $\mathbf{M}_U$ is determined for every $x \in U$.

The existence and the continuity of heat kernels have been studied by several authors. In [**5**], the existence of quasi continuous versions of heat kernel (i.e. transition density) has been proven under ultracontractivity. Grigor'yan has shown the corresponding result only assuming local ultracontractivity in [**22**]. In [**17**], the existence of jointly continuous heat kernels has been shown for the case of Dirichlet forms induced by resistance forms under ultracontractivity. The following theorem establishes the existence of jointly continuous heat kernel for the Dirichlet form induced by a resistance form without ultracontractivity and, at the same time, gives an upper diagonal estimate of the heat kernel. The main theorem of this section is the following.

THEOREM 10.4. *Assume that* $\overline{B_R(x,r)}$ *is compact for any* $x \in X$ *and* $r > 0$. *Let* U *be a non-empty open subset of* X. *Then there exits* $p_U(t,x,y) : (0, + \times X \times X \to [0,+\infty)$ *which satisfies the following conditions:*
(TD1) $p_U(t,x,y)$ *is continuous on* $(0,+\infty) \times X \times X$. *Define* $p_U^{t,x}(y) = p_U(t,x,y)$. *Then* $p_U^{t,x} \in \mathcal{D}_U$ *for any* $(t,x) \in (0,+\infty) \times X$.
(TD2) $p_U(t,x,y) = p_U(t,y,x)$ *for any* $(t,x,y) \in (0,+\infty) \times X \times X$.
(TD3) *For any non-negative (Borel)-measurable function* u *and any* $x \in X$,

$$(p_t^U u)(x) = \int_X p_U(t,x,y)u(y)\mu(dy) \tag{10.2}$$

(TD4) *For any* $t, s > 0$ *and any* $x, y \in X$,

$$p_U(t+s,x,y) = \int_X p_U(t,x,z)p_U(s,y,z)\mu(dz). \tag{10.3}$$

Furthermore, let A *be a Borel subset of* X *which satisfies* $0 < \mu(A) < +\infty$. *Define* $\overline{R}(x,A) = \sup_{y \in A} R(x,y)$ *for any* $x \in X$. *Then*

$$p_U(t,x,x) \leq \frac{2\overline{R}(x,A)}{t} + \frac{\sqrt{2}}{\mu(A)} \tag{10.4}$$

for any $x \in X$ *and any* $t > 0$.

The proof of the upper heat kernel estimate (10.4) is fairly simple. Originally, the same result has been obtained by more complicated discussion in [**6**] and [**41**]. A simplified argument, which is essentially the same as ours, for random walks can be found in [**8**].

REMARK. In fact, we have the following inequality which is slightly better than (10.4). For any $\epsilon > 0$,

$$p_U(t,x,x) \leq \left(1 + \frac{1}{\epsilon}\right)\frac{\overline{R}(x,A)}{t} + \frac{\sqrt{1+\epsilon}}{\mu(A)}. \tag{10.5}$$

This inequality implies that

$$\lim_{t \to +\infty} p_U(t,x,x) \leq \frac{1}{\mu(X)}.$$

for any $x \in X$.

DEFINITION 10.5. $p_U(t,x,y)$ is called the transition density and/or the heat kernel associated with the Dirichlet form $(\mathcal{E}_U, \mathcal{D}_U)$ on $L^2(X,\mu)$.

COROLLARY 10.6. *Assume that* $\overline{B_R(x,r)}$ *is compact for any* $x \in X$ *and any* $r > 0$. *Let* U *be a non-empty open subset of* X. *Then*

$$\lim_{t \downarrow 0} tp_U(t,x,x) = 0$$

for any $x \in X$.

PROOF. Choose $A = B_R(x,r)$. By (10.4), it follows that $tp_U(t,x,x) \leq 3r$ for sufficiently small t. $\square$

Now we begin our proof of Theorem 10.4 which consists of several lemmas. First we deal with the case where $\overline{U}$ is compact.

LEMMA 10.7. *If* $\overline{U}$ *is compact, then we have* $p_U(t,x,y) : (0,+\infty) \times X \times X$ *which satisfies* (TD1), (TD2), (TD3) *and* (TD4).

PROOF. Let H_U be the non-negative self-adjoint operator on $L^2(U, \mu_U)$ associated with the Dirichlet form $(\mathcal{E}_U, \mathcal{D}_U)$. Then by Lemma 9.7, H_U has compact resolvent. Hence, there exists a complete orthonormal system $\{\varphi_n\}_{n\geq 1}$ of $L^2(U, \mu_U)$ and $\{\lambda_n\}_{n\geq 1} \subseteq [0, +\infty)$ such that $\varphi_n \in \mathrm{Dom}(H_U) \subseteq \mathcal{D}_U$, $H_U\varphi_n = \lambda_n\varphi_n$, $\lambda_n \leq \lambda_{n+1}$ and $\lim_{n\to+\infty} \lambda_n = +\infty$.
Claim 1:

$$\sum_{n\geq 1} \frac{1}{\lambda_n + 1} < +\infty \tag{10.6}$$

Proof of Claim 1: By Lemma 9.6, for any $x \in U$, there exists $g^x_{1,U} \in \mathcal{D}_U$ such that $\mathcal{E}_1(g^x_{1,U}, u) = u(x)$ for any $u \in \mathcal{D}_U$. Since

$$\varphi_n(x) = \mathcal{E}_1(g^x_{1,U}, \varphi_n) = (\lambda_n + 1)\int_U g^x_{1,U}\varphi_n d\mu_U,$$

we have $g^x_{1,U} = \sum_{n\geq 1} \frac{\varphi_n(x)}{\lambda_n+1}\varphi_n$ in $L^2(U, \mu_U)$. Hence

$$g^x_{1,U}(x) = \mathcal{E}_1(g^x_{1,U}, g^x_{1,U}) = \sum_{n\geq 1} \frac{\varphi_n(x)^2}{\lambda_n + 1} \tag{10.7}$$

On the other hand, by the same argument as in the proof of Theorem 9.9, there exists $c_U > 0$ such that

$$|\mathcal{E}_1(u, g^x_{1,U})| \leq |u(x)| \leq ||u||_{\infty,K} \leq c_U\sqrt{\mathcal{E}_1(u, u)}$$

for any $u \in \mathcal{D}_U$, where $K = \overline{U}$. This implies that $\mathcal{E}_1(g^x_{1,U}, g^x_{1,U}) \leq c_U$. Combining this with (10.7), we see that $g^x_{1,U}(x)$ is uniformly bounded on U. Hence integrating (10.7) with respect to x, we obtain (10.6) by the monotone convergence theorem.
Claim 2: $||\varphi_n||_\infty \leq \sqrt{D\lambda_n}$ for any $n \geq 2$, where $D = \sup_{x,y\in U} R(x, y)$.
Proof of Claim 2: By (3.1),

$$|\varphi_n(x) - \varphi_n(y)|^2 \leq \mathcal{E}(\varphi_n, \varphi_n)R(x, y) = \lambda_n R(x, y). \tag{10.8}$$

We have two cases. First if $U \neq X$, then $\varphi_n(y) = 0$ for any $y \in U^c$. Hence (10.8) implies the claim. Secondly, if $U = X$, then (X, R) is compact. It follows that $\lambda_1 = 0$ and φ_1 is constant on X. Hence $\int_X \varphi_n(x)\mu(dx) = 0$ for any $n \geq 2$. For any $x \in X$, we may find $y \in X$ so that $\varphi_n(x)\varphi_n(y) \leq 0$. Since $|\varphi_n(x)|^2 \leq |\varphi_n(x) - \varphi_n(y)|^2$, (10.8) yields the claim.
Claim 3: $\sum_{n\geq 1} e^{-\lambda_n t}\varphi_n(x)\varphi_n(y)$ converges uniformly on $[T, +\infty) \times X \times X$ for any $T > 0$.
Proof of Claim 3: Note that $e^{-a} \leq 2/a^2$ for any $a > 0$. This fact with Claim 2 shows that $|e^{-\lambda_n t}\varphi_n(x)\varphi_n(y)| \leq 2/(\lambda_n t^2)$. Using Claim 1, we immediately obtain Claim 3.

Now, let $\tilde{p}_U(t, x, y) = \sum_{n\geq 1} e^{-\lambda_n t}\varphi_n(x)\varphi_n(y)$. By Claim 3, $\tilde{p}$ is continuous on $(0, +\infty) \times X \times X$. Also, $\tilde{p}_U$ is the integral kernel of the strongly continuous semigroup $\{T^U_t\}_{t>0}$ associated with the Dirichlet form $(\mathcal{E}_U, \mathcal{D}_U)$ on $L^2(U, \mu_U)$. Let A be a Borel set. Then

$$(T^U_t \chi_A)(x) = \int_X \tilde{p}_U(t, x, y)\chi_A(y)\mu(dy).$$

In particular $T^U_t\chi_A$ is continuous on X. By [**21**, Theorem 4.2.3], $p^U_t\chi_A$ is a quasi continuous version of $T^U_t\chi_A$. Since any quasi continuous function is continuous by Proposition 9.13, we have $(T^U_t\chi_A)(x) = (p^U_t\chi_A)(x)$ for any $x \in X$. Letting

$p_U(t,x,y) = \tilde{p}_U(t,x,y)$, we have (TD3). The rest of the desired properties are straightforward. □

The following facts are well-known in general setting. See [**21**] for example. In this particular situation, they can be checked by the eigenfunction expansion of the heat kernel above.

LEMMA 10.8. *Assume that $\overline{U}$ is compact.*
(1) For any $t > 0$ and any $x, y \in X$,

$$\frac{\partial p_U}{\partial t}(t,x,y) = -\mathcal{E}(p_U^{t/2,x}, p_U^{t/2,y}).$$

(2) For any $t, s > 0$ and any $x \in X$,

$$\mathcal{E}(p_U^{t,x}, p_U^{s,x}) \le \frac{2}{t+s} p_U\left(\frac{t+s}{2}, x, x\right).$$

LEMMA 10.9. *If $\overline{U}$ is compact, then* (10.4) *holds for any Borel subset A of X which satisfies $0 < \mu(A) < +\infty$.*

PROOF. Since $\int_A p_U(t,x,y)\mu(dy) \le \int_X p_U(t,x,y)\mu(dy) \le 1$, there exists $y_* \in A$ such that $p_U(t,x,y_*) \le 1/\mu(A)$. By this fact along with Lemma 10.8-(2),

$$\frac{1}{2}p_U(t,x,x)^2 \le p_U(t,x,y_*)^2 + |p_U(t,x,x) - p_U(t,x,y_*)|^2$$
$$\le \frac{1}{\mu(A)^2} + \overline{R}(x,A)\mathcal{E}(p_U^{t,x}, p_U^{t,x}) \le \frac{1}{\mu(A)^2} + \frac{\overline{R}(x,A)}{t} p_U(t,x,x)$$

Solving this with respect to $p_U(t,x,x)$, we have

$$p_U(t,x,x) \le \frac{\overline{R}(x,A)}{t} + \left(\frac{2}{\mu(A)^2} + \frac{\overline{R}(x,A)^2}{t^2}\right)^{\frac{1}{2}} \le \frac{2\overline{R}(x,A)}{t} + \frac{\sqrt{2}}{\mu(A)}.$$

□

REMARK. To get (10.5), we only need to use

$$p_U(t,x,x)^2 \le (1+\epsilon)p_U(t,x,y)^2 + \left(1 + \frac{1}{\epsilon}\right)|p_U(t,x,x) - p_U(t,x,y)|^2$$

in place of the inequality with $\epsilon = 1$ in the above proof.

Thus we have shown Theorem 10.4 if $\overline{U}$ is compact.

PROOF OF THEOREM 10.4. If $\overline{U}$ is compact, then we have completed the proof. Assume that $\overline{U}$ is not compact. Fix $x_* \in X$ and set $U_n = B_R(x_*, n) \cap U$ for any $n = 1, 2, \ldots$. Note that $\overline{U_n}$ is compact. Write $p_n(t,x,y) = p_{U_n}(t,x,y)$.
Claim 1 $\;p_n(t,x,y) \le p_{n+1}(t,x,y)$ for any $x, y \in X$ and any $n \ge 1$.
Proof of Claim 1. Let $\tau_n = \tau_{U_n}$. Then $\tau_n \le \tau_{n+1}$ for any n. Hence

$$(p_t^{U_n}u)(x) = E_x(\chi_{t<\tau_n}u(X_t)) \le E_x(\chi_{t<\tau_{n+1}}u(X_t)) = (p_t^{U_{n+1}}u)(x)$$

for any non-negative measurable function u and any $x \in X$. By (TD3), we deduce Claim 1.

Let A be a Borel subset of X which satisfies $0 < \mu(A) < +\infty$. By (TD4) and (10.4), we have

$$p_n(t,x,y) \le \sqrt{p_n(t,x,x)}\sqrt{p_n(t,y,y)} \le \left(\frac{2\overline{R}(x,A)}{t} + \frac{\sqrt{2}}{\mu(A)}\right)^{\frac{1}{2}}\left(\frac{2\overline{R}(y,A)}{t} + \frac{\sqrt{2}}{\mu(A)}\right)^{\frac{1}{2}} \tag{10.9}$$

for any $x \in X$, any $t > 0$ and any n. Hence $p_n(t,x,y)$ is uniformly bounded and monotonically nondecreasing as $n \to +\infty$. This shows that $p_n(t,x,y)$ converges as $n \to +\infty$. If $p(t,x,y) = \lim_{n\to+\infty} p_n(t,x,y)$, then $p(t,x,y)$ satisfies the same inequality as (10.9). In particular (10.4) holds for $p(t,x,x)$. Also, we immediately verity (TD2) and (TD4) for $p(t,x,y)$ from corresponding properties of $p_n(t,x,y)$. About (TD3), let u be a non-negative Borel-measurable function. Then by (TD3) for $p_n(t,x,y)$,

$$(p_t^{U_n}u)(x) = E_x(\chi_{\{t<\tau_n\}}u(X_t)) = \int_X p_n(t,x,y)u(y)\mu(dy)$$

for any $x \in X$. The monotone convergence theorem shows that

$$E_x(\chi_{t<\tau_U}u(X_t)) = \int_X p(t,x,y)u(y)\mu(dy).$$

Since the left-hand side of the about equality equals $(p_t^U u)(x)$, we have (TD3).

Finally we show (TD1). Fix $(t,x,y) \in (0,+\infty) \times X \times X$. Define $V = (t-\epsilon, t+\epsilon) \times B_R(x,r) \times B_R(y,r)$, where $r > 0$ and $0 < \epsilon < t$. (10.4) shows that

$$C = \sup_{(s,x',y') \in V_1, n\ge 1}\left(\sqrt{\frac{p_n(s,x',x')}{s}} + \sqrt{\frac{p_n(s,y',y')}{s}}\right) < +\infty,$$

where $V_1 = ((t-\epsilon)/2, t+\epsilon) \times B_R(x,r) \times B_R(y,r)$. By Lemma 10.8, for any $(s,a,b) \in V$ and any $n \ge 1$,

$$\begin{aligned}
&|p_n(t,x,y) - p_n(s,a,b)| \\
\le& |p_n(t,x,y) - p_n(t,x,b)| + |p_n(t,x,b) - p_n(t,a,b)| + |p_n(t,a,b) - p_n(s,a,b)| \\
\le& \sqrt{\mathcal{E}(p_{U_n}^{t,x}, p_{U_n}^{t,x})R(y,b)} + \sqrt{\mathcal{E}(p_{U_n}^{t,b}, p_{U_n}^{t,b})R(x,a)} + |t-s|\left|\frac{\partial p_n}{\partial t}(t',a,b)\right| \\
\le& \sqrt{\frac{p_n(t,x,x)R(y,b)}{t}} + \sqrt{\frac{p_n(t,b,b)R(x,a)}{t}} + 2|t-s|\frac{\sqrt{p_n(t'/2,a,a)p(t'/2,b,b)}}{t'} \\
\le& C\sqrt{R(x,a)} + C\sqrt{R(y,b)} + C^2|t-s|
\end{aligned}$$

where t' is a value between t and s. Letting $n \to +\infty$, we have

$$|p(t,x,y) - p(s,a,b)| \le C\sqrt{R(x,a)} + C\sqrt{R(y,b)} + C^2|t-s|.$$

Hence $p(t,x,y)$ is continuous on $(0,+\infty) \times X \times X$. By (TD4), $p_U^{t,x} \in L^2(X,\mu)$ for any $t > 0$ and any $x \in X$. Using (TD3) and (TD4), we see

$$p_U^{t,x} = p_{t/2}^U(p_U^{t/2,x}) = T_t(p_U^{t/2,x})$$

for any $t > 0$ and any $x, y \in X$, where $\{T_t\}_{t>0}$ is the strongly continuous semigroup associated with the Dirichlet form $(\mathcal{E}_U, \mathcal{D}_U)$ on $L^2(X,\mu)$. Hence $p_U^{t,x} \in \mathcal{D}$ for any $t > 0$ and any $x \in X$. $\square$

At the end of this section, we give a fundamental relations between the Green function, Laplacian and the transition density. By Theorem 10.3, $(\mathcal{E}_U, \mathcal{D}_U)$ is a regular Dirichlet form on $L^2(U, \mu|_U)$. ($\mathcal{D}_U$ and $\mathcal{E}_U$ are defined in Definition 10.2.)

THEOREM 10.10. *Assume that $U \subset X$, $\overline{U}$ is compact and that $U \neq X$. Let $B = X \backslash U$. Then*

$$g_B(x,y) = \int_0^\infty p_U(t,x,y)dt \tag{10.10}$$

for any $x, y \in X$. Moreover, let H_U be the nonnegative self-adjoint operator on $L^2(U, \mu_U)$ associated with the Dirichlet form $(\mathcal{E}_U, \mathcal{D}_U)$. Define $G_U f$ for $f \in L^2(U, \mu_U)$ by

$$(G_U f)(x) = \int_U g_B(x,y)f(y)\mu(dy).$$

for any $x \in U$. Then $\mathrm{Im}(G_U) = \mathrm{Dom}(H_U)$ and G_U is the inverse of H_U.

By the next corollary, we can identify $g_B(x,y)$ with the (0-order) Green function associated with the Dirichlet form $(\mathcal{E}_U, \mathcal{D}_U)$. See Bass [**11**, Sections II.3 & II.4]. Note that (10.10) and (10.11) are the counterparts of [**11**, II (4.5)Corollary and II (3.14)] respectively.

COROLLARY 10.11. *Under the same assumptions as in Theorem 10.10,*

$$E_x(\chi_{t<\tau_U} f(X_s)) = \int_U g_B(x,y)f(y)\mu(dy). \tag{10.11}$$

for any nonnegative Borel measurable function $f : X \to \mathbb{R}$ and any $x \in X$. In particular,

$$E_x(\tau_U) = \int_U g_B(x,y)\mu(dy). \tag{10.12}$$

(10.12) connects the mean exit time with the Green function. It plays an important role in the heat kernel estimate, in particular, in the proof of Lemma 18.1.

PROOF OF THEOREM 10.10. We use the same notations as in the proof of Lemma 10.7. Since $B \neq \emptyset$, (RF2) implies that $\mathcal{E}_U(u,u) > 0$ for any $u \in \mathcal{D}_U$. Therefore, $\lambda_1 > 0$. Note that $g_B^x \in \mathcal{F}(B) = \mathcal{D}_U$. We have

$$\varphi_n(x) = \mathcal{E}(g_B^x, \varphi_n) = \int_U g_B^x(y)(H_U\varphi_n)(y)\mu(dy) = \lambda_n \int_U g_B^x(y)\varphi_n(y)\mu(dy). \tag{10.13}$$

Hence $g_B^x = \sum_{n \geq 1} (\lambda_n)^{-1}\varphi_n(x)\varphi_n$ in $L^2(U, \mu_U)$. This implies

$$g_B(x,y) = \mathcal{E}(g_B^x, g_B^y) = \sum_{n \geq 1} \frac{\varphi_n(x)\varphi_n(y)}{\lambda_n} \tag{10.14}$$

for any $x, y \in X$. (Note that if $x \notin X$, then both sides of the above equality is 0.) By the proof of Lemma 10.7,

$$p_U(t,x,y) = \sum_{n \geq 0} e^{-\lambda_n t}\varphi_n(x)\varphi_n(y),$$

where the infinite sum is uniformly convergent on $[T, +\infty) \times X \times X$ for any $T > 0$. Moreover, we have $|e^{-\lambda_n t}\varphi_n(x)\varphi_n(y)| \leq 2/(\lambda_n t^2)$ by the proof of the Claim 3 of Lemma 10.7. This shows that

$$\int_T^\infty p_U(t,x,y)dt = \sum_{n\geq 1} \frac{e^{-\lambda_n T}}{\lambda_n}\varphi_n(x)\varphi_n(y).$$

Hence

$$\Big| \int_T^\infty p_U(t,x,y)dt - \sum_{n\geq 1} \frac{\varphi_n(x)\varphi_n(y)}{\lambda_n} \Big| \leq \sqrt{\sum_{n\geq 1} \frac{(1-e^{-\lambda_n T})\varphi_n(x)^2}{\lambda_n}} \sqrt{\sum_{n\geq 1} \frac{(1-e^{-\lambda_n T})\varphi_n(y)^2}{\lambda_n}}$$

By the monotone convergence theorem, letting $T \downarrow 0$, we obtain (10.10).

Now, note that

$$\mathrm{Dom}(H_U) = \Big\{u \Big| u \in L^2(U, \mu_U), u = \sum_{n\geq 1} \alpha_n\varphi_n, \sum_{n\geq 1}(\lambda_n\alpha_n)^2 < +\infty\Big\}$$

and that $H_U u = \sum_{n\geq 1} \lambda_n\alpha_n\varphi_n$ if $u = \sum_{n\geq 1}\alpha_n\varphi_n$. Let $f = \sum_{n\geq 1} a_n\varphi_n \in L^2(U, \mu_U)$. Then

$$(G_U f)(x) = \int_U g_B^x f d\mu = \sum_{n\geq 1} \frac{a_n}{\lambda_n}\varphi_n(x).$$

By the above facts, we see that $G_U f \in \mathrm{Dom}(H_U)$ and $H_U G_U f = f$. Similarly, it is easy to see that $G_U H_U u = u$ for any $u \in \mathrm{Dom}(H_U)$. Thus G_U is the inverse of G_U. □

Proof of Corollary 10.11. By (10.1), it follows that

$$\int_U p_U(t,x,y)f(y)\mu(dy) = E_x(\chi_{t<\tau_U} f(X_s)).$$

Integrating this from 0 to $+\infty$ with respect to t and using the Fubini theorem, we have (10.11). (10.12) follows by letting $f = 1$. □

Part 2

Quasisymmetric metrics and volume doubling measures

The main subject of this part is the notion of quasisymmetric maps, which has been introduced in [**51**] as certain generalization of quasiconformal mappings of the complex plane. The results in this part will play an indispensable role in the next part, where we will modify the original resistance metric quasisymmetrically to obtain a metric which is suitable for describing asymptotic behaviors of the associated heat kernel.

At the first section, we present several notions, whose combinations are shown to be equivalent to being quasisymmetric in the second section, where we give the precise definition of quasisymmetry in Definition 12.1. Establishing such an equivalence, we resolve the notion of being quasisymmetric into geometric and analytic components. In the latter two sections, we discuss relations between a metric and a measure. Under the volume doubling property of a measure, we will construct a quasisymmetric metric which satisfies prescribed relation between a measure and a metric.

Since [**51**], quasisymmetric maps and related subjects have been studied deeply by many authors. See Heinonen [**32**] for example. Some of the results in this section may be included in the preceding articles. However, we give all the proofs since it is difficult to find an exact reference from such a huge literature.

11. Semi-quasisymmetric metrics

In this section, we introduce several notions associated with quasisymmetric mappings and clarify their relations.

NOTATION. Let X be a set and let d be a distance on X. $\overline{B}_d(x,r)$ is the closed ball, i.e. $\overline{B}_d(x,r) = \{y|y \in X, d(x,y) \leq r\}$. For any $A \subseteq X$, $\mathrm{diam}(A,d)$ is the diameter of A with respect to d defined by $\mathrm{diam}(A,d) = \sup_{x,y \in A} d(x,y)$. Moreover, we set $d_*(x) = \sup_{y \in X} d(x,y)$ for any $x \in X$.

In the rest of this section, we assume that d and ρ are distances on a set X.

The following notion "semi-quasisymmetric" is called "weakly quasisymmetric" in [**51**] and can be traced back to [**14**] and [**33**]. See [**51**] for details.

DEFINITION 11.1. ρ is said to be semi-quasisymmetric with respect to d, or $(\mathrm{SQS})_d$ for short, if and only if there exist $\epsilon \in (0,1)$ and $\delta > 0$ such that $\rho(x,z) < \epsilon\rho(x,y)$ whenever $d(x,z) < \delta d(x,y)$.

In the above definition, we may assume $\delta < 1$ without loss of generality.

PROPOSITION 11.2. *If ρ is $(\mathrm{SQS})_d$, then the identity map from (X,d) to (X,ρ) is continuous.*

This fact has been obtained in [**51**].

PROOF. Assume that $d(x_n,x) \to 0$ as $n \to +\infty$ and $\rho(x_n,x) \to a$ as $n \to +\infty$, where $a > 0$. Choose $\epsilon_1 \in (\epsilon,1)$. Then $\rho(x_{n+m},x) > \epsilon_1\rho(x_n,x)$ and $d(x_{n+m},x) < \delta d(x_n,x)$ for sufficiently large n and m. By $(\mathrm{SQS})_d$, it follows that $\rho(x_{n+m},x) < \epsilon\rho(x_n,x)$. This contradiction implies the desired conclusion. □

PROPOSITION 11.3. *Assume that (X,d) is uniformly perfect. Then ρ is $(\mathrm{SQS})_d$ if and only if, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\rho(x,z) < \epsilon\rho(x,y)$ whenever $d(x,z) < \delta d(x,y)$.*

PROOF. Assume that ρ is $(\mathrm{SQS})_d$. We will show that $d(x,z) < (c\delta)^n d(x,y)$ implies $\rho(x,z) < \epsilon^n \rho(x,y)$ by induction, where c is the constant appearing in Definition 7.1. The case $n = 1$ is obvious. Assume that it is true for n. Suppose that $d(x,z) < (c\delta)^{n+1} d(x,y)$. Since (X,d) is uniformly perfect, there exits $y' \in X$ such that $c(c\delta)^n d(x,y) \le d(x,y') < (c\delta)^n d(x,y)$. By induction assumption, $\rho(x,y') < \epsilon^n \rho(x,y)$. Also since $d(x,z) < \delta d(x,y')$. we have $\rho(x,z) < \epsilon\rho(x,y')$. Therefore $\rho(x,z) < \epsilon^{n+1}\rho(x,y)$.

The converse is obvious. $\square$

Next we consider a geometric interpretation of semi-quasisymmetry. We say that ρ is semi-quasiconformal with respect to d, $(\mathrm{SQC})_d$ for short, if ρ-balls are equivalent to d-balls with a uniform distortion. (The precise definition is given in Definition 11.4.) $(\mathrm{SQS})_d$ implies $(\mathrm{SQC})_d$ but not vise versa. To get an "if and only if" assertion, we need a kind of uniform distortion condition regarding annuli instead of balls, called annulus semi-quasiconformality. To give a precise statement, we introduce the following notions.

DEFINITION 11.4. (1) Define $\overline{d}_\rho(x,r) = \sup_{y \in B_\rho(x,r)} d(x,y)$ for $x \in X$ and $r > 0$. d is said to be doubling with respect to ρ if and only if there exist $\alpha > 1$ and $c > 0$ such that $\overline{d}_\rho(x,\alpha r) \le c\overline{d}_\rho(x,r) < +\infty$ for any $r > 0$ and any $x \in X$.
(2) ρ is said to be semi-quasiconformal with respect to d, or $(\mathrm{SQC})_d$ for short, if and only if $\overline{d}_\rho(x,r) < +\infty$ for any $x \in X$ and any $r > 0$ and there exists $\delta \in (0,1)$ such that $B_d(x, \delta\overline{d}_\rho(x,r)) \subseteq B_\rho(x,r)$ for any $x \in X$ and $r > 0$.
(3) ρ is said to be annulus semi-quasiconformal with respect to d, or $(\mathrm{ASQC})_d$ for short, if and only if $\overline{d}_\rho(x,r) < +\infty$ for any $x \in X$ and $r > 0$ and, for any $\epsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that $d(x,y) \ge \delta\overline{d}_\rho(x,r)$ whenever $\rho(x,y) \ge \epsilon r$.
(4) ρ is called weak annulus semi-quasiconformal with respect to d, or $(\mathrm{wASQC})_d$ for short, if and only if $\overline{d}_\rho(x,r) < +\infty$ for any $x \in X$ and $r > 0$ and there exist $\epsilon \in (0,1)$ and $\delta \in (0,1)$ such that $d(x,y) \ge \delta\overline{d}_\rho(x,r)$ whenever $\rho(x,y) \ge \epsilon r$.

REMARK. (1) If d is doubling with respect to ρ, then

$$\overline{d}_\rho(x,ar) \le c_0 a^\omega \overline{d}_\rho(x,r)$$

for any $r > 0, a \ge 1$ and $x \in X$, where c_0 and ω are positive constants which are independent of x, a and r. Hence the value of a itself is not essential. An easy choice of a is two, and this is why we call this notion "doubling".
(2) Note that $B_\rho(x,r) \subseteq \overline{B}_d(x, \overline{d}_\rho(x,r))$. Hence ρ is $(\mathrm{SQC})_d$ if and only if, for any $x \in X$ and $r > 0$, there exist R_1 and R_2 such that $B_d(x,R_1) \subseteq B_\rho(x,r) \subseteq B_d(x,R_2)$ and $R_1 \ge CR_2$, where $C \in (0,1)$ is independent of x and r. Therefore in this case, a ρ-ball is equivalent to a d-ball with a uniformly bounded distortion.
(3) Assume that $\overline{d}_\rho(x,r) < +\infty$ for any x and r. Then $(\mathrm{ASQC})_d$ is equivalent to the following statement: for any $\epsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that $d(x,y) \ge \delta\overline{d}_\rho(x,r)$ whenever $r > \rho(x,y) \ge \epsilon r$. Also $(\mathrm{ASQC})_d$ implies that a ρ-annulus $B_\rho(x,r) \backslash B_\rho(x,\epsilon r)$ is contained in a d-annulus $B_d(x,(1+\gamma)\overline{d}_\rho(x,r)) \backslash B_d(x,\delta\overline{d}_\rho(x,r))$ for any $\gamma > 0$.

THEOREM 11.5. *Assume that both (X,d) and (X,ρ) are uniformly perfect and that $\overline{d}_\rho(x,r) < +\infty$ for any $x \in X$ and $r > 0$. Then the following four conditions are equivalent.*
(a) *ρ is* $(\mathrm{SQS})_d$.

(b) *d is doubling with respect to ρ and ρ is* $(\mathrm{SQC})_d$.
(c) *ρ is* $(\mathrm{ASQC})_d$.
(d) *ρ is* $(\mathrm{wASQC})_d$.

PROOF. *(a)* $\Rightarrow$ *(b)*: First we show that d is doubling with respect to ρ. Since (X, ρ) is uniformly perfect, $B_\rho(x,r)\backslash B_\rho(x,cr) \neq \emptyset$ unless $X\backslash B_\rho(x,r) = \emptyset$, where c is independent of x and r. By Proposition 11.3, we may assume $\epsilon < c^2$. Now by $(\mathrm{SQS})_d$, $\rho(x,z)/\epsilon \geq \rho(x,y)$ implies $d(x,z) \geq \delta d(x,y)$.
Claim: Suppose $r/\sqrt{\epsilon} > \rho(x,y)$. Then $\rho(x,z)/\epsilon > \rho(x,y)$ for some $z \in B_\rho(x,r)$.
Proof of the claim: If $X\backslash B_\rho(x,r) \neq \emptyset$, then there exists $z \in X$ such that $r > \rho(x,z) \geq cr$. Hence $\rho(x,z)/\epsilon \geq cr/\epsilon > r/\sqrt{\epsilon} > \rho(x,y)$. In case $X = B_\rho(x,r)$, let $\rho_*(x) = \sup_{x' \in X} \rho(x,x')$. Then $\rho_*(x)/\epsilon > \rho_*(x) \geq \rho(x,y)$. Hence there exists $z \in B_\rho(x,r) = X$ such that $\rho(x,z)/\epsilon > \rho(x,y)$. Thus we have shown the claim.
If $\rho(x,z)/\epsilon > \rho(x,y)$, $(\mathrm{SQS})_d$ implies $d(x,z) \geq \delta d(x,y)$. By the above claim, we obtain that $\overline{d}_\rho(x,z) \geq \delta\overline{d}_\rho(x,y/\sqrt{\epsilon})$. Hence d is doubling with respect to ρ.

Next we show that ρ is $(\mathrm{SQC})_d$. Suppose that $d(x,z) < \delta\overline{d}_\rho(x,r)$. Then there exists $y \in B_\rho(x,r)$ such that $d(x,z) < \delta d(x,y)$. Hence by $(\mathrm{SQS})_d$, $\rho(x,z) < \epsilon\rho(x,y) < r$ and hence $z \in B_\rho(x,r)$.
(b) $\Rightarrow$ *(c)*: Let $\epsilon \in (0,1)$. By $(\mathrm{SQC})_d$, $B_\rho(x,\epsilon r) \supseteq B_d(x, \delta\overline{d}_\rho(x,\epsilon r))$. Since d is doubling with respect to ρ, $\overline{d}_\rho(x,\epsilon r) > c'\overline{d}_\rho(x,r)$, where c' is independent of x and r. Therefore, $B_\rho(x,\epsilon r) \supseteq B_d(x, \delta c'\overline{d}_\rho(x,r))$. This immediately implies $(\mathrm{ASQC})_d$.
(c) $\Rightarrow$ *(d)*: This is obvious.
(d) $\Rightarrow$ *(a)*: Let $\rho(x,z) \geq \epsilon r$. Then $\epsilon^{-n+1}r \leq \rho(x,z) < \epsilon^{-n}r$ for some $n \geq 0$. By $(\mathrm{ASQC})_d$, $\delta\overline{d}_\rho(x,r) \leq \delta\overline{d}_\rho(x,\epsilon^{-n}r) \leq d(x,z)$. Hence, $d(x,z) < \delta\overline{d}_\rho(x,r)$ implies $\rho(x,z) < \epsilon r$. Now suppose $d(x,z) < \delta d(x,y)$. Since $\delta d(x,y) \leq \delta\overline{d}_\rho(x,\rho(x,y))$, we have $\rho(x,y) < \epsilon\rho(x,y)$. □

Next we present useful implications of $(\mathrm{SQS})_d, (\mathrm{SQC})_d$ and $(\mathrm{ASQC})_d$.

DEFINITION 11.6. ρ is said to decay uniformly with respect to d if and only if (i) $\mathrm{diam}(X,d) < +\infty$ and there exist $r_* > \mathrm{diam}(X,d)$ and $(a,\lambda) \in (0,1)^2$ such that $\overline{\rho}_d(x,\lambda r) \leq a\overline{\rho}_d(x,r)$ for any $x \in X$ and $r \in (0, r_*]$
or
(ii) $\mathrm{diam}(X,d) = +\infty$ and there exists $(a,\lambda) \in (0,1)^2$ such that $\overline{\rho}_d(x,\lambda r) \leq a\overline{\rho}_d(x,r)$ for any $x \in X$ and $r > 0$.

PROPOSITION 11.7. *Assume that (X,d) is uniformly perfect and ρ is* $(\mathrm{SQS})_d$. *Then ρ decays uniformly with respect to d. More precisely, if* $\mathrm{diam}(X,d) < +\infty$, *then, for any $r_* > 0$, there exists $(a,\lambda) \in (0,1)^2$ such that $\overline{\rho}_d(x,\lambda r) \leq a\overline{\rho}_d(x,r)$ for any $x \in X$ and $r \in (0,r_*]$.*

REMARK. If ρ is $(\mathrm{SQS})_d$, then $B_d(x,\delta d(x,y)) \subseteq B_\rho(x,\epsilon\rho(x,y))$. Hence if $r < \delta d_*(x)$, we have $\overline{\rho}_d(x,r) < +\infty$. Note that $d_*(x) \geq \mathrm{diam}(X,d)/2$.

PROOF. Since ρ is $(\mathrm{SQS})_d$, there exist $\epsilon \in (0,1)$ and $\delta \in (0,1)$ such that $\rho(x,z) < \epsilon\rho(x,y)$ whenever $d(x,z) < \delta d(x,y)$. Also there exists $c \in (0,1)$ such that $B_d(x,r)\backslash B_d(x,cr) \neq \emptyset$ unless $X = B_d(x,r)$. Suppose that $\mathrm{diam}(X,d) < +\infty$. Choose $n \geq 1$ so that $c^{n-1}r_* < \mathrm{diam}(X,d)/2$. Since $d_*(x) \geq \mathrm{diam}(X,d)/2$, it follows that $X \neq B_d(x,c^{n-1}r)$ for any $r \in (0,r_*]$. Therefore, there exists $y \in X$ such that $c^n r \leq d(x,y) < c^{n-1}r$. If $d(x,z) < c^n\delta r$, we have $d(x,z) < \delta d(x,y)$. Hence, $\rho(x,z) < \epsilon\rho(x,y)$. This shows that $\overline{\rho}_d(x,c^n\delta r) \leq \epsilon\rho(x,y) \leq \epsilon\overline{\rho}_d(x,r)$. Similar argument suffices as well in the case where $\mathrm{diam}(X,d) = +\infty$. □

PROPOSITION 11.8. *Assume that d is doubling with respect to ρ and ρ is $(\mathrm{SQC})_d$. Let μ be a Borel regular measure on X. Then μ is $(\mathrm{VD})_\rho$ if μ is $(\mathrm{VD})_d$.*

Combining this proposition with Theorem 11.5, we see that the volume doubling property is inherited from d to ρ if ρ is $(\mathrm{SQS})_d$ under uniform perfectness.

PROOF. Since d is $(\mathrm{VD})_\rho$,

$$B_\rho(x, 2r) \subseteq B_d(x, 2\overline{d}_\rho(x, 2r)) \subseteq B_d(x, c'\overline{d}_\rho(x, r)),$$

where $c' > 1$. If μ is $(\mathrm{VD})_d$, then

$$\mu(B_d(x, c'\overline{d}_\rho(x, r))) < c\mu(B_d(x, \delta\overline{d}_\rho(x, r))).$$

Moreover, by $(\mathrm{SQC})_d$, $B_d(x, \delta\overline{d}_\rho(x, r)) \subseteq B_\rho(x, r)$. Thus, we have

$$\mu(B_\rho(x, 2r)) \leq c\mu(B_d(x, \delta\overline{d}_\rho(x, r)) \leq c\mu(B_\rho(x, r)).$$

$\square$

The following lemma is quite similar to Theorem 11.5 but is a little stronger since it does not assume that (X, d) is uniformly perfect. We will take advantage of this stronger statement later in the proofs of Proposition 11.10 and Lemma 13.9.

LEMMA 11.9. *Assume that (X, ρ) is uniformly perfect. If ρ is $(\mathrm{ASQC})_d$, then d is doubling with respect to ρ.*

PROOF. By the assumption, $B_\rho(x, r)\backslash B_\rho(x, cr) \neq \emptyset$ unless $B_\rho(x, cr) = X$ for some $c \in (0, 1)$. Let $\epsilon = c^2$. By $(\mathrm{ASQC})_d$, for some $\delta \in (0, 1)$, $d(x, y) \geq \delta\overline{d}_\rho(x, r)$ when $\rho(x, y) \geq \epsilon r$. If $B_\rho(x, r) = B_\rho(x, cr)$, then $\overline{d}_\rho(x, r) = \overline{d}_\rho(x, cr) \geq \delta\overline{d}_\rho(x, r)$. If $B_\rho(x, r)\backslash B_\rho(x, cr) \neq \emptyset$, then there exists $y \in X$ such that $\epsilon r \leq \rho(x, y) \leq cr$. This also implies that $\overline{d}_\rho(x, cr) \geq \delta\overline{d}_\rho(x, r)$. Hence we have the doubling property of $\overline{d}_\rho$. $\square$

PROPOSITION 11.10. *Let (X, ρ) be uniformly perfect. Assume that $\overline{d}_\rho(x, r) < +\infty$ for any $x \in X$ and $r > 0$. If ρ is $(\mathrm{ASQC})_d$, then*

$$d(x, y) \asymp \overline{d}_\rho(x, \rho(x, y))$$

for any $x, y \in X$.

PROOF. There exist $\epsilon \in (0, 1)$ and $\delta \in (0, 1)$ such that $\epsilon r \leq \rho(x, y) < r$ implies $d(x, y) \geq \delta\overline{d}_\rho(x, r)$. Choose $\alpha > 1$ so that $\alpha\epsilon < 1$. For any $y \in X$, we have $\epsilon\alpha\rho(x, y) \leq \rho(x, y) < \alpha\rho(x, y)$. Hence $d(x, y) \geq \delta\overline{d}_\rho(x, \alpha\rho(x, y)) \geq \delta\overline{d}_\rho(x, \rho(x, y))$. By Lemma 11.9, d is doubling with respect to ρ. Therefore, $c_2\overline{d}_\rho(x, \rho(x, y)) \leq \overline{d}_\rho(x, \alpha\rho(x, y)) \leq d(x, y)$, where c_2 only depends on α. $\square$

12. Quasisymmetric metrics

In this section, we will introduce the notion of being quasisymmetric and relate it to the notions obtained in the last section.

d and ρ are distances on a set X through this section.

DEFINITION 12.1. ρ is said to be quasisymmetric, or QS for short, with respect to d if and only if there exists a homeomorphism h from $[0, +\infty)$ to itself such that $h(0) = 0$ and, for any $t > 0$, $\rho(x, z) < h(t)\rho(x, y)$ whenever $d(x, z) < td(x, y)$. We write $\rho \underset{\mathrm{QS}}{\sim} d$ if ρ is quasisymmetric with respect to d.

The followings are basic properties of quasisymmetric distances.

PROPOSITION 12.2. *Assume that ρ is quasisymmetric with respect to d. Then*
(1) *d is quasisymmetric with respect to ρ.*
(2) *The identity map from (X, d) to (X, ρ) is a homeomorphism.*
(3) *(X, d) is uniformly perfect if and only if (X, ρ) is uniformly perfect.*
(4) *(X, d) is bounded if and only if (X, ρ) is bounded.*
(5) *Define $\overline{d}_\rho(x, r) = \sup_{y \in B_\rho(x,r)} d(x, y)$ and $\overline{\rho}_d(x, r) = \sup_{y \in B_d(x,r)} \rho(x, y)$. Then $\overline{d}_\rho(x, r)$ and $\overline{\rho}_d(x, r)$ are finite for any $x \in X$ and any $r > 0$.*

Those statements, in particular (1) and (3), have been obtained in the original paper [**51**].

PROOF. (1) Note that $\rho(x, z) \geq h(t)\rho(x, y)$ implies $d(x, z) \geq td(x, y)$. Hence if $h(t)^{-1}\rho(x, z) > \rho(x, y)$, then $2t^{-1}d(x, z) > d(x, y)$. Set $g(s) = 2/h^{-1}(1/t)$. Then $g(s)$ is a homeomorphism from $[0, +\infty)$ to itself and $g(s)d(x, z) > d(x, y)$ whenever $t\rho(x, z) > \rho(x, y)$. Thus d is QS with respect to ρ.
(2) If $\rho \underset{\text{QS}}{\sim} d$, then ρ is $(\text{SQS})_d$ and d is $(\text{SQS})_\rho$. Now, Proposition 11.2 suffices.
(3) There exists $\delta \in (0, 1)$ such that $B_d(x, r/\delta)\backslash B_d(x, r) \neq \emptyset$ if $B_d(x, r) \neq X$ by the uniform perfectness. Choose $t_* \in (0, 1)$ so that $h(t_*) < 1$. Suppose $B_\rho(x, r) \neq X$. There exists $y \in X$ such that $\rho(x, y) > r$. Let $r = \delta t_* d(x, y)$. Since $r < d(x, y)$, $B_d(x, r) \neq \emptyset$. Hence there exists $y_1 \in X$ such that $\delta t_* d(x, y) \leq d(x, y_1) < t_* d(x, y)$. Since $\rho \underset{\text{QS}}{\sim} d$, we have $\lambda_1 \rho(x, y) < \rho(x, y_1) < \lambda_2 \rho(x, y)$, where $0 < \lambda_1 = h(2/(\delta t_*)) < \lambda_2 = h(t_*) < 1$. In the same way, we have y_2 which satisfies $\lambda_1 \rho(x, y_1) < \rho(x, y_2) < \lambda_2 \rho(x, y_1)$. Inductively, we may construct $\{y_n\}_{n \geq 1}$ such that $\lambda_1 \rho(x, y_n) < \rho(x, y_{n+1}) < \lambda_2 \rho(x, y_n)$. Choose m so that $\rho(x, y_{m+1}) < r \leq \rho(x, y_m)$. Then $y_m \in B_\rho(x, r/\lambda_1)\backslash B_\rho(x, r)$. Hence (X, ρ) is uniformly perfect.
(4), (5) Obvious. □

By (1) of the above proposition, $\underset{\text{QS}}{\sim}$ is an equivalence relation.

The following theorem relates the notion of begin semi-quasisymmetric with being quasisymmetric. It has essentially been obtained in [**51**, Theorem 3.10], where the notion of "uniformly perfect" is called "homogeneously dense".

THEOREM 12.3. *Assume that both (X, d) and (X, ρ) are uniformly perfect. Then ρ is QS with respect to d if and only if ρ is $(\text{SQS})_d$ and d is $(\text{SQS})_\rho$.*

PROOF. If ρ is QS with respect to d, then it is straightforward to see that d and ρ are semi-quasisymmetric with respect to each other. Conversely, assume that d and ρ are semi-quasisymmetric with respect to each other. Then by Proposition 11.3, we may construct homeomorphisms $h_1 : [0, \delta_1] \to [0, \epsilon_1]$ and $h_2 : [0, \delta_2] \to [0, \epsilon_2]$ which satisfy
(i) $h_1(0) = 0, h_2(0) = 0$,
(ii) $\rho(x, z) < h_1(\delta)\rho(x, y)$ whenever $d(x, z) < \delta d(x, y)$ for any $\delta \in (0, \delta_1]$ and
(iii) $d(x, z) < h_2(\delta)d(x, y)$ whenever $\rho(x, z) < \delta\rho(x, y)$ for any $\delta \in (0, \delta_2]$.
Define

$$h_3(t) = \begin{cases} 2/(h_2)^{-1}(1/t) & \text{for } t \in [1/\delta_2, +\infty), \\ 2/\epsilon_2 & \text{for } t \in [0, 1/\delta_2]. \end{cases}$$

Then $\rho(x, z) < h_3(\delta)\rho(x, y)$ whenever $d(x, z) < \delta d(x, y)$ for any $\delta \in (0, +\infty)$. There is no difficulty to find a homeomorphism $h : [0, +\infty) \to [0, +\infty)$ with $h(0) = 0$

which satisfies that $h(t) \geq h_1(t)$ for any $t \in [0, \delta_1]$ and that $h(t) \geq h_3(t)$ for any $t \in [\delta_1, +\infty)$. Obviously $d(x, z) < td(x, y)$ implies $\rho(x, z) < h(t)\rho(x, y)$ for any $t > 0$. Therefore, ρ is QS with respect to d. $\square$

Combining this theorem with Theorem 11.5, we can produce several equivalent conditions for quasisymmetry under uniform perfectness.

The next corollary is a modified version of Proposition 11.8.

COROLLARY 12.4. *Assume that (X, d) is uniformly perfect and that $\rho \underset{\mathrm{QS}}{\sim} d$. Let μ be a Borel regular measure on (X, d). Then μ is* $(\mathrm{VD})_d$ *if and only if it is* $(\mathrm{VD})_\rho$.

PROOF. By Proposition 12.2-(3), (X, ρ) is uniformly perfect. Hence by Theorem 12.3, ρ is $(\mathrm{SQS})_d$ and d is $(\mathrm{SQS})_\rho$. Theorem 11.5 shows that d is doubling with respect to ρ and ρ is $(\mathrm{SQC})_d$. By Proposition 11.8, if μ is $(\mathrm{VD})_d$, then μ is $(\mathrm{VD})_\rho$. The converse follows by exchanging d and ρ. $\square$

13. Relations of measures and metrics

To obtain a heat kernel estimate, one often show a certain kind of relations concerning a measure and a distance. The typical example in the following relation:

$$d(x, y)\mu(B_\rho(x, \rho(x, y))) \asymp \rho(x, y)^\beta, \tag{13.1}$$

where $d(x, y)$ is the resistance metric (may be written as $R(x, y)$), ρ is a distance used in the heat kernel estimate and β is a positive exponent. The left hand side corresponds to the mean exit time from a ρ-ball. We generalize such a relation and study it in the light of quasisymmetry in the present section.

Throughout this section, d and ρ are distances on a set X which give the same topology on X. μ is a Borel regular measure on (X, d). We assume that $0 < \mu(B_d(x, r)) < +\infty$ and $0 < \mu(B_\rho(x, r)) < +\infty$ for any $x \in X$ and any $r > 0$.

NOTATION. We set $V_d(x, r) = \mu(B_d(x, r))$ and $V_\rho(x, r) = \mu(B_\rho(x, r))$.

Let $H : (0, +\infty)^2 \to (0, +\infty)$ satisfy the following two conditions:

(H1) if $0 < s_1 \leq s_2$ and $0 < t_1 \leq t_2$, then $H(s_1, t_1) \leq H(s_2, t_2)$,

(H2) for any $(a, b) \in (0, +\infty)^2$, define

$$h(a, b) = \sup_{(s,t)\in(0,+\infty)^2} \frac{H(as, bt)}{H(s, t)}.$$

Then $h(a, b) < +\infty$ for any $(a, b) \in (0, +\infty)^2$ and there exists $c_0 > 0$ such that $h(a, b) < 1$ for any $(a, b) \in (0, c_0)^2$.

Also $g : (0, +\infty) \to (0, +\infty)$ is a monotonically increasing function satisfying $g(t) \downarrow 0$ as $t \downarrow 0$ and the doubling property, i.e. there exists $c > 0$ such that $g(2t) \leq cg(t)$ for any $t > 0$.

REMARK. The condition (H1) is the monotonicity of the function H. If (H2) is satisfied, then

$$h(a, b)^n H(s, t) \geq H(a^n s, b^n t)$$

for any a, b, s, t and n. Since $h(a, b) < 1$ for $(a, b) \in (0, c_0)^2$, this inequality yields a kind of homogeneity of the decay of $H(s, t)$ as $(s, t) \to (0, 0)$.

We will study several relations between conditions concerning d, ρ, μ, H and g.

Definition 13.1. (1) We say that the condition (DM1) holds if and only if there exists $\eta : (0,1] \to (0,+\infty)$ such that η is monotonically nondecreasing, $\eta(t) \downarrow 0$ as $t \downarrow 0$ and

$$\eta(\lambda)\frac{g(\rho(x,y))}{H(d(x,y),V_\rho(x,\rho(x,y)))} \geq \frac{g(\lambda\rho(x,y)))}{H(d(x,y),V_\rho(x,\lambda\rho(x,y)))}$$

for any $x, y \in X$ and any $\lambda \in (0,1]$.
(2) We say that the condition (DM2) holds if and only if

$$H(d(x,y),V_\rho(x,\rho(x,y))) \asymp g(\rho(x,y))$$

for any $x, y \in X$.
(3) We say that the condition (DM3) holds if and only if there exist $r_* > \mathrm{diam}(X,\rho)$ such that

$$H(\overline{d}_\rho(x,r),V_\rho(x,r)) \asymp g(r)$$

for any $x \in X$ and any $r \in (0,r_*]$.

The relation (DM2) can be seen as a generalization of the above mentioned relation (13.1), where $H(s,t) = st$ and $g(r) = r^\beta$. The relation (DM1) looks too complicated but it is shown to be necessary if $d \underset{\mathrm{QS}}{\sim} \rho$ and (DM2) is satisfied. See Corollary 13.3.

Remark. If $\mathrm{diam}(X,\rho) = +\infty$, then we remove the statement "there exists $r_* > \mathrm{diam}(X,\rho)$ such that" and replace "$r \in (0,r_*]$" by "$r > 0$" in (3) of the above definition.

In the next section, we are going to construct a distance ρ on X which satisfies all three conditions (DM1), (DM2) and (DM3) with $g(r) = r^\beta$ for sufficiently large β under a certain assumptions. See Theorem 14.1 for details.

The next theorem gives the basic relations. Much clearer description from quasisymmetric point of view can be found in the corollary below.

Theorem 13.2. *Assume that (X,ρ) is uniformly perfect, that $\lim_{s\downarrow 0} h(s,1) = \lim_{t\downarrow 0} h(1,t) = 0$ and that there exists $c_* > 0$ such that $\mu(X) \leq c_* V_\rho(x,\rho_*(x))$ for any $x \in X$, where $\rho_*(x) = \sup_{y\in X}\rho(x,y)$.*
(1) *μ is $(\mathrm{VD})_\rho$ if (DM1) and (DM2) hold.*
(2) *(DM1) and (DM2) hold if and only if (DM3) holds, d decays uniformly with respect to ρ and ρ is $(\mathrm{ASQC})_d$.*

Remark. If $\mathrm{diam}(X,\rho) = +\infty$, then $\rho_*(x) = +\infty$ for any $x \in X$. In this case, we define $B_\rho(x,+\infty) = X$ and $V_\rho(x,+\infty) = \mu(X)$. Hence letting $c_* = 1$, we always have $\mu(X) \leq c_* V_\rho(x,\rho_*(x))$.

On the other hand, if $\mathrm{diam}(X,\rho) < +\infty$, then $\mathrm{diam}(X,\rho)/2 \leq \rho_*(x) \leq \mathrm{diam}(X,\rho)$. In this case, $X = \overline{B}(x,\rho_*(x))$.

Corollary 13.3. *In addition to the assumptions in Theorem 13.2, suppose that (X,d) is uniformly perfect. Then the following four conditions are equivalent:*
(a) *(DM1) and (DM2) hold.*
(b) *$\rho \underset{\mathrm{QS}}{\sim} d$ and (DM2) holds.*
(c) *$\rho \underset{\mathrm{QS}}{\sim} d$ and (DM3) holds.*
(d) *(DM3) holds, d decays uniformly with respect to ρ and ρ is $(\mathrm{SQS})_d$.*
Moreover, if any of the above conditions is satisfied, then μ is $(\mathrm{VD})_d$ and $(\mathrm{VD})_\rho$.

The rest of this section is devoted to proving the above theorem and the corollary.

LEMMA 13.4. *If* (DM1) *and* (DM2) *are satisfied, than, for any* $\epsilon > 0$*, there exists* $\delta > 0$ *such that* $d(x,z) < \epsilon d(x,y)$ *whenever* $\rho(x,z) < \delta\rho(x,y)$*. In particular,* d *is* $(\mathrm{SQS})_\rho$.

PROOF. Assume that $d(x,z) \geq \epsilon d(x,y)$ and that $\rho(x,z) \leq \rho(x,y)$. Let $\lambda = \rho(x,z)/\rho(x,y)$. Then by (DM2),

$$\begin{aligned} c_2 g(\lambda g(x,y)) &\geq H(d(x,z), V_\rho(x,\rho(x,z))) \\ &\geq H(\epsilon d(x,y), V_\rho(x,\lambda\rho(x,y))) \\ &\geq h(1/\epsilon,1)^{-1} H(d(x,y), V_\rho(x,\lambda\rho(x,y))). \end{aligned}$$

Hence

$$c_3 \leq \frac{g(\lambda\rho(x,y))}{H(d(x,y), V_\rho(x,\lambda\rho(x,y)))},$$

where c_3 is a positive constant which depends only on ϵ. This combined with (DM1) and (DM2) implies that $0 < c_4 \leq \eta(\lambda)$, where c_4 depends only on ϵ. Hence, there exists $\delta > 0$ such that $\rho(x,z) \geq \delta\rho(x,y)$. Thus we have the contraposition of the statement. □

LEMMA 13.5. *Assume* (DM1) *and that* $\lim_{t\downarrow 0} h(1,t) = 0$*. Then, for any* $\lambda > 0$*, there exists* $a > 0$ *such that* $V_\rho(x,\lambda\rho(x,y)) \geq aV_\rho(x,\rho(x,y))$ *for any* $x, y \in X$.

REMARK. Note that $h(1,t^n) \leq h(1,t)^n$. Hence $\lim_{t\downarrow 0} h(1,t) = 0$ if and only if there exists $t_* \in (0,1)$ such that $h(1,t_*) < 1$.

PROOF. For $\lambda \geq 1$, we may choose $a = 1$. Suppose that $\lambda \in (0,1)$. Then by (DM1) and the doubling property of g,

$$\begin{aligned} h\Big(1, \frac{V_\rho(x,\lambda\rho(x,y))}{V_\rho(x,\rho(x,y))}\Big) &\geq \frac{H(d(x,y), V_\rho(x,\lambda\rho(x,y)))}{H(d(x,y), V_\rho(x,\rho(x,y)))} \\ &\geq \eta(\lambda)\frac{g(\lambda\rho(x,y))}{g(\rho(x,y))} \geq c_\lambda > 0, \end{aligned}$$

where c_λ depends only on λ. Since $\lim_{t\downarrow 0} h(1,t) = 0$, we have the desired conclusion. □

LEMMA 13.6. *Assume that* (X,ρ) *is uniformly perfect and that* $\lim_{t\downarrow 0} h(1,t) = 0$*. If* d *is* $(\mathrm{SQS})_\rho$ *and* (DM2) *holds, then, for any sufficiently small* $\lambda \in (0,1)$*, there exists* $a > 0$ *such that* $V_\rho(x,\lambda\rho(x,y)) \geq aV_\rho(x,\rho(x,y))$ *for any* $x, y \in X$.

PROOF. Since d is $(\mathrm{SQS})_\rho$, there exist $\epsilon \in (0,1)$ and $\delta_0 \in (0,1)$ such that $d(x,z) < \epsilon d(x,y)$ whenever $\rho(x,z) < \delta_0\rho(x,y)$. If $\delta \leq \delta_0$, then $d(x,z) < \epsilon d(x,y)$ whenever $\rho(x,z) < \delta\rho(x,y)$. Also since (X,ρ) is uniformly perfect, there exists $c \in (0,1)$ such that $B_\rho(x,r)\backslash B_\rho(x,cr) \neq \emptyset$ if $B_\rho(x,r) \neq X$. Let x and $y \in X$. Then we may choose $z \in B_\rho(x,\delta r)\backslash B_\rho(x,c\delta r)$, where $r = \rho(x,y)$. Note that $d(x,z) <$

$\epsilon d(x,y) < d(x,y)$. By the doubling property of g and (DM2),

$$\begin{aligned} c' g(\rho(x,z)) &\geq c_2 g(\rho(x,z)/(c\delta)) \geq c_2 g(r) \geq H(d(x,y), V_\rho(x,r)) \\ &\geq H(d(x,z), V_\rho(x,r)) \geq h\Big(1, \frac{V_\rho(x,\rho(x,z))}{V_\rho(x,\rho(x,y))}\Big)^{-1} H(d(x,z), V_\rho(x,\rho(x,z))) \\ &\geq c_1 h\Big(1, \frac{V_\rho(x,\delta\rho(x,y))}{V_\rho(x,\rho(x,y))}\Big)^{-1} g(\rho(x,z)). \end{aligned}$$

Therefore, it follows that

$$h\Big(1, \frac{V_\rho(x,\delta\rho(x,y))}{V_\rho(x,\rho(x,y))}\Big) \geq c_3.$$

Since $\lim_{t\downarrow 0} h(1,t) = 0$, we have $V_\rho(x,\delta\rho(x,y)) \geq aV_\rho(x,\rho(x,y))$. Letting $\lambda = \delta$, we have the desired statement. $\square$

From Lemmas 13.5 and 13.6, we have the following lemma.

Lemma 13.7. *Assume that (X,ρ) is uniformly perfect, that $\lim_{t\downarrow 0} h(1,t) = 0$, and that there exists $c_* > 0$ such that $\mu(X) \leq c_* V_\rho(x,\rho_*(x))$ for any $x \in X$. If either* (DM1) *is satisfied or d is* $(\mathrm{SQS})_\rho$ *and* (DM2) *is satisfied, then μ is volume doubling with respect to ρ.*

Proof. $B_\rho(x,r)\backslash B_\rho(x,cr) \neq \emptyset$ unless $B_\rho(x,cr) = X$ by uniform perfectness. Suppose $r < \rho_*(x) = \sup_{y\in X} \rho(x,y)$. Choose λ so that $0 < \lambda < c$. Then there exists $y \in X$ such that $cr \leq \rho(x,y) < r$. By Lemmas 13.5 and 13.6, we have $V_\rho(x,\lambda\rho(x,y)) \geq aV_\rho(x,\rho(x,y))$ in either case. This immediately implies $V_\rho(x,\lambda r) \geq aV_\rho(x,cr)$. Therefore, if $r < c\rho_*(x)$, then $V_\rho(x,\lambda' r) \geq aV_\rho(x,r)$, where $\lambda' = \lambda/c < 1$. If $\mathrm{diam}(X,\rho) = +\infty$, then we have finished the proof. Otherwise, $\rho_*(x) < +\infty$ for any $x \in X$.

If $r \in [c\rho_*(x), \rho_*(x))$, there exists $y \in X$ such that $r \leq \rho(x,y) \leq \rho_*(x)$. Lemma 13.5 implies that $V_\rho(x,\lambda\rho(x,y)) \geq aV_\rho(x,\rho(x,y)) \geq aV_\rho(x,r)$. Since $r/c \geq \rho_*(x) \geq \rho(x,y)$, we have $V_\rho(x,\lambda' r) \geq V_\rho(x,\lambda\rho(x,y)) \geq aV_\rho(x,r)$.

Finally, let $r \geq \rho_*(x)$. Then there exists $y \in X$ such that $\rho_*(x)/2 < \rho(x,y) \leq \rho_*(x)$ and $V_\rho(x,\rho_*(x))/2 \leq V_\rho(x,\rho(x,y))$. By Lemma 13.5, $V_\rho(x,\lambda'\rho(x,y)) \geq a'V_\rho(x,\rho(x,y))$, where a' is independent fo x and y. Hence

$$\begin{aligned} \frac{a'c_*}{2} V_\rho(x,r) = \frac{a'c_*}{2}\mu(X) \leq \frac{a'}{2} V_\rho(x,\rho_*(x)) &\leq a'V_\rho(x,\rho(x,y)) \\ &\leq V_\rho(x,\lambda'\rho(x,y)) \leq V_\rho(x,\lambda' r) \end{aligned}$$

$\square$

Proof of Theorem 13.2-(1). Combining Lemmas 13.4 and 13.7, we immediately obtain Theorem 13.2-(1). $\square$

Lemma 13.4 implies the next fact.

Lemma 13.8. *Assume that (X,ρ) is uniformly perfect and that μ is volume doubling with respect to ρ. If* (DM1) *and* (DM2) *are satisfied and* $\lim_{s\downarrow 0} h(s,1) = 0$, *then ρ is* $(\mathrm{ASQC})_d$.

PROOF. First we suppose that $\mathrm{diam}(X,\rho) < +\infty$. Lemma 13.4 implies that d is $(\mathrm{SQS})_\rho$. Let $r_* > \mathrm{diam}(X,\rho)$. By Proposition 11.7-(1), there exist $\lambda \in (0,1)$ and $a \in (0,1)$ such that $\overline{d}_\rho(x,\lambda r) \leq a\overline{d}_\rho(x,r)$ for any $x \in X$ and any $r \in (0, r_*]$. Let $r \in (0, r_*]$. (Note that $\overline{d}_\rho(x, \lambda^n r) \leq a^n \overline{d}_\rho(x,r)$. Hence λ can be arbitrarily small.) Then

$$\overline{d}_\rho(x,r) = \sup\{d(x,y)|y \in B_\rho(x,r)\backslash B_\rho(x,\lambda r)\}. \tag{13.2}$$

Since μ is $(\mathrm{VD})_\rho$, there exists $\alpha > 0$ such that $\alpha V_\rho(x,\lambda r) \geq V_\rho(x,r)$. Now choose $x, y \in B_\rho(x,r)\backslash B_\rho(x,\lambda r)$. Then $\lambda\rho(x,y) \leq \rho(x,z) \leq \rho(x,y)/\lambda$. Therefore,

$$\begin{aligned} c_1 g(\rho(x,z)) &\leq H(d(x,z), V_\rho(x,\rho(x,z))) \leq H(d(x,z), V_\rho(x, \rho(x,y)/\lambda))) \\ &\leq H(d(x,z), \alpha V_\rho(x,\rho(x,y))) \leq c_2 h(1,\alpha) h\Big(\frac{d(x,z)}{d(x,y)}, 1\Big) g(\rho(x,y)). \end{aligned}$$

This along with the doubling property of g shows that

$$h\Big(\frac{d(x,z)}{d(x,y)}, 1\Big) \geq c_3 > 0,$$

where c_3 is independent of x, y and z. Since $h(s,1) \downarrow 0$ as $s \downarrow 0$, there exists $\delta > 0$ such that $d(x,z) \geq \delta d(x,y)$. By (13.2), we see that $d(x,z) \geq \delta\overline{d}_\rho(x,r)$. Hence $B_\rho(x,r)\backslash B_\rho(x,\lambda r) \subseteq \overline{B}_\rho(x, \overline{d}_\rho(x,r))\backslash B_\rho(x, \delta\overline{d}_\rho(x,r))$. Next we consider the case where $r > r_*$. Note that $r_*\mathrm{diam}(X,d) \geq \rho_*(x)$. Hence $B_\rho(x,r) = B_\rho(x,r_*) = X$ and $\overline{d}_\rho(x,r) = \overline{d}_\rho(x,r_*) = d_*(x)$. Also, $\rho(x,z) \leq r_*$ for any $z \in X$. Therefore if $\lambda r \leq \rho(x,y) < r$, then $\lambda r_* \leq \rho(x,z) < r_*$ and hence $d(x,z) \geq \delta\overline{d}_\rho(x,r_*) = \delta\overline{d}_\rho(x,r_*)$. This completes the proof when $\mathrm{diam}(X,\rho) < +\infty$. Using similar argument, we immediately obtain the case where $\mathrm{diam}(X,\rho) = +\infty$. □

LEMMA 13.9. *Assume that (X,ρ) is uniformly perfect, that μ is $(\mathrm{VD})_\rho$ and that ρ is $(\mathrm{ASQC})_d$. If* (DM2) *holds, then, for any $r_* > \mathrm{diam}(X,\rho)$,*

$$H(\overline{d}_\rho(x,r), V_\rho(x,r)) \asymp g(r)$$

for any $x \in X$ and any $r \in (0,r_*]$. In particular, (DM3) *holds.*

REMARK. If $\mathrm{diam}(X,\rho) = +\infty$, then we remove ", for any $r_* > \mathrm{diam}(X,\rho)$," and replace "$r \in (0,r_*]$" by "$r > 0$" in the statement of the above lemma.

PROOF. There exists $c \in (0,1)$ such that $B_\rho(x,r)\backslash B_\rho(x,cr) \neq \emptyset$ if $X \neq B_\rho(x,cr)$ by the uniform perfectness. Recall that $\rho_*(x) \geq \mathrm{diam}(X,\rho)/2$ for any $x \in X$. Hence if $c^{n-1}r_* < \mathrm{diam}(X,\rho)/2$, then $c^{n-1}r < \rho_*(x)$ for any $r \in (0,r_*]$. Therefore $X \neq B_\rho(x, c^{n-1}r)$. So, we have $y \in X$ satisfying $c^n r \leq \rho(x,y) < c^{n-1}r$. By Proposition 11.10 and (DM2),

$$H(\overline{d}_\rho(x,\rho(x,y)), V_\rho(x,\rho(x,y)) \asymp g(\rho(x,y)).$$

On the other hand, Lemma 11.9 implies that d is doubling with respect to ρ. This and the doubling property of μ show that

$$H(\overline{d}_\rho(x,\rho(x,y)), V_\rho(x,\rho(x,y))) \asymp H(\overline{d}_\rho(x,r), V_\rho(x,r)).$$

Moreover, by the doubling property of g, $c_6 g(r) \leq g(\rho(x,y)) \leq g(r)$. Combining the last three inequalities, we immediately obtain the desired statement. □

LEMMA 13.10. *Assume that (X,ρ) is uniformly perfect and that ρ is $(\mathrm{ASQC})_d$. Then* (DM3) *implies* (DM2).

PROOF. By Proposition 11.10,

$$\overline{d}_\rho(x, \rho(x, y)) \asymp d(x, y).$$

Hence, letting $r = \rho(x, y)$, we obtain

$$H(\overline{d}_\rho(x, r), V_\rho(x, r)) \asymp H(d(x, y), V_\rho(x, r))$$

This immediately implies (DM2). $\square$

LEMMA 13.11. *Assume that* (X, ρ) *is uniformly perfect and* $\lim_{s\downarrow 0} h(s, 1) = 0$. *If* ρ *is* $(\mathrm{ASQC})_d$, *d decays uniformly with respect to ρ and* (DM3) *holds, then* (DM1) *holds.*

PROOF. Since d decays uniformly with respect to ρ, there exists $c_0 > 0$ and $\tau > 0$ such that $\overline{d}_\rho(x, \lambda r) \le c_0\lambda^\tau \overline{d}_\rho(x, r)$ for any $x \in X$ and $r \in (0, r_*]$. (If $\mathrm{diam}(X, \rho) = +\infty$, we always replace $(0, r_*]$ by $(0, +\infty)$ in this proof.) Let $r = \rho(x, y)$. By Proposition 11.10 and (DM3),

$$\begin{aligned} c_1 g(\lambda r) \le H(\overline{d}_\rho(x, \lambda r), V_\rho(x, \lambda r)) &\le H(c_0\lambda^\tau \overline{d}_\rho(x, r), V_\rho(x, \lambda r)) \\ &\le h(c_3\lambda^\tau, 1) H(d(x, y), V_\rho(x, \lambda r)), \end{aligned}$$

where $c_3 > 0$ is independent of x and y. Moreover, by Lemma 13.10, we have (DM2). Hence $H(d(x, y), V_\rho(x, r))/g(r)$ is uniformly bounded. So, there exists $c_4 > 0$ such that

$$c_4 h(c_0\lambda^\tau, 1) \ge \frac{H(d(x, y), V_\rho(x, \rho(x, y)))}{g(\rho(x, y))} \frac{g(\lambda\rho(x, y))}{H(d(x, y), V_\rho(x, \lambda\rho(x, y)))}$$

for any x and y. Since $\lim_{\lambda\downarrow 0} h(c_0\lambda^\tau, 1) = 0$, (DM1) follows by letting $\eta(\lambda) = c_4 h(c_0\lambda^\tau, 1)$. $\square$

PROOF OF THEOREM 13.2-(2). Assume (DM1) and (DM2). By Lemma 13.8, ρ is $(\mathrm{ASQC})_d$. By Lemma 13.4, d is $(\mathrm{SQS})_\rho$. This along with Proposition 11.7 implies that d decays uniformly with respect to ρ. Now (DM3) follows by using Lemma 13.9.

Lemmas 13.10 and 13.11 suffice for the converse direction. $\square$

PROOF OF COROLLARY 13.3. $(a) \Rightarrow (b)$ Using Lemma 13.4 and 13.8 and applying Theorem 11.5, we see that d and ρ are semi-quasisymmetric with respect to each other. Hence by Theorem 12.3, $\rho \underset{\mathrm{QS}}{\sim} d$.
$(b) \Rightarrow (c)$ Since $\rho \underset{\mathrm{QS}}{\sim} d$, Theorems 11.5 and 12.3 show that d is $(\mathrm{SQS})_\rho$ and that ρ is $(\mathrm{ASQC})_d$. By Lemma 13.7, μ is $(\mathrm{VD})_\rho$. Therefore Lemma 13.9 yields (DM3).
$(c) \Rightarrow (d)$ Since $\rho \underset{\mathrm{QS}}{\sim} d$, Theorems 11.5 and 12.3 show that d is $(\mathrm{SQS})_\rho$ and that ρ is $(\mathrm{ASQC})_d$. Then by Proposition 11.7, d decays uniformly with respect to ρ.
$(d) \Rightarrow (a)$ This immediately follows by Theorem 13.2. $\square$

14. Construction of quasisymmetric metrics

The main purpose of this section is to construct a distance ρ which satisfy the conditions (DM1) and (DM2) in Section 13 in the case where $g(r) = r^\beta$.

In this section, (X, d) is a metric space and μ is a Borel regular measure on (X, d) which is volume doubling with respect to d. We also assume that $0 <$

$\mu(B_d(x,r)) < +\infty$ for any $x \in X$ and $r > 0$. Let $H : (0,+\infty)^2 \to (0,+\infty)$ satisfy (H1) and (H2) as in Section 13.

THEOREM 14.1. *Assume that* (X,d) *is uniformly perfect and that* μ *is* $(\mathrm{VD})_d$. *For sufficiently large* $\beta > 0$, *there exists a distance* ρ *on* X *such that* $\rho \underset{\mathrm{QS}}{\sim} d$ *and* (DM3) *holds with* $g(r) = r^\beta$.

REMARK. If $\rho \underset{\mathrm{QS}}{\sim} d$ and (X,d) is uniformly perfect, then Proposition 12.2 and Corollary 12.4 imply that (X,ρ) is uniformly perfect and that μ is $(\mathrm{VD})_\rho$.

Our distance ρ satisfies the condition (c) of Corollary 13.3. If $\lim_{s\downarrow 0} h(s,1) = \lim_{t\downarrow 0} h(1,t) = 0$, then we have all the assumption of the corollary and hence obtain the statements (a) through (d). In particular, d, μ and ρ satisfy (DM1) and (DM2) with $g(r) = r^\beta$. In particular, letting $H(s,t) = st$, we establish the existence of a distance which is quasisymmetric to the resistance metric and satisfies (13.1) if μ is volume doubling with respect to the resistance metric. This fact plays an important role in the next part.

EXAMPLE 14.2. (1) If $H(s,t) = t$, (DM3) is

$$\mu(B_\rho(x,r)) \asymp r^\beta.$$

Hence in this case, Theorem 14.1 implies the following well-known theorem: if (X,d) is uniformly perfect and μ is $(\mathrm{VD})_d$, then there exists a metric ρ such that the metric measure space (X,ρ,μ) is Ahlfors regular. ((X,ρ,μ) is called a metric measure space if and only if (X,ρ) is a metric space and μ is a Borel-regular measure on (X,ρ). A metric measure space (X,ρ,μ) is called Ahlfors α-regular if and only if $\mu(B_\rho(x,r)) \asymp r^\alpha$ for any $x \in X$ and any $r \le \mathrm{diam}(X,d)$.) See Heinonen [**32**, Chapter 14] and Semmes [**48**, Section 4.2] for details.
(2) Let $F : (0,+\infty) \to (0,+\infty)$ be monotonically nondecreasing. Suppose that there exist positive constants c_1, τ_1 and τ_2 such that

$$F(xy) \le c_1 \max\{x^{\tau_1}, x^{\tau_2}\} F(y)$$

for any $x, y \in (0,+\infty)$. Define $H(s,t) = F(s^p t^q)$. If $p \ge 0$, $q \ge 0$ and $(p,q) \ne (0,0)$, then H satisfies (H1) and (H2). In fact,

$$H(as,bt) = F(a^p b^q s^p t^q) \le c_1 \max\{(a^p b^q)^{\tau_1}, (a^p b^q)^{\tau_2}\} H(s,t).$$

Hence $h(a,b) \le c_1 \max\{(a^p b^q)^{\tau_1}, (a^p b^q)^{\tau_2}\}$.

To prove Theorem 14.1, we need several preparations.

NOTATION. Define $v(x,y) = V_d(x, d(x,y)) + V_d(y, d(x,y))$. Also define

$$\varphi(x,y) = \begin{cases} H(d(x,y), v(x,y)) & \text{if } x \ne y, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\varphi(x,y) = \varphi(y,x) \ge 0$ and that $\varphi(x,y) = 0$ implies $x = y$.

Hereafter in this section, we always assume that (X,d) is uniformly perfect and that μ is $(\mathrm{VD})_d$. By the volume doubling property, we have the following lemma.

LEMMA 14.3. *For any* $x, y \in X$,

$$v(x,y) \asymp V_d(x, d(x,y)).$$

Lemma 14.4. *Define*

$$f_{\tau_1,\tau_2}(t) = \begin{cases} t^{\tau_1} & \text{if } t \in (0,1), \\ t^{\tau_2} & \text{if } t \geq 1. \end{cases}$$

Then there exist positive constants c_1, τ_1 and τ_2 such that

$$V_d(x, \delta d(x,y)) \leq c_1 f_{\tau_1,\tau_2}(\delta) V_d(x, d(x,y))$$

for any $x, y \in X$ and any $\delta > 0$.

Proof. If $\delta \geq 1$, this is immediate from the volume doubling property. Since (X,d) is uniformly perfect, there exists $c \in (0,1)$ such that $B_d(x,r) \neq X$ implies $B_d(x,r) \backslash B_d(x,cr) \neq \emptyset$. Let $r = d(x,y)$. Choose $z \in B_d(x,r/2) \backslash B_d(x,cr/2)$. It follows that $V_d(x,cr/4) + V_d(z,cr/4) \leq V_d(x,r)$. Now by the volume doubling property, $V_d(z,cr/4) \geq aV_d(x,cr/4)$, where a is independent of x, z and r. Hence $V_d(x,cr/4) \leq (1+a)^{-1} V_d(x,r)$. This shows the desired inequality when $\delta \in (0,1)$. □

Lemma 14.5. *There exists a homeomorphism $\xi : [0,+\infty) \to [0,+\infty)$ such that $\xi(0) = 0$ and $\varphi(x,z) < \xi(t)\varphi(x,y)$ whenever $d(x,z) < td(x,y)$.*

Proof. Assume that $d(x,z) < td(x,y)$. Write $f = f_{\tau_1,\tau_2}$. Then by (H1), (H2) and the above lemmas,

$$\begin{aligned} \varphi(x,z) = H(d(x,z), v(x,z)) &\leq H(td(x,y), Mc_1 f(t) V_d(x, d(x,y))) \\ &\leq H(td(x,y), M^2 c_1 f(t) v(x,y)) \leq h(1, M^2 c_1) h(t, f(t)) H(d(x,y), v(x,y)). \end{aligned}$$

By the definition of $h(a,b)$, it follows that $h(t, f(t))$ is monotonically nondecreasing. Also if $t < c_0$, then $h(t, t^{\tau_1}) < 1$. Since $h(t^n, t^{n\tau_1}) \leq h(t, t^{\tau_1})^n$ for $n \geq 0$, we see that $h(t, f(t)) \to 0$ as $t \downarrow 0$. Therefore, there exists a homeomorphism $\xi : [0,+\infty) \to [0,+\infty)$ such that $\xi(0) = 0$ and $\xi(t) \geq h(1, M^2 c_1) h(t, f(t))$ for any $t > 0$. □

Definition 14.6. $f : X \times X \to [0,+\infty)$ is called a quasidistance on X if and only if f satisfies the following three conditions:
(QD1) $f(x,y) \geq 0$ for any $x, y \in X$. $f(x,y) = 0$ if and only if $x = y$.
(QD2) $f(x,y) = f(y,x)$ for any $x, y \in X$.
(QD3) There exists $K > 0$ such that $f(x,y) \leq K(f(x,z) + f(z,y))$ for any $x, y, z \in X$.

Lemma 14.7. *$\varphi(x,y)$ is a quasidistance.*

Proof. Since $d(x,y) \leq d(x,z) + d(z,y)$, either $d(x,y) \leq d(x,z)/2$ or $d(x,y) \leq d(z,y)$. Assume that $d(x,y) \leq d(x,z)/2$. Then Lemma 14.5 implies that $\varphi(x,y) \leq \xi(1/2)\varphi(x,z) \leq \xi(1/2)(\varphi(x,z) + \varphi(z,y))$. □

Lemma 14.8. *If $f : X \times X \to [0,+\infty)$ is a quasidistance on X, then there exists $\epsilon_0 > 0$ such that f^ϵ is equivalent to a distance for any $\epsilon \in (0, \epsilon_0]$, i.e.*

$$f(x,y)^\epsilon \asymp \rho_\epsilon(x,y)$$

for any $x, y \in X$, where ρ_ϵ is a distance on X.

See Heinonen [**32**, Proposition 14.5] for the proof of this lemma.

LEMMA 14.9. *For sufficiently large $\beta > 0$, there exists a distance ρ on X such that $\rho \underset{\mathrm{QS}}{\sim} d$ and*

$$\varphi(x, y) \asymp \rho(x, y)^{\beta} \tag{14.1}$$

for any $x, y \in X$.

PROOF. By Lemmas 14.7 and 14.8, if β is large enough, then there exists a distance ρ which satisfies (14.1). By Lemma 14.5, $d(x, z) < td(x, y)$ implies $\rho(x, z) < c\xi(t)^{1/\beta}\rho(x, y)$ for some $c > 0$. Hence $\rho \underset{\mathrm{QS}}{\sim} d$. $\square$

Since $\rho \underset{\mathrm{QS}}{\sim} d$, d and ρ define the same topology on X. Also since (X, d) is uniformly perfect, so is (X, ρ). Then, Theorem 12.3 shows that d and ρ are semi-quasisymmetric with respect to each other. So we may enjoy the results in Theorem 11.5 in the rest of discussions.

LEMMA 14.10. *For any $x \in X$ and any $r > 0$,*

$$V_\rho(x, r) \asymp V_d(x, \overline{d}_\rho(x, r)). \tag{14.2}$$

PROOF. Since ρ is $(\mathrm{SQC})_d$,

$$B_d(x, c\overline{d}_\rho(x, r)) \subseteq B_\rho(x, r) \subseteq B_d(x, c'\overline{d}_\rho(x, r)).$$

This and the volume doubling property of μ imply (14.2). $\square$

PROOF OF THEOREM 14.1. The rest is to show (DM3). Since (X, ρ) is uniformly perfect, there exists $c \in (0, 1)$ such that $B_\rho(x, r)\backslash B_\rho(x, cr) \neq \emptyset$ unless $B_\rho(x, r) = X$. We will consider the case when $\mathrm{diam}(X, \rho) < +\infty$. Let $r_* > \mathrm{diam}(X, \rho)$. Choose $n \geq 1$ so that $c^n r < \mathrm{diam}(X, \rho)/2$. Note that $\mathrm{diam}(X, \rho)/2 \leq \rho_*(x)$. Hence if $r \in (0, r_*]$, then $c^n r < \rho_*(x)$. Therefore there exists $y \in X$ such that $c^{n+1} r \leq \rho(x, y) < c^n r$. By $(\mathrm{ASQC})_d$, there exists $\delta > 0$ such that $d(x, z) \geq \delta\overline{d}_\rho(x, r)$ for any $r > 0$ and any $z \in B_\rho(x, r)\backslash B_\rho(x, cr)$. This along with the doubling property of $\overline{d}_\rho(x, r)$ implies that $\overline{d}_\rho(x, r) \geq d(x, y) \geq \delta\overline{d}_\rho(x, c^n r) \geq c'\overline{d}_\rho(x, r)$. Then by Lemma 14.3. the volume doubling property of μ, we have (14.1) and (14.2),

$$\begin{aligned}(c^n r)^\beta \geq \rho(x, y)^\beta &\geq c_3 H(d(x, y), c_4 V_d(x, d(x, y))) \\ &\geq c_3 H(c'\overline{d}_\rho(x, r), c_5 V_d(x, \overline{d}_\rho(x, r))) \geq c_6 H(\overline{d}_\rho(x, r), V_\rho(x, r)).\end{aligned}$$

Similarly,

$$\begin{aligned}(c^{n+1} r)^\beta \leq \rho(x, y)^\beta &\leq c_7 H(d(x, y), c_8 V_d(x, d(x, y))) \\ &\leq c_7 H(\overline{d}_\rho(x, r), V_d(x, \overline{d}_\rho(x, r))) \leq c_8 H(\overline{d}_\rho(x, r), V_\rho(x, r)).\end{aligned}$$

Thus we obtain (DM3) if $\mathrm{diam}(X, \rho) < +\infty$. The other case follows by almost the same argument. $\square$

Part 3

Volume doubling measures and heat kernel estimates

In this part, we will show results on heat kernel estimates, which answer the two questions in the introduction, under the foundation laid by the previous two parts. The first question is how and when we can find a metric which is suitable for describing asymptotic behaviors of a heat kernel. As an answer, Theorem 15.11 shows that if the underlying measure is volume doubling with respect to the resistance metric, then we can get a good (on-diagonal, at least) heat kernel estimate by quasisymmetric modification of the resistance metric. The second question concerns jumps. Namely, what kind of jumps can we allow to get a good heat kernel estimate? Theorem 15.11 also gives an answer to this question, saying that the annulus comparable condition (with the volume doubling property) is sufficient and necessary for a good heat kernel estimate.

15. Main results on heat kernel estimates

In this section, we present the main results on heat kernel estimates. There will be three main theorems, 15.6, 15.10 and 15.11. The first one gives a good (on-diagonal and lower near diagonal) heat kernel estimate if (ACC) holds, the measure is $(\mathrm{VD})_R$ and the distance is quasisymmetric with respect to the resistance metric R. The second one provides geometrical and analytical equivalent conditions for having a good heat kernel estimate. As an application, we will recover the two-sided off-diagonal heat kernel estimates for the Brownian motions on homogeneous random Sierpinski gaskets obtained by Barlow and Hambly in [**7**]. See Section 24 for details. Finally in the third theorem, $(\mathrm{VD})_R$ and (ACC) ensure the existence of a distance d which is quasisymmetric to R and under which a good heat kernel estimate holds.

Proofs of Propositions and Theorems in this section are given in latter sections in this part.

Throughout this section, $(\mathcal{E}, \mathcal{F})$ is a regular resistance form on a set X and R is the associated resistance metric on X. We assume that (X, R) is separable, complete, uniformly perfect and locally compact. Let μ be a Borel regular measure on (X, R) which satisfies $0 < \mu(B_R(x, r)) < +\infty$ for any $x \in X$ and any $r > 0$. Under those assumptions, if $\mathcal{D}$ is the closure of $\mathcal{F} \cap C_0(X)$ with respect to the $\mathcal{E}_1$-norm, then $(\mathcal{E}, \mathcal{D})$ is a regular Dirichlet form $L^2(X, \mu)$. Let $(\{X_t\}_{t>0}, \{P_x\}_{x \in X})$ be the Hunt process associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(X, \mu)$.

As we have shown in Section 7, if (X, R) is complete and μ is $(\mathrm{VD})_R$, then $\overline{B_R(x, r)}$ is compact for any $x \in X$ and any $r > 0$. Hence under $(\mathrm{VD})_R$, Theorem 10.4 implies the existence of a jointly continuous heat kernel (i.e. transition density) $p(t, x, y)$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(X, \mu)$.

Definition 15.1. Let d be a distance on X giving the same topology as R. Define $\overline{R}_d(x, r) = \sup_{y \in B_d(x, r)} R(x, y)$, $V_d(x, r) = \mu(B_d(x, r))$ and

$$h_d(x, r) = \overline{R}_d(x, r) V_d(x, r)$$

for any $r > 0$ and any $x \in X$.

Lemma 15.2. *For each $x \in X$, $\overline{R}_d(x, r)$ and $V_d(x, r)$ are monotonically nondecreasing left-continuous function on $(0, +\infty)$. Moreover $\lim_{r \downarrow 0} \overline{R}_d(x, r) = 0$ and $\lim_{r \downarrow 0} V_d(x, r) = \mu(\{x\})$.*

By the above lemma, $h_d(x, r)$ is monotonically nondecreasing left-continuous function on $(0, +\infty)$ and $\lim_{r \downarrow 0} h_d(x, r) = 0$.

Definition 15.3. Let d be a distance on X which gives the same topology as R. We say that the Einstein relation with respect d, $(\mathrm{EIN})_d$ for short, holds if and only if

$$E_x(\tau_{B_d(x,r)}) \asymp h_d(x,r), \tag{$(\mathrm{EIN})_d$}$$

for any $x \in X$ and $r > 0$ with $X \neq B_d(x,r)$.

The name "Einstein relation" has been used by several authors . See [**26**] and [**50**] for example.

We have two important equivalences between the resistance estimate, the annulus comparable condition and the Einstein relation.

Proposition 15.4. *Assume that d is a distance on X and $d \underset{\mathrm{QS}}{\sim} R$. Then* (RES) *is equivalent to*

$$R(x, B_d(x,r)^c) \asymp \overline{R}_d(x,r) \tag{15.1}$$

for any $x \in X$ and any $r > 0$ with $B_d(x,r) \neq X$.

Proposition 15.5. *Assume that μ is $(\mathrm{VD})_R$, that d is a distance on X and that $d \underset{\mathrm{QS}}{\sim} R$. Then* (RES), (ACC) *and* $(\mathrm{EIN})_d$ *are equivalent to one another.*

The proofs of the above propositions are in Section 18.

Now we have the first result on heat kernel estimate.

Theorem 15.6. *Assume that* (ACC) *holds. Suppose that μ has volume doubling property with respect to R. Then, there exists a jointly continuous heat kernel $p(t,x,y)$ associated with the Dirichlet form $(\mathcal{E},\mathcal{D})$ on $L^2(X,\mu)$. Moreover, if a distance d on X is quasisymmetric with respect to R, then* $(\mathrm{EIN})_d$ *holds and*

$$\frac{c_1}{V_d(x,r)} \le p(h_d(x,r),x,y) \tag{15.2}$$

and

$$p(h_d(x,r),x,x) \le \frac{c_2}{V_d(x,r)} \tag{15.3}$$

for any $x \in X$, any $r > 0$ and any $y \in X$ with $d(x,y) \le c_3 \min\{r, \mathrm{diam}(X,d)\}$.

(15.2) will imply (15.5), which is the counterpart of the near diagonal lower estimate $(\mathrm{NDL})_{\beta,d}$. In the case of diffusions, if the distance is not geodesic, the lower near diagonal estimate is known as a substitute for the off-diagonal lower sub-Gaussian estimate.

Note that $R \underset{\mathrm{QS}}{\sim} R$ and $\overline{R}_d(x,r) \asymp r$ if (X,R) is uniformly perfect. Hence, $h_d(x,r) = rV_R(x,r)$ and the above theorem shows

$$p(rV_R(x,r),x,x) \asymp \frac{1}{V_R(x,r)}.$$

This has essentially been obtained in [**41**].

To state the next theorem, we need several notions and results on monotonically non-decreasing functions on $(0,+\infty)$ and their inverse.

Definition 15.7. Let $f : (0,+\infty) \to (0,+\infty)$.

(1) f is said to be doubling if there exists $c > 0$ such that $f(2t) \le cf(t)$ for any $t \in (0,+\infty)$.

(2) f is said to decay uniformly if and only if there exists $(\delta, \lambda) \in (0,1)^2$ such that $f(\delta t) \leq \lambda f(t)$ for any $t \in (0, +\infty)$.
(3) f is said to be a monotone function with full range if and only if f is monotonically non-decreasing, $\lim_{t\downarrow 0} f(t) = 0$ and $\lim_{t\to+\infty} f(t) = +\infty$. For a monotone function with full range on $(0, +\infty)$, we define $f^{-1}(y) = \sup\{x|f(x) \leq y\}$ and call f^{-1} the right-continuous inverse of f.

LEMMA 15.8. *Let $f : (0, +\infty) \to (0, +\infty)$ be a monotone function with full range.* (1) *If f is doubling, then f^{-1} decays uniformly and $f(f^{-1}(y)) \asymp y$ for any $y \in (0, +\infty)$.*
(2) *If f decays uniformly, then f^{-1} is doubling and $f^{-1}(f(x)) \asymp x$ for any $x \in (0, +\infty)$.*

This lemma is rather elementary and we omit its proof.

The following definition is a list of important relations or properties between a heat kernel, a measure and a distance.

DEFINITION 15.9. Let d be a distance of X giving the same topology as R and let $g : (0, +\infty) \to (0, +\infty)$ be a monotone function with full range.
(1) A heat kernel $p(t, x, y)$ is said to satisfy on-diagonal heat kernel estimate of order g with respect to d, $(\mathrm{DHK})_{g,d}$ for short, if and only if

$$p(t, x, x) \asymp \frac{1}{V_d(x, g^{-1}(t))}$$

for any $x \in X$ and any $t > 0$, where g^{-1} is the right-continuous inverse of g.
(2) A heat kernel $p(t, x, y)$ is said to have the doubling property. (KD) for short, if and only if there exists $c_1 > 0$ such that

$$p(t, x, x) \leq c_1 p(2t, x, x).$$

for any $x \in X$ and any $t > 0$.
(3) We say that $(\mathrm{DM1})_{g,d}$ holds if and only if there exists $\eta : (0, 1] \to [0, +\infty)$ such that η is monotonically nondecreasing, $\lim_{t\downarrow 0} \eta(t) = 0$ and

$$\frac{g(\lambda d(x, y))}{V_d(x, \lambda d(x, y))} \leq \frac{g(d(x, y))}{V_d(x, d(x, y))}\eta(\lambda)$$

for any $x, y \in X$ and any $\lambda \in (0, 1]$.
(4) We say that $(\mathrm{DM2})_{g,d}$ holds if and only if

$$R(x, y)V_d(x, d(x, y)) \asymp g(d(x, y))$$

for any $x, y \in X$.

The conditions $(\mathrm{DM1})_{g,d}$ and $(\mathrm{DM2})_{g,d}$ corresponds to (DM1) and (DM2) with $H(s, t) = st$ respectively. $(\mathrm{DM2})_{g,d}$ is the counterpart of $\mathrm{R}(\beta)$ introduced defined in [**6**] and introduced in the introduction. It relates the mean exit time with $g(d(x, y))$ through $(\mathrm{EIN})_d$.

REMARK. If $\mathrm{diam}(X, d)$ is bounded, it is enough for g to be only defined on $(0, \mathrm{diam}(X, d))$, for example, to describe $(\mathrm{DM2})_{g,d}$. In such a case, the value of g for $[\mathrm{diam}(X, d), +\infty)$ does not make any essential differences. One can freely extend $g : (0, \mathrm{diam}(X, d)) \to (0, +\infty)$ to $g : (0, +\infty) \to (0, +\infty)$ so that g satisfies required conditions as being doubling, decaying uniformly or being strictly monotone.

Here is our second theorem giving equivalent conditions for a good heat kernel.

THEOREM 15.10. *Assume that* (ACC) *holds. Let d be a distance on X giving the same topology as R and let $g : (0,+\infty) \to (0,+\infty)$ be a monotone function with full range and doubling. Then the following statements* (a), (b), (c) *and* $(\mathrm{HK})_{g,d}$ *are equivalent.*

(a) *(X,d) is uniformly perfect,* $(\mathrm{DM1})_{g,d}$ *and* $(\mathrm{DM2})_{g,d}$ *hold.*

(b) *$d \underset{\mathrm{QS}}{\sim} R$ and* $(\mathrm{DM2})_{g,d}$ *holds.*

(c) *$d \underset{\mathrm{QS}}{\sim} R$ and, for any $x \in X$ and any $r \le \mathrm{diam}(X,d)$,*

$$h_d(x,r) \asymp g(r) \tag{15.4}$$

$(\mathrm{HK})_{g,d}$ *$d \underset{\mathrm{QS}}{\sim} R$, g decays uniformly, a jointly continuous heat kernel $p(t,x,y)$ associated with the Dirichlet form $(\mathcal{E},\mathcal{D})$ on $L^2(X,\mu)$ exists and satisfies* (KD) *and* $(\mathrm{DHK})_{g,d}$.

Moreover, if any of the above conditions holds, then there exist positive constants c and c' such that

$$\frac{c'}{V_d(x,g^{-1}(t))} \le p(t,x,y) \tag{15.5}$$

for any $y \in B_d(x, cg^{-1}(t))$. Furthermore, assume that $\Phi(r) = g(r)/r$ is a monotone function with full range and decays uniformly. We have the following off-diagonal estimates:

Case 1: If $(\mathcal{E},\mathcal{F})$ has the local property, then

$$p(t,x,y) \le \frac{c_1}{V_d(x,g^{-1}(t))} \exp\left(-c_2\left(\frac{d(x,y)}{\Phi^{-1}(t/d(x,y))}\right)\right) \tag{15.6}$$

for any $x,y \in X$ and any $t > 0$, where $c_1, c_2 > 0$ are independent of x,y and t.

Case 2: Assume that $d(x,y)$ has the chain condition, i.e. for any $x,y \in X$ and any $n \in \mathbb{N}$, there exist $x_0,\dots,x_n$ such that $x_0 = x, x_n = y$ and $d(x_i,x_{i+1}) \le Cd(x,y)/n$ for any $i = 0,\dots,n-1$, where $C > 0$ is independent of x,y and n. Then,

$$\frac{c_4}{V_d(x,g^{-1}(t))} \exp\left(-c_5\left(\frac{d(x,y)}{\Phi^{-1}(t/d(x,y))}\right)\right) \le p(t,x,y) \tag{15.7}$$

for any $x,y \in X$ and any $t > 0$, where $c_3, c_4 > 0$ are independent of x,y and t.

REMARK. Note that if $d = R$, then the above theorem says $(\mathrm{DM2})_{g,R}$ implies $(\mathrm{DM1})_{g,R}$. Moreover, in this case, $(\mathrm{DM2})_{g,R}$ shows the uniform volume doubling property given by Kumagai in [**41**]. A Borel regular measure μ on a metric space (X,d) is said to satisfy uniform volume doubling property if and only if there exists a doubling function f such that

$$\mu(B_d(x,r)) \asymp f(r)$$

for any $x \in X$ and any $r > 0$. In fact, he has shown the above theorem in this special case including the off-diagonal estimates when $(\mathcal{E},\mathcal{F})$ satisfies the local property.

The above theorem is useful to show a heat kernel estimate for a specific example. In the next section, we will apply this theorem to (traces of) α-stable processes on $\mathbb{R}$ for $\alpha \in (1, 2]$. Also, in Section 24, we will apply (15.6) and (15.7) to homogeneous random Sierpinski gaskets and recover the off-diagonal heat kernel estimate obtained by Barlow and Hambly in [**7**].

The next theorem assures the existence of a distance d which satisfies the conditions in Theorem 15.10 for certain g if μ is $(\mathrm{VD})_R$ and (ACC) holds.

THEOREM 15.11. *Assume that (X, R) is uniformly perfect. Then the following conditions* (C1), (C2), ..., (C6) *are equivalent.*
(C1) (ACC) *holds and μ is* $(\mathrm{VD})_R$.
(C2) *μ is* $(\mathrm{VD})_R$ *and* $(\mathrm{EIN})_R$ *holds.*
(C3) (ACC) *holds and there exist a distance d on X and $\beta > 1$ such that* $(\mathrm{HK})_{g,d}$ *with $g(r) = r^\beta$ is satisfied.*
(C4) *There exist a distance d on X and $\beta > 1$ such that $d \underset{\mathrm{QS}}{\sim} R$ and*

$$E_x(\tau_{B_d(x,r)}) \asymp r^\beta \asymp h_d(x, r) \tag{15.8}$$

for any $x \in X$ and any $r > 0$ with $B_d(x, r) \neq X$.
(C5) *μ is* $(\mathrm{VD})_R$. *If d is a distance on X and $d \underset{\mathrm{QS}}{\sim} R$, then* $(\mathrm{EIN})_d$ *holds.*
(C6) *μ is* $(\mathrm{VD})_R$. *There exists a distance d on X such that $d \underset{\mathrm{QS}}{\sim} R$ and* $(\mathrm{EIN})_d$ *holds.*

Moreover, if any of the above conditions holds, then we can choose the distance d in (C3) *and* (C4) *so that*

$$d(x, y)^\beta \asymp R(x, y)(V_R(x, R(x, y)) + V_R(y, R(x, y))) \tag{15.9}$$

for any $x, y \in X$.

Both the volume doubling property and the Einstein relation are known to be necessary to obtain a good heat kernel estimate. Hence the implication (C2) $\Rightarrow$ (C3) shows that (ACC) is also necessary to get a reasonable two-sided diagonal heat kernel estimate.

REMARK. By Theorem 7.12, we may replace (ACC) by (RES) in (C1) and (C3).

We have a simpler statement in the local case. Recall that (X, R) is assumed to be uniformly perfect. Using Corollary 7.13, we have the next corollary.

COROLLARY 15.12. *Assume that $(\mathcal{E}, \mathcal{F})$ has the local property. Then the following conditions* (C1)' *and* (C3)' *are equivalent:*
(C1)' *μ is* $(\mathrm{VD})_R$.
(C3)' *There exist a distance d on X and $\beta > 1$ such that* $(\mathrm{HK})_{g,d}$ *with $g(r) = r^\beta$ holds.*

Moreover, if any of the above conditions is satisfied, then we have the counterpart of the near diagonal lower estimate, (15.5), *and off-diagonal sub-Gaussian upper estimate* (15.6).

Next we apply the above theorems to the Dirichlet form associated with a trace of a resistance form $(\mathcal{E}, \mathcal{F})$ on X. Since $(\mathcal{E}, \mathcal{F})$ is assumed to be regular, $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ is a regular resistance form by Theorem 8.4. Let Y be a closed subset of X which is uniformly perfect. Assume that $(\mathcal{E}, \mathcal{F})$ satisfies (RES). By Theorem 8.6, $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ satisfies (RES) as well. Recall that $R|_Y$ is the restriction of R to $Y \times Y$ and it coincides with the resistance metric associated with $(\mathcal{E}|_Y, \mathcal{F}|_Y)$. Let ν be a

Borel regular measure on $(Y, R|_Y)$ which satisfy $0 < \mu(B_R(x,r) \cap Y) < +\infty$ for any $x \in Y$ and $r > 0$. If ν is $(\mathrm{VD})_{R|_Y}$, then (ACC) for $(Y, R|_Y)$ follows by Theorem 7.12. Therefore, the counterpart of Theorems 15.6, 15.10 and 15.11 hold for $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ with (ACC) granted. (Note that the conditions (a), (b), (c) and $(\mathrm{HK})_{g,d}$ imply the volume doubling property.) In particular, we have the following result.

THEOREM 15.13. *Let μ be a Borel regular measure on (X,R) that satisfies $0 < \mu(B_R(x,r)) < +\infty$ for any $x \in X$ and any $r > 0$. Assume that* (ACC) *holds for $(\mathcal{E},\mathcal{F})$ and that there exists a distance d on X such that $d \underset{\mathrm{QS}}{\sim} R$ and $(\mathrm{HK})_{g,d}$ with $g(r) = r^\beta$ is satisfied. Let Y be a non-empty closed subset of X and let $(\mathcal{E}|_Y, \mathcal{D}|_Y)$ be the regular Dirichlet form on $L^2(Y,\nu)$ induced by $(\mathcal{E}|_Y, \mathcal{F}|_Y)$. If $(Y, R|_Y)$ is uniformly perfect and there exist $\gamma > 0$ and a Borel regular measure ν on $(Y, R|_Y)$ such that*

$$\mu(B_d(x,r)) \asymp r^\gamma \nu(B_d(x,r) \cap Y) \tag{15.10}$$

for any $x \in Y$ and any $r > 0$ with $B_d(x,r) \neq X$, then it follows that $\beta > \gamma$, that there exists a jointly continuous heat kernel $p_\nu^Y(t,x,y)$ associated with the regular Dirichlet form $(\mathcal{E}|_Y, \mathcal{D}|_Y)$ on $L^2(Y,\nu)$ and that

$$p_\nu^Y(t,x,x) \asymp \frac{1}{\nu(B_d(x, t^{1/(\beta-\gamma)}) \cap Y)} \tag{15.11}$$

for any $x \in X$ and any $t > 0$. In particular, if $\mu(B_d(x,r)) \asymp r^\alpha$ for any $x \in X$ and any $r > 0$ with $B_d(x,r) \neq X$, then

$$p_\nu^Y(t,x,x) \asymp t^{-\frac{\alpha-\gamma}{\beta-\gamma}} \tag{15.12}$$

for any $x \in Y$ and any $t > 0$ with $B_d(x, t^{1/(\beta-\gamma)}) \neq X$.

If $\mu(B_r(x,d)) \asymp r^\alpha$, then the Hausdorff dimension of (X,d) is α and $\mu(A) \asymp \mathcal{H}^\alpha(A)$ for any Borel set A, where $\mathcal{H}^\alpha$ is the α-dimensional Hausdorff measure of (X,d). Hence, (X,d) is Ahlfors α-regular set. In such a case, (15.10) implies that $(Y, d|_Y)$ is Ahlfors $(\alpha-\gamma)$-regular set.

We will apply the above theorem for the traces of the standard resistance form on the Sierpinski gasket in Example 20.11.

16. Example: the α-stable process on $\mathbb{R}$

In this section, we will apply the results in the last section to the resistance form associated with the α-stable process on $\mathbb{R}$ for $\alpha \in (1,2]$. For $\alpha = 2$, the α-stable process is the Brownian motion on $\mathbb{R}$. We denote the Euclidean distance on $\mathbb{R}$ by d_E.

DEFINITION 16.1. (1) For $\alpha \in (0,2)$, define

$$\mathcal{F}^{(\alpha)} = \left\{ u \middle| u \in C(\mathbb{R}), \int_{\mathbb{R}^2} \frac{(u(x)-u(y))^2}{|x-y|^{1+\alpha}} dxdy < +\infty \right\} \tag{16.1}$$

and

$$\mathcal{E}^{(\alpha)}(u,v) = \int_{\mathbb{R}^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+\alpha}} dxdy \tag{16.2}$$

for any $u, v \in \mathcal{F}^{(\alpha)}$. Moreover, define $\mathcal{D}^{(\alpha)} = \mathcal{F}^{(\alpha)} \cap L^2(\mathbb{R}, dx)$.
(2) For $\alpha = 2$, define

$$\mathcal{F}_0^{(2)} = \{u | u \in C^1(\mathbb{R}), \int_{\mathbb{R}} (u'(x))^2 dx < +\infty\}$$

and

$$\mathcal{E}^{(2)}(u, v) = \int_{\mathbb{R}} u'(x)v'(x)dx$$

for any $u, v \in \mathcal{F}_0^{(2)}$.

For $\alpha = 2$, $(\mathcal{E}^{(2)}, \mathcal{F}_0^{(2)})$ does not satisfy (RF2). To make a resistance form, we need to take a kind of closure of $(\mathcal{E}^{(2)}, \mathcal{F}_0^{(2)})$. The next proposition is elementary. We omit its proof.

Proposition 16.2. *If $\{u_n\}_{n\geq 1} \subseteq \mathcal{F}_0^{(2)}$ satisfies $\mathcal{E}^{(2)}(u_n - u_m, u_n - u_m) \to 0$ as $n, m \to +\infty$ and $u_n(0) \to a$ as $n \to +\infty$ for some $a \in \mathbb{R}$, then $\{u_n\}_{n\geq 1}$ converges compact uniformly to $u \in C(\mathbb{R})$ as $n \to +\infty$.*

Definition 16.3. We use $\mathcal{F}^{(2)}$ to denote the collection of all the limits u in the sense of Proposition 16.2. Define

$$\mathcal{E}^{(2)}(u, v) = \lim_{n\to+\infty} \mathcal{E}^{(2)}(u_n, v_n)$$

for any $u, v \in \mathcal{F}^{(2)}$, where $\{u_n\}_{n\geq 1}$ and $\{v_n\}_{n\geq 1}$ are the sequences convergent to u and v respectively in the sense of Proposition 16.2. Also set $\mathcal{D}^{(2)} = \mathcal{F}^{(2)} \cap L^2(\mathbb{R}, dx)$.

It is well-known that, for $\alpha \in (0, 2]$, $(\mathcal{E}^{(\alpha)}, \mathcal{D}^{(\alpha)})$ is a regular Dirichlet form on $L^2(\mathbb{R}, dx)$ and the associated non-negative self-adjoint operator on $L^2(\mathbb{R}, dx)$ is $(-\Delta)^{\alpha/2}$, where $\Delta = d^2/dx^2$ is the Laplacian. The corresponding Hunt process is called the α-stable process on $\mathbb{R}$. See [**40, 15**] for example. Note that $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ has a natural scaling property. More precisely, for $u \in \mathcal{F}^{(\alpha)}$, define $u_t(x) = u(tx)$ for any $t > 0$. Then,

$$\mathcal{E}^{(\alpha)}(u_t, u_t) = t^{\alpha-1}\mathcal{E}^{(\alpha)}(u, u)$$

for any $t > 0$. Combining this scaling property with [**24**, Theorem 8.1], we have the following.

Proposition 16.4. *For $\alpha \in (1, 2]$, $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ is a regular resistance form on $\mathbb{R}$. The corresponding resistance metric $R^{(\alpha)}(x, y) = \gamma_\alpha |x - y|^{\alpha-1}$ for any $x, y \in \mathbb{R}$, where γ_α is independent of x and y.*

By this proposition, for $\alpha \in (1, 2]$, if $\mathcal{D}_\mu^{(\alpha)} = L^2(\mathbb{R}, \mu) \cap \mathcal{F}^{(\alpha)}$, then $(\mathcal{E}^{(\alpha)}, \mathcal{D}_\mu^{(\alpha)})$ is a regular Dirichlet form on $L^2(\mathbb{R}, \mu)$ for any Radon measure μ on $\mathbb{R}$.

Theorem 16.5. *$(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(a)})$ satisfies the annulus comparable condition* (ACC) *for $\alpha \in (1, 2]$.*

Proof. By the scaling property with the invariance under parallel translations, there exist $c_1, c_2 > 0$ such that

$$R^{(\alpha)}(x, B(x, r)^c) = c_1 r$$
$$R^{(\alpha)}(x, B(x, r)^c \cap \overline{B(x, 2r)}) = c_2 r,$$

where $B(x, r) = B_{R^{(\alpha)}}(x, r)$. Now, it is obvious that (ACC) holds. □

Due to this theorem, we can apply Theorems 15.6 and 15.11 to get an estimate of the heat kernel associated with the Dirichlet form $(\mathcal{E}^{(\alpha)}, \mathcal{D}^{(\alpha)}_\mu)$ on $L^2(\mathbb{R}, \mu)$ if μ has the volume doubling property with respect to the Euclidean distance. (Note that $R^{(\alpha)}$ is a power of the Euclidean distance.) As a special case, we have the following proposition.

Proposition 16.6. *Define $p^{(\alpha)}_\delta(t, x, y)$ as the heat kernel associated with the Dirichlet form $(\mathcal{E}^{(\alpha)}, \mathcal{D}^{(\alpha)})$ on $L^2(\mathbb{R}, x^\delta dx)$ for $\delta > -1$. For $\alpha \in (1, 2]$,*

$$p^{(\alpha)}_\delta(t, 0, 0) \asymp t^{-\frac{\delta+1}{\delta+\alpha}}$$

for any $t > 0$.

Recall that $(-\Delta)^{\alpha/2}$ is the associated self-adjoint operator for $\delta = 0$. Hence $p^{(\alpha)}_0(t, x, y) = P^\alpha(t, |x - y|)$, where $P^\alpha(t, \cdot)$ is the inverse Fourier transform of $e^{-ct|x|^\alpha}$ for some $c > 0$. This immediately imply that $p^{(\alpha)}_0(t, x, x) = a/t^{1/\alpha}$ for some $a > 0$.

Next we consider the trace of $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ on the Cantor set. Let K be the Cantor set, i.e.

$$K = \left\{ \sum_{m=1}^{\infty} \frac{i_m}{3^m} \middle| i_1, i_2, \ldots \in \{0, 2\} \right\}.$$

The Hausdorff dimension h of (K, d_E) is $\log 2/\log 3$. Let ν be the h-dimensional normalized Hausdorff measure. Define

$$K_{i_1 \ldots i_m} = \left\{ \sum_{k=1}^{m} \frac{i_k}{3^k} + \frac{1}{3^m} \sum_{n=1}^{\infty} \frac{j_n}{3^n} \middle| j_1, j_2, \ldots \in \{0, 2\} \right\}$$

for any $i_1, \ldots, i_m \in \{0, 2\}$. Then $\nu(K_{i_1 \ldots i_m}) = 2^{-m}$. Hence $\nu(B(x, r)) \asymp r^h$ for any $r \in [0, 1]$ and any $x \in K$. It is easy to see that ν has the volume doubling property with respect to d_E. Also $(K, d_E|_K)$ is uniformly perfect. Recall that $R^{(\alpha)} = \gamma_\alpha (d_E)^{\alpha - 1}$. Also we have

$$\mu(B_{d_E}(x, r)) \asymp r^{1-h} \nu(B_{d_E}(x, r) \cap K)$$

Since K is compact and any $u \in \mathcal{F}^{(\alpha)}$ is continuous, $\mathcal{F}^{(\alpha)}|_K \subseteq L^2(K, \nu)$. Hence $(\mathcal{E}^{(\alpha)}|_K, \mathcal{F}^{(\alpha)}|_K)$ is a regular Dirichlet form on $L^2(K, \nu)$.

Using Theorem 15.13, we have the following result.

Theorem 16.7. *Let $\alpha \in (1, 2]$. There exists a jointly continuous heat kernel $p^{(\alpha)}_K(t, x, y)$ on $(0, +\infty) \times K^2$ associated with the Dirichlet form $(\mathcal{E}^{(\alpha)}|_K, \mathcal{F}^{(\alpha)}|_K)$ on $L^2(K, \nu)$. Moreover,*

$$p^{(\alpha)}_K(t, x, x) \asymp t^{-\eta} \tag{16.3}$$

for any $t \in (0, 1]$ and any $x \in K$, where $\eta = \frac{\log 2}{(\alpha - 1)\log 3 + \log 2}$.

If $\alpha = 2$, the process associated with $(\mathcal{E}^{(2)}|_K, \mathcal{F}^{(2)}|_K)$ on $L^2(K, \nu)$ is called the generalized diffusion on the Cantor set. Fujita has studied the heat kernel associated with the generalized diffusion on the Cantor set extensively in [**20**]. He has obtained (16.3) for this case by a different method.

17. Basic tools in heat kernel estimates

The rest of this part is devoted to proving the theorems in Section 15. In this section, we review the general methods of estimates of a heat kernel and make necessary modifications to them. The results in this section have been developed by several authors, for example, [**1**], [**38**] and [**22**].

In this section, (X, d) is a metric space and $(\mathcal{E}, \mathcal{D})$ is a regular Dirichlet form on $L^2(X, \mu)$, where μ is a Radon measure on X. (We do not assume that $(\mathcal{E}, \mathcal{D})$ is derived from a resistance form.) We assume that there exists a jointly continuous heat kernel (i.e. transition density) $p(t, x, y)$ associated with this Dirichlet form.

First we introduce a result on diagonal-lower estimate of a heat kernel. The Chapman-Kolmogorov equation implies the following fact.

Lemma 17.1. *For any Borel set $A \subseteq X$, any $t > 0$ and any $x \in X$,*

$$\frac{P_x(X_t \in A)^2}{\mu(A)} \leq p(2t, x, x).$$

The next lemma can be extracted from [**22**, Proof of Theorem 9.3].

Lemma 17.2. *Let $h : X \times (0, +\infty) \to [0, +\infty)$ satisfy the following conditions* (HKA), (HKB) *and* (HKC):
(HKA) *For any $x \in X$, $h(x, r)$ is a monotonically nondecreasing function of r and $\lim_{r \downarrow 0} h(x, r) = 0$.*
(HKB) *There exists $a_1 > 0$ such that $h(x, 2r) \leq a_1 h(x, r)$ for any $x \in X$ and any $r > 0$.*
(HKC) *There exists $a_2 > 0$ such that $h(x, r) \leq a_2 h(y, r)$ for any $x, y \in X$ with $d(x, y) \leq r$.*

Assume that there exist positive constants c_1, c_2 and $r_* \in (0, +\infty) \cup \{+\infty\}$ such that

$$c_1 h(x, r) \leq E_x(\tau_{B_d(x,r)}) \leq c_2 h(x, r) \tag{17.1}$$

for any $x \in X$ and any $r \in (0, r_*]$. Then,
(1) *There exist $\epsilon \in (0, 1)$ and $c > 0$ such that*

$$E_x(e^{-\lambda \tau_{B_d(x,r)}}) \leq \epsilon \tag{17.2}$$

whenever $\lambda h(x, r) \leq c$ and $r \in (0, r_*/2]$.
(2) *For any $r \in (0, r_*/2]$ and any $t > 0$,*

$$P_x(\tau_{B_d(x,r)} \leq t) \leq \epsilon e^{\frac{ct}{h(x,r)}}, \tag{17.3}$$

where ϵ and c are the same as in (1).

Combining the above two lemmas, we immediately obtain the following theorem.

Theorem 17.3. *Under the same assumptions of Lemma 17.1, there exist positive constants α and δ such that*

$$\frac{\alpha}{\mu(B_d(x, r))} \leq p(\delta h(x, r), x, x)$$

for any $x \in X$ and any $r \in (0, r_*/2]$. Moreover, if μ has the volume doubling property with respect to d and $h(x, \lambda r) \leq \eta h(x, r)$ for any $x \in X$ and any $r \in (0, r_*]$, *where* λ *and* η *belong to* $(0,1)$ *and are independent of* x *and* r*, then there exist* $\alpha' > 0$ *and* $c_* \in (0,1)$ *such that*

$$\frac{\alpha'}{\mu(B_d(x,r))} \leq p(h(x,r),x,x)$$

for any $x \in X$ *and* $r \in (0, c_* r_*]$.

Next we give a result on off-diagonal upper estimate.

Hereafter, $h(x,r)$ is assumed to be independent on $x \in X$. We write $h(r) = h(x,r)$.

The following line of reasoning has essentially been developed in the series of papers by Barlow and Bass [**2, 3, 4**]. It has presented in [**1**] in a concise and organized manner. Here we follow a sophisticated version in [**22**]. Generalizing the discussion in [**22**, Proof of Theorem 9.1], we have the following lemma.

Lemma 17.4. *Let* $(\mathcal{E},\mathcal{D})$ *be local. Also let* $h : (0,+\infty) \to (0,+\infty)$ *be a monotone function with full range, continuous, strictly increasing and doubling.*
(1) *If there exist* $\epsilon \in (0,1)$ *and* $c > 0$ *such that* (17.2) *holds for any* $r \in (0, r_*]$ *and any* $x \in X$ *with* $\lambda h(r) \geq c$*, then, for any* $q > 0$,

$$E_x(e^{-\lambda \tau_{B_d(x,r)}}) \leq c_1 \exp\left(- c_2 \frac{r}{h^{-1}(c/\lambda)}\right) \tag{17.4}$$

for any $\lambda > 0$ *and any* $r \in (0, qr_*]$*, where* $c_1 = \epsilon^{-2\max\{1,q\}}$ *and* $c_2 = -\log \epsilon$.
(2) *Moreover, assume that* $\Psi(r) = h(r)/r$ *is a monotone function with full range and strictly increasing. If* (17.4) *holds for any* $\lambda > 0$ *and any* $r \in (0,R]$*, then, for any* $\delta \in (0,1)$*, any* $t > 0$ *and any* $r \in (0,R]$,

$$P_x(\tau_{B_d(x,r)} \leq t) \leq c_1 \exp\left(- \frac{c_3 r}{\Psi^{-1}(c_4 t/r)}\right), \tag{17.5}$$

where $c_3 = c_2(1-\delta)$ *and* $c_4 = c/(c_2\delta)$.

Remark. The local property of a Dirichlet form is equivalent to that the associated Hunt process is a diffusion. See [**21**, Theorem 4.5.1] for details.

Proof. (1) First assume that $r/h^{-1}(c/\lambda) \geq 2$ and $r_*/h^{-1}(c/\lambda) \geq 2$. Then there exists $n \in \mathbb{N}$ such that

$$\frac{r}{h^{-1}(c/\lambda)} \geq n \geq \frac{r}{2h^{-1}(c/\lambda)} \geq \frac{r}{r_*}. \tag{17.6}$$

If n is the maximum natural number satisfying (17.6), then $n \geq r/h^{-1}(c/\lambda) - 1$. Since $\lambda h(r/n) \geq c$ and $r/n \in (0, r_*]$, argument in [**22**, Proof of Theorem 9.1] works and implies

$$E_x(e^{-\lambda \tau_{B_d(x,r)}}) \leq \epsilon^n \leq \frac{1}{\epsilon} \exp\left(- \frac{c_2 r}{h^{-1}(c/\lambda)}\right). \tag{17.7}$$

If $r/h^{-1}(x/\lambda) \leq 2$, then

$$E_x(e^{-\lambda \tau_{B_d(x,r)}}) \leq 1 \leq \frac{1}{\epsilon^2} \exp\left(- \frac{c_2 r}{h^{-1}(c/\lambda)}\right). \tag{17.8}$$

Finally if $r_*/h^{-1}(c/\lambda) \leq 2$, then $r/h^{-1}(c/\lambda) \leq qr_*/h^{-1}(x/\lambda) \leq 2q$ for any $r \in (0, qr_*]$. Hence

$$E_x(e^{-\lambda \tau_{B_d(x,r)}}) \leq 1 \leq \frac{1}{\epsilon^{2q}} \exp\left(- \frac{c_2 r}{h^{-1}(c/\lambda)}\right). \tag{17.9}$$

Combining (17.7), (17.8) and (17.9), we obtain the desired inequality.
(2) By [**22**, Proof of Theorem 9.1],

$$P_x(\tau_{B_d(x,r)} \leq t) \leq e^{\lambda t} E_x(e^{-\lambda \tau_{B_d(x,r)}}) \leq c_1 \exp\left(\lambda t - \frac{c_2 r}{h^{-1}(c/\lambda)}\right) \tag{17.10}$$

for any $t > 0$, any $\lambda > 0$ and any $r \in (0, R]$. Let $\lambda = \frac{\delta c_2 r}{t\Psi^{-1}(ct/(\delta c_2 r))}$. Then $\lambda t = \delta \frac{c_2 r}{h^{-1}(c/\lambda)} = \frac{\delta c_2 r}{\Psi^{-1}(ct/(\delta c_2 r))}$. Hence we have (17.5). □

THEOREM 17.5. *Let $(\mathcal{E}, \mathcal{D})$ be local. Also let $h : (0, +\infty) \to (0, +\infty)$ be a monotone function with full range, continuous, strictly increasing, doubling and decays uniformly. Assume that $\Psi(r) = h(r)/r$ is a monotone function with full range and strictly increasing and that μ is $(\mathrm{VD})_d$. If there exist $c_1, c_3, c_4 > 0$ such that* (17.5) *holds for any $t > 0$ and any $r \in (0, R]$ and*

$$p(t, x, x) \leq \frac{c_5}{\mu(B_d(x, h^{-1}(t)))}$$

for any $t \in (0, t_*]$ and any $x \in X$, then there exist c_6 and c_7 such that

$$p(t, x, y) \leq \frac{c_6}{\mu(B_d(x, h^{-1}(t)))} \exp\left(- c_7 \frac{d(x, y)}{\Psi^{-1}(2c_4 t/d(x, y))}\right) \tag{17.11}$$

for any $t \in (0, t_*]$ and any $x, y \in X$ with $d(x, y) \leq R$.

The next two lemmas are technically the keys in proving the above theorem. The first one is well-known. See [**22**, Lemma 11.1].

LEMMA 17.6. *Assume that μ is $(\mathrm{VD})_d$. There exist $c_0 > 0$ and $\alpha > 0$ such that*

$$\mu(B_d(x, r_1)) \leq c_0 (r_1/r_2)^\alpha \mu(B_d(x, r_2))$$

for any $r_1 \geq r_2 > 0$ and

$$\mu(B_d(x, r)) \leq c_0 \Big(1 + \frac{d(x, y)}{r}\Big)^\alpha \mu(B_d(y, r))$$

for any $x, y \in X$ and $r > 0$. In particular, there exists $M > 0$ such that

$$\mu(B_d(x, r)) \leq M \mu(B_d(y, r))$$

if $d(x, y) \leq r$.

LEMMA 17.7. *Let Ψ be a monotone function with full range, strictly increasing and continuous. Set $h(r) = r\Psi(r)$. For any $\gamma > 0$, any $\epsilon > 0$, any $s > 0$ and any $r > 0$,*

$$1 + \frac{r}{h^{-1}(s)} \leq \max\{\epsilon^{-1}, 1 + \gamma\} \exp\left(\frac{\epsilon r}{\Psi^{-1}(\gamma s/r)}\right). \tag{17.12}$$

PROOF. Set $x = r/h^{-1}(s)$. If $0 \leq x \leq \gamma$, then (17.12) holds. Assume that $x \geq \gamma$. Then

$$\psi^{-1}\left(\frac{\gamma s}{r}\right) = \Psi^{-1}\left(\frac{\gamma}{x}\Psi\left(\frac{r}{x}\right)\right) \leq \frac{r}{x}.$$

This implies

$$\exp\left(\frac{\epsilon r}{\Psi^{-1}(\gamma s/r)}\right) \geq \exp \epsilon x \geq 1 + \epsilon x.$$

Hence we have (17.12). □

PROOF OF THEOREM 17.5. We can prove this theorem by modifying the discussion in [**22**, Section 12.3]. Note that the counterpart of [**22**, (12.20)], which is one of the key ingredients of the discussion in [**22**], is obtained by Lemma 17.7. □

Next we give an off-diagonal lower estimate. For our theorem, the local property of the Dirichlet form is not required but the estimate may not be best possible without the local property, i.e. if the Hunt process associated with the Dirichlet form has jumps. One can find the original form on this theorem in [**1**].

THEOREM 17.8. *Let $\Psi : (0, +\infty) \to (0, +\infty)$ be a monotone function with full range, strictly increasing and continuous. Set $h(r) = r\Psi(r)$. Assume that μ is $(\mathrm{VD})_d$ and that $d(x, y)$ satisfies the chain condition defined in Case 2 of Theorem 15.10. Also assume that there exist c_1 and c_2 such that*

$$\frac{c_1}{V_d(x, h^{-1}(t))} \le p(t, x, y) \tag{17.13}$$

for any $t \in (0, t_*]$ and any $x, y \in X$ with $d(x, y) \le c_2 h^{-1}(t)$. Then

$$\frac{c_3}{V_d(x, h^{-1}(t))} \exp\left(- c_4 \frac{d(x, y)}{\Psi^{-1}(c_5 t/d(x, y))}\right) \le p(t, x, y) \tag{17.14}$$

for any $t \in (0, t_*]$ and any $x, y \in X$.

LEMMA 17.9. *Let $C, D, T \in (0, +\infty)$. Then $D \le Ch^{-1}(T)$ if and only if $D/C \le \Psi^{-1}(TC/D)$. Also $D \ge Ch^{-1}(T)$ if and only if $D/C \ge \Psi^{-1}(TC/D)$.*

The ideas of the following proof is essentially found in [**1**]. We modify a version in [**38**].

PROOF OF THEOREM 17.8. If $d(x, y) \le c_2 h^{-1}(t)$, then (17.13) implies (17.14). So we may assume that $d(x, y) \ge ch^{-1}(t)$, where $c = \min\{c_2/(6C), 1/(2C)\}$, without loss of generality. By Lemma 17.9, we have

$$\frac{d(x, y)}{c\Psi^{-1}(ct/d(x, y))} \ge 1.$$

Therefore, there exists $n \in \mathbb{N}$ such that

$$\frac{d(x, y)}{2c\Psi^{-1}(2ct/d(x, y))} \le n \le \frac{d(x, y)}{c\Psi^{-1}(ct/d(x, y))}. \tag{17.15}$$

Note that (17.15) is equivalent to

$$ch^{-1}\Big(\frac{t}{n}\Big) \le \frac{d(x, y)}{n} \le 2ch^{-1}\Big(\frac{t}{n}\Big). \tag{17.16}$$

Now we use the classical chaining argument. (See [**1**] for example.) Note that

$$p(t, x, y) = \int_{X^{n-1}} p\Big(\frac{t}{n}, x, z_1\Big)p\Big(\frac{t}{n}, z_1, z_2\Big)\cdots p\Big(\frac{t}{n}, z_{n-1}, y\Big)\mu(dz_1)\cdots\mu(dz_{n-1}).$$

By the chain condition, we may choose a sequence $\{x_i\}_{i=0}^n$ such that $x_0 = x, x_n = y$ and $d(x_i, x_{i+1}) \le D/n$ for any $i = 0, 1, \ldots, n-1$, where $D = Cd(x, y)$. Define $B_i = B_d(x_i, D/n)$ for $i = 1, \ldots, n-1$. If $z_i \in B_i$ and $z_{i+1} \in B_{i+1}$, then $d(z_i, z_{i+1}) \le 3D/n$. By (17.16), $3D/n \le c_2 h^{-1}(t/n)$ and $D/n \le h^{-1}(t/n)$, (17.13) and Lemma 17.6 yield

$$p(t, z_i, z_{i+1}) \ge \frac{c_1}{V_d(z_i, h^{-1}(t/n))} \ge \frac{c_1}{MV_d(x_i, h^{-1}(t/n))},$$

Hence

$$
\begin{aligned}
&p(t,x,y)\\
&\geq \int_{B_1\times\ldots\times B_{n-1}} p\Big(\frac{t}{n},x,z_1\Big)p\Big(\frac{t}{n},z_1,z_2\Big)\cdots p\Big(\frac{t}{n},z_{n-1},y\Big)\mu(dz_1)\cdots\mu(dz_{n-1})\\
&\geq (c_1/M)^n\frac{1}{V_d(x,h^{-1}(t))}\prod_{i=1}^{n-1}\frac{V_d(x_i,D/n)}{V_d(x_i,h^{-1}(t/n))}\\
&\geq (c_1/M)^n\frac{1}{V_d(x,h^{-1}(t))}\prod_{i=1}^{n-1}\frac{V_d(x_i,D/n)}{V_d(x_i,h^{-1}(t/n))}
\end{aligned}
$$

By Lemma 17.6 and (17.16),

$$
\frac{V_d(x_i,D/n)}{V_d(x_i,h^{-1}(t/n))}\geq (c_0)^{-1}\Big(\frac{D}{nh^{-1}(t/n)}\Big)^{\lambda}\geq {c_0}^{-1}(cC)^{\lambda}.
$$

Therefore there exists $L>1$ such that

$$
p(t,x,y)\geq\frac{L^{-n}}{V_d(x,h^{-1}(t))}.
$$

Now the desired estimate follows immediately from (17.15). $\square$

18. Proof of Theorem 15.6

We assume the same prerequisites on a resistance form $(\mathcal{E},\mathcal{F})$ and the associated resistance metric R as in Section 15.

LEMMA 18.1. *Let A be an open set containing $x\in X$. Assume that $A\neq X$ and that $\overline{A}$ is compact. Then, for any $\gamma\in(0,1)$,*

$$
(1-\gamma)R(x,A^c)V_R(x,\gamma R(x,A^c))\leq E_x(\tau_A)\leq R(x,A^c)\mu(A)
$$

PROOF. Set $B=A^c$. Note that $E_x(\tau_A)=\int_A g_B(x,y)\mu(dy)$ by Corollary 10.11. Since $g_B(x,y)\leq g_B(x,x)=R(x,B)$, the upper estimate is obvious. If $y\in B_R(x,\gamma R(x,B))$, then (GF4) implies that $g_B^x(y)\geq(1-\gamma)g_B^x(x)$. Therefore,

$$
E_x(\tau_A)\geq\int_{B_R(x,\gamma R(x,B))}g_B^x(y)\mu(dy)\geq(1-\gamma)R(x,B)V_R(x,\gamma R(x,B)).
$$

$\square$

PROPOSITION 18.2. *Assume that $d\underset{\mathrm{QS}}{\sim}R$.*

(1) *There exists $\delta>0$ such that $B_d(x,\delta\overline{d}_R(x,r))\subseteq B_R(x,r)\subseteq B_d(x,\overline{d}_R(x,r))$ and $B_R(x,\delta\overline{R}_d(x,r))\subseteq B_d(x,r)$ for any $x\in X$ and any $r>0$, where $\overline{d}_R(x,r)=\sup_{y\in B_R(x,r)}d(x,y)$.*

(2) *There exists $c>0$ such that $\overline{R}_d(x,2r)\leq c\overline{R}_d(x,r)$ for any $x\in X$ and any $r>0$.*

(3) *If $\mathrm{diam}(X,d)<+\infty$, then, for any $r_*>0$, there exist $\lambda\in(0,1)$ and $\delta\in(0,1)$ such that $\overline{R}_d(x,\lambda r)\leq\delta\overline{R}_d(x,r)$ for any $x\in X$ and any $r\in(0,r_*]$. If $\mathrm{diam}(X,d)=+\infty$, then we have the same statement with $r_*=+\infty$.*

(4) *If μ is $(\mathrm{VD})_R$, then it is $(\mathrm{VD})_d$.*

PROOF. If $d \underset{\text{QS}}{\sim} R$, then by Theorem 12.3, d is $(\text{SQS})_R$ and R is $(\text{SQS})_d$. Since (X, R) is assumed to be uniformly perfect, Proposition 12.2-(3) implies that (X, d) is uniformly perfect. Hence we may apply Theorem 11.5. Note that the statement (a) of Theorem 11.5 holds.

(1) By the statement (b) of Theorem 11.5, d is $(\text{SQC})_R$ and R is $(\text{SQC})_d$.
(2) By the statement (b) of Theorem 11.5, R is doubling with respect to d.
(3) Proposition 11.7 suffices to deduce the desired result. □

PROOF OF PROPOSITION 15.4. Define $\widetilde{R}(x,r) = R(x, B_R(x,r)^c)$.
Assume (RES). By Proposition 18.2-(1),

$$B_R(x, \delta\overline{R}_d(x,r)) \subseteq B_d(x,r) \subseteq B_R(x, \overline{R}_d(x,r)). \tag{18.1}$$

Hence by (RES),

$$\widetilde{R}(x,r) \geq \widetilde{R}(x, \delta\overline{R}_d(x,r)) \geq c\delta\overline{R}_d(x,r).$$

If $B_R(x, \overline{R}_d(x,r)) \neq X$, then (RES) also shows

$$\widetilde{R}(x,r) \leq \widetilde{R}(x, \overline{R}_d(x,r)) \leq c\overline{R}_d(x,r).$$

In case $X = B_R(x, \overline{R}_d(x,r))$, we have $\text{diam}(X,R)/2 \leq \overline{R}_d(x,r)$. Hence

$$\widetilde{R}(x,r) \leq \text{diam}(X,R) \leq 2\overline{R}_d(x,r).$$

Conversely assume that $\widetilde{R}(x,r) \asymp \overline{R}_d(x,r)$ for any $x \in X$ and any $r > 0$ with $B_d(x,r) \neq X$. By (18.1),

$$c_1\overline{R}_d(x,r) \leq \widetilde{R}(x, \overline{R}_d(x,r)) \quad \text{and} \quad \widetilde{R}(x, \delta\overline{R}_d(x,r)) \leq c_2\overline{R}_d(x,r). \tag{18.2}$$

On the other hand, by Proposition 18.2-(1) and (2), there exists $\eta \in (0,1)$ such that

$$\eta\theta(r) \leq \delta\overline{R}_d(x, \delta\overline{d}_R(x,r)) \leq r \leq \theta(r)$$

for any $x \in X$ and $r > 0$, where $\theta(r) = \overline{R}_d(x, \overline{d}_R(x,r))$. Hence by (18.2),

$$r \leq \theta(r) \leq \widetilde{R}(x, \theta(r)) \leq \widetilde{R}(x, r/\eta)$$
$$\widetilde{R}(x, \delta r) \leq \widetilde{R}(x, \delta\theta(r)) \leq c_2\theta(r) \leq c_2 r/\eta.$$

This suffices for (RES). □

PROOF OF PROPOSITION 15.5. Assume (RES). Let $\gamma \in (0,1)$. By Proposition 15.4 and the volume doubling property of μ, we obtain

$$\begin{aligned}\mu(B_R(x, \gamma R(x, B_d(x,r)^c))) &\geq \mu(B_R(x, \gamma c_3\overline{R}_d(x,r))) \\ &\geq c'\mu(B_R(x, \overline{R}_d(x,r))) \geq c'\mu(B_d(x,r)).\end{aligned}$$

By Lemmas 18.1 and 15.4 , $c'(1-\gamma)c_3h_d(x,r) \leq E_x(\tau_{B_d(x,r)}) \leq c_4h_d(x,r)$.
Conversely, assume $(\text{EIN})_d$. By Lemma18.1,

$$c_1\overline{R}_d(x,r) \leq R(x, B_d(x,r)^c). \tag{18.3}$$

Also Lemma 18.1 and the volume doubling property of μ yield

$$c_2R(x, B_d(x,r)^c)\mu(B_R(x, R(x, B_d(x,r)^c))) \leq \overline{R}_d(x,r)V_d(x,r). \tag{18.4}$$

By (18.3), it follows that

$$V_d(x,r) \leq V_R(x, \overline{R}_d(x,r)) \leq c_3V_R(x, c_1\overline{R}_d(x,r)) \leq V_R(x, B_d(x,r)^c).$$

This and (18.4) show that $c_4 R(x, B_d(x,r)^c) \leq \overline{R}_d(x,r)$. Thus we obtain (15.1). Now Proposition 15.4 implies (RES). $\square$

PROOF OF THEOREM 15.6. By Lemma 7.9, $B_R(x,r)$ is totally bounded for any $x \in X$ and any $r > 0$. Hence $\overline{B_R(x,r)}$ is compact. By Theorem 10.4, there exists a jointly continuous heat kernel $p(t,x,y)$. Since $d \underset{\text{QS}}{\sim} R$, $B_d(x,r)$ is compact for any $x \in X$ and any $r > 0$. Using (10.4) with $A = B_d(x,r)$ and letting $t = h_d(x,r)$, we have

$$p(h_d(x,r), x, x) \leq \frac{2+\sqrt{2}}{V_d(x,r)}.$$

The rest is to show $(\mathrm{EIN})_d$ and the lower estimate of the heat kernel. Since μ is $(\mathrm{VD})_R$, (X,R) has the doubling property. Hence by Theorem 7.12, (ACC) implies (RES). Proposition 15.5 shows $(\mathrm{EIN})_d$. Next we show that $h_d(x,r)$ satisfies the conditions (HKA), (HKB) and (HKC) in Lemma 17.2. (HKA) is immediate. (HKB) follows from Proposition 18.2-(2) and (4). Note that $\mathrm{diam}(B_d(x,r), R) \geq \overline{R}_d(x,r) \geq \mathrm{diam}(B_d(x,r), R)/2$. If $d(x,y) \leq r$, then $B_d(y,r) \leq B_d(x,2r)$. Hence,

$$\overline{R}(y,r) \leq \mathrm{diam}(B_d(y,r), R) \leq \mathrm{diam}(B_d(x,2r)) \leq 2\overline{R}(x,2r).$$

By Proposition 18.2-(2), we have (HKC). Now assume that $\mathrm{diam}(X,R) = +\infty$. Then we have $(\mathrm{EIN})_d$. By Proposition 18.2-(3), Theorem 17.3 shows that, for some $c > 0$,

$$\frac{c}{V_d(x,r)} \leq p(h_d(x,r), x, x) \tag{18.5}$$

for any $x \in X$ and any $r > 0$. Next we consider the case where $\mathrm{diam}(X,R) < +\infty$. If $B_d(x,r) = X$, then $r \geq \mathrm{diam}(X,d)/2$. Therefore, assumptions of Lemma 17.2 hold with $r_* = \mathrm{diam}(X,d)/3$. Hence by Theorem 17.3, (18.5) is satisfied for any $x \in X$ and any $r \in (0, \alpha\mathrm{diam}(X,d)]$, where α is independent of x. Next we show

$$\frac{1}{\mu(X)} \leq \inf_{x \in X, r \geq \alpha\mathrm{diam}(X,d)} p(h_d(x,r), x, x) \tag{18.6}$$

and

$$\sup_{x \in X, r \geq \alpha\mathrm{diam}(X,d)} \frac{1}{V_d(x,r)} < +\infty. \tag{18.7}$$

Letting $A = X$ in Lemma 17.1, we have $\mu(X)^{-1} \leq p(t,x,x)$ for any $x \in X$ and any $t > 0$. This yields (18.6). Let $r_1 = \alpha\mathrm{diam}(X,d)$. Since $X \subseteq B_R(x,r)$ for some $r > 0$, it follows that X is compact. Hence we may choose $N > 0$ so that, for any $x, y \in X$, there exists $\{x_i\}_{i=1,\ldots,N+1} \subseteq X$ such that $x_1 = x, x_{N+1} = y$ and $d(x_i, x_{i+1}) \leq r_1$ for any $i = 1, \ldots, N-1$. Since μ has the volume doubling property with respect to d, there exists $a_1 > 0$ such that $V_d(y_1, r_1) \leq a_1 V_d(y_2, r_1)$ for any $y_1, y_2 \in X$ with $d(x,y) \leq r_1$. Hence, for any $x, y \in X$, $V_d(x, r_1) \leq (a_1)^N V_d(y, r_1)$. This shows (18.7). Thus we have obtained (18.6) and (18.7). Therefore, there exists $C > 0$ such that

$$\frac{C}{V_d(x,r)} \leq p(h_d(x,r), x, x)$$

for any $x \in X$ and any $r \geq \alpha\mathrm{diam}(X,d)$. Hence changing c, we have (18.5) for any $x \in X$ and any $r > 0$ in this case as well.

Now, by Proposition 18.2-(3), there exists $\lambda \in (0,1)$ such that $\overline{R}_d(x, \lambda r) \leq (c/4)\overline{R}_d(x,r)$ for any $x \in X$ and any $r \leq \mathrm{diam}(X,d)$, where c is the constant appearing in (18.5). Since $\overline{R}_d(x,r) = \mathrm{diam}(X,d)$ for any $r \geq \mathrm{diam}(X,d)$, we see that $R(x,y) \leq (c/4)\overline{R}_d(x,r)$ if $d(x,y) \leq \lambda \min\{r, \mathrm{diam}(X,d)\}$. Let $T = h_d(x,r)$. Then, this and (18.5) imply

$$|p(T,x,x) - p(T,x,y)|^2 \leq R(x,y)\mathcal{E}(p^{T,x}, p^{T,x}) \leq \frac{R(x,y)p(T,x,x)}{T}$$
$$\leq \frac{c\overline{R}_d(x,r)p(T,x,x)}{4\overline{R}_d(x,r)V_d(x,r)} = \frac{1}{4}\frac{c}{V_d(x,r)}p(T,x,x) \leq \frac{1}{4}p(T,x,x)^2.$$

Hence,

$$p(h_d(x,r),x,y) \geq \frac{p(h_d(x,r),x,x)}{2} \geq \frac{1}{2}\frac{c}{V_d(x,r)}.$$

Thus we have shown Theorem 15.6. □

19. Proof of Theorems 15.10, 15.11 and 15.13

The proofs of Theorems 15.10 and 15.11 depend on the results in Sections 13 and 14. We use those results by letting $H(s,t) = st$. Note that all the assumptions on H in Sections 13 and 14 are satisfied for this particular H.

Proof of Theorem 15.10. The equivalence between (a), (b) and (c) is immediate form the corresponding part of Corollary 13.3. Next assume that (a), (b) and (c) hold. By Corollary 13.3, μ has the volume doubling property with respect to d and R. Also by Lemma 14.4 and Proposition 18.2-(2), there exists $\lambda, \delta \in (0,1)$ such that $h_d(x,r) \leq \lambda h_d(x,\delta r)$ for any $x \in X$ and $r > 0$. Hence by (15.4), g decays uniformly. Lemma 15.8 implies that g^{-1} is doubling and decays uniformly. Now apply Theorem 15.6. There exists a jointly continuous heat kernel $p(t,x,y)$. Furthermore, combining (15.2), (15.3) and (15.4) along with the volume doubling property and the above mentioned properties of g and g^{-1}, we obtain

$$\frac{1}{V_d(x,g^{-1}(t))} \asymp p(t,x,x)$$

for any $t \leq cg(\mathrm{diam}(X,d))$ and any $x \in X$, where c is independent of x and r. The same arguments as in the proof of Theorem 15.6, in particular, (18.6) and (18.7) show $(\mathrm{DHK})_{g,d}$ for $t \geq cg(\mathrm{diam}(X,d))$. Now (KD) is straightforward by the volume doubling property. Thus we have obtained $(\mathrm{HK})_{g,d}$.

Conversely, assume $(\mathrm{HK})_{g,d}$. (KD) and $(\mathrm{DHK})_{g,d}$ imply that μ has the volume doubling property with respect to d. Since $d \underset{\mathrm{QS}}{\sim} R$, we have the volume doubling property of μ with respect to R. Also (X,d) is uniformly perfect. By Theorem 15.6, we have (15.2) and (15.3). Comparing those with $(\mathrm{DHK})_{g,d}$, we see that

$$V_d(x, g^{-1}(h_d(x,r))) \asymp V_d(x,r) \tag{19.1}$$

for any $x \in X$ and any $r > 0$. Set $r_* = \mathrm{diam}(X,d)$. Note that $B_d(x,r) \neq X$ for any $r < r_*/2$. By Lemma 14.4, for any $\delta > 1$, there exists $\lambda \in (0,1)$ such that $V_d(x,\lambda r) \leq \delta^{-1}V_d(x,r)$ for any $r < r_*/2$. This along with (19.1) shows that $r \asymp g^{-1}(h_d(x,r))$ for any $r < \lambda r_*/2$. This and Lemma 15.8 imply (15.4) for $r < \lambda r_*/2$. Let us think about $r \in [\lambda r_*/2, r_*]$. If $r_* < +\infty$, then (X,d) is compact and so is (X,R). Therefore, $\overline{R}_d(x,r) \leq \mathrm{diam}(X,R)$ and

$V_d(x,r) \leq \mu(X) < +\infty$. Let $r \in [\lambda r_*/2, r_*]$. By the volume doubling property, $V_d(x,r) \geq V_d(x, \lambda r_*/2) \geq cV_d(x,r_*) = c\mu(X)$. Also, by Proposition 7.6-(2), we have $\overline{R}_d(x,r) \geq \overline{R}_d(x,\lambda r_*/2) \geq cV_d(x,r_*) \geq c'\mathrm{diam}(X,R)/2$. Hence $cc'\mathrm{diam}(X,R)\mu(X)/2 \leq h_d(x,r) \leq \mathrm{diam}(X,R)\mu(X)$ for any $r \in [\lambda r_*/2, r_*]$. Also $g(\lambda r_*/2) \leq g(r) \leq g(r_*)$ for any $r \in [\lambda r_*/2, r_*]$. Therefore, adjusting constants, we obtain (15.4) for $r \in (0, r_*]$. Thus the condition (c) has been verified.

(15.5) follows from its counterpart (15.2).

The rest is off-diagonal estimates. Note that both g and Φ are doubling and decay uniformly. Then by Lemma19.1 below, we may replace g and Φ by h and Ψ which are continuous and strictly increasing. For the upper off-diagonal estimate, since $d \underset{\mathrm{QS}}{\sim} R$, Theorem 15.6 implies (17.1) with $r_* = \mathrm{diam}(X,d)/2$. Then by Lemmas 17.2 and 17.4, we obtain (17.5) for any $t > 0$ and any $r \in (0, \mathrm{diam}(X,d))$. Applying Theorem 17.5, replacing h and Ψ by g and Φ, and using the doubling properties, we obtain (15.6) for any $x, y \in X$ and any $t > 0$. Finally, since we have (15.5), Theorem 17.8 shows an off-diagonal lower estimate, which easily implies (15.7) by similar argument as in the case of upper off-diagonal estimate. □

Lemma 19.1. *Suppose that $g : (0,+\infty) \to (0,+\infty)$ is a monotone function with full range and doubling. Then there exists $h : (0,+\infty) \to (0,+\infty)$ such that h is continuous and strictly monotonically increasing on $(0,+\infty)$ and $g(r) \asymp h(r)$ for any $r \in (0,+\infty)$. Moreover, if g decays uniformly, then $g^{-1}(t) \asymp h^{-1}(t)$ for any $t \in (0,+\infty)$.*

Proof. Assume that $g(2r) \leq cg(r)$ for any r. Set $\theta(r) = 1 + (1 + e^{-r})^{-1}$. Note that θ is strictly monotonically increasing and $1 < \theta(r) < 2$ for any r. Let $G(r) = \theta(r)g(r)$. Then H is strictly monotonically increasing. There exists a continuous function $F : (0,+\infty) \to (0,+\infty)$ such that $F(G(r)) = r$ for any $r > 0$. Define $f(x) = \theta(x)F(x)$. Then f is strictly monotonically increasing and continuous and so is the inverse of f, which is denoted by h. Since $f(G(r)) = \theta(G(r))F(G(r)) = \theta(G(r))r$, we have $\theta(r)g(r) = h(\theta(G(r))r)$. This implies $h(r)/2 \leq g(r) \leq ch(r)$.

Now assume that g decays uniformly. Then so does h. By Lemma 15.8, h^{-1} is doubling. Since $h(r)/2 \leq g(r) \leq ch(r)$, we have $h^{-1}(t/c) \leq g^{-1}(t) \leq h^{-1}(2t)$ for any $t \in (0,+\infty)$. Hence the doubling property of h^{-1} shows $h^{-1}(t) \asymp g^{-1}(t)$ for any $t \in (0,+\infty)$. □

Proof of Theorem 15.11. By Theorem 15.6, (C1) implies (C5). Note that $R \underset{\mathrm{QS}}{\sim} R$. Since (X,R) is uniformly perfect, it follows that $\overline{R}_R(x,r) \asymp r$. Hence (C5) implies (C2). Obviously (C2) implies (C6). By Proposition 15.5, (C6) implies (C1).

(C1) ⇒ (C3) and (C4): Assume (C1). Then by Theorem 14.1, there exists a metric d which satisfies the condition (c) of Theorem 15.10 with some $\beta > 1$. Therefore, we have (C3) by Theorem 15.10. Also, (C4) follows by Theorem 15.6.

(C3) ⇒ (C1): Assume (C3). Then, $(\mathrm{DHK})_{g,d}$ and (KD) imply the volume doubling property of μ with respect to d. Since $d \underset{\mathrm{QS}}{\sim} R$, μ has the volume doubling property with respect to R as well. Hence we have (C1).

(C4) ⇒ (C1): Assume (C4). Then $(2r)^\beta \asymp \overline{R}_d(x,2r)V_d(x,2r)$. By Proposition 18.2-(2), $\overline{R}_d(x,2r) \asymp \overline{R}_d(x,r)$. Hence μ is $(\mathrm{VD})_d$. Since $d \underset{\mathrm{QS}}{\sim} R$, μ is $(\mathrm{VD})_R$. Also we have $(\mathrm{EIN})_d$. Hence Proposition 15.5 shows (ACC). Thus (C1) is verified.

Finally, (15.9) follows by the process of construction of d in Section 14, in particular, by (14.1). $\square$

Proof of Theorem 15.13. Let $g(r) = r^{\beta}$ and let $h(r) = r^{\beta-\gamma}$. By Theorem 15.10, $(\mathrm{HK})_{g,d}$ shows $(\mathrm{DM2})_{g,d}$ and $(\mathrm{DM1})_{g,d}$. Using (15.10), we obtain $(\mathrm{DM2})_{h,d|_Y}$ and $(\mathrm{DM1})_{h,d|_Y}$, where we replace μ by ν. Since we have (ACC) for $(\mathcal{E}|_Y, \mathcal{F}|_Y)$, Theorem 15.10 implies the counterpart of $(\mathrm{HK})_{h,d|_Y}$. Thus we have (15.11). $\square$

Part 4

Random Sierpinski gaskets

The main purpose of this part is to apply theorems in the last part to resistance forms on random Sierpinski gaskets. The notion of random (recursive) self-similar set has introduced in [**44**], where basic properties like the Hausdorff dimension have been studied. Analysis on random Sierpinski gaskets has been developed in a series of papers by Hambly[**27, 28, 29**]. He has defined "Brownian motion" on a random Sierpinski gasket associated with a natural resistance form and studied asymptotic behaviors of associated heat kernel and eigenvalue counting function. He has found possible fluctuations in those assymptotics, which have later confirmed in [**30**].

In this part, we will first establish a sufficient and necessary condition for a measure to be volume doubling with respect to the resistance metric associated with the natural resistance form in Theorem 23.2. This result is a generalization of the counterpart in [**39**] on self-similar sets. Using this result, we show that a certain class of random self-similar measure always has the volume doubling property with respect to the resistance metric, so that we may apply theorems on heat kernel estimates in the last part. Note that Hambly has used the Hausdorff measure associated with the resistance metric, which is not a random self-similar measure in general. In fact, in Section 25, we show that the Hausdorff measure is not volume doubling with respect to the resistance metric almost surely. On the contrary, in the homogeneous case, the Hausdorff measure is a random self-similar measure and is shown to satisfy the volume doubling condition in Section 24. Applying Theorem 15.10, we will recover the two-sided off-diagonal heat kernel estimate in [**7**]. See Theorem 24.7.

Throughout this part, we fix $p_1 = \sqrt{-1}, p_2 = -\sqrt{3}/2 - \sqrt{-1}/2$ and $p_3 = \sqrt{3}/2 - \sqrt{-1}/2$ and set $V_0 = \{p_1, p_2, p_3\}$. Note that $p_1 + p_2 + p_3 = 0$ and that V_0 is the set of vertices of a regular triangle. Let T be the convex hull of V_0. We will always identify $\mathbb{R}^2$ with $\mathbb{C}$ if no confusion may occur.

Define $f_i : \mathbb{C} \to \mathbb{C}$ by $f_i(z) = (z-p_i)/2+p_i$ for $i = 1, 2, 3$. The self-similar set K associated with $\{f_1, f_2, f_3\}$ is the Sierpinski gasket, i.e. K is the unique nonempty compact set which satisfies $K = f_1(K) \cup f_2(K) \cup f_3(K)$. Recall the definition of self-similar sets in Example 7.2. To distinguish K from other generalized Sierpinski gasket, we call K the original Sierpinski gasket.

20. Generalized Sierpinski gasket

In this section, as a basic component of random Sierpinski gasket, we define a family of self-similar sets in $\mathbb{R}^2$ which can be considered as a modification of the original Sierpinski gasket. Then making use of the theory in [**36**], we briefly review the construction of resistance forms on those sets. Also, in Example 20.11, we apply Theorem 15.13 to subsets of the original Sierpinski gasket and obtain heat kernel estimates for the traces onto those subsets.

The following is a standard set of definitions and notations regarding self-similar sets.

Definition 20.1. Let S be a finite set.
(1) We define $W_m(S) = S^m = \{w_1w_2\cdots w_m | w_j \in S \text{ for } j = 1, \ldots, m\}$ for $m \geq 1$ and $W_0(S) = \{\emptyset\}$. Also $W_*(S) = \cup_{m\geq 0} W_m(S)$. For any $w \in W_*(S)$, the length of w, $|w|$, is defined to be m where $w \in W_m(S)$. For any $w = w_1w_2\cdots w_m \in W_m(S)$,

define $[w]_0 = \emptyset$ and

$$[w]_n = \begin{cases} w_1w_2\cdots w_n & \text{if } 1 \le n < m, \\ w & \text{if } n \ge m. \end{cases}$$

(2) $\Sigma(S)$ is defined by $\Sigma(S) = S^{\mathbb{N}} = \{\omega_1\omega_2\ldots|\omega_j \in S \text{ for any } j \in \mathbb{N}\}$. For any $\omega = \omega_1\omega_2\ldots \in \Sigma(S)$, define $[\omega]_0 = \emptyset$ and $[\omega]_n = \omega_1\omega_2\cdots\omega_n$ for any $n \ge 1$. For $w \in W_*(S)$, define $\Sigma_w(S) = \{\omega|\omega \in \Sigma(S), [\omega]_{|w|} = w\}$ and define $\sigma_w : \Sigma(S) \to \Sigma(S)$ by $\sigma_w(\omega) = w\omega$.

A generalized Sierpinski gasket is defined as a self-similar set which preserves some of the good properties possessed by the original Sierpinski gasket.

Definition 20.2. Let K be a non-empty compact subset of $\mathbb{R}^2$ and let $S = \{1,\ldots,N\}$ for some integer $N \ge 3$. Also let $F_i(x) = \alpha_iA_ix + q_i$ for any $i \in S$, where $\alpha_i \in (0,1)$, $A_i \in O(2)$, where $O(2)$ is the 2-dimensional orthogonal matrices, and $q_i \in \mathbb{R}^2$. Recall the definitions of p_1, p_2, p_3, V_0 and T at the beginning of this part. $(K, S, \{F_i\}_{i\in S})$ is called a generalized Sierpinski gasket, GSG for short, if and only if the following four conditions are satisfied:
(GSG1) $K = \cup_{i\in S}F_i(K)$,
(GSG2) $F_i(p_i) = p_i$ for $i = 1,2,3$,
(GSG3) $F_i(T) \subseteq T$ for any $i \in S$ and $F_i(T) \cap F_j(T) \subseteq F_i(V_0) \cap F_j(V_0)$ for any $i, j \in S$ with $i \ne j$,
(GSG4) For any $i, j \in \{1,2,3\}$, there exist $i_1,\ldots,i_m$ such that $i_1 = i$, $i_m = j$ and $F_{i_k}(V_0) \cap F_{i_{k+1}}(V_0) \ne \emptyset$ for all $k = 1,\ldots,m-1$.
Write $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$. For any $w = w_1w_2\cdots w_m \in W_*(S)\backslash W_0(S)$, we define $F_w = F_{w_1} \circ \ldots \circ F_{w_m}$ and $K_w = F_w(K)$. Also $V_m(\mathcal{L}) = \cup_{w\in W_m(S)}F_w(V_0)$.

By (GSG1), (GSG2) and (GSG3), the results in [**36**, Sections 1.2 and 1.3] show that a GSG $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is a post critically finite self-similar structure whose post critical set is V_0. Also by (GSG4), K is connected.

Using the results in [**36**, Section 1.4], we have the following proposition.

Proposition 20.3. *Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a GSG and let $(\mu_s)_{s\in S} \in (0,1)^S$ satisfy $\sum_{s\in S}\mu_s = 1$. Then there exists a Borel regular measure μ on K such that $\mu(K_w) = \mu_{w_1}\mu_{w_2}\cdots\mu_{w_m}$ for any $w = w_1w_2\cdots w_m \in W_*(S)$.*

Definition 20.4. The Borel regular measure μ in Proposition 20.3 is called the self-similar measure on K with weight $(\mu_s)_{s\in S}$.

Next we give a brief survey on the construction of a (self-similar) resistance form on a GSG. See [**36**] for details.

Proposition 20.5. *Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a GSG and let $(D, \mathbf{r}) \in \mathcal{LA}(V) \times (0,+\infty)^S$. Then there exists a Laplacian $L_m \in \mathcal{LA}(V_m(\mathcal{L}))$ such that, for any $u, v \in \ell(V_m(\mathcal{L}))$,*

$$\mathcal{E}_{L_m}(u,v) = \sum_{w\in W_m(S)} \frac{1}{r_w}\mathcal{E}_D(u\circ F_w, v\circ F_w),$$

where $\mathbf{r} = (r_i)_{i\in S}$ and $r_w = r_{w_1}\cdots r_{w_m}$ for $w = w_1w_2\cdots w_m \in W_m(S)$.

Definition 20.6. Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a GSG and let $(D, \mathbf{r}) \in \mathcal{LA}(V) \times (0,+\infty)^S$. Define $L_m(D,\mathbf{r}) \in \mathcal{LA}(V_m(\mathcal{L}))$ as the Laplacian L_m obtained in Proposition 20.5. $(D,\mathbf{r})$ is called a regular harmonic structure on $\mathcal{L}$ if and only if $\mathbf{r} \in (0,1)^S$ and $\{(V_m(\mathcal{L}), L_m(D,\mathbf{r}))\}_{m\ge 0}$ is a compatible sequence.

By [**36**, Proposition 3.1.3], $\{(V_m(\mathcal{L}), L_m(D, \mathbf{r})\}_{m\geq 0}$ is a compatible sequence if and only if

$$\mathcal{E}_D(u,u) = \inf\{\mathcal{E}_{L_1}(v,v)|v \in \ell(V_1(\mathcal{L})), v|_{V_0} = u\}$$

for any $u \in \ell(V_0)$. Hence we only have to verify finite number of equations to show that $(D, \mathbf{r})$ is a regular harmonic structure.

Now let $(D, \mathbf{r})$ be a regular harmonic structure. By Theorem 3.13, the compatible sequence $\mathcal{S} = \{(V_m(\mathcal{L}), L_m(D, \mathbf{r})\}_{m\geq 0}$ produces a resistance form $(\mathcal{E}, \mathcal{F})$ on X, where X is the completion of $(V_*, R_{\mathcal{S}})$. By the results in [**36**, Sections 3.2 and 3.3], we may identify $X = K$ and obtain the following proposition.

Proposition 20.7. *Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a GSG and let $(D, \mathbf{r})$ be a regular harmonic structure on $\mathcal{L}$. There exists a resistance form $(\mathcal{E}, \mathcal{F})$ on K which satisfies the following properties (a), (b) and (c):*
(a) The associated resistance metric R gives the same topology as the restriction of the Euclidean metric does,
(b) $L_{(\mathcal{E},\mathcal{F}),V_m} = L_m(D, \mathbf{r})$ for any $m \geq 0$,
(c) For any $u \in \mathcal{F}$, $u \circ F_s \in \mathcal{F}$ for any $s \in S$ and

$$\mathcal{E}(u,u) = \sum_{s\in S} \frac{1}{r_s} \mathcal{E}(u\circ F_s, u\circ F_s). \tag{20.1}$$

In particular, (K, R) is compact and $(\mathcal{E}, \mathcal{F})$ is a regular resistance form.

The equation (20.1) is called the self-similarity of the resistance form $(\mathcal{E}, \mathcal{F})$.

Recall that the chain condition of a distance is required to get a (lower) off-diagonal estimate of a heat kernel. In [**35**], we have obtained a condition for the existence of a shortest path metric which possesses the chain condition. The next definitions and the following theorems are essentially included in [**35**].

Definition 20.8. Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a GSG.
(1) For $p, q \in V_n(\mathcal{L})$, $(x_1, \ldots, x_m)$ is called an n-path between p and q if $x_1 = p, x_m = q$ and for any $i = 1, \ldots, m-1$, there exists $w \in W_n(S)$ such that $x_i, x_{i+1} \in F_w(V_0)$.
(2) $\mathcal{L}$ is said to admit symmetric self-similar geodesics if and only if there exists $\gamma \in (0, 1)$ such that

$$\gamma^{-1} = \min\{m-1|(x_1, \ldots, x_m) \text{ is a 1-path between } p \text{ and } q\}$$

for any $p, q \in V_0$ with $p \neq q$. γ is called the symmetric geodesic ratio of $\mathcal{L}$.

Definition 20.9. Let (X, d) be a metric space. For $x, y \in X$, a continuous curve $g : [0, d(x, y)] \to X$ is called a geodesic between x and y if and only if $d(g(s), g(t)) = |s - t|$ for any $s, t \in [0, 1]$. If there exists a geodesic between x and y for any $x, y \in X$, then d is called a geodesic metric on X.

Obviously, a geodesic metric satisfies the chain condition. The following theorem shows the existence of geodesic metric.

Theorem 20.10. *Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be a GSG. Assume that $\mathcal{L}$ admits symmetric self-similar geodesics. Then there exists a geodesic metric d on K which gives the same topology as the Euclidean metric does on K and*

$$d(F_i(x), F_i(y)) = \gamma d(x, y)$$

for any $x, y \in K$ and any $i \in S$, where γ is the symmetric geodesic ratio of $\mathcal{L}$. Moreover, for any $p, q \in V_n(S)$,

$$d(p,q) = \gamma^n \min\{m-1|(x_1,\ldots,x_m) \text{ is an } n\text{-path between } p \text{ and } q\}.$$

PROOF. We can verify all the conditions in [**35**, Theorem 4.3] and obtain this theorem. □

We present two examples which will used as a typical component of random Sierpinski gaskets in the following sections.

EXAMPLE 20.11 (the original Sierpinski gasket). Let $f_i(z) = (z-p_i)/2 + p_i$ and let K be the original Sierpinski gasket. Set $S = \{1,2,3\}$. Then $(K, S, \{f_i\}_{i\in S})$ is a generalized Sierpinski gasket. We write $\mathcal{L}_{SG} = (K, S, \{f_i\}_{i\in S})$. Define

$$D_h = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -(1+h) & h \\ 1 & h & -(1+h) \end{pmatrix}$$

for $h > 0$. By [**36**, Exercise 3.1], there exists a unique (r_1, r_2, r_3) such that $(D_h, (r_1, r_2, r_3))$ is a harmonic structure for each $h > 0$. Also the unique (r_1, r_2, r_3) satisfies $r_2 = r_3$ and $(D_h, r_1, r_2, r_3))$ is regular. We write $r_i = r_i^{SG}$ for $i = 1,2,3$. Note that r_i^{SG} depends on h in fact.

Hereafter in this example, we set $h = 1$. Then $r_1^{SG} = r_2^{SG} = r_3^{SG} = 3/5$. Set $\mathbf{r} = (3/5, 3/5, 3/5)$. Let μ be the self-similar measure on K with weight $(1/3, 1/3, 1/3)$. Let $(\mathcal{E}, \mathcal{F})$ be the regular resistance form on K associated with $(D_1, \mathbf{r})$ and let R be the associated resistance metric on K. Then by Barlow-Perkins [**9**], it has been known that the heat kernel $p(t,x,y)$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K,\mu)$ satisfies

$$c_1 t^{-\frac{d_S}{2}} \exp\left(-c_2\left(\frac{|x-y|^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right) \le p(t,x,y) \le c_3 t^{-\frac{d_S}{2}} \exp\left(-c_4\left(\frac{|x-y|^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right) \tag{20.2}$$

for any $t \in (0,1]$ and any $x, y \in K$, where $d_s = \log 9/\log 5$ and $d_w = \log 5/\log 2$. The exponents d_s and d_w are called the spectral dimension and the walk dimension of the Sierpinski gasket respectively. In this case, $\mathcal{L}_{SG}$ admits symmetric self-similar geodesics with the geodesic ratio $1/2$. The resulting geodesic metric on K is equivalent to the Euclidean metric.

Next we consider the traces of $(\mathcal{E}, \mathcal{F})$ on Ahlfors regular subsets of K. It is known that

$$R(x,y) \asymp d_E(x,y)^{(\log 5 - \log 3)/\log 2}$$

for any $x, y \in K$, where $d_E(x,y) = |x-y|$. Hence $d_E \underset{\mathrm{QS}}{\sim} R$. Also,

$$\mu(B_{d_E}(x,r)) \asymp r^{d_H} \tag{20.3}$$

for any $x \in K$ and $r \in (0,1]$, where $d_H = \log 3/\log 2$ is the Hausdorff dimension of (K, d_E). Let Y be a closed Ahlfors δ-regular subset of K, i.e. there exists a Borel regular measure ν on Y such that $\nu(B_d(x,r) \cap Y) \asymp r^\delta$ for any $x \in Y$ and any $r \in (0,1]$. Then by (20.2) and (20.3), we may verify all the assumptions

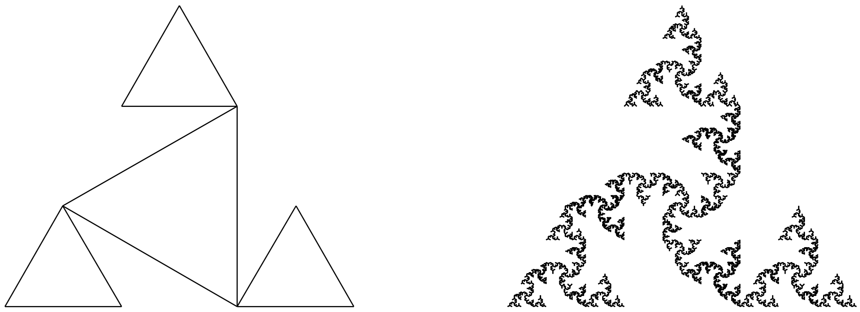


FIGURE 2. the Sierpinski spiral

of Theorem 15.13. Thus there exists a jointly continuous heat kernel $p_\nu^Y(t,x,y)$ associated with the regular Dirichlet form $(\mathcal{E}|_Y, \mathcal{F}|_Y)$ on $L^2(Y,\nu)$ and

$$p_\nu^Y(t,x,x) \asymp t^{-\eta}$$

for any $x \in K$ and any $t \in (0,1]$, where $\eta = \frac{\delta \log 2}{\log 5 - \log 3 + \delta \log 2}$. In particular, if Y is equal to the line segment $\overline{p_2 p_3}$, then $\delta = 1$ and $\eta = \log 2 / \log{(10/3)}$.

EXAMPLE 20.12 (the Sierpinski spiral). For $i = 1,2,3$, define $h_i(z) = (z - p_i)/3 + p_i$ for any $z \in \mathbb{C}$. Also define $h_4(z) = -z/\sqrt{-3}$. The unique non-empty compact subset K of $\mathbb{C}$ satisfying $K = \cup_{i=1,2,3,4} h_i(K)$ is called the Sierpinski spiral, the S-spiral for short. See Figure 2. Let $S = \{1,2,3,4\}$. Then $(K, S, \{h_i\}_{i \in S})$ is a generalized Sierpinski gasket. We use $\mathcal{L}_{SP}$ to denote this generalize Sierpinski gasket associated with the S-spiral. Let D_h be the same as in Example 20.11 for $h > 0$. Define $r_1^{SP} = (h-\gamma)/(h+1)$, $r_2^{SP} = (1-\gamma h)/(h+1)$, $r_3^{SP} = (1-\gamma)/2$ and $r_4^{SP} = \gamma$. Then $(D_h, (r_i^{SP})_{i \in S})$ is a regular harmonic structure for $\gamma \in (0, \min\{h, 1/h\})$. Let $(\mathcal{E}, \mathcal{F})$ be the regular resistance form on K associated with $(D_h, (r_i^{SP})_{i \in S})$ and let R be the resistance distance induced by $(\mathcal{E}, \mathcal{F})$. Note that K is a dendrite, i.e. for any two points $x, y \in K$, there is a unique simple path between x and y. It follows that R is a geodesic metric. The Hausdorff dimension d_H of (K, R) is given by the unique d which satisfies

$$\sum_{i=1}^{4} (r_i^{SP})^d = 1.$$

By Theorem 23.8, any self-similar measure on K has the volume doubling property with respect to R. (Note that a generalize Sierpinski gasket itself is a special random Sierpinski gasket. Also for the S-spiral, all the adjoining pair are trivial, i,e. $((j_1, i_1), (j_2, i_2))$ is an adjoining pair if and only if $j_1 = j_2$ and $i_1 = i_2$. See Definition 23.7 for the definition of adjoining pair.) In particular, letting ν be the self-similar measure with weight $((r_i^{SP})^{d_H})_{i \in S}$, we have

$$R(x,y) V_R(x, R(x,y)) \asymp R(x,y)^{d_H + 1}.$$

By Theorem 15.10, the heat kernel $p(t,x,y)$ associated with the Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\nu)$ satisfies

$$c_1 t^{-\frac{d_H}{d_H+1}} \exp\left(-c_2\left(\frac{R(x,y)^{d_H+1}}{t}\right)^{\frac{1}{d_H}}\right) \le p(t,x,y) \le c_3 t^{-\frac{d_H}{d_H+1}} \exp\left(-c_4\left(\frac{R(x,y)^{d_H+1}}{t}\right)^{\frac{1}{d_H}}\right)$$

for any $t \in (0,1]$ and any $x,y \in K$. Note that the S-spiral admits symmetric self-similar geodesics with the ratio $1/3$ and this geodesic metric coincides with the resistance metric R when $h=1$ and $\gamma = 1/3$.

21. Random Sierpinski gasket

In this section, we will give basic definitions and notations for random (recursive) Sierpinski gaskets. Essentially the definition is the same as in [**44, 27, 28, 29**]. However, we will not introduce the randomness until Section 25.

For $j = 1,\dots,M$, let $\mathcal{L}_j = (K(j), S_j, \{F_i^j\}_{i\in S_j})$ be a generalized Sierpinski gasket, where $S_j = \{1,\dots,N_j\}$. Set $N = \max_{j=1,\dots,M} N_j$ and define $S = \{1,\dots,N\}$. Those generalized Sierpinski gaskets $\mathcal{L}_1,\dots,\mathcal{L}_M$ are the basic components of our random Sierpinski gasket.

Definition 21.1. Let $W_* \subseteq W_*(S)$ and let $\Gamma : W_* \to \{1,\dots,M\}$. (W_*,Γ) is called a random Sierpinski gasket generated by $\{L_1,\dots,\mathcal{L}_M\}$ if and only if the following properties are satisfied:
(RSG) $\emptyset \in W_*$ and, for $m \ge 1$, $w = w_1w_2\cdots w_m \in W_m(S)$ belongs to W_* if and only if $[w]_{m-1} \in W_*$ and $w_m \in S_{\Gamma([w]_{m-1})}$.

Strictly speaking, to call (W_*,Γ) a "random" Sierpinski gasket, one need to introduce a randomness in the choice of $\Gamma(w)$ for every w, i.e. a probability measure on the collections of (W_*,Γ). We will do so in the final section, Section 25. Until then, we study each (W_*,Γ) respectively without randomness.

Note that (W_*,Γ) is not a geometrical object. The set $K(W_*,\Gamma) \subseteq \mathbb{R}^2$ defined in Proposition 21.3-(2) is the real geometrical object considered as the random self-similar "set" generated by (W_*,Γ).

Definition 21.2. Let (W_*,Γ) be a random Sierpinski gasket generated by $\{L_1,\dots,\mathcal{L}_M\}$. Define $W_m = W_* \cap W_m(S)$.
(1) Define $F_\emptyset = I$, where I is the identity map from $\mathbb{R}^2$ to itself. For any $m \ge 1$ and $w = w_1w_2\cdots w_m \in W_m$, define $F_w : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F_w = F_{w_1}^{\Gamma([w]_0)} \circ F_{w_2}^{\Gamma([w]_1)} \circ \dots \circ F_{w_m}^{\Gamma([w]_{m-1})}.$$

(2) $\Sigma(W_*,\Gamma) = \{w_1w_2\dots | w_1w_2\dots \in \Sigma(S), w_1\dots w_m \in W_m \text{ for any } m \ge 1\}$.
(3) Define $T_m(W_*,\Gamma) = \cup_{w\in W_m} F_w(T)$ and $V_m(W_*,\Gamma) = \cup_{w\in W_m} F_w(V_0)$ for any $m \ge 0$.

The following results are basic properties of random Sierpinski gaskets which are analogous to self-similar sets.

Proposition 21.3. *Let (W_*,Γ) be a random Sierpinski gasket generated by $\{L_1,\dots,\mathcal{L}_M\}$.*
(1) $\cap_{m\ge0} T_m(W_*,\Gamma)$ *equals to the closure of* $\cup_{m\ge0} V_m(W_*,\Gamma)$ *with respect to the*

Euclidean metric.
(2) *Define* $K(W_*, \Gamma) = \cap_{m \geq 0} T_m$ *and* $K_w(W_*, \Gamma) = K(W_*, \Gamma) \cap F_w(T)$ *for any* $w \in W_*$. *Then,* $K_w(W_*, \Gamma) \cap K_v(W_*, \Gamma) = F_w(V_0) \cap F_v(V_0)$ *for any* $w, v \in W_*$ *with* $\Sigma_w(S) \cap \Sigma_v(S) = \emptyset$.
(3) *Let* $\omega = \omega_1\omega_2 \ldots \in \Sigma(W_*, \Gamma)$. *Then*

$$K_{[\omega]_m}(W_*, \Gamma) \supseteq K_{[\omega]_{m+1}}(W_*, \Gamma)$$

for any $m \geq 0$ *and* $\cap_{m \geq 1} K_{[\omega]_m}(W_*, \Gamma)$ *is a single point. If we denote this single point by* $\pi_{W_*,\Gamma}(\omega)$, *then the map* $\pi_{W_*,\Gamma} : \Sigma(W_*, \Gamma) \to K(W_*, \Gamma)$ *is continuous and onto. For any* $k = 1, 2, 3$, $(\pi_{W_*,\Gamma})^{-1}(p_k) = \{(k)^\infty\}$, *where* $(k)^\infty = kkk \ldots \in \Sigma(S)$.
(4) *For any* $x \in K(W_*, \Gamma)$, *set* $n(x) = \#(\pi^{-1}_{W_*,\Gamma}(x))$. *Then* $1 \leq n(x) \leq 5$. *Moreover* $n(x) \geq 2$ *if and only if there exist* $w \in W_*$, $i_1, \ldots, i_{n(x)} \in S_{\Gamma(w)}$ *with* $i_m \neq i_n$ *for any* $m \neq n$ *and* $k_1, \ldots, k_{n(x)} \in \{1, 2, 3\}$ *such that*

$$\pi^{-1}_{W_*,\Gamma}(x) = \{wi_m(k_m)^\infty | m = 1, \ldots, n(x)\}.$$

Next we try to describe the self-similarity of random Sierpinski gasket.

Definition 21.4. Let (W_*, Γ) be a random Sierpinski gasket generated by $\{\mathcal{L}_1, \ldots, \mathcal{L}_M\}$.
(1) For any $w \in W_*$, define $W^w_* = \{v | wv \in W_*\}$ and $\Gamma^w : W^w_* \to \{1, \ldots, M\}$ by $\Gamma^w(v) = \Gamma(wv)$ for any $v \in W^w_*$.
(2) A subset $\Lambda \subseteq W_*$ is called a partition of W_* if and only if $\Sigma(W_*, \Gamma) \subseteq \cup_{w \in \Lambda} \Sigma_w(S)$ and $\Sigma_{w(1)}(S) \cap \Sigma_{w(2)}(S) = \emptyset$ for any $w(1), w(2) \in \Lambda$ with $w(1) \neq w(2)$.

The following theorem gives the self-similarity of random Sierpinski gasket. (21.1) is the counterpart of the ordinary self-similarity $K = \cup_{i=1}^N F_i(K)$.

Proposition 21.5. *Let* (W_*, Γ) *be a random Sierpinski gasket generated by* $\{\mathcal{L}_1, \ldots, \mathcal{L}_M\}$. *For any* $w \in W_*$, (W^w, Γ^w) *is a random Sierpinski gasket generated by* $\{\mathcal{L}_1, \ldots, \mathcal{L}_M\}$, $K_w(W_*, \Gamma) = F_w(K(W^w_*, \Gamma^w))$ *and* $F_w \circ \pi_{W^w_*,\Gamma^w} = \pi_{W_*,\Gamma} \circ \sigma_w$. *Moreover, if* Λ *is a partition of* W_*, *then*

$$K(W_*, \Gamma) = \bigcup_{w \in \Lambda} K_w(W_*, \Gamma^w) = \bigcup_{w \in \Lambda} F_w(K(W^w_*, \Gamma^w)). \tag{21.1}$$

The following proposition describes the topological structure of a random Sierpinski gasket.

Proposition 21.6. *Let* $K_{m,x}(W_*, \Gamma) = \cup_{w \in W_m, x \in K_w(W_*,\Gamma)} K_w(W_*, \Gamma)$. *Then* $K_{m,x}(W_*, \Gamma)$ *is a neighborhood of* x *and* $\sup_{x \in K(W_*,\Gamma)} \mathrm{diam}(K_{m,x}, d_E) \to 0$ *as* $m \to +\infty$, *where* d_E *is the Euclidean distance.*

Proof. Set $A_{m,x} = \cup_{w \in W_m, x \notin K_w(W_*,\Gamma)} K_w(W_*, \Gamma)$. Then $A_{m,x}$ is compact and $x \notin A_{m,x}$. Hence $\alpha = \min_{y \in A_{m,x}} |x - y| > 0$. Write $K_{m,x} = K_{m,x}(W_*, \Gamma)$. For any $s \in (0, \alpha)$, $B_{d_E}(x, s) \cap K(W_*, \Gamma) \subseteq K_{m,x}$. Hence $K_{m,x}$ is a neighborhood of x. Let $\overline{L}$ be the maximum of the Lipschitz constants of F^j_i for $j \in \{1, \ldots, M\}$ and $i \in S_j$. Then $\mathrm{diam}(K_w(W_*, \Gamma), d_E) \leq \overline{L}^m \mathrm{diam}(T, d_E)$. Thus $\sup_{x \in K(W_*,\Gamma)} \mathrm{diam}(K_w(W_*, \Gamma)) \leq \overline{L}^m \mathrm{diam}(T, d_E) \to 0$ as $m \to +\infty$. □

Figure 3 shows two random Sierpinski gaskets generated by $\{\mathcal{L}_{SG}, \mathcal{L}_{SP}\}$.

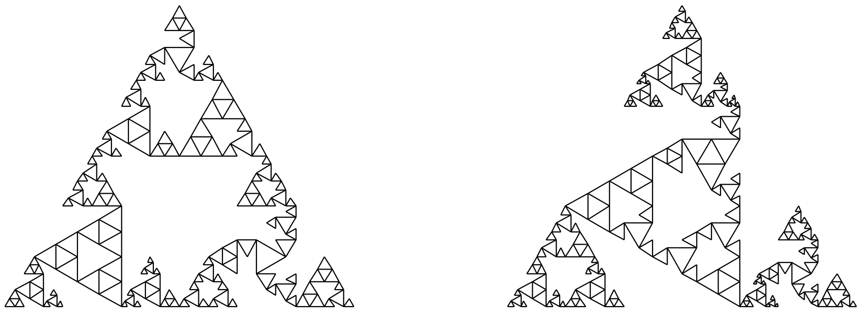


FIGURE 3. Random Sierpinski gaskets

22. Resistance forms on Random Sierpinski gaskets

The main purpose of this section is to introduce the construction of a (random self-similar) resistance form on a random Sierpinski gasket. We follow the method of construction given in [**28**]. Furthermore, we are going to study the resistance metric associated with the constructed resistance form.

In this section, we fix a random Sierpinski gasket (W_*, Γ) generated by $\{\mathcal{L}_j\}_{j=1}^M$, where $\mathcal{L}_j = (K(j), S_j, \{F_i^j\}_{i \in S_j})$ and $S_j = \{1, \ldots, N_j\}$. We write T_m, V_m, K, K_w and π in place of $T_m(W_*, \Gamma), V_m(W_*, \Gamma)$ and so on.

Let $(D, \mathbf{r}^{(j)})$ be a regular harmonic structure for each $j \in \{1, \ldots, M\}$. Set $\mathbf{r}^{(j)} = (r_i^{(j)})_{i \in S_j}$. (Note that D is independent of j.) Define $\overline{r} = \max\{r_i^{(j)} | j \in \{1, \ldots, M\}, i \in S_j\}$ and $\underline{r} = \min\{r_i^{(j)} | j \in \{1, \ldots, M\}, i \in S_j\}$.

We first construct a series of resistance forms and/or Laplacians on $\{V_m\}_{m \geq 0}$ as in the case of a generalized Sierpinski gasket.

DEFINITION 22.1. Let (W_*, Γ) be a random Sierpinski gasket generated by $\{\mathcal{L}_1, \ldots, \mathcal{L}_M\}$. For any $w \in W_m$, define $r_w = r_{w_1}^{\Gamma([w]_0)} r_{w_2}^{\Gamma([w]_1)} \cdots r_{w_m}^{\Gamma([w]_{m-1})}$. (We set $r_\emptyset = 1$.) Define a symmetric bilinear form $\mathcal{E}_m$ on $\ell(V_m)$ by

$$\mathcal{E}_m(u, v) = \sum_{w \in W_m} \frac{1}{r_w} \mathcal{E}_D(u \circ F_w, v \circ F_w)$$

for any $u, v \in \ell(V_m)$. We use L_m to denote the symmetric linear operator from $\ell(V_m)$ to itself satisfying $\mathcal{E}_m(u, v) = -(u, L_m v)_{V_m}$ for any $u, v \in \ell(V_m)$.

Since each $(D, \mathbf{r}^{(j)})$ is a harmonic structure, we have the following fact immediately.

PROPOSITION 22.2. *$\mathcal{E}_m$ is a resistance form on V_m for any $m \geq 1$ and L_m is a Laplacian on V_m. Moreover, $\{(V_m, L_m)\}_{m \geq 0}$ is a compatible sequence.*

Let $\mathcal{S} = \{(V_m, L_m)\}_{m \geq 0}$ be the compatible sequence obtained in Proposition 22.2. Then we have a resistance form $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}_{\mathcal{S}})$ on V_* by Theorem 3.13. Let $R_{\mathcal{S}}$ be the associated resistance metric on V_*. Note that if $x, y \in V_m$, then $R_{\mathcal{S}}(x, y)$

is equal to the resistance metric with respect to the resistance form $(\mathcal{E}_m, \ell(V_m))$ on V_m. We use this fact in the followings. Let (X, R) be the completion of $(V_*, R_{\mathbb{S}})$ as in Theorem 3.13. We are going to identify X with K and show that R gives the same topology as the restriction of the Euclidean metric on K does. Hereafter we use $\mathcal{E}, \mathcal{F}$ and $\mathcal{R}$ to denote $\mathcal{E}_{\mathbb{S}}, \mathcal{F}_{\mathbb{S}}$ and $R_{\mathbb{S}}$ if no confusion may occur.

The following definition is an analogue of the notion of scales in [**39**].

Definition 22.3. For $s \in (0, 1)$ define

$$\Lambda_s = \{w | w \in W_* \backslash W_0, r_{[w]_{|w|-1}} > s \geq r_w\}$$

and $\Lambda_1 = \{\emptyset\}$. For any $x \in X$ and any $s \in (0, 1]$,

$$\Lambda_{s,x} = \{w | w \in \Lambda_s, x \in K_w\}, \qquad K_s(x) = \cup_{w \in \Lambda_{s,x}} K_w,$$

$$\Lambda^1_{s,x} = \{w | w \in \Lambda_s, K_w \cap K_s(x) \neq \emptyset\} \text{ and } \qquad U_s(x) = \cup_{w \in \Lambda^1_{s,x}} K_w.$$

Also $Q_s(x) = \cup_{w \in \Lambda_s \backslash \Lambda^1_{s,x}} K_w$, $C_s(x) = U_s(x) \cap Q_s(X)$.

We think of K_w's for $w \in \Lambda_s$ a "ball" of radius s with respect to the resistance metric. Also, $U_s(x)$ is regarded as a s-neighborhood of x. Such a viewpoint will be justified in Corollary 22.8. First we show that $\{U_s(x)\}_{s>0}$ is a fundamental system of neighborhoods with respect to the Euclidean metric.

Lemma 22.4. *Let d_E be the restriction of Euclidean metric on K. In this lemma, we use the topology of K induced by d_E.*
(1) *$K_s(x)$ and $U_s(x)$ are compact.*
(2) *$U_s(x)$ is a neighborhood of x with respect to d_E. Moreover,*

$$\lim_{s \downarrow 0} \sup_{x \in K} \operatorname{diam}(U_s(x), d_E) = 0.$$

(3) *$C_s(x) \subseteq \cup_{w \in \Lambda^1_{s,x}} F_w(V_0)$ and $C_s(x)$ is the topological boundary of $U_s(x)$.*

Proof. (1) and (3) are immediate. About (2), for $w = w_1 w_2 \cdots w_m \in \Lambda_s$, since $r_{w_1 w_2 \cdots w_{m-1}} > s \geq r_w$, it follows that $\overline{r}^{m-1} \geq s \geq \underline{r}^m$. Hence

$$\frac{\log s}{\log \underline{r}} \leq m \leq \frac{\log s}{\log \overline{r}} + 1 \tag{22.1}$$

Let $\underline{m}(s)$ be the integral part of $\log s / \log \underline{r}$ and let $\overline{m}(s)$ be the integral part of $\log s / \log \overline{r} + 2$. Then (22.1) implies $U_s(x) \supseteq K_{\overline{m}(s),x}$. By Proposition 21.6, $U_s(x)$ is a neighborhood of x. Also by (22.1),

$$\operatorname{diam}(U_s(x), d_E) \leq 4 \sup_{w \in \Lambda_s} \operatorname{diam}(K_w, d_E) \leq 4 \sup_{x \in K} \operatorname{diam}(K_{\overline{m}(s),x}).$$

Now Proposition 21.6 yields the desired result. $\square$

In the next lemma, we show that the diameter of K_w for $w \in \Lambda_s$ is roughly s.

Lemma 22.5. (1) *There exists $c_0 > 0$ such that $\sup_{x,y \in K_w \cap V_*} R(x, y) \leq c_0 r_w$ for any $w \in W_*$.*
(2) *There exists $c_1 > 0$ such that $R(x, y) \leq c_1 s$ for any $s \in (0, 1]$, any $x \in V_*$ and any $y \in U_s(x) \cap V_*$.*

Proof. (1) First we enumerate two basic facts.
Fact 1: $r_w \leq (\overline{r})^{|w|}$.
Fact 2: Define $R_* = \max\{R(x, y) | x, y \in V_0\}$. Then $R(x, y) \leq r_w R_*$ for any $x, y \in F_w(V_0)$.

Assume that $x \in F_w(V_0)$ and $y \in F_w(V_1(\mathcal{L}_{\Gamma(w)}))$ for some $w \in W_*$. Note that $F_w(V_1(\mathcal{L}_{\Gamma(w)})) = \cup_{i \in S_{\Gamma(w)}} F_{wi}(V_0)$. Recall that we set $N = \max_{j=1,\dots,M} N_j$, where $N_j = \#(S_j)$. Since $\#(V_1(\mathcal{L}_{\Gamma(w)})) \leq 3N$, we may find $m \leq 3N$, $i_1, \dots, i_m \in S_{\Gamma(w)}$ and $x_0, x_1, \dots, x_m \in F_w(V_1(\mathcal{L}_{\Gamma(w)}))$ satisfying $x_0 = x$, $x_m = y$ and $x_{k-1}, x_k \in F_{wi_k}(V_0)$ for any $k = 1, \dots, m$. By the above facts,

$$R(x,y) \leq \sum_{k=1}^{m} R(x_{k-1}, x_k) \leq 3NR_*\overline{r}r_w.$$

Now, let $x \in F_w(V_0)$ and let $y \in K_w \cap V_*$. Then $y \in F_{wv}(V_0)$ for some $wv \in W_*$. Choose $y_i \in F_{[wv]_{|w|+i}}$ for $i = 1, \dots, |v|-1$. Set $y_0 = x$ and $y_{|v|} = y$. By the above argument,

$$\begin{aligned} R(x,y) \leq \sum_{i=0}^{|v|-1} R(y_i, y_{i+1}) &\leq \sum_{i=0}^{|v|-1} 3NR_*\overline{r}r_{[wv]_{|w|+i}} \\ &\leq \sum_{i=0}^{\infty} 3NR_*(\overline{r})^{i+1}r_w = \frac{3NR_*\overline{r}r_w}{1-\overline{r}}. \end{aligned}$$

This shows that $\sup_{x,y \in K_w \cap V_*} R(x,y) \leq 6NR_*\overline{r}(1-\overline{r})^{-1}r_w$.
(2) Let $y \in U_s(x) \cap V_*$. There exist $w(1), w(2) \in \Lambda^1_{s,x}$ and $z \in F_{w(1)}(V_0) \cap F_{w(2)}(V_0)$ such that $x \in K_{w(1)} \cap V_*$ and $y \in K_{w(2)} \cap V_*$. By (1), $R(x,y) \leq R(x,z) + R(z,y) \leq c_0(r_{w(1)} + r_{w(2)}) \leq 2c_0 s$. □

Next lemma is the heart of the series of discussions. It shows that $U_s(x)$ contains a resistance ball of radius cs, where c is independent of s.

LEMMA 22.6. *There exists $c_2 > 0$ such that $R(x,y) \geq c_2 s$ for any $s \in (0,1]$, any $x \in V_*$ and any $y \in Q_s(x) \cap V_*$.*

PROOF. Set $K_* = K_s(x) \cap V_*$ and $Q_* = Q_s(x) \cap V_*$.
Claim 1 Let $z \in (K_s(x) \cup Q_s(x))^c \cap V_*$. For any $a, b, c \in \mathbb{R}$, there exists $u \in \mathcal{F}$ such that $u|_{K_*} = a, u|_{Q_*} = b$ and $u(z) = c$.
Proof of Claim 1 Set $m_* = \max_{w \in \Lambda^1_{s,x}} |w|$. We may choose $m \geq m_*$ so that $z \in F_w(V_0)$, $K_w \cap Q_s(x) = \emptyset$ and $K_w \cap K_s(x) = \emptyset$ for some $w \in W_m$. Considering the resistance form $(\mathcal{E}_m, \ell(V_m))$, we find $\tilde{u} \in \ell(V_m)$ such that $\tilde{u}|_{K_s(x) \cap V_m} = a, \tilde{u}|_{Q_s(x) \cap V_m} = b$ and $\tilde{u}(z) = c$. Since $(\mathcal{E}, \mathcal{F})$ is the limit of the compatible sequence (V_m, L_m), the harmonic extension of $\tilde{u}$, $h_{V_m}(\tilde{u})$, possesses the desired properties.
Claim 2 Let $\mathcal{F}_{s,x} = \{u | u \in \mathcal{F},\ u|_{K_*}$ and $u|_{Q_*}$ are constants$\}$. Then $(\mathcal{E}, \mathcal{F}_{s,x})$ is a resistance form on $(V_* \backslash (K_s(x) \cap Q_s(X))) \cup \{K_*\} \cup \{Q_*\}$.
Proof of Claim 2 By Claim 1, we see that $(K_*)^{\mathcal{F}} = K_*$. Theorem 4.3 implies that $(\mathcal{E}, \mathcal{F}^{K_*})$ is a resistance form on $(V_* \backslash K_s(x)) \cup \{K_*\}$. Again by Claim 1, $(Q_*)^{\mathcal{F}^{K_*}} = Q_*$. Using Theorem 4.3, we verify Claim 2.
Claim 3 Let $R_*(\cdot,\cdot)$ be the resistance metric associated with $(\mathcal{E}, \mathcal{F}_{s,x})$. Then $R_*(K_*, Q_*) \geq c_2 s$ for any $x \in V_*$ and any $s \in (0,1]$, where c_2 is independent of x and s.
Proof of Claim 3 Let $\widetilde{V} = \cup_{w \in \Lambda^1_{s,x} \backslash \Lambda_{s,x}} F_w(V_0)$. Define $V = (\widetilde{V} \backslash (K_s(x) \cup Q_s(x))) \cup \{K_0\} \cup \{Q_0\}$. Note that V is naturally regarded as a subset of $(V_* \backslash (K_s(x) \cap$

$Q_s(X))) \cup \{K_*\} \cup \{Q_*\}$. Also, $\Phi : \widetilde{V} \to V$ is defined by

$$\Phi(x) = \begin{cases} x & \text{if } x \notin K_s(x) \cup Q_s(x), \\ K_0 & \text{if } x \in K_s(x), \\ Q_0 & \text{if } x \in Q_s(X). \end{cases}$$

Let

$$\mathcal{E}_V(u,v) = \sum_{w \in \Lambda^1_{s,x} \backslash \Lambda_{s,x}} \frac{1}{r_w} \mathcal{E}_D(u \circ \Phi, v \circ \Phi)$$

for any $u, v \in \ell(V)$. Then $(\mathcal{E}_V, \ell(V))$ is a resistance form on V. If $R_V(\cdot,\cdot)$ is the resistance metric associated with $(\mathcal{E}_V, \ell(V))$, then $R_V(K_0, Q_0) = R_*(K_*, Q_*)$. Let us consider $R_V(K_0, Q_0)$. Any path of resistors between K_0 and Q_0 corresponds to $(r_w)^{-1}\mathcal{E}_D(u \circ \Phi, v \circ \Phi)$ for some $w \in \Lambda^1_{s,x} \backslash \Lambda_{s,x}$. Let $w \in \Lambda^1_{s,x} \backslash \Lambda_{s,x}$. If $F_w(V_0) \cap K_s(x)$ or $F_w(V_0) \cap Q_s(x)$ is empty, then this part does not contribute to the effective resistance between K_0 and Q_0. So assume that both $p_w = K_s(x) \cap F_w(V_0)$ and $q_w = Q_s(x) \cap F_w(V_0)$ are non-empty. Let $r(w)$ be the effective resistance between p_w and q_w with respect to the resistance form derived from the resistance form $(r_w)^{-1}\mathcal{E}_D(\cdot,\cdot)$ on $F_w(V_0)$. Since the choice of p_w and q_w in $F_w(V_0)$ is finite, it follows that $\alpha_1 r_w \leq r(w) \leq \alpha_2 r_w$, where α_1 and α_2 are independent of x, s and w. Since $r_w \geq \underline{r}s$, we have $\alpha_3 s \leq r(w) \leq \alpha_2 s$, where $\alpha_3 = \alpha_1 \underline{r}$. Now $R_V(K_0, Q_0)$ is the resistance of the parallel circuit with the resistors of resistances $r(w)$. Since $\#(\Lambda^{s,x}_1)$ is uniformly bounded with respect to x and s, in fact 45 is a sufficient upper bound, we have

$$\alpha_4 s \leq R_V(K_0, Q_0) \leq \alpha_5 s,$$

where α_4 and α_5 are independent of x and s. This completes the proof of Claim 3.

Since $R_*(K_*, Q_*) \leq R(x, y)$ for any $y \in Q_s(x)$, Claim 3 suffices for the proof of this lemma. □

Combining all the lemmas, we finally obtain our main theorem of this section.

THEOREM 22.7. *The resistance metric R and the Euclidean metric d_E give the same topology on V_*. Moreover, the identity map on V_* is extended to a homeomorphism between the completions of (V_*, R) and (V_*, d_E).*

PROOF. If $\{x_n\}_{n \geq 1} \subseteq V_*$ and $x \in V_*$, then the following three conditions (A), (B) and (C) are equivalent.

(A) $\lim_{n \to +\infty} R(x_n, x) = 0$

(B) For any $s > 0$, there exists $N > 0$ such that $x_n \in U_s(x)$ for any $n \geq N$.

(C) $\lim_{n \to +\infty} |x_n - x| \to 0$.

In fact, by Lemmas 22.5-(2) and 22.6, (A) is equivalent to (B). Lemma 22.4-(2) shows that (B) is equivalent to (C).

Hence, the identity map between (V_*, R) and (V_*, d_E) is homeomorphism. Next assume that $\{x_n\}_{n \geq 1}$ is a d_E-Cauchy sequence. Let $x \in K$ be the limit of $\{x_n\}_{n \geq 1}$ with respect to d_E. Since $U_s(x)$ is a neighborhood of x with respect to d_E by Lemma 22.4-(2), $x_n \in U_s(x)$ for sufficiently large n. Lemma 22.5-(2) shows that $\{x_n\}_{n \geq 1}$ is an R-Cauchy sequence. Conversely assume that $\{x_n\}_{n \geq 1}$ is not a d_E-Cauchy sequence. There exist $\delta > 0$ and subsequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$ such that $|x_{n_i} - x_{m_i}| \geq \delta$ for any $i \geq 1$. By Lemma 22.5-(2), we may choose $s \in (0, 1]$ so that $\mathrm{diam}(U_s(x), d_E) < \delta$. This shows that $x_{n_i} \notin U_s(x_{m_i})$. By Lemma 22.6, it follows that $R(x_{n_i}, x_{m_i}) \geq c_2 s$. Hence $\{x_n\}_{n \geq 1}$ is not an R-Cauchy sequence.

Thus we have shown that the completions of (V_*, R) and (V_*, d_E) are naturally homeomorphic. $\square$

By this theorem, we are going to identify the completion of (V_*, R) with K. In other words, the resistance metric R is naturally extended to K. Using [**36**, Theorem 2.3.10], we think of $(\mathcal{E}, \mathcal{F})$ as a resistance form on K and R as the associated resistance metric from now on. Note that (K, R) is compact and hence $(\mathcal{E}, \mathcal{F})$ is regular.

By the identification described above, Lemmas 22.4, 22.5 and 22.6 imply that $U_s(x)$ is comparable with the resistance ball of radius s.

COROLLARY 22.8. *There exist* $\alpha_1, \alpha_2 > 0$ *such that*

$$B_R(x, \alpha_1 s) \subseteq U_s(x) \subseteq B_R(x, \alpha_2 s)$$

for any $x \in X$ *and any* $s \in (0, 1]$.

Since $(\mathcal{E}, \mathcal{F})$ is a resistance form on K and (K, R) is compact, we immediately obtain the following result.

COROLLARY 22.9. $(\mathcal{E}, \mathcal{F})$ *is a local regular resistance form on* K. *Moreover, let* μ *be a Borel regular measure on* (K, R) *with* $\mu(K) < +\infty$. *Then* $(\mathcal{E}, \mathcal{F})$ *is a local regular Dirichlet form on* $L^2(K, \mu)$.

DEFINITION 22.10. $(\mathcal{E}, \mathcal{F})$ constructed in this section is called the resistance form on K associated with $((D, \mathbf{r}^{(j)}))_{j=1,\dots,M}$. $(\mathcal{E}, \mathcal{F})$ is also called the Dirichlet form associated with $((D, \mathbf{r}^{(j)}))_{j=1,\dots,M}$.

23. Volume doubling property

In this section, we will give a criterion for the volume doubling property of a measure with respect to the resistance metric in Theorem 23.2. For random self-similar measures, we will obtain a simpler condition in Theorem 23.8.

As in the last section, (W_*, Γ) is a random Sierpinski gasket generated by $\{\mathcal{L}_1, \dots, \mathcal{L}_M\}$, $(D, \mathbf{r}^{(j)})$ is a regular harmonic structure on $\mathcal{L}_j$ for any j and $(\mathcal{E}, \mathcal{F})$ is the resistance form on K associated with $\{(D, \mathbf{r}^{(j)})\}_{j=1,\dots,M}$. We continue to use the same notations as in the previous section.

The first theorem is immediate from Theorem 9.4 and Corollary 22.9.

THEOREM 23.1. *Let* μ *be a finite Borel regular measure on* K. *Then* $(\mathcal{E}, \mathcal{F})$ *is a local regular Dirichlet form on* $L^2(K, \mu)$.

The following theorem gives a necessary and sufficient condition for the volume doubling property with respect to the resistance metric. It is a generalization of [**39**, Theorem 1.3.5]. The conditions (EL) and (GE) correspond to (ELm) and (GE) in [**39**] respectively.

THEOREM 23.2. *Let* μ *be a finite Borel regular measure on* K. μ *has the volume doubling property with respect to the resistance distance* R *if and only if the following two conditions* (GE) *and* (EL) *are satisfied:*
(GE) *There exists* $c_1 > 0$ *such that* $\mu(K_w) \leq c_1 \mu(K_v)$ *for any* $w, v \in \Lambda_s$ *with* $K_w \cap K_v \neq \emptyset$ *and any* $s \in (0, 1]$.
(EL) *There exists* $c_2 > 0$ *such that* $\mu(K_{wi}) \geq c_2 \mu(K_w)$ *for any* $w \in W_*$ *and any* $i \in S_{\Gamma(w)}$.

We need two lemmas to prove this theorem.

LEMMA 23.3. *Let μ be a finite Borel regular measure on K. μ has the volume doubling property with respect to the resistance metric R if and only if there exist $\alpha \in (0,1)$ and $c > 0$ such that $\mu(U_s(x)) \leq c\mu(U_{\alpha s}(x))$ for any $x \in X$ and any $s \in (0,1]$.*

PROOF. By Corollary 22.8, $B_R(x, \alpha_1 s) \subseteq U_s(x) \subseteq B_R(x, \alpha_2 s)$. Assume that $\mu(U_s(x)) \leq c\mu(U_{\alpha s}(x))$. Choose n so that $\alpha^n \alpha_2 < \alpha_1$. Then

$$\mu(B_R(x, \alpha_1 s)) \leq \mu(U_s(x)) \leq c^n \mu(U_{\alpha^n s}(x)) \leq \mu(B_R(x, \alpha^n \alpha_2 s)).$$

Hence μ has the volume doubling property with respect to R.

Conversely, assume that $\mu(B_R(x,s)) \leq c_* \mu(B_R(x, \delta s))$ for some $c_* > 0$ and $\delta \in (0,1)$. Choose n so that $\delta^n \alpha_2 < \alpha_1$. Then

$$\mu(U_s(x)) \leq \mu(B_R(x, \alpha_2 s)) \leq (c_*)^n \mu(B_R(x, \delta^n \alpha_2 s)) \leq (c_*)^n \mu(U_{\delta^n \alpha_2 (\alpha_1)^{-1} s}(x)).$$

Letting $\alpha = \delta^n \alpha_2 (\alpha_1)^{-1}$, we have the desired statement. □

LEMMA 23.4. *Let $s \in (0,1]$ and let $w \in \Lambda_s$. If $\alpha \leq \underline{r}^2$, then there exists $x \in K_w$ such that $U_s(x) \subseteq K_w$.*

PROOF. Set $w = w_1 w_2 \cdots w_m$, where $m = |w|$. Choose k and l so that $k, l \in \{1,2,3\}$, $k \neq w_m$ and $l \neq k$. Note that $K_{wkl} \cap F_w(V_0) = \emptyset$. Since $r_{w_1 w_2 \cdots w_{m-1}} > s$, it follows that $r_{wk} > \underline{r}^2 s$. If $\alpha \leq \underline{r}^2$, then $r_{wklv} \in \Lambda_{\alpha s}$ for some $v \in W_*(S)$. Set $w_* = wklv$. Choose $x \in K_{w_*} \backslash F_{w_*}(V_0)$. By Proposition 21.3-(2) and (4), $\Lambda_{\alpha s, x} = \{w_*\}$ and $[w']_m = w$ for any $w' \in \Lambda^1_{\alpha s, x}$. Hence $U_{\alpha s}(x) \subseteq K_w$. □

PROOF OF THEOREM 23.2. Assume (GE) and (EL). Fix $\alpha \in (0,1)$. Let $w \in \Lambda_{s,x}$ and let $wv \in \Lambda_{\alpha s, x}$. For any $w' \in \Lambda^1_{s,x}$, there exists $w'' \in \Lambda_{s,x}$ such that $K_{w''} \cap K_{w'} \neq \emptyset$ and $K_{w''} \cap K_w \neq \emptyset$. Hence by (GE), $\mu(K_{w''}) \leq (c_1)^2 \mu(K_w)$. Since $\#(\Lambda^1_{s,x}) \leq 45$,

$$\mu(U_s(x)) \leq 45 (c_1)^2 \mu(K_w). \tag{23.1}$$

(In the above argument we only need that $\#(\Lambda^1_{s,x})$ is uniformly bounded. Since V_0 is a regular triangle and every $K(W_*, \Gamma) \subseteq T$, we may deduce that the uniform bound is no greater than 45.) Now, since $wv \in \Lambda_{\alpha s}$ and $w \in \Lambda_s$,

$$\alpha s < r_w r_{v_*} \leq s r_{v_*} \leq s \overline{r}^{|v|-1},$$

where $v_* = [v]_{|v|-1}$. Letting m_* be the integral part of $\frac{\log \alpha}{\log \overline{r}} + 2$, we have $|v| \leq m_*$. Note that m_* only depends on α. By (EL), $\mu(K_{wv}) \geq (c_2)^{m_*} \mu(K_w)$. Hence (23.1) shows that

$$\mu(U_{\alpha s}) \geq (c_2)^{m_*} \mu(K_w) \geq (c_2)^{m_*} (c_1)^{-2} \frac{1}{45} \mu(U_s(x)).$$

By Lemma 23.3, μ has the volume doubling property with respect to R.

Next assume that (GE) do not hold. For any $C > 0$, there exist $s \in (0,1]$ and $w, v \in \Lambda_s$ with $K_w \cap K_v \neq \emptyset$ such that $\mu(K_v) \geq C\mu(K_w)$. Let $\alpha \in (0, \underline{r}^2]$. By Lemma 23.4, $U_{\alpha s}(x) \subseteq K_w$ for some $x \in K_w$. Since $v \in \Lambda^1_{s,x}$,

$$\mu(U_s(x)) \geq (1 + C)\mu(K_w) \geq (1 + C)\mu(U_{\alpha s}(x)).$$

Lemma 23.3 shows that μ does not have the volume doubling property with respect to R.

Finally, if (EL) do not hold, then for any $\epsilon > 0$ there exist $w \in W_*$ and $i \in S_{\Gamma(w)}$ such that $\mu(K_{wi}) \leq \epsilon\mu(K_w)$. Set $s = r_w$. Let $\alpha \in (0, \underline{r}^3]$. Then $\alpha s \leq \underline{r}^3 s \leq \underline{r}^2 r_{wi}$. By Lemma 23.4, there exists $x \in K_{wi}$ such that $U_{\alpha s}(x) \subseteq K_{wi}$. Now,

$$\mu(U_{\alpha s}(x)) \leq \mu(K_{wi}) \leq \epsilon\mu(K_w) \leq \epsilon\mu(U_s(x)).$$

Using Lemma 23.3, we see that μ does not have the volume doubling property with respect to R. $\square$

Next we introduce the notion of random self-similar measures, which is a natural generalization of self-similar measures on ordinary self-similar sets. See Definition 20.4 for the definition of self-similar measures on generalized Sierpinski gaskets.

PROPOSITION 23.5. *Let $\mu^{(j)} = (\mu_i^{(j)})_{i \in S_j} \in (0,1)^{S_j}$ satisfy $\sum_{i \in S_j} \mu_i^{(j)} = 1$ for each $j = 1, \ldots, M$. Define $\mu_w = \mu_{w_1}^{\Gamma([w]_0)} \mu_{w_2}^{\Gamma([w]_1)} \cdots \mu_{w_m}^{\Gamma([w]_{m-1})}$ for any $w = w_1 w_2 \cdots w_m \in W_*$. Then there exists a unique Borel regular probability measure μ on K such that $\mu(K_w) = \mu_w$ for any $w \in W_*$. Moreover, μ satisfies the condition* (EL) *in Theorem 23.2*

Note that the Hausdorff measure associated with the resistance metric, which has been studied in [**28, 29, 30**] is not a random self-similar measure in general except for the homogeneous case.

DEFINITION 23.6. The Borel regular probability measure μ in Proposition 23.5 is called the random self-similar measure on (W_*, Γ) generated by $(\mu^{(1)}, \ldots, \mu^{(M)})$.

In the next definition, we introduce a notion describing relations of neighboring K_w's for $w \in \Lambda_s$ in order to apply Theorem 23.2.

DEFINITION 23.7. A pair $((j_1, i_1), (j_2, i_2)) \in \{(j, i) | j = 1, \ldots, M, i \in \{1, 2, 3\}\}^2$ is called an adjoining pair for (W_*, Γ) if and only if there exist $w, v \in W_*$ such that $wi_1, vi_2 \in \Lambda_s$ for some $s \in (0, 1]$, $w \neq v$, $j_1 = \Gamma(w), j_2 = \Gamma(v)$ and $\pi(w(i_1)^\infty) = \pi(v(i_2)^\infty)$.

THEOREM 23.8. *Let μ be a random self-similar measure on (W_*, Γ) generated by $(\mu^{(1)}, \ldots, \mu^{(M)})$. μ has the volume doubling property with respect to the resistance metric R if*

$$\frac{\log \mu_{i_1}^{(j_1)}}{\log r_{i_1}^{(j_1)}} = \frac{\log \mu_{i_2}^{(j_2)}}{\log r_{i_2}^{(j_2)}} \tag{23.2}$$

for any adjoining pair $((j_1, i_1), (j_2, i_2))$ for (W_*, Γ).

Before proving this theorem, we give an example where the condition of the above theorem is realized.

EXAMPLE 23.9. Let $\mathcal{L}_1 = \mathcal{L}_{SG}$ and let $\mathcal{L}_2 = \mathcal{L}_{SP}$, where $\mathcal{L}_{SG}$ and $\mathcal{L}_{SP}$ are the original Sierpinski gasket and the Sierpinski spiral respectively introduced in Section 20. Set $S_1 = \{1, 2, 3\}$ and $S_2 = \{1, 2, 3, 4\}$. Define $H = \{(h, \gamma) | 0 < h, \gamma \in (0, \min\{h, 1/h\})\}$. Fix $(h, \gamma) \in H$ and set $r_i^{(1)} = r_i^{SG}$ for $i \in S_1$ and $r_i^{(2)} = r_i^{SP}$ for $i \in S_2$. (Recall that r_i^{SG} only depends on h and r_i^{SP} depend on h and γ. See Examples 20.11 and 20.3.) Denote $\mathbf{r}^{(j)} = (r_i^{(j)})_{j \in S_j}$ for $j = 1, 2$. Define α_* by the unique α satisfying $\sum_{i \in S_1} (r_i^{(1)})^\alpha = 1$. Note that α_* depends only on h. When

$h = 1$, then $r_i^{(1)} = 3/5$ for any $i \in S_1$ and hence $\alpha_* = \log 3/(\log 5 - \log 3)$. Let $\mu_i^{(1)} = (r_i^{(1)})^{\alpha_*}$ for $i \in S_1$. Define

$$H_0 = \{(h, \gamma) | (h, \gamma) \in H, \sum_{i=1,2,3} (r_i^{(2)})^{\alpha_*} < 1\}.$$

If $h = 1$, $r_i^{(2)} = (1 - \gamma)/2$ for any $i \in \{1, 2, 3\}$. This implies $(1, \gamma) \in H_0$ for any $\gamma \in (0, 1)$. Hence H_0 is a non-empty open subset of $\mathbb{R}^2$. Let $\mu_i^{(2)} = (r_i^{(2)})^{\alpha_*}$ for any $i \in \{1, 2, 3\}$ and let $\mu_4^{(2)} = 1 - \sum_{i=1}^3 \mu_i^{(2)}$. Applying Theorem 23.8, we have the following proposition:

Proposition Assume that $(h, \gamma) \in H_0$. Let (W_*, Γ) be any random Sierpinski gasket generated by generated by $\{\mathcal{L}_1, \mathcal{L}_2\}$. Let μ_* be the random self-similar measure generated by $((\mu_i^{(j)})_{i \in S_j})_{j=1,2}$ and let R be the resistance metric on $K = K(W_*, \Gamma)$ associated with $((D_h, \mathbf{r}^{(j)}))_{j=1,2}$. Then μ_* has the volume doubling property with respect to R.

The rest of this section is devoted to the proof of Theorem 23.8.

Proof of Theorem 23.8. Let $\omega(1)$ and $\omega(2)$ belong to $\Sigma(W_*, \Gamma)$. Write $\omega(i) = \omega(i)_1 \omega(i)_2 \ldots$ for $i = 1, 2$. Assume that $\omega(1) = w i_1 (k)^\infty, \omega(2) = w i_2 (l)^\infty \in \Sigma(W_*, \Gamma)$, where $w \in W_* \backslash W_0$, $i_1 \neq i_2 \in S_{\Gamma(w)}$, $k, l \in \{1, 2, 3\}$ and $\pi(\omega(1)) = \pi(\omega(2))$. Set $r_{i,n} = r_{\omega(i)_n}^{(\Gamma([\omega(i)]_{n-1}))}$ and $\mu_{i,n} = \mu_{\omega(i)_n}^{(\Gamma([\omega(i)]_{n-1}))}$ for $i = 1, 2$ and $n \geq 1$. Define $\{m_n\}_{n \geq 0}$ and $\{M_n\}_{n \geq 0}$ inductively by

$$\begin{cases} m_0 = M_0 & = |w| \\ m_{n+1} & = \inf\{m | m > m_n, r_{[\omega(1)]_m} = r_{[\omega(2)]_{m'}} \text{ for some } m'\} \\ M_{n+1} & = \inf\{m | m > M_n, r_{[\omega(1)]_{m'}} = r_{[\omega(2)]_m} \text{ for some } m'\} \end{cases}$$

(If $\inf\{m | m > m_n, r_{[\omega(1)]_m} = r_{[\omega(2)]_{m'}}$ for some $m'\} = \emptyset$, then we define $m_N = M_N = +\infty$ for all $N \geq n + 1$.) Also define $s_n = r_{[\omega(1)]_{m_n}}$ for $n \geq 0$. (If $m_n = +\infty$, then define $s_n = 0$.) Note that $s_n = r_{[\omega(1)]_{m_n}} = r_{[\omega(2)]_{M_n}}$.

Claim 1 Let $n \geq 1$. Then there exists α_n such that $\mu_{1,m} = (r_{1,m})^{\alpha_n}$ for any $m = m_n + 1, \ldots m_{n+1}$ and $\mu_{2,m} = (r_{2,m})^{\alpha_n}$ for any $m = M_n + 1, \ldots, M_{n+1}$.

Proof of Claim 1 For sufficiently small $\epsilon > 0$, $[w(1)]_{m_n+1}, [w(2)]_{M_n+1} \in \Lambda_{s_n+\epsilon}$. Hence $(\Gamma([w(1)]_{m_n}), k), (\Gamma([w(2)]_{M_n}), l)$ is an adjoint pair. By (23.2),

$$\frac{\log \mu_{1,m_n+1}}{\log r_{1,m_n+1}} = \frac{\log \mu_{2,M_n+1}}{\log r_{2,M_n+1}} \tag{23.3}$$

Set $\alpha_n = \log \mu_{1,m_n+1} / \log r_{1,m_n+1}$. Let $m_n + 1 \leq m < m_{n+1}$. Then there exists $m' \in [M_n, M_{n+1} - 1]$ such that $r_{[w(2)]_{m'}} < r_{[w(1)]_m} < r_{[w(2)]_{m'+1}}$. Set $s_* = r_{[w(1)]_m}$. Then we see that $[w(1)]_m, [w(2)]_{m'+1} \in \Lambda_{s_*}$ and $[w(1)]_{m+1}, [w(2)]_{m'+1} \in \Lambda_{s_*+\epsilon}$ for sufficiently small $\epsilon > 0$. Hence $((\Gamma([w(1)]_{m-1}), k), (\Gamma([w(2)]_{m'}), l))$ and $((\Gamma([w(1)]_m, k), (\Gamma([w(2)]_{m'}, l))$ are adjoint pairs. Using (23.2), we see that

$$\frac{\log \mu_{1,m}}{\log r_{1,m}} = \frac{\log \mu_{2,m'+1}}{\log r_{2,m'+1}} = \frac{\log \mu_{1,m+1}}{\log r_{1,m+1}}. \tag{23.4}$$

By similar argument,

$$\frac{\log \mu_{2,m}}{\log r_{2,m}} = \frac{\log \mu_{2,m+1}}{\log r_{2,m+1}} \tag{23.5}$$

for any $m = M_n + 1, \dots, M_{n+1} - 1$. The equations (23.3), (23.4) and (23.5) immediately imply the claim. (End of Proof of Claim 1)

Claim 2 Set $s_* = \min\{r_{wi_1}, r_{wi_2}\}$. Define $m_* = \min\{m | s_* > r_{[\omega(1)]_m} \geq s_1\}$ and $M_* = \min\{m' | s_* > r_{[\omega(2)]_{m'}} \geq s_1\}$. There exists $\alpha_0 > 0$ such that $\mu_{1,m} = (r_{1,m})^{\alpha_0}$ and $\mu_{2,m'} = (r_{2,m'})^{\alpha_0}$ for any $m = m_*, \dots, m_1$ and any $m' = M_*, \dots, M_1$.

Proof of Claim 2 If $m_1 = m_0 + 1$, then $s_* = s_1$. Hence we have Claim 2. Similarly, if $M_1 = M_0 + 1$, then we have Claim 2. Thus we may assume that $m_1 \geq m_0 + 2$ and $M_1 \geq M_0 + 2$. Then $[\omega(1)]_{m_1}, [\omega(2)]_{M_1} \in \Lambda_{s_1}$, and so it follows that $(\Gamma([\omega(1)]_{m_1-1}), k), (\Gamma([\omega(2)]_{M_1-1}), l))$ is an adjoining pair. By (23.2),

$$\frac{\log \mu_{1,m_1}}{\log r_{1,m_1}} = \frac{\log \mu_{2,M_1}}{\log r_{2,M_1}} \tag{23.6}$$

Let $m \in \{m_*, \dots, m_1 - 1\}$. Then there exists $m' \in [M_0 + 1, M_1 - 1]$ such that $r_{[w(2)]_{m'}} < r_{[w(1)]_m} < r_{[w(2)]_{m'+1}}$. Using similar argument as in the proof of Claim 1, we obtain counterparts of (23.4) and (23.5). These equalities along with (23.6) yield the claim. (End of Proof of Claim 2)

Claim 3 Define $L = \min\{n | n \in \mathbb{N}, \overline{r}^n < \underline{r}\}$. If $[\omega(1)]_m, [\omega(2)]_{m'} \in \Lambda_s$ for some $s \in [s_1, s_0)$, then

$$(\underline{\mu})^L (\underline{r})^{\alpha_0(L+1)} \mu_{[\omega(1)]_m} \leq \mu_{[\omega(2)]_{m'}} \leq (\underline{\mu})^{-L} (\underline{r})^{-\alpha_0(L+1)} \mu_{[\omega(1)]_m}, \tag{23.7}$$

where $\underline{\mu} = \min\{\mu_i^{(j)} | j \in \{1, \dots, M\}, i \in \{1, 2, 3\}\}$.

Proof of Claim 3 Assume that $r_{wi_1} = s_*$. Note that $m_* = m_0 + 2$. First we consider the case where $s \in [s_*, s_0)$. It follows that $m = m_0 + 1$ and $wi_1 \in \Lambda_s$. Since $r_{[\omega(2)]_{m'-1}} \geq s_* = r_{wi_1}$, we have $r_{2,M_0+1} \cdots r_{2,m'-1} \geq r_{1,m_0+1}$. This shows that $\overline{r}^{m'-M_0-1} \geq \underline{r}$. Therefore, $m' - M_0 \leq L$. Now $\mu_w \leq \mu_{wi_1} \leq \mu_w \underline{\mu}$ and $\mu_w \leq \mu_{[\omega(2)]_{m'}} \leq \mu_w (\underline{\mu})^{m'-M_0} \leq \mu_w (\underline{\mu})^L$. This immediately imply Claim 3 in this case. Next suppose $s \in [s_1, s_*)$. By Claim 2, $\mu_{1,m_0+2} \cdots \mu_{1,m} = (r_{1,m_0+2} \cdots r_{1,m})^{\alpha_0}$ and $\mu_{2,M_*} \cdots \mu_{2,m'} = (r_{2,M_*} \cdots r_{2,m'})^{\alpha_0}$. On the other hand, $r_{[\omega(1)]_{m-1}} > s \geq r_{[\omega(1)]_m}$. Hence

$$\frac{s}{r_{wi_1}} \geq r_{1,m_0+2} \cdots r_{1,m} \geq \frac{\underline{r} s}{r_{wi_1}}.$$

This implies

$$\frac{\mu_w s^{\alpha_0}}{(r_w)^{\alpha_0}} \frac{1}{\underline{r}} \geq \mu_{[\omega(1)]_m} \geq \frac{\mu_w s^{\alpha_0}}{(r_w)^{\alpha_0}} \underline{r}^{\alpha_0} \underline{\mu}. \tag{23.8}$$

Similarly, we have

$$\frac{s}{r_{[\omega(2)]_{M_*-1}}} \geq r_{2,M_*} \cdots r_{2,m'} \geq \frac{\underline{r} s}{r_{[\omega(2)]_{M_*-1}}}$$

and hence

$$\frac{\mu_w s^{\alpha_0}}{(r_w)^{\alpha_0}} \frac{1}{\underline{r}^{\alpha_0 L}} \geq \mu_{[\omega(2)]_{m'}} \geq \frac{\mu_w s^{\alpha_0}}{(r_w)^{\alpha_0}} \underline{\mu}^L \underline{r}^{\alpha_0} \tag{23.9}$$

Combining (23.8) and (23.9), we have the claim. If $r_{wi_2} = s_*$, then an analogous discussion yields the claim as well. (End of Proof of Claim 3)

Claim 4 Define $\alpha_* = \max\{\log \mu_i^{(j)} / \log r_i^{(j)} | j \in \{1, \dots, M\}, i \in \{1, 2, 3\}\}$. If $[\omega(1)]_m, [\omega(2)]_{m'} \in \Lambda_s$ for some $s \in (0, s_0)$, then

$$(\underline{\mu})^L (\underline{r})^{\alpha_*(L+2)} \mu_{[\omega(1)]_m} \leq \mu_{[\omega(2)]_{m'}} \leq (\underline{\mu})^{-L} (\underline{r})^{-\alpha_*(L+2)} \mu_{[\omega(1)]_m} \tag{23.10}$$

Proof of Claim 4 By Claim 1, $\mu_{1,m_n+1}\cdots\mu_{2,m_{n+1}} = (r_{1,m_n+1}\cdots r_{1,m_{n+1}})^{\alpha_n} = (r_{2,M_n+1}\cdots r_{2,M_{n+1}})^{\alpha_n} = \mu_{2,M_n+1}\cdots\mu_{2,M_{n+1}}$. for any $n \geq 1$. Suppose $s \in [s_{p+1}, s_p)$ for some $p \geq 1$. Then

$$\frac{\mu_{[\omega(1)]_p}}{\mu_{[\omega(2)]_p}} = \frac{\mu_{1,m_0+1}\cdots\mu_{1,m_1}}{\mu_{2,M_0+1}\cdots\mu_{2,M_1}} \times \frac{\mu_{1,m_p+1}\cdots\mu_{1,m}}{\mu_{2,M_p+1}\cdots\mu_{2,m'}}.$$

By Claim 3, we have an estimate of the first part of the right-hand side of the above equality. For the second part, $\mu_{1,m_p+1}\cdots\mu_{1,m} = (r_{1,m_p+1}\cdots r_{1,m'})^{\alpha_p}$. On the other hand, $s/s_p \geq r_{1,m_p+1}\cdots r_{1,m} \geq \underline{r}s/s_p$. Hence we have

$$\Big(\frac{s}{s_p}\Big)^{\alpha_p} \geq \mu_{1,m_p+1}\cdots\mu_{1,m} \geq (\underline{r})^{\alpha_p}\Big(\frac{s}{s_p}\Big)^{\alpha_p}.$$

Similarly,

$$\Big(\frac{s}{s_p}\Big)^{\alpha_p} \geq \mu_{2,M_p+1}\cdots\mu_{2,m'} \geq (\underline{r})^{\alpha_p}\Big(\frac{s}{s_p}\Big)^{\alpha_p}.$$

Hence $(\mu_{1,m_p+1}\cdots\mu_{1,m})/(\mu_{2,M_p+1}\cdots\mu_{2,m'}) \leq (\underline{r})^{-\alpha_p} \leq (\underline{r})^{-\alpha_*}$. Combining this with Claim 3, we obtain Clam 4. (End of Proof of Claim 4)

Finally, we prove the theorem. If $w(1), w(2) \in \Lambda_s$ for some $s \in (0,1]$, $w(1) \neq w(2)$ and $K_{w(1)} \cap K_{w(2)} \neq \emptyset$, then $w(1) = [\omega(1)]_m$ and $w(2) = [\omega(2)]_{m'}$ for some $\omega(1) = wi_1(k)^\infty, \omega(2) = wi_2(l)^\infty \in \Sigma(W_*, \Gamma)$, where $w \in W_*\backslash W_0$, $i_1 \neq i_2 \in S_{\Gamma(w)}$, $k, l \in \{1,2,3\}$ and $\pi(\omega(1)) = \pi(\omega(2))$. By Claim 4, $\mu(K_{w(2)}) \leq (\underline{\mu})^{-L}(\underline{r})^{-\alpha_*(L+2)}\mu(K_{w(1)})$. Hence we have (GE). Proposition 23.5 shows that μ satisfies (EL). Using Theorem 23.2, we see that μ has the volume doubling property with respect to R. □

24. Homogeneous case

In this section, we treat a special class of random Sierpinski gasket called homogeneous random Sierpinski gaskets. In this case, the Hausdorff measure is a random self-similar measure and it is always volume doubling with respect to the resistance metric. The associated diffusion process has been extensively studied in [**27, 29, 7**]. Most of the results in this section are the reproduction of their works from our view point.

As in the previous sections, $\mathcal{L}_j = (K(j), S_j, \{F_j^i\}_{i\in S_j})$ is a generalized Sierpinski gasket for $j = 1, \ldots, M$ and $S_j = \{1, \ldots, N_j\}$.

Definition 24.1. Let (W_*, Γ) be a random Sierpinski gasket generated by $\{\mathcal{L}_1, \ldots, \mathcal{L}_M\}$.
(1) (W_*, Γ) is called homogeneous if and only if $\Gamma(w) = \Gamma(v)$ for any $w, v \in W_m$ and for any $m \geq 0$.
(2) Let (W_*, Γ) be homogeneous. For $m \geq 1$, we define $\Gamma_m = \Gamma(w)$ for $w \in W_{m-1}$. Set $\nu_i^{(j)} = (N_j)^{-1}$ for any $j = 1, \ldots, M$ and any $i \in S_j$. The random self-similar measure ν on (W_*, Γ) generated by $\{(\nu_i^{(1)})_{i\in S_1}, \ldots, (\nu_i^{(M)})_{i\in S_M}\}$ is called the canonical measure on (W_*, Γ).

This canonical measure coincides with the measure used in [**27, 28, 29**]. We will show in Theorem 24.5 that the canonical measure is equivalent to the Hausdorff measure associated with the resistance metric.

Throughout this section, (W_*, Γ) is a homogeneous random Sierpinski gasket. Let $(D, \mathbf{r}^{(j)})$ be a regular harmonic structure on $\mathcal{L}_j$ for each $j = 1, \ldots, M$. We will also require homogeneity for the resistance scaling ratio $\mathbf{r}^{(j)}$. Namely, the following

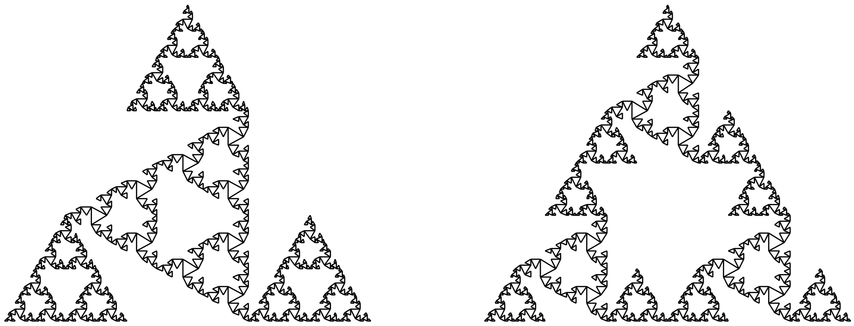


FIGURE 4. Homogeneous random Sierpinski gaskets

condition (HG):

(HG) $r^{(j)}_{i_1} = r^{(j)}_{i_2}$ for any j and any $i_1, i_2 \in \mathcal{S}_j$.

is assumed hereafter in this section. Under (HG), we write $r^{(j)}_i = r^{(j)}$.

PROPOSITION 24.2. *Assume* (HG).

(1) *Define* $r(m) = r^{(\Gamma_1)} \cdots r^{(\Gamma_m)}$. *Then* $\Lambda_s = W_m$ *for* $s \in (r(m-1), r(m)]$.

(2) *Let* ν *be the canonical measure on* (W_*, Γ). *Then* $\nu(K_w) = \#(W_m)^{-1} = (N_{\Gamma_1} \cdots N_{\Gamma_m})^{-1}$ *for any* $w \in W_m$.

Note that $j_1 = j_2$ for any adjoining pair $((j_1, i_1), (j_2, i_2))$ in the case of a homogeneous random Sierpinski gasket. Hence by Theorem 23.8, we immediately obtain the following result.

THEOREM 24.3. *Assume* (HG). *The canonical measure* ν *has the volume doubling property with respect to the resistance metric* R *on* $K = K(W_*, \Gamma)$ *associated with* $((D, \mathbf{r}^{(j)}))_{j=1,\ldots,M}$.

We can describe more detailed structure of the canonical measure ν in terms of the resistance metric.

DEFINITION 24.4. Define $\psi(s) = \#(\Lambda_s)^{-1}$ for any $s \in (0, 1]$. For $s \geq 1$, we define $\psi(s) = \psi(1)$. For any $\delta > 0$ and any $A \subseteq K$, we define

$$\mathcal{H}^{\psi}_{\delta}(A) = \inf\big\{\sum_{i\geq 1} \psi(\mathrm{diam}(U_i, R)) | A \subseteq \cup_{i\geq 1} U_i, \mathrm{diam}(U_i, R) \leq \delta \text{ for any } i \geq 1\big\}$$

and $\mathcal{H}^{\psi}(A) = \lim_{\delta\downarrow 0} \mathcal{H}^{\psi}_{\delta}(A)$. $\mathcal{H}^{\psi}$ is called the ψ-Hausdorff measure of (K, R).

It is known that $\mathcal{H}^{\psi}$ is a Borel regular measure. See [**46**]. The next theorem shows that ν is equivalent to the ψ-Hausdorff measure.

THEOREM 24.5. *Assume* (HG). *The canonical measure* ν *is equivalent to the* ψ*-dimensional Hausdorff measure* $\mathcal{H}^{\psi}$ *of* (K, R). *More precisely, there exist* $c_1, c_2 > 0$ *such that*

$$c_1\nu(A) \leq \mathcal{H}^{\psi}(A) \leq c_2\nu(A) \tag{24.1}$$

for any Borel set $A \subseteq K$ *and*

$$c_1\psi(r) \le \nu(B_R(x,r)) \le c_2\psi(r) \tag{24.2}$$

for any $x \in X$ *and any* $r \in (0,1]$.

PROOF. Note that ψ has the doubling property, i.e. $\psi(2s) \le c\psi(s)$ for any s, where c is independent of s. Now, if $w \in \Lambda_s$, then $\nu(K_w) = \psi(s)$. Since $1 \le \#(\Lambda^1_{s,x}) \le 45$,

$$\psi(s) \le \nu(U_s(x)) \le 45\psi(s) \tag{24.3}$$

By Corollary 22.8, $\nu(B_R(x,\alpha_1 s)) \le \nu(U_s(x)) \le \nu(B_R(x,\alpha_2 s))$. This along with (24.3) and the doubling property of ψ implies (24.2).

Next we show (24.1). By Lemma 22.5-(1), $\alpha_3 r_w \le \mathrm{diam}(K_w,R) \le \alpha_4 r_w$ for any $w \in W_*$, where α_3 and α_4 are independent of w. Since $\psi(r_w) = \nu(K_w)$, we have $\alpha_6\nu(K_w) \le \psi(\mathrm{diam}(K_w,R)) \le \alpha_7\nu(K_w)$. Let A be a compact subset of K. Define $\Lambda_s(A) = \{w|w \in \Lambda_s, K_w \cap A \neq \emptyset\}$ and $K_s(A) = \cup_{w\in\Lambda_s(A)}K_w$. Then $\cap_{n\ge1}K_{1/n}(A) = A$. Note that $\max_{w\in\Lambda_s}\mathrm{diam}(K_w,R) \le \alpha_8 s$. Hence,

$$\mathcal{H}^{\psi}_{\alpha_8 s}(A) \le \sum_{w\in\Lambda_s(A)}\psi(\mathrm{diam}(K_w,R)) \le \sum_{w\in\Lambda_s(A)}\alpha_7\nu(K_w) \le \alpha_7\nu(K_s(A)).$$

Letting $s \downarrow 0$, we obtain $\mathcal{H}^{\psi}(A) \le \alpha_7\nu(A)$ for any compact set A. Since both $\mathcal{H}^{\psi}$ and ν are Borel regular, $\mathcal{H}^{\psi}(A) \le \alpha_7\nu(A)$ for any Borel set A. Finally, let A be a Borel set. Suppose $A \subseteq \cup_{i\ge1}U_i$. Choose $x_i \in U_i$. Then by (24.2)

$$\nu(A) \le \sum_{i\ge1}\nu(U_i) \le \sum_{i\ge1}\mu(B_R(x,\mathrm{diam}(U_i,R))) \le c_2\sum_{i\ge1}\psi(\mathrm{diam}(U_i,R))$$

This shows $\nu(A) \le c_2\mathcal{H}^{\psi}(A)$. Thus we have (24.1). □

By (24.3), we have the uniform volume doubling property, which has been defined in [**41**]. By the above theorem, we have

$$R(x,y)V_R(x,R(x,y)) \asymp R(x,y)\psi(R(x,y)),$$

Hence Theorem 15.10 implies the following theorem. (In fact, since we have the uniform volume doubling property, [**41**, Theorem 3.1] suffices to have (24.4) and (24.5). Recall the remark after Theorem 15.10.)

THEOREM 24.6. *Let* $(\mathcal{E},\mathcal{F})$ *be the regular local Dirichlet form on* $L^2(K,\nu)$ *associated with* $((D,\mathbf{r}^{(j)}))_{j=1,\dots,M}$. *Assume* (HG). *There exists a jointly continuous heat kernel* $p(t,x,y)$ *associated with the Dirichlet form* $(\mathcal{E},\mathcal{F})$ *on* $L^2(K,\nu)$. *Define* $g(r) = r\psi(r)$. *(Note that* $\psi(g^{-1}(t)) \asymp t/g^{-1}(t)$.*) Then*

$$p(t,x,x) \asymp \frac{g^{-1}(t)}{t} \tag{24.4}$$

for any $t > 0$ *and any* $x \in K$ *and*

$$p(t,x,y) \le \frac{c_1g^{-1}(t)}{t}\exp\left(-c_2\left(\frac{R(x,y)}{\psi^{-1}(t/R(x,y))}\right)\right) \tag{24.5}$$

for any $t > 0$ *and any* $x, y \in K$.

By (24.4), the fluctuation from a power law in the on-diagonal behavior of heat kernels given in [**27, 29**] is now understood as the fluctuation of $\psi(r)$ versus r^{α}, where α is the Hausdorff dimension of (K, R).

Unfortunately, the resistance metric is not (equivalent to a power of) a geodesic metric in general, and hence (24.5) may not be best possible. To construct a geodesic metric, we define the notion of n-paths for homogeneous random Sierpinski gaskets in the similar way as in Definition 20.8.

THEOREM 24.7. *Assume that $\mathcal{L}_j$ admits symmetric self-similar geodesics with the ratio γ_j. Set $\gamma(m) = \gamma_{\Gamma_1} \cdots \gamma_{\Gamma_j}$. Then there exists a geodesic metric d on K which satisfies*

$$d(p, q) = \gamma(n) \min\{m - 1 | (x_1, \ldots, x_m) \text{ is an } n\text{-path between } p \text{ and } q\}$$

for any $p, q \in V_n$. Moreover, assume (HG) *and that $r^{(j)}/N_j < \gamma_j$ for any $j = 1, \ldots, M$. Set $\nu(m) = \#(W_m)^{-1}$ and $T_m = \nu(m)r(m)$ for any $m \geq 0$. Define $\beta_m = \log T_m / \log \gamma(m)$ and*

$$h(s) = \begin{cases} s^{\beta_m} & \text{if } T_m \leq s < T_{m-1}, \\ s^2 & \text{if } t \geq 1. \end{cases}$$

Then

$$\frac{c_1}{V_d(x, h^{-1}(t))} \exp\left(- c_2\left(\frac{d(x,y)}{\Phi^{-1}(t/d(x,y))} \right) \right) \leq p(t,x,y) \leq \frac{c_3}{V_d(x, h^{-1}(t))} \exp\left(- c_4\left(\frac{d(x,y)}{\Phi^{-1}(t/d(x,y))} \right) \right), \tag{24.6}$$

where $\Phi(s) = h(s)/s$.

The above two-sided off-diagonal estimate is essentially same as that obtained by Barlow and Hambly in [**7**]. More precisely, they have shown (24.7) and (24.8) given below.

We will prove this theorem at the end of this section.

REMARK. (24.6) has equivalent expressions. Set $\alpha_m = \log \nu(m) / \log \gamma(m)$. Then (24.6) is equivalent to

$$\frac{c_5}{t^{\alpha_m/\beta_m}} \exp\left(- c_6\left(\frac{d(x,y)^{\beta_n}}{t} \right)^{\frac{1}{\beta_n - 1}} \right) \leq p(t,x,y) \leq \frac{c_7}{t^{\alpha_m/\beta_m}} \exp\left(- c_8\left(\frac{d(x,y)^{\beta_n}}{t} \right)^{\frac{1}{\beta_n - 1}} \right). \tag{24.7}$$

if $T_m \leq t < T_{m-1}$ and $T_n/\gamma(n) \leq t/d(x,y) < T_{n-1}/\gamma(n-1)$.

Also (24.6) is equivalent to

$$\frac{c_9}{\nu(m)} \exp\left(- c_{10}\frac{T_m}{T_n} \right) \leq p(t,x,y) \leq \frac{c_{11}}{\nu(m)} \exp\left(- c_{12}\frac{T_m}{T_n} \right). \tag{24.8}$$

if $T_m \leq t < T_{m-1}$ and $T_n/\gamma(n) \leq t/d(x,y) < T_{n-1}/\gamma(n-1)$.

Example 24.8. As in Example 23.9, let $\mathcal{L}_1 = \mathcal{L}_{SG}$ and let $\mathcal{L}_2 = \mathcal{L}_{SP}$. We consider homogeneous random Sierpinski gasket generated by $\{\mathcal{L}_1, \mathcal{L}_2\}$. See Figure 4. Fix $h = 1$ and set $\mathbf{r}^{(1)} = (3/5, 3/5, 3/5)$ and $\mathbf{r}^{(2)} = (1/3, 1/3, 1/3, 1/3)$. Then $(D_1, \mathbf{r}^{(j)})$ is a regular harmonic structure on $\mathcal{L}_j$ for $j = 1, 2$. Note that $((D_1, \mathbf{r}^{(j)}))_{j=1,2}$ satisfies the assumption (HG). In this case, $\nu_i^{(1)} = 1/3$ for $i \in S_1$ and $\nu_i^{(2)} = 1/4$ for $i \in S_2$. Also in this case, by Examples 20.11 and 20.12, both $\mathcal{L}_1$ and $\mathcal{L}_2$ admit symmetric self-similar geodesics with the geodesic ratio $\gamma_1 = 1/2$ and $\gamma_2 = 1/3$ respectively. Since $r^{(1)}/N_1 = 1/5 < \gamma_1$ and $r^{(2)}/N_2 = 1/12 < \gamma_2$, we have (24.6), (24.7) and (24.8).

The rest of this section is devoted to proving Theorem 24.7. The existence of a geodesic distance d is shown by similar argument in the proof of [**35**, Theorem 4.3]. Using the same discussion, we obtain the following lemma as well.

Lemma 24.9. *Define* $\overline{\gamma}(m) = \max_{w \in W_m} \mathrm{diam}(K_w, d)$. *Then* $\overline{\gamma}(m) \asymp \gamma(m)$.

Lemma 24.10. *For* $x, y \in K$, *define* $M(x, y) = \inf\{m | y \notin U_{r(m)}(x)\}$. *Then*
(1) *For some* $m_* \in \mathbb{N}$,

$$1 \leq \inf\{n | (x_0, \ldots, x_n) \text{ is an } M(x,y)\text{-path and there exist } w(1), w(2) \in W_{M(x,y)} \text{ such that } x, x_0 \in K_{w(1)} \text{ and } x_n, y \in K_{w(2)}\} \leq m_*$$

for any $x, y \in K$,
(2) $R(x, y) \asymp r(M(x, y))$,
(3) $d(x, y) \asymp \gamma(M(x, y))$

Proof. Note that $\Lambda_{r(k)} = W_k$ for any k. Since $y \in U_{r(m-1)}(x) \backslash U_{r(m)}(x)$ for $m = M(x, y)$, we have (1). Combining (1) and Corollary 22.8, we obtain (2). (3) follows from Lemma 24.9 and (1). □

Lemma 24.11. $d \underset{\mathrm{QS}}{\sim} R$.

Proof. By Lemma 24.10-(2), for any $n \geq 1$, there exists $\delta_n > 0$ such that $R(x, z) \leq \delta_n R(x, y)$ implies $M(x, z) \geq M(x, y) + n$. Fix $\epsilon \in (0, 1)$. By Lemma 24.10-(2), for sufficiently large n, $d(x, z) \leq \epsilon d(x, y)$ whenever $M(x, z) \geq M(x, y) + n$. Hence d is $(\mathrm{SQS})_R$. The similar discussion shows that R is $(\mathrm{SQS})_d$. Hence we have $d \underset{\mathrm{QS}}{\sim} R$ by Theorem 12.3. □

Lemma 24.12. $V_d(x, d(x, y)) \asymp \nu(M(x, y))$.

Proof. Since ν is $(\mathrm{VD})_R$ and $d \underset{\mathrm{QS}}{\sim} R$, ν is $(\mathrm{VD})_d$. Lemma 24.10-(3) implies $V_d(x, d(x, y)) \asymp V_d(x, \gamma(M(x, y))$. Set $m = M(x, y)$. Note that $B_d(x, \gamma(m)) \subseteq U_{r(m)}(x)$. Hence

$$V_d(x, \gamma(m)) \leq \nu(U_{r(m)}(x)) \leq \#(\Lambda^1_{r(m),x})\nu(m) \leq C\nu(m),$$

where C is independent of x and m. On the other hand, if $w \in W_m$ and $x \in K_w$, then $K_w \subseteq B_d(x, \overline{\gamma}(m))$. Combining this with Lemma 24.9, we obtain $\nu(m) = \nu(K_w) \leq V_d(x, \overline{\gamma}(m)) \leq c V_d(x, \gamma(m))$. Thus, we have the desired result. □

Proof of Theorem 24.7. By Lemmas 24.10 and 24.12, we obtain

$$R(x, y) V_d(x, d(x, y)) \asymp T_{M(x,y)} \asymp h(d(x, y)).$$

Hence $(\mathrm{DM2})_{h,g}$ follows. At the same time, h is a monotone function with full range and doubling. Now we have the condition (b) of Theorem 15.10. Moreover, since $r^{(j)}/N_j < \gamma_j$ for any j, Φ is a monotone function with full range and decays uniformly. Hence Theorem 15.10 implies (24.6). $\square$

25. Introducing randomness

Finally in this section, we introduce randomness in the random Sierpinski gaskets. As mentioned before, the Hausdorff measure associated with the resistance metric is almost surely not (equivalent to) a random self-similar measure.

As in the previous section, we fix a family of generalized Sierpinski gaskets $\{\mathcal{L}_1, \ldots, \mathcal{L}_M\}$. Let $\mathcal{L}_j = (K(j), S_j, \{F_i^j\}_{i \in S_j})$, where $S_j = \{1, \ldots, N_j\}$. Set $N = \max_{j=1,\ldots,M} N_j$ and $S = \{1, \ldots, N\}$ as before.

Definition 25.1. Let $\Omega =$

$$\{(W_*, \Gamma) | (W_*, \Gamma) \text{ is a random Sierpinski gasket generated by } \{\mathcal{L}_1, \ldots, \mathcal{L}_M\}\}.$$

Define $\Omega_{w,j} = \{(W_*, \Gamma) | (W_*, \Gamma) \in \Omega, w \in W_*, \Gamma(i) = j\}$ for $i \in W_*(S)$ and $j \in \{1, \ldots, M\}$. Let $\mathcal{B}_m$ be the σ-algebra generated by $\{\Omega_{w,j} | w \in \cup_{n=0}^{m-1} W_n(S), j \in \{1, \ldots, M\}\}$ and define $\mathcal{B} = \cup_{m \geq 1} \mathcal{B}_m$.

For $\omega = (W_*, \Gamma) \in \Omega$, we write $W_*(\omega) = W_*, \Gamma(\omega) = \Gamma$, $K^\omega = K(W_*, \Gamma)$, $W_m^\omega = W_m(W_*, \Gamma)$ and so on.

According to [**28, 29**], we have the following fact.

Proposition 25.2. *Let $(\nu_j)_{j=1,\ldots,M} \in (0,1)^M$ satisfy $\sum_{j=1}^M \nu_j = 1$. Then there exists a probability measure P on $(\Omega, \mathcal{B})$ such that $\{\Omega_{w,j} | w \in W_*(S), j \in \{1, \ldots, M\}\}$ is independent and $P(\Gamma(w) = j | w \in W_*) = \nu_j$ for any $w \in W_*(S)$ and any $j \in \{1, \ldots, M\}$.*

We fix such a probability measure P on $(\Omega, \mathcal{B})$ as in Proposition 25.2.

Now let $(D, \mathbf{r}^{(j)})$ be a regular harmonic structure on $\mathcal{L}_j$ for $j = 1, \ldots, M$. We use $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ to denote the resistance form on K^ω associated with $((D, \mathbf{r}^{(j)}))_{j=1,\ldots,M}$ for $\omega \in \Omega$. In [**28**], Hambly has introduced a probability measure μ on K^ω which is natural from the view point of the resistance metric in the following way.

Definition 25.3. Let $\omega = (W_*, \Gamma) \in \Omega$. Choose $x_w \in K_w^\omega$ for $w \in W_*$. For $n \geq 1$, define

$$\mu_n = \sum_{w \in W_m^\omega} \frac{(r_w)^{-1}}{\sum_{v \in W_m^\omega} (r_v)^{-1}} \delta_{x_w},$$

where δ_x is the Dirac's point mass. Let $\mu = \mu^\omega$ be one of the accumulating points of $\{\mu_n\}$ in the weak sense.

Note that since K^ω is compact, $\{\mu_n\}$ has accumulating points. This measure μ^ω is known to be equivalent to the proper dimensional Hausdorff measure and it is not a random self-similar measure for P-a. s. $\omega \in \Omega$. See [**28, 29**] for details. In [**28, 30**], Hambly and Kumagai have shown some fluctuations in the asymptotic behavior of heat kernels associated with the Dirichlet form $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ on $L^2(K^\omega, \mu^\omega)$ for P-a. s. $\omega \in \Omega$. In particular, by [**30**, Theorem 5.5], we have the following theorem.

Theorem 25.4. *μ^ω is not $(\mathrm{VD})_R$ for P-a. s. ω.*

As in the homogeneous case, a fluctuation of the diagonal behavior of heat kernels from a power law has been shown in [**30**] as well. By the above theorem, however, the fluctuation in this case may be caused by the lack of volume doubling property. (Recall that the volume doubling property always holds in the homogeneous case.) Hence those two fluctuations in homogeneous and non-homogeneous cases are completely different in nature.

PROOF. Using [**30**, Theorem 5.5], we see that (GE) do not hold for P-a. s. ω. Hence by Theorem 23.2, μ^ω is not $(\mathrm{VD})_R$ for P-a. s. ω. $\square$

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Assumptions, Conditions and Properties in Parentheses

List of Notations

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