Time changes of the Brownian motion: Poincaré inequality, heat kernel estimate and protodistance

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Abstract

In this paper, time changes of the Brownian motions on generalized Sierpinski carpets including n-dimensional cube $[0,1]^n$ are studied. Intuitively time change corresponds to alteration to density of the medium where the heat flows. In case of the Brownian motion on $[0,1]^n$, density of the medium is homogeneous and represented by the Lebesgue measure. Our study includes densities which are singular to the homogeneous one. We establish a rich class of measures called measures having weak exponential decay. This class contains measures which are singular to the homogeneous one such as Liouville measures on $[0,1]^2$ and self-similar measures. We are going to show the existence of time changed process and associated jointly continuous heat kernel for this class of measures. Furthermore, we obtain diagonal lower and upper estimates of the heat kernel as time tends to 0. In particular, to express the principal part of the lower diagonal heat kernel estimate, we introduce "protodistance" associated with the density as a substitute of ordinary metric. If the density has the volume doubling property with respect to the Euclidean metric, the protodistance is shown to produce metrics under which upper off-diagonal sub-Gaussian heat kernel estimate and lower near diagonal heat kernel estimate will be shown.

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1. Introduction

The reflected Brownian motion on the n-dimensional cube $[0,1]^n$ is associated with the Dirichlet form

$$\mathcal{E}(u,v) = \int_{[0,1]^n} \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j}\nu_*(dx) = -\int_{[0,1]^n} u\Delta v \nu_*(dx),$$

where ν_* is the Lebesgue measure and $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial {x_j}^2}$ is the Laplacian. In this case, we regard $[0,1]^n$ as a homogeneous medium and consider associated heat flow on it. By introducing inhomogeneous density of a medium, speed of heat flow changes according to the given density at each point while the paths stay the same as the original Brownian motion. To be precise, if $f : [0,1]^n \to [0,\infty)$ gives a density relative to the Lebesgue measure ν_*, then our (inhomogeneous) medium is represented by the measure $f(x)\nu_*(dx)$ and the corresponding Laplacian is identified with $f^{-1}\Delta$. Furthermore, we may even consider a density μ which is singular to the Lebesgue measure ν_*. Such a change of density of a medium, whether it is absolutely continuous to the Lebesgue measure or not, is called a time change of the original process, namely, the Brownian motion. In general, time change may not be possible with respect to every measure and the first step of the study is to determine whether time change is possible or not with respect to a given measure. The abstract theory of time change has been developed in the framework of Dirichlet forms by many authors. See [**20**] and [**15**] for example.

In recent years, there has been much interest in time change of the Brownian motion on $\mathbb{R}^2$ with respect to Liouville measures, which is also called Liouville quantum gravities, associated with Gaussian free fields. In [**21, 22**], the authors have constructed time changes of the Brownian motion with respect to Liouville measures and called them Liouville Brownian motions. The existence of jointly continuous heat kernels associated Liouville Brownian motions, which are called Liouville heat kernels, has been shown as well. The study of asymptotic properties of Liouville heat kernels has been initiated in [**37**] and the “spectral dimension” has been show to be one in [**1**]. As is seen in Theorem 1.1, we will establish a class of measures on $[0,1]^2$ containing Liouville measures and recover some of the results on Liouville measures obtained in [**37**] and [**1**].

In this paper, we study time changes of Brownian motions on generalized Sierpinski carpets. The Brownian motion on a generalized Sierpinski carpet has been constructed and studied by Barlow and Bass [**4, 5, 6, 7, 8, 9**]. As a special case, it includes the reflected Brownian motions on $[0,1]^n$. A generalized Sierpinski carpet K is defined as the invariant set of a collection of finite number of contractions $\{F_i\}_{i\in S}$, i.e.

$$F_i(x) = \frac{1}{l}(x - x_i) + x_i \quad \text{and} \quad K = \bigcup_{i\in S} F_i(K),$$

where S is a finite set and $l \geq 2$ is an integer. We will give the precise definition of generalized Sierpinski carpets in Section 2. Let ν_* be the normalized Hausdorff measure of K. Combining Barlow-Bass's results with the uniqueness of the Brownian motion shown in [**10**], we now know that the local regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\nu_*)$ associated with the Brownian motion is self-similar in the following sense:

$$\mathcal{E}(u,v) = \sum_{i\in S} \frac{1}{r_*}\mathcal{E}(u\circ F_i, v\circ F_i) \tag{1.1}$$

for any $u, v \in \mathcal{F}$. The constant r_* is called the resistance scaling ratio whose value is 2^{n-2} if $K = [0,1]^n$ and $l = 2$. Given the density μ of a medium, which is a Borel regular probability measure in general, we are going to study the following questions:

A When is time change possible?
B Dose time changed process possess a continuous heat kernel?
C What are asymptotic behaviors of the process/the heat kernel?
D Is there any intrinsic "metric" suitable to describe the time changed process?

In this direction, Barlow-Kumagai [**11**] has studied time changes of the Brownian motions on generalized Sierpinski carpets when the density is a self-similar measure. They have determined the condition when time change is possible, shown the existence of jointly continuous heat kernel and studied the pointwise asymptotic behavior of the hear kernel as time tends to 0. In their case, the self-similarity of the measure has played important roles in the study and made their analysis possible.

If $r_* \in (0,1)$, then the quadratic form $(\mathcal{E}, \mathcal{F})$ is known to be a resistance form, which is extensively studied in [**35**]. In this case, there exists intrinsic metric called the resistance metric. For any Borel regular radon measure, time change is possible and the associated jointly continuous heat kernel exists. In particular, if a measure has the volume doubling property with respect to the resistance metric, one can construct a metric which is quasisymmetric to the resistance metric and the heat kernel satisfies sub-Gaussian estimates as (1.7) and (1.8). As a next step, we will address the case when $r_* \geq 1$ in this paper.

Naturally, we are going to start with the question (A). The key roll to answer this question is played by the (0-order) Green function $g(x,y)$. Note that in order to consider time change, we have to modify the original domain $\mathcal{F}$ of the quadratic form $\mathcal{E}$ and obtain a new domain $\mathcal{F}_\mu$ so that $(\mathcal{E}, \mathcal{F}_\mu)$ may become a Dirichlet form on $L^2(K,\mu)$. On the other hand, the Green function, which is the integral kernel of the Dirichlet Laplacian, is unchanged even after time change. By introducing

$$h(x,y) = \begin{cases} -\log|x-y| + C & \text{if } r_* = 1 \\ |x-y|^{-\log r_*/\log l} & \text{if } r_* > 1, \end{cases}$$

which has the same singularity as the Green function, we are going to give a useful sufficient condition in terms of $h(x,y)$ to ensure that time change is possible in Theorem 6.9. In the later sections, the sufficient condition is applied to many examples of interest.

To establish the existence and the continuity of a heat kernel of time changed process, the main tool of our approach is the Poincaré inequality with respect to a given measure μ, that is,

$$\mathcal{E}(u,u) \geq \frac{c_1}{h_\mu(\emptyset)^2}\int_K (u(y) - (u)_\mu)^2 \mu(dy), \tag{1.2}$$

where $(u)_\mu = \frac{1}{\mu(K)} \int_K u(y)\mu(dy)$,

$$h_\mu(\emptyset) = \sup_{x\in K} \int_K h(x,y)\mu(dy)$$

and c_1 is independent of μ and u. In case of self-similar measures studied in [**11**], the Poincarè inequality can be obtained straightforward by combining the self-similarities of both the Dirichlet form, (1.1), and the measure. Without self-similarity of the measure μ, first we need to employ the method developed by Bass in [**13**] to show a weaker version of (1.2). Then by a tricky argument using self-similarity (1.1) of the form, we will manage to get the strong version (1.2) of the Poincaré inequality in Section 9.

Making use of the Poincaré inequality (1.2), we will show Nash type inequality which leads to the existence and the continuity of heat kernel in Section 10. In view of those results, we are going to establish a class of measures called measures having weak exponential decay in Section 11. In particular, a Borel regular probability measure on $[0,1]^2$ has weak exponential decay if and only if there exist $c_1, c_2, \alpha_1, \alpha_2 > 0$ such that

$$c_1 r^{\alpha_1} \le \mu(B_*(x,r)) \le c_2 r^{\alpha_2}, \tag{1.3}$$

for any $x \in [0,1]^2$ and $r \in (0,1]$, where $B_*(x,r) = \{y | y \in [0,1]^2, |x-y| < r\}$. For measures having weak exponential decay, time changed processes do exist, the associated heat semigroups are ultracontractive, and time changed processes possess jointly continuous heat kernels. The collection of measure having weak exponential decay is a rich class containing all the measures having the volume doubling property with respect to the Euclidean metric. It also contains many measures without the volume doubling property. For example it contains the class of statistically random measures studied by Falconer [**18**]. See Section 14 for details. Moreover, due to [**21**, Theorem 2.2] and [**1**, Lemma 3.1], one can confirm that the condition (1.3) is satisfied by Liouville measures on $[0,1]^2$, which has been already mentioned above. See [**21, 22, 37, 1**] regarding the recent studies on Liouville measures.

Concerning asymptotic behaviors of the hear kernel, we will have uniform upper estimate by means of the Nash type inequality in Theorem 10.5. This upper estimate turns out to be the best one when we assume further conditions like (1.4) and (1.5) in Theorem 1.1 on μ. For Liouville measures, however, our general result is not as sharp as what are obtained in the recent works in [**37**] and [**1**]. To study a pointwise lower diagonal estimate of the heat kernels and to give our partial answer to the above question (D), we introduce the quantity $\delta_\mu(x,y)$ called the protodistance[1]. In general a protodistance is not symmetric in x and y. Moreover it does not satisfy triangle inequality. Nevertheless the "ball" $B_{\delta_\mu}(x,r) = \{y | \delta_\mu(x,y) < r\}$ appears naturally in our lower diagonal heat kernel estimate. Namely, we will show in Section 12 that $1/\mu(B_{\delta_\mu}(x,t))$ is the principal part of a lower estimate of $p_\mu(t,x,x)$ for μ-a.e. $x \in K$. See Theorem 1.1 as well. The protodistance $\delta_\mu(x,y)$ roughly corresponds (but is not exactly equal) to $|x-y|^{-\log r_*/\log l}\mu(B_*(x,|x-y|))$, which is denoted by $Q_\mu(x,y)$. In case μ has the volume doubling property, our protodistance is actually bi-Lipschitz equivalent to both $Q_\mu(x,y)$ and a power of

[1] Our protodistance is not related to the notion of protometric given by Deza and Chebotarev in [**16**]

certain metric under which one obtains sub-Gaussian heat kernel estimate. See Theorem 1.2 for example. We present the following theorem for the case of time changes of 2-dimensional Brownian motion as a showcase of our results without the volume doubling property.

THEOREM 1.1. *Let μ be a Borel regular probability measure on $[0,1]^2$. If μ has weak exponential decay, then time change with respect to μ is possible, the time changed process possesses jointly continuous heat kernel $p_\mu(t,x,y)$ on $(0,\infty)\times K\times K$, and there exist $\gamma_*>0$, $\{T_x\}_{x\in[0,1]^2}$ and $c_1>0$ such that $T_x>0$ for μ-a.e. $x\in[0,1]^2$ and*

$$\frac{c_1}{t|\log t|^9}\le\frac{c_1}{\mu(B_{\delta_\mu}(x,\gamma_* t))|\log t|^9}\le p_\mu(t,x,x)$$

for any $t\in(0,T_x]$. Furthermore, if there exists a monotonically non-increasing function $f:(0,\infty)\to[1,\infty)$ such that for any $x\in K$ and $r>0$

$$\mu(B_*(x,2r))\le f(r)\mu(B_*(x,r)), \tag{1.4}$$

where $B_*(x,r)=\{y||x-y|<r\}$ and

$$\lim_{r\downarrow 0}\frac{\log f(r)}{\log r}=0, \tag{1.5}$$

then

$$\lim_{t\downarrow 0}-\frac{\log p_\mu(t,x,x)}{\log t}=1$$

for any $x\in K$.

See Section 13 for the proof of this theorem. It is not known whether (1.4) and (1.5) hold for Liouville measures or not.

Finally from Section 15, we study the case when the density μ has the volume doubling property with respect to the Euclidean metric, i.e.

$$\mu(B_*(x,2r))\le C\mu(B_*(x,r))$$

for any $x\in K$ and $r>0$, where C is independent of x and r. By the preceding works, for example, [**27, 34, 28**], the volume doubling property has been known to be indispensable to sub-Gaussian heat kernel estimates. This is the case in our framework as well. What matters is to find a suitable metric in order to show additional conditions leading to sub-Gaussian heat kernel estimates. Our candidate of such a metric is the protodistance even though it is not a metric. As is mentioned above, however, with the volume doubling property, the protodistance δ_μ has simpler expression Q_μ and is bi-Lipschitz equivalent to a power of certain metric, which is, in fact, the desired metric. More precisely, in case of time changes of the Brownian motion on $[0,1]^n$ for example, our results can be stated as follows;

THEOREM 1.2. *Let μ be a Borel regular measure on $[0,1]^n$. Assume that there exist $c>0$ and $\epsilon>0$ such that*

$$Q_\mu(x,z)\le c\left(\frac{|x-z|}{|x-y|}\right)^\epsilon Q_\mu(x,y) \tag{1.6}$$

whenever $x,y,z\in[0,1]^n$ and $|x-y|\ge|x-z|$ and that μ has the volume doubling property with respect to the Euclidean metric. Define

$$\mathfrak{B}_\mu=\{\beta|(Q_\mu)^{1/\beta}\text{ is bi-Lipschitz equivalent to a metric on }K\}.$$

Then $\mathfrak{B}_\mu = [\beta_*, \infty)$ *or* $\mathfrak{B}_\mu = (\beta_*, \infty)$ *for some* $\beta_* \geq 2$*. Furthermore, for any* $\beta \in \mathfrak{B}_\mu$*, if* d *is a metric which is bi-Lipschitz equivalent to* $(Q_\mu)^{1/\beta}$*, then* d *is quasisymmetric to the Euclidean metric and there exist* $c_1, c_2, c_3, c_4 > 0$ *such that*

$$p_\mu(t,x,y) \leq \frac{c_1}{\mu(B_d(x,t^{1/\beta}))} \exp\left(- c_2 \Big(\frac{d(x,y)^\beta}{t} \Big)^{\frac{1}{\beta-1}} \right). \tag{1.7}$$

for any $x, y \in K$ *and* $t \in (0,\infty)$*. Furthermore if* $d(x,y)^\beta \leq c_3 t$*, then*

$$\frac{c_4}{\mu(B_d(x,t^{1/\beta}))} \leq p_\mu(t,x,y). \tag{1.8}$$

This theorem is obtained as a special case of combination of Theorems 15.7 and 15.11. The condition (1.6) only requires mild decay of μ and it is always fulfilled under the volume doubling condition if $n = 2$. The lower heat kernel estimate (1.8) is called near diagonal lower estimate. It is known that the near diagonal lower estimate is the best possible substitute of off-diagonal lower sub-Gaussian estimate

$$\frac{c_5}{\mu(B_d(x,t^{1/\beta}))} \exp\left(- c_6 \Big(\frac{d(x,y)^\beta}{t} \Big)^{\frac{1}{\beta-1}} \right) \leq p_\mu(t,x,y) \tag{1.9}$$

when the metric does not satisfy the chain condition introduced in Section 15. In fact, if the metric d does satisfy the chain condition, then the volume doubling property of the density μ and (1.8) imply (1.9). See [**34**] for example. In light of the above theorem and the remark about the lower estimate (1.9), we will raise an open problem concerning legitimate definition of "walk" dimension and the "intrinsic" metric associated with a given density μ in Section 15.

The followings are conventions in notations in this paper.

(1) The lower case c (with or without a subscript) represents a constant which is independent of the variables in question and may have different values from place to place (even in the same line).

(2) The constants $c^n_{k.l}, c_{k.l}$ and $m_{k.l}$ where $k, l, n \in \mathbb{N}$ are constants appearing first time in the equation $(k.\, l)$. For example, $c^1_{5.2}, c^2_{5.2}, c^3_{5.2}$ and $c^4_{5.2}$ are constants appearing in (5.2). In particular, $m_{k.l}$ is used for non-negative integer.

(3) For a metric space (X, d), we define $C(X)$ as the collection of continuous functions on X.

2. Generalized Sierpinski carpets

In this section, we introduce the definition of generalized Sierpinski carpet and give fundamental geometric and topological properties of them. The following definition is given by Barlow-Bass[**9**].

Definition 2.1. Let $H_0 = [0,1]^n$, where $n \in \mathbb{N}$, and let $l \in \mathbb{N}$ with $l \geq 2$. Set $\mathcal{Q} = \{\prod_{i=1}^n [(k_i - 1)/l, k_i/l] \,|\, (k_1, \ldots, k_n) \in \{1, \ldots, l\}^n\}$. For any $Q \in \mathcal{Q}$, define $F_Q : H_0 \to H_0$ by $F_Q(x) = x/l + a_Q$, where we choose a_Q so that $F_Q(H_0) = Q$. Let $S \subseteq \mathcal{Q}$ and let $\mathrm{GSC}(n,l,S)$ be the self-similar set with respect to $\{F_Q\}_{Q \in S}$, i.e. $\mathrm{GSC}(n,l,S)$ is the unique nonempty compact set satisfying

$$\mathrm{GSC}(n,l,S) = \bigcup_{Q \in S} F_Q(\mathrm{GSC}(n,l,S)).$$

Set $H_1(S) = \cup_{Q \in S} F_Q(H_0)$. $\mathrm{GSC}(n,l,S)$ is called a generalized Sierpinski carpet if and only if the following four conditions (GSC1), ..., (GSC4) are satisfied:

(GSC1) (Symmetry) $H_1(S)$ is preserved by all the isometries of the unit cube H_0.
(GSC2) (Connected) $\mathrm{int}(H_1(S))$ is connected.
(GSC3) (Non-diagonality) For any $x \in H_1(S)$, there exists $r_0 > 0$ such that $\mathrm{int}(H_1(S) \cap B(x,r))$ is nonempty and connected for any $r \in (0, r_0)$, where $B(x,r) = \{y|y \in \mathbb{R}^n, |x-y| < r\}$.
(GSC4) (Border included) The line segment between 0 and $(1, 0, \ldots, 0)$ is contained in $H_1(S)$.

We define d_* as the restriction of the Euclidean metric of $\mathbb{R}^n$ on the generalized Sierpinski carpet $\mathrm{GSC}(n,l,S)$.

Example 2.2. The standard plane Sierpinski carpet is equal to $\mathrm{GSC}(2,3,S)$, where $S = \mathcal{Q} - \{[1/3, 2/3]^2\}$. Also $[0,1]^n = \mathrm{GSC}(n,l,\mathcal{Q})$ for any $l \geq 2$.

In the rest of this paper, we fix $n \in \mathbb{N}$, $l \geq 2$ and $S \subseteq \mathcal{Q}$ and assume that $\mathrm{GSC}(n,l,S)$ is a generalized Sierpinski carpet. If no confusion may occur, we use K to denote $\mathrm{GSC}(n,l,S)$.

The followings are a standard set of notations on self-similar sets.

Definition 2.3. Let $m \geq 0$. For $w = (w_1, \ldots, w_m) \in \mathcal{Q}^m$, we write $w = w_1 \ldots w_m$ and define $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$ and $H_w = F_w(H_0)$. Moreover, we set

$$\Sigma = S^{\mathbb{N}} = \{\omega | \omega = \omega_1 \omega_2 \ldots, \omega_i \in S \text{ for any } i \in \mathbb{N}\}$$

and

$$W_m = S^m = \{w_1 \ldots w_m | w_i \in S \text{ for } i = 1, \ldots, m\}.$$

In particular, we write $W_0 = \{\emptyset\}$. Set $W_* = \cup_{m \geq 0} W_m$. For $w \in W_*$, we define $|w| = m$ if $w \in W_m$. Define $F_\emptyset$ as the identity map. Moreover, for any $w = w_1 \ldots w_m \in W_*$, define

$$\Sigma_w = \{\omega | \omega = \omega_1 \omega_2 \ldots \in \Sigma, \omega_i = w_i \text{ for any } i \in \{1, \ldots, m\}\}$$

and

$$K_w = F_w(K).$$

The following proposition is well-known. See [**32**, Theorem 1.2.3] for example.

Proposition 2.4. *$\cap_{m \geq 1} K_{\omega_1 \ldots \omega_m}$ is a single point for any $\omega = \omega_1 \omega_2 \ldots \in \Sigma$. Denote the single point by $\pi(\omega)$. Then π is a continuous surjection.*

In fact, the triple $\mathcal{L} = (K, S, \{F_s\}_{s \in S})$ consists a self-similar structure defined in [**32**, Section 1.2]. Here we recall some of basic notions introduced in [**32**] associated with a self-similar structure.

Definition 2.5. The critical set $\mathcal{C}$ and the post critical set $\mathcal{P}$ associated the self-similar structure $\mathcal{L}$ are defined as

$$\mathcal{C} = \bigcup_{Q_1, Q_2 \in S, Q_1 \neq Q_2} \pi^{-1}(K_{Q_1} \cap K_{Q_2})$$

and

$$\mathcal{P} = \bigcup_{m \geq 1} \pi^{-m}(\mathcal{C}).$$

Furthermore, we define $V_0 = \pi(\mathcal{P})$.

The set V_0 is though of as the "boundary" of the self-similar set K. In [**32**, Section 1.2], it is shown that if $w, v \in W_*$ and $\Sigma_w \cap \Sigma_v = \emptyset$, then

$$K_w \cap K_v \subseteq F_w(V_0) \cap F_v(V_0). \tag{2.10}$$

In case of generalized Sierpinski carpets, the boundary V_0 is equal to $K \cap \partial H_0$, where ∂H_0 is the topological boundary of $H_0 = [0,1]^n$ as a subset of $\mathbb{R}^n$.

Proposition 2.6. *Let $I_{i,j} = \{(x_1, \ldots, x_n) | (x_1, \ldots, x_n) \in H_0, x_i = j\}$ for $i = 1, \ldots, n$ and $j = 0, 1$. Define $B_{i,j} = K \cap I_{i,j}$ and $S_{i,j} = \{Q | Q \in S, Q \cap I_{i,j} \neq \emptyset\}$. Then for any $(i,j) = \{1, \ldots, n\} \times \{0,1\}$,*

$$B_{i,j} = \bigcup_{Q \in S_{i,j}} F_Q(B_{i,j}) \quad \text{and} \quad V_0 = \bigcup_{i=1,\ldots,n, j=0,1} B_{i,j}$$

Notation. For a finite set A, $\#(A)$ is the number of elements of A.

Note that $I_{i,j}$ is congruent to $[0,1]^{n-1}$ and

$$\partial H_0 = \bigcup_{i=1,2,\ldots,n, j=0,1} I_{i,j}.$$

Moreover B_{i_1,j_1} and B_{i_2,j_2} are isometric under the natural isometry between I_{i_1,j_1} and I_{i_2,j_2} for any i_1, i_2, j_1, j_2 and hence $\#(S_{i_1,j_1}) = \#(S_{i_2,j_2})$. Define $N = \#(S)$ and $N_B = \#(S_{i,j})$. Then as $S_{i,j} \subseteq S$, it follows that $N > N_B > 1$.

One can easily see the following fact by (2.10).

Lemma 2.7. *Define $V_m = \cup_{w \in W_m} F_w(V_0)$ and $V_* = \cup_{m \geq 0} V_m$. Then for any $x \in K \backslash V_*$, $\pi^{-1}(x)$ is a single point. Moreover,*

$$\sup_{x \in K} \#(\pi^{-1}(x)) \leq 2^n.$$

This lemma shows that the self-similar structure $\mathcal{L} = (K, S, \{F_s\}_{s \in S})$ is strongly finite. See [**34**, Definition 1.2.1] for the definition of strongly finiteness. Furthermore, we have the following fact proven in [**34**, Proposition 3.4.3].

Proposition 2.8. *The self-similar structure $\mathcal{L} = (K, S, \{F_s\}_{s \in S})$ associated with generalized Sierpinski carpet is rationally ramified.*

See [**34**, Definition 1.5.10] for the definition of rationally ramified self-similar structure. This fact enable us to apply results in [**34**] in the following sections.

Definition 2.9. Let $\Gamma \subseteq W_*$. Define

$$K(\Gamma) = \cup_{w \in \Gamma} K_w$$
$$\partial K(\Gamma) = K(\Gamma) \cap \overline{K \backslash K(\Gamma)},$$
$$K^o(\Gamma) = K(\Gamma) \backslash \partial K(\Gamma).$$

Γ is said to be independent if and only if $\Sigma_w \cap \Sigma_v = \emptyset$ for any $w, v \in \Gamma$ with $w \neq v$. If Γ is independent and $\cup_{w \in \Gamma} \Sigma_w = \Sigma$, then Γ is called a partition of Σ.

Definition 2.10. Let $U \subseteq K$. We define $\Gamma_m^k(U) \subseteq W_m$ and $V_m^k(U) \subseteq K$ for $k = 0, 1, \ldots$ inductively by

$$\Gamma_m^0(U) = \{w | w \in W_m, K_w \cap U \neq \emptyset\},$$
$$V_m^k(U) = K(\Gamma_m^k(U)) \quad \text{and} \quad \Gamma_m^{k+1}(U) = \Gamma_m^0(V_m^k(U)).$$

In particular, if $U = \{x\}$ for some $x \in K$, then we write $\Gamma_m(x) = \Gamma_m^1(U)$ and $V_m(x) = V_m^1(U)$.

Remark. $\#(\Gamma_m(x)) \leq 4^n$.

By the above definition and Lemma 2.7, we immediately obtain the next lemma.

Lemma 2.11. *Let μ be a Radon measure on K. If $\Gamma \subseteq W_*$ is independent, then*

$$\int_{K(\Gamma)} f(x)\mu(dx) \leq \sum_{w\in\Gamma} \int_{K_w} f(x)\mu(dx) \leq 2^n \int_{K(\Gamma)} f(x)\mu(dx)$$

for any non-negative function $f \in L^1(K,\mu)$.

To finish this section, we introduce the notion of self-similar measures.

Proposition 2.12. *Let $(\mu_i)_{i\in S} \in (0,1)^S$ satisfy $\sum_{i\in S}\mu_i = 1$. Then there exists a unique Borel regular probability measure μ on K such that*

$$\mu(K_{w_1\dots w_m}) = \mu_{w_1}\cdots\mu_{w_m}$$

for any $w_1\dots w_m \in W_*$. The measure μ is called the self-similar measure with weight $(\mu_i)_{i\in S}$.

3. Standing assumptions and notations

In this section, we note assumptions and notations which are persistent through this paper. As remarked in the last section, we fix $n \in \mathbb{N}, l \geq 2$ and $S \subseteq \mathcal{Q}$ and assume that $\mathrm{GSC}(n,l,S)$ is a generalized Sierpinski carpet. We write $K = \mathrm{GSC}(n,l,S)$ and $N = \#(S)$. Also $\mathcal{L}$ is the self-similar structure associated with K, i.e. $\mathcal{L} = (\mathrm{GSC}(n,l,S), S, \{F_Q\}_{Q\in S})$.

Notation. (1) We use d_* to denote the restriction of the Euclidean metric to K.
(2) For a metric d on K, we define $B_d(x,r)$ as the ball with center x and radius r with respect to d, i.e. $B_d(x,r) = \{y|y \in K, d(x,y) < r\}$. In particular, we write $B_*(x,r) = B_{d_*}(x,r)$.
(2) Define ν_* as the self-similar measure with weight $(1/N,\dots,1/N)$. Define $d_H = \log N/\log l$. Then d_H is the Hausdorff dimension of K with respect to d_* and ν_* is the normalized d_H-dimensional Hausdorff measure.
(3) Let μ be a Borel regular measure on K. We use $||f||_{\mu,p}$ to denote the L^p-norm of $f \in L^p(K,\mu)$. If no confusion may occur, we omit μ in $||f||_{\mu,p}$ and write simply $||f||_p$.

4. Gauge function

In this section, we introduce the notion of a gauge function which was formulated and extensively studied in [**34**] in order to study geometry of self-similar sets. In this paper, gauge functions will play an essential role as a fundamental tool to characterize underlying geometry associated with time change of the Brownian motion. See Section 10 for example.

Definition 4.1. Let $\mathbf{g} : W_* \to (0,1]$. We say that $\mathbf{g}$ is a gauge function on W_* if and only if the following two conditions (G1) and (G2) hold:
(G1) $\mathbf{g}(\emptyset) = 1$ and $0 < \mathbf{g}(wi) \leq \mathbf{g}(w)$ for any $i \in S$ and $w \in W_*$.
(G2) $\sup_{w\in W_m} \mathbf{g}(w) \to 0$ as $m \to 0$
In addition, if
(EL) there exist $\lambda_1, \lambda_2 \in (0,1)$ and $c_1 > 0$ such that $\mathbf{g}(wv) \leq c_1(\lambda_1)^{|v|}\mathbf{g}(w)$ for

any $w, v \in W_*$ and $\mathbf{g}(wi) \geq \lambda_2 \mathbf{g}(wi)$ for any $i \in S$,
then the gauge function $\mathbf{g}$ is said to be elliptic.

If $\mathbf{g}$ is a gauge function, we think of $\{\mathbf{g}(w)\}_{w \in W_*}$ as the "diameters" of K_w's with respect to the gauge function $\mathbf{g}$ even though there may not exist any metric such that $\mathbf{g}(w)$ coincides with the diameter of K_w with respect to the metric.

Note that there exists a natural gauge function associated with a Borel regular probability measure on K. By elementary arguments, we may easily verify the following fact.

PROPOSITION 4.2. *Let μ be a Borel regular probability measure on K. Assume that $\mu(\{x\}) = 0$ for any $x \in K$ and that $\mu(K_w) > 0$ for any $w \in W_*$. Define $\mu(w) = \mu(K_w)$ for any $w \in W_*$. Then $\mu : W_* \to (0, 1]$ is a gauge function.*

DEFINITION 4.3. The gauge function constructed in Proposition 4.2 from a probability measure μ is called the gauge function associated with the measure μ. Furthermore, μ is said to be elliptic if the associated gauge function is elliptic.

In Proposition 4.2, we have done abuse of notation by using μ to denote the gauge function associated with a measure μ. We do this if no confusion can occur.

Next we define a kind of "balls" associated with a gauge function.

DEFINITION 4.4. Let $\mathbf{g}$ be a gauge function on W_*. Define

$$\Lambda_\rho^{\mathbf{g}} = \{w | w = w_1 \dots w_m \in W_*, \mathbf{g}(w_1 \dots w_{m-1}) > \rho \geq \mathbf{g}(w)\}$$

for $\rho \in (0, 1]$ and call $\{\Lambda_\rho^{\mathbf{g}}\}_{\rho \in (0,1]}$ the scale of W_* associated with the gauge function $\mathbf{g}$. For $x \in K$ and $\rho \in (0, 1]$, define

$$\begin{aligned}
\Lambda_\rho^{g}(x) &= \{w | w \in \Lambda_\rho^{\mathbf{g}}, x \in K_w\},\\
K^{\mathbf{g}}(x, \rho) &= \bigcup_{w \in \Lambda_\rho^{\mathbf{g}}(x)} K_w,\\
\Lambda_{\rho,1}^{\mathbf{g}}(x) &= \{w | w \in \Lambda_\rho^{\mathbf{g}}, K_w \cap K_\rho^{\mathbf{g}}(x) \neq \emptyset\},\\
U^{\mathbf{g}}(x, \rho) &= \bigcup_{w \in \Lambda_{\rho,1}^{\mathbf{g}}(x)} K_w.
\end{aligned}$$

Moreover, a gauge function $\mathbf{g}$ is said to be locally finite if

$$\sup_{x \in X, \rho \in (0,1]} \#(\Lambda_\rho^{\mathbf{g}}(x)) < +\infty. \tag{LF}$$

The set $\Lambda_\rho^{\mathbf{g}}$ is the collection of K_w's whose "diameter" under the gauge function $\mathbf{g}$ is almost ρ and the set $U^{\mathbf{g}}(x, \rho)$ is a kind of "ball" with center x and radius ρ. Under certain conditions on gauge function, there exists a distance such that $U^{\mathbf{g}}(x, \rho)$ is (equivalent to) the real ball with respect to the distance. See Section 17 for details.

The following proposition is immediate from the above definition.

PROPOSITION 4.5. *If $\mathbf{g}$ is a gauge function on W_*, then $\Lambda_\rho^{\mathbf{g}}$ is a partition of Σ.*

EXAMPLE 4.6. (1) For $w \in W_*$, define $\mathbf{g}_*(w) = l^{-|w|}$. $\mathbf{g}_*$ is a locally finite elliptic gauge function on W_*. Write $\Lambda_\rho^* = \Lambda_\rho^{\mathbf{g}_*}$, $K^*(x, \rho) = K^{\mathbf{g}_*}(x, \rho)$, $\Lambda_{\rho,1}^*(x) = \Lambda_{\rho,1}^{\mathbf{g}_*}(x)$ and $U^*(x, \rho) = U^{\mathbf{g}_*}(x, \rho)$ for any $\rho \in (0, 1]$ and $x \in K$. Note that there exist $c_1, c_2 > 0$ such that $B_*(x, c_1\rho) \subseteq U^{\mathbf{g}_*}(x, \rho) \subseteq B_*(x, c_2\rho)$ for any $x \in K$ and any $\rho \in (0, 1]$. In this sense, the gauge function $\mathbf{g}_*$ corresponds to d_*, which is the restriction of the Euclidean metric on K. More precisely, according to Definition 17.3, the gauge function $\mathbf{g}_*$ will be said to be adapted to the metric d_*. Note that

$$\Gamma_m(x) = \Lambda^*_{l^{-m},1}(x) \quad \text{and} \quad V_m(x) = U^*(x, l^{-m}). \tag{4.1}$$

for any $x \in K$ and any $m \geq 0$.
(2) The gauge function associated with the self-similar measure ν_* is given by $\nu_*(w) = \nu_*(K_w) = N^{-|w|}$. Recall that $N = \#(S)$. This gauge function ν_* is elliptic. Moreover, for any $w \in W_*$,

$$\mathbf{g}_*(w)^{d_H} = \nu_*(w).$$

In the next definition, we formulate two degrees of similarity among subsets of words.

Definition 4.7. Let Γ_1 and Γ_2 be independent finite subsets of W_*.
(1) We say that Γ_1 and Γ_2 are similar if and only if there exist a bijective map $\psi : \Gamma_1 \to \Gamma_2$ and a similitude $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(x) = l^{-M}x + a$ for some $(M, a) \in \mathbb{Z} \times \mathbb{R}^n$, $\varphi(K(\Gamma_1)) = K(\Gamma_2)$ and $\varphi(K_w) = K_{\psi(w)}$ for any $w \in \Gamma_1$. ψ is called an isomorphism between Γ_1 and Γ_2 and φ is called the similitude associated with ψ. Set

$$n(\Gamma_1, \Gamma_2) = M.$$

We write $\Gamma_1 \sim \Gamma_2$ if and only if Γ_1 and Γ_2 are similar.
(2) We say that Γ_1 and Γ_2 are similar up to their boundaries, or B-similar for short, if and only if Γ_1 and Γ_2 are similar and $\varphi(K^o(\Gamma_1)) = K^o(\Gamma_2)$, where $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is the similitude associated with an isomorphism $\psi : \Gamma_1 \to \Gamma_2$ between Γ_1 and Γ_2. In this case, ψ is called a B-isomorphism between Γ_1 and Γ_2 and φ is called the B-similitude associated with ψ. We write $\Gamma_1 \underset{B}{\sim} \Gamma_2$ if and only if Γ_1 and Γ_2 are B-similar.

The following lemma is straightforward by the above definition.

Proposition 4.8. *Both the relations* $\sim$ *and* $\underset{B}{\sim}$ *are equivalence relations on independent finite subsets of* W_*.

The following theorem will be the key to the proofs of Lemmas 7.5 and 20.3. It ensures finiteness of the equivalent classes under $\sim$ and $\underset{B}{\sim}$ for special subsets of W_*.

Theorem 4.9. *Let* $\mathbf{g}$ *be a gauge function. Assume that* $\mathbf{g}$ *is elliptic and locally finite. Then* $\{\Lambda^{\mathbf{g}}_{\rho,1}(x) | x \in K, \rho \in (0,1]\}/\!\sim$ *and* $\{\Lambda^{\mathbf{g}}_{\rho,1}(x) | x \in K, \rho \in (0,1]\}/\underset{B}{\sim}$ *are finite sets.*

Proof. Note that the self-similar structure $\mathcal{L}$ associated with a generalized Sierpinski carpet is strongly finite and rationally ramified. Therefore, by [**34**, Theorem 2.2.7], $\mathbf{g}$ is intersection type finite. (See [**34**, Definition 2.2.3] for the definition of being intersection type finite.) Since $\Lambda^{\mathbf{g}}_{\rho_1,1}(x) \sim \Lambda^{\mathbf{g}}_{\rho_2,1}(y)$ if and only if $(\rho_1, x) \underset{1}{\sim} (\rho_2, y)$, where $\underset{1}{\sim}$ is defined in [**34**, Definition 2.2.11], the finiteness of $\{\Lambda^{\mathbf{g}}_{\rho,1}(x) | x \in K, \rho \in (0,1]\}/\!\sim$ follows from [**34**, Theorem 2.2.13]. Note that $\Lambda^{\mathbf{g}}_{\rho_1,1}(x) \underset{B}{\sim} \Lambda^{\mathbf{g}}_{\rho_2,1}(y)$ if and only if $\Lambda^{\mathbf{g}}_{\rho_1,1}(x) \sim \Lambda^{\mathbf{g}}_{\rho_2,1}(y)$ and $\varphi(\partial U^{\mathbf{g}}(x, \rho_1)) = \partial U^{\mathbf{g}}(y, \rho_2)$, where φ is the similitude associated with an isomorphism between $\Lambda^{\mathbf{g}}_{\rho_1,1}(x)$ and $\Lambda^{\mathbf{g}}_{\rho_2,1}(y)$. Since $\mathbf{g}$ is intersection type finite, once an equivalence class of $\Lambda^{\mathbf{g}}_{\rho,1}(x)$ is fixed, then there exist only finite number of possibilities in choosing the boundary of $U^{\mathbf{g}}(\rho, x)$. Hence the number of equivalence classes of $\{\Lambda^{\mathbf{g}}_{\rho,x} | x \in K, \rho \in (0,1]\}$ under $\underset{B}{\sim}$ is finite as well. $\square$

5. The Brownian motion and the Green function

In this section, we are going to review the basic results on the Brownian motions on generalized Sierpinski carpets by Barlow-Bass[**4, 5, 6, 7, 8, 9**] and study properties of the associated Green function and Dirichlet heat kernels. As stated in the last section, K is always a generalized Sierpinski carpet, ν_* is the normalized Hausdorff measure and d_* is the (restriction of) Euclidean metric. The following theorem is a collection of Barlow-Bass's results.

THEOREM 5.1. *There exist $r_* \in (0, N)$ and a local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \nu_*)$ such that $u \circ F_Q \in \mathcal{F}$ for any $u \in \mathcal{F}$ and $Q \in S$ and*

$$\mathcal{E}(u, v) = \frac{1}{r_*} \sum_{Q \in S} \mathcal{E}(u \circ F_Q, v \circ F_Q). \tag{5.1}$$

for any $u \in \mathcal{F}$. The diffusion process $(\{X_t\}_{t>0}, \{P_x\}_{x \in K})$ associated with this Dirichlet form is called the Brownian motion of K. Moreover, there is a jointly continuous transition density/heat kernel $p(t, x, y)$ associated with the Brownian motion, i.e. $p(t, x, y)$ is positive and continuous on $(0, \infty) \times K \times K$ and, for any bounded Borel measurable function $f : K \to \mathbb{R}$,

$$E_x(f(X_t)) = \int_K p(t, x, y) f(y) \nu_*(dy)$$

for any $x \in K$ and any $t > 0$. Let

$$d_S = 2\frac{\log N}{\log N - \log r_*} \quad \text{and} \quad d_w = \frac{\log N - \log r_*}{\log l}.$$

Then there exist $c^1_{5.2}, c^2_{5.2}, c^3_{5.2}, c^4_{5.2} > 0$ such that

$$\begin{aligned} \frac{c^1_{5.2}}{t^{d_s/2}} \exp\left(- c^2_{5.2} \left(\frac{|x-y|^{d_w}}{t} \right)^{\frac{1}{d_w - 1}} \right) &\le p(t, x, y) \\ &\le \frac{c^3_{5.2}}{t^{d_s/2}} \exp\left(- c^4_{5.2} \left(\frac{|x-y|^{d_w}}{t} \right)^{\frac{1}{d_w - 1}} \right) \end{aligned} \tag{5.2}$$

for any $t \in (0, 1]$ and $x, y \in K$. Moreover, $(\mathcal{E}, \mathcal{F})$ satisfies elliptic Harnack inequality with respect to d_*, i.e. there exists $c > 0$ such that if u is positive and harmonic on $B_*(x, 2r)$, then

$$\sup_{y \in B_{d_*}(x,r)} u(y) \le c \inf_{y \in B_{d_*}(x,r)} u(y). \tag{5.3}$$

The estimate (5.2) is called sub-Gaussian off-diagonal heat kernel estimate. The constants d_S and d_w are called the spectral dimension and the walk dimension of the generalized Sierpinski carpet respectively. In [**9**], Barlow and Bass have shown the transition density estimate (5.2) for the Brownian motion on a generalized Sierpinski carpet constructed as certain scaling limit of the Brownian motions on certain domains in $\mathbb{R}^n$ approximating to the generalized Sierpinski carpet. Later in [**10**], the self-similarity of the Dirichlet form $(\mathcal{E}, \mathcal{F})$, (5.1), has been established along with the uniqueness of a local regular Dirichlet form with local symmetries. In this paper, $(\mathcal{E}, \mathcal{F})$ is always the unique local regular Dirichlet form on $L^2(K, \nu_*)$ associated with the Brownian motion given in the above theorem. The constant r_* in (5.1) is called the resistance scaling ratio.

As is mentioned in the introduction, if $r_* \in (0, 1)$, then $(\mathcal{E}, \mathcal{F})$ is a resistance form. In such a case, time change has been studied extensively in [**35**]. In this paper, we will study the remaining case. Namely, we always assume that $r_* \geq 1$ hereafter.

By [**9**], we have additional properties of $(\mathcal{E}, \mathcal{F})$ and $p(t, x, y)$ as follows.

Proposition 5.2. *Let H be the non-negative self-adjoint operator associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \nu_*)$ and let $T_t = e^{-Ht}$.*
(1) *$\{T_t\}_{t>0}$ is ultracontractive.*
(2) *There exist $\{\lambda_i^*\}_{i\geq 1}$ and $\{\psi_i\}_{i\geq 1} \subseteq L^2(K, \nu_*)$ such that $\lambda_1^* = 0$, $0 < \lambda_i^* \leq \lambda_{i+1}^*$ for any $i \geq 2$, $\lim_{i\to\infty} \lambda_i^* = \infty$, $\psi_i \in \mathrm{Dom}(H) \cap C(K)$, $\{\psi_i\}_{i\geq 1}$ is a complete orthonormal system of $L^2(K, \nu_*)$ and $H\psi_i = \lambda_i^* \psi_i$ for any i.*
(3)

$$p(t, x, y) = \sum_{i=1}^{\infty} e^{-\lambda_i^* t} \psi_i(x) \psi_i(y) \tag{5.4}$$

where the right-hand side converges uniformly and absolutely on $[L, \infty) \times K \times K$ for any $L > 0$.
(4)

$$\sup_{t\geq 1, (x,y)\in K^2} p(t, x, y) < +\infty.$$

(5) *If $u \in C(K)$, then $||T_t u - u||_\infty \to 0$ as $t \downarrow 0$, where $||f||_\infty = \sup_{x\in K} |f(x)|$ for $f : K \to \mathbb{R}$.*

Next we define the γ-order resolvent kernel $g_\gamma(x, y)$.

Definition 5.3. Let $\gamma > 0$. Define

$$g_\gamma(x, y) = \int_0^\infty e^{-\gamma t} p(t, x, y) dt$$

The resolvent kernel g_γ has singularities at $x = y$. The order of the singularities of g_γ is given by the following function $h(x, y)$.

Definition 5.4. Define

$$\alpha = \frac{\log r_*}{\log l}$$

and

$$h(x, y) = \begin{cases} |x - y|^{-\alpha} & \text{if } \alpha > 0, \\ -\log \dfrac{|x - y|}{\sqrt{n}e} & \text{if } \alpha = 0. \end{cases}$$

Recall that we always assume $r_* \geq 1$. As a consequence, it follows that $\alpha \geq 0$.

Remark. Let $\alpha = 0$. Note that for any $x, y \in K$, $|x - y| \leq \sqrt{n}$ and hence $h(x, y) \geq 1$. Define $h_*(x, y) = \max\{-\log |x - y|, 1\}$. Then there exist $c_1, c_2 > 0$ such that

$$c_1 h_*(x, y) \leq h(x, y) \leq c_2 h_*(x, y)$$

for any $x, y \in K$.

LEMMA 5.5. *For any $\gamma > 0$, there exists $c(\gamma) > 0$ such that*

$$g_\gamma(x, y) \leq c(\gamma)h(x, y)$$

for any $x, y \in K$.

PROOF. This is immediate form (5.2) and (4) of Proposition 5.2. $\square$

Let U be an open subset of K. Next we introduce the Brownian motion which is killed upon exiting U. Define $\mathcal{D}_U = \{u | u \in \mathcal{F} \cap C(K), u|_{K\backslash U} \equiv 0\}$. We define $\mathcal{F}_U$ as the closure of $\mathcal{D}_U$ with respect to the inner-product $\mathcal{E}(u, v) + \int_K uvd\nu_*$. Note that $\mathcal{F}_U \subseteq \mathcal{F}$ and that $u(x) = 0$ for ν_*-a.e. $x \in K\backslash U$. Hence $\mathcal{F}_U$ is regarded as a subspace of $L^2(U, \nu_*|_U)$. Define $\mathcal{E}_U(u, v) = \mathcal{E}(u, v)$ for any $u, v \in \mathcal{F}_U$. Using the results in [**20**, Section 4.4], we see that $(\mathcal{E}_U, \mathcal{F}_U)$ is a local regular Dirichlet form on $L^2(K, \nu_*)$. We denote the diffusion process associated with the Dirichlet form $(\mathcal{E}_U, \mathcal{F}_U)$, which is called the Brownian motion killed upon exiting U, by $(\{X_t^U\}_{t>0}, \{P_x^U\}_{x\in K})$ and the corresponding expectation by $\{E_x^U\}$.

LEMMA 5.6. *Let $\Gamma \subset W_*$ be finite. Assume that $K^o(\Gamma)$ is connected. Let $U = K^o(\Gamma)$. Then the Brownian motion killed upon exiting U has a jointly continuous transition density $p_U(t, x, y)$ on $(0, \infty) \times K \times K$ which satisfies:*
(a)

$$0 < p_U(t, x, y) \leq p(t, x, y)$$

for any $(t, x, y) \in (0, \infty) \times U \times U$.
(b) $p_U(t, x, y) = 0$ *if either $x \notin U$ or $y \notin U$.*
Moreover if $U \neq K$, then

$$g^U(x, y) = \int_0^\infty p_U(t, x, y)dt$$

is continuous on $K \times K$ and positive on $U \times U$. There exists $c_{5.5} > 0$ such that

$$g^U(x, y) \leq c_{5.5}h(x, y) \tag{5.5}$$

for any $x, y \in K$.

The function g^U defined in the above lemma is called the Green function of the domain U.

REMARK. The constant $c_{5.5}$ only depends on Γ. To clarify the dependence, we use $c_{5.5}(\Gamma)$ in place of $c_{5.5}$, if necessary.

DEFINITION 5.7. For a measurable set $U \subseteq K$, τ_U is the exit time from U defined by $\tau_U = \inf\{t | t > 0, X_t \notin U\}$.

PROOF OF LEMMA 5.6. The existence of jointly continuous transition density is due to [**9**, Proposition 6.15]. In fact, the case considered in [**9**, Proposition 6.15] corresponds to the case when Γ is a single word. One can easily adapt, however, the arguments in the proof of [**9**, Proposition 6.14] to our situation. By a similar modification of the arguments in [**9**], it follows that

$$p_U(t, x, y) = \sum_{i\geq 1} e^{-\lambda_i^U t}\psi_i^U(x)\psi_i^U(y)$$

where $\{\lambda_i^U\}_{i\geq 1}$ is a monotonically increasing sequence of non-negative numbers with $\lim_{i\to\infty} \lambda_i^U = +\infty$ and ψ_i^U is an eigenfunction with the eigenvalue λ_i^U of the self-adjoint operator associated with the Dirichlet form $(\mathcal{E}_U, \mathcal{F}_U)$ on $L^2(K, \nu_*)$. Note that the support of ψ_i^U is in $K(\Gamma)$. Moreover, ψ_i^U is continuous on K and $\{\psi_i^U\}_{i\geq 1}$ is a complete orthonormal system of $L^2(K, \nu_*|_U)$. Now, if $p_U(t,x,x) = 0$ for some $x \in U$ and some $s > 0$, then $\psi_i^U(x) = 0$ for any $i \geq 1$. This implies $p_U(t,x,x) = 0$ for any $t > 0$. Since $\int_K p_U(t,x,y)^2 \nu_*(dy) = p(2t,x,x)$, it follows that $p(t,x,y) = 0$ for any $y \in K$ and any $t > 0$. On the other hand, the same argument as in the proof of [**9**, Proposition 6.20] shows that $p_U(t,x,y) > 0$ if $|x-y|$ is small enough. Therefore, $p_U(t,x,x) > 0$ for any $t > 0$ and any $x \in U$. Now, the same discussion as in the proof of [**32**, Proposition 5.1.10] yields the positivity of $p_U(t,x,y)$ if both x and y belong to U.

Next assume that $K \neq U$. Then $\nu_*(K\backslash U) > 0$. This implies

$$\inf_{x\in U} \int_{K\backslash U} p(t,x,y)\nu_*(dy) > 0.$$

Denote the above infimum by $\delta(t)$. Then

$$P_x(\tau_U > t) = \int_K p_U(t,x,y)\nu_*(dy) \leq 1 - \int_{K\backslash U} p(t,x,y)\nu_*(dy) \leq 1 - \delta(t) < 1.$$

By the Markov property,

$$\int_K p_U(kt,x,y)\mu(dy) \leq (1-\delta(t))^k.$$

for any $x \in K$. Hence as $k \to \infty$,

$$p_U(2kt,x,x) = \int_K p_U(kt,x,y)^2\mu(dy) \leq c\int_K p_U(kt,x,y)\mu(dy) \to 0,$$

where $c = \sup_{x,y\in K} p_U(t,x,y)$. On the other hand, if $\lambda_i^U = 0$ for some i, then $p_U(kt,x,x) \geq \psi_i^U(x)^2$ for any $k \geq 0$. Therefore, we conclude that $\lambda_1^U > 0$. This shows that there exists $\lambda > 0$ such that

$$p_U(t,x,y) \leq Ce^{-\lambda t}$$

for any $x, y \in K$ and $t \geq 1$. Combining this fact with the transition density estimate (5.2), we obtain the continuity and the estimate (5.5) of $g^U(x,y)$. □

Strictly speaking, $p_U(t,x,y)$ and $g^U(x,y)$ is defined only when U is an open set. We abuse notations, however, and write $p_{K(\Gamma)}(t,x,y) = p_{K^o(\Gamma)}(t,x,y)$ and $g^{K(\Gamma)}(x,y) = g^{K^o(\Gamma)}(x,y)$.

In the rest of this section, we are going to investigate properties of the heat kernel $p_{B_*(x,R)}(t,x,y)$ and the Green function $g^{B_*(x,R)}(x,y)$ near the diagonal part $\{(x,x)|x \in K^2\}$.

LEMMA 5.8. *There exist $c_{5.6}^1 > 0$ and $c_{5.6}^2 \in (0, \frac{1}{2}]$ such that if $R \leq \mathrm{diam}(K, d_*)$, $x \in K$ and $|x-y| \leq c_{5.6}^2 R$, then*

$$\frac{c_{5.6}^1}{t^{d_S/2}} \leq p_{B_*(x,R)}(t,x,y) \tag{5.6}$$

for any $t \in \Big[|x-y|^{d_w}, 2(c_{5.6}^2 R)^{d_w}\Big]$.

PROOF. Let $\overline{R} = \mathrm{diam}(K, d_*)$. By the heat kernel estimate (5.2), standard arguments as in [**24**] and [**34**] imply that there exist $c_{5.7}^1, c_{5.7}^2 > 0$ such that

$$P_x(\tau_{B_*(x,r)} \leq t) \leq c_{5.7}^1 \exp\left(-c_{5.7}^2\Big(\frac{r^{d_w}}{t}\Big)^{\frac{1}{d_w-1}}\right) \tag{5.7}$$

for any $r \in (0, \overline{R}]$ and $t > 0$. If $R \le \overline{R}$ and $y \in B_*(x, R/2)$, then by [**24**, Theorem 10.4], it follows that

$$p(t,x,y) \le p_{B_*(x,R)}(t,x,y) + P_x(\tau_{B_*(x,R)} \le t/2) \sup_{s\in[t/2,t]} \sup_{v \in B_*(x,R+\epsilon)} p(t,v,y) \\ + P_y(\tau_{B_*(y,R/2)} \le t/2) \sup_{s\in[t/2,t]} \sup_{u \in B_*(y,R/2+\epsilon)} p(t,x,u).$$

Using (5.2) and (5.7) and letting $\epsilon \to 0$, we see that there exist positive constants $c^1_{5.8}$ and $c^2_{5.8}$ determined by $c^3_{5.2}, c^1_{5.7}$ and $c^2_{5.7}$ such that

$$(5.8)\quad \frac{c^1_{5.2}}{t^{d_s/2}} \exp\left(- c^2_{5.2}\left(\frac{|x-y|^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right) \\ \le p_{B_*(x,R)}(t,x,y) + \frac{c^1_{5.8}}{t^{d_s/2}} \exp\left(- c^2_{5.8}\left(\frac{R^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right)$$

for any $t \in (0, 1]$. Choose positive δ so that

$$(5.9)\quad c^1_{5.8} \exp\left(- c^2_{5.8}\delta^{\frac{d_w}{d_w-1}}\right) \le \frac{c^1_{5.2}}{2} \exp\left(- c^2_{5.2}\right)$$

$$(5.10)\quad \max\left\{(\overline{R})^{d_w}, 2^{d_w-1}\right\} \le \delta.$$

Set $c^1_{5.6} = \frac{c^1_{5.2}}{2} \exp\left(- c^2_{5.2}\right)$. By (5.8) and (5.9), if $\frac{R^{d_w}}{t} \ge \delta$ and $\frac{|x-y|^{d_w}}{t} \le 1$, i.e. if $t \in \left[|x-y|^{d_w}, \frac{R^{d_w}}{\delta}\right]$, then (5.6) holds. Set $c^2_{5.6} = \left(\dfrac{1}{2\delta}\right)^{1/d_w}$. Since $0 < R \le \overline{R}$, (5.10) implies that $\frac{R^{d_w}}{\delta} = 2(c^2_{5.6}R)^{d_w} \le 1$. Also by (5.10) we have $c^2_{5.6} \le 1/2$. □

LEMMA 5.9. *Let $\overline{R} = \mathrm{diam}(K, d_*)$.*
(1) *Suppose $\alpha > 0$. There exists $c_{5.11} > 0$ such that if $R \le \overline{R}$, $x \in K$ and $|x-y| \le c^2_{5.6}R$, then*

$$(5.11)\quad c_{5.11} h(x,y) \le g^{B_*(x,R)}(x,y).$$

(2) *Suppose $\alpha = 0$. There exists $c_{5.12} > 0$ such that if $R \le \overline{R}$, $x \in K$ and $|x-y| \le c^2_{5.6}R$, then*

$$(5.12)\quad c_{5.12} h\left(\frac{x}{c^2_{5.6}R}, \frac{y}{c^2_{5.6}R}\right) \le g^{B_*(x,R)}(x,y)$$

PROOF. If $|x-y| \le c^2_{5.6}R$, then $2|x-y|^{d_w} \le 2(c^2_{5.6}R)^{d_w}$. Hence by Lemma 5.9, (5.6) holds for $t \in [|x-y|^{d_w}, 2|x-y|^{d_w}]$. This implies

$$(5.13)\quad g^{B_*(x,R)}(x,y) = \int_0^\infty p_{B_*(x,R)}(t,x,y)dt \ge \int_{|x-y|^{d_w}}^{2|x-y|^{d_w}} \frac{c^1_{5.6}}{t^{d_S/2}}dt$$

Assume $\alpha > 0$. Recall that $\alpha = d_w(d_S/2 - 1)$. Hence by (5.13), we immediately obtain (5.11). If $\alpha = 0$, then $d_S = 2$ and (5.13) implies $g^{B_*(x,R)}(x,y) \ge \log 2$. Since (5.6) holds for $t \in [|x-y|^{d_w}, 2(c^2_{5.6}R)^{d_w}]$, we see that

$$g^{B_*(x,R)}(x,y) \ge \int_{|x-y|^{d_w}}^{2(c^2_{5.6}R)^{d_w}} \frac{c^1_{5.6}}{t}dt = -d_w \log \frac{|x-y|}{c^2_{5.6}R} + \log 2.$$

Now (5.12) follows by a routine calculation. □

6. Time change of the Brownian motion

In this section, we are going to study when time change of the Brownian motion is possible with respect to given measure μ. The main tool is the potential theory of Dirichlet forms developed, for example, in [**20**]. As in the last section, $K = \mathrm{GSC}(n, l, S)$ is a generalized Sierpinski carpet and $(\mathcal{E}, \mathcal{F})$ is the Dirichlet form associated with the Brownian motion on K. Moreover, we only consider the case $r_* \geq 1$ and hence $\alpha = \log r_* / \log l \geq 0$.

In the rest of this paper, we assume the following.

Assumption 6.1. $l \geq 4$

This is a technical assumption to make several statements, conditions and proofs simple. Even if $l = 2$ or 3, by modifying technical arguments properly, all the theorems in the rest of this paper hold without any change.

Under the above assumption, we have the following fact which makes our discussion slightly simpler.

Lemma 6.2. *For any $s \in S$, there exists $s' \in S$ such that $K_s \cap K_{s'} = \emptyset$.*

The quantity $h_\mu(w)$ defined below plays a crucial role in this paper. Since $h(x, y)$ has the same singularity as the Green function, $h_\mu(w)$ corresponds roughly to the exit time from K_w. Accordingly, the square of $h_\mu(\emptyset)$ will be shown to behave as the Poincaré constant in Theorem 9.1.

Definition 6.3. Let μ be a Borel regular probability measure on K. We define

$$h_\mu(w) = \sup_{x \in K_w} \int_{K_w} h(x, y)\mu(dy).$$

for $w \in W_*$. Moreover, define a Borel regular probability measure μ_w on K by $\mu_w(A) = \mu(F_w(A))/\mu(K_w)$ for any Borel set $A \subseteq K$. Set

$$\begin{aligned}\mathcal{M}_P(K) = \{\mu | \mu &\text{ is a Borel regular probability measure on } K,\\ &\mu(K_w) > 0 \text{ for any } w \in W_*,\ \mu(\{x\}) = 0 \text{ for any } x \in K\\ &\qquad\qquad\qquad\qquad\text{and } h_\mu(\emptyset) < +\infty\}\end{aligned}$$

Note that if $\mu \in \mathcal{M}_P(K)$, then $h_\mu(w) < +\infty$ for any $w \in W_*$.

We immediately have the following lemmas by direct calculations.

Lemma 6.4. (1) *If $\alpha > 0$, then $h(F_w(x), F_w(y)) = (r_*)^{|w|} h(x, y)$ for any $w \in W_*$ and $x, y \in K$.*
(2) *If $\alpha = 0$, then*

$$h(x, y) = h(F_w(x), F_w(y)) - |w| \log l$$

for any $w \in W_*$ and $x, y \in K$.

Lemma 6.5.

$$h_{\mu_w}(\emptyset) = \begin{cases} \dfrac{1}{(r_*)^{|w|}\mu(K_w)} h_\mu(w) & \text{if } \alpha > 0,\\ \dfrac{1}{\mu(K_w)} h_\mu(w) - |w| \log l & \text{if } \alpha = 0 \end{cases}$$

for any $\mu \in \mathcal{M}_P(K)$ and $w \in W_*$.

PROOF. Note that

$$\int_K h(x,y)\mu_w(dy) = \frac{1}{\mu(K_w)}\int_{K_w} h(x,(F_w)^{-1}(y))\mu(dy).$$

Now Lemma 6.4 suffices. □

LEMMA 6.6. *Let ν be a Borel regular measure on K with $\nu(K) < +\infty$. If $\int_K h(x,y)\nu(dx)\nu(dy) < +\infty$, then ν is of energy finite integrals or equivalently, belongs to the class S_0, which is defined in* [**20**, Section 2.2]. *Moreover, if $h_\nu(\emptyset) < +\infty$, then $\nu \in S_{00}$ and, for any compact subset M of K,*

$$\mathrm{Cap}(M) \geq \frac{\nu(M)}{\sup_{x\in M}\int_M g_1(x,y)\nu(dy)}, \tag{6.1}$$

where Cap(·) *is the* 1*-capacity defined in* [**20**, Section 2.1].

PROOF. Set $(G_\gamma\nu)(x) = \int_K g_\gamma(x,y)\nu(dy)$ and let $a_i = \int_K (G_\gamma\nu)(x)\psi_i(x)\nu_*(dx)$, where ψ_i is an eigenfunction of H appearing in Proposition 5.2. Then by Fubini's theorem,

$$\begin{aligned}a_i = \int_0^\infty \int_K\int_K & e^{-\gamma t}p(t,x,y)\psi_i(x)\nu_*(dx)\nu(dy)dt \\ &= \int_0^\infty \int_K e^{-(\gamma+\lambda_i^*)}\psi_i(y)\nu(dy) = \frac{1}{\gamma+\lambda_i^*}\int_K \psi_i(y)\nu(dy).\end{aligned}$$

Since the convergence in (5.4) is uniform, it follows that

$$\int_L^\infty e^{-\gamma t}p(t,x,y)dt\nu(dx)\nu(dy) = \sum_{i=1}^\infty \frac{e^{-(\gamma+\lambda_i^*)L}}{\gamma+\lambda_i^*}\left(\int_K \psi_i(x)\nu(dx)\right)^2.$$

Letting $L \downarrow 0$, we obtain

$$\int_K g_\gamma(x,y)\nu(dx)\nu(dy) = \sum_{i=1}^\infty \frac{1}{\gamma+\lambda_i^*}\left(\int_K \psi_i(x)\nu(dx)\right)^2 < +\infty$$

This implies

$$\sum_{i=1}^\infty \lambda_i^*(a_i)^2 \leq \int_K g_\gamma(x,y)\nu(dx)\nu(dy) < +\infty.$$

Therefore, $G_\gamma\nu \in \mathcal{F}$. Let $u = \sum_{i\geq 1} b_i\psi_i \in L^2(K,\nu_*)$.

$$\mathcal{E}_\gamma(G_\gamma\nu, T_tu) = \sum_{i\geq 1}(\lambda_i^*+\gamma)a_ib_ie^{-\lambda_i^*t} = \sum_{i\geq 1}\int_K e^{-\lambda_i^*t}b_i\psi_i(x)\nu(dx)$$

Now by the same argument as in the proof of Lemma 10.9, there exist $a, c > 0$ such that $||\psi_i||_\infty \leq c(\lambda_i^*)^a$ for any $i \geq 1$. Note that $|b_i| \leq ||u||_2$ for any $i \geq 1$ by the Schwartz inequality. This implies that $\sum_{i\geq 1} e^{-\lambda_i^*t}b_i\psi_i$ converges uniformly on K for any $t > 0$. Therefore,

$$\mathcal{E}_\gamma(G_\gamma\nu, T_tu) = \int_K (T_tu)(x)\nu(dx)$$

In particular if $u \in \mathcal{F}\cap C(K)$, then by Proposition 5.2-(5), we have

$$\mathcal{E}_\gamma(G_\gamma\nu, u) = \int_K u(x)\nu(dx).$$

By [**20**, (2.2.2)], we see that $\nu \in S_0$. If $\sup_{x\in K}\int_K h(x,y)\nu(dy) < +\infty$, then Lemma 5.5 shows that $G_\gamma\nu$ is bounded for any $\gamma > 0$. This immediately implies $\nu \in S_{00}$. If M is compact, then ν_M belongs to S_{00} as well, where $\nu_M(A) = \nu(A\cap M)$ for any ν-measurable set A. By [**20**, Problem 2.2.2], we obtain (6.1). □

DEFINITION 6.7. Define $\mathcal{M}_P^{TC}(K)$ by

$$\mathcal{M}_P^{TC}(K) = \{\mu|\mu \in \mathcal{M}_P(K) \text{ and } P_x(\tau_Y) = 0 \text{ for any } x \in K, \\ \text{where } Y \text{ is the quasi-support of } \mu.\}$$

Now we use the theory of Dirichlet form and see that time change is possible if $\mu \in \mathcal{M}_P^{TC}(K)$.

THEOREM 6.8. *If $\mu \in \mathcal{M}_P^{TC}(K)$, then we have a local regular Dirichlet form $(\mathcal{E}, \mathcal{F}_\mu)$ on $L^2(K,\mu)$ corresponding the time change of the Brownian motion. More precisely, let $\mathcal{F}_\mu$ be the completion of $\mathcal{F}\cap C(K)$ with respect to the inner product $\mathcal{E}_{\mu,1}(u,v) = \mathcal{E}(u,v) + \int_K uvd\mu$. Then $(\mathcal{E}, \mathcal{F}_\mu)$ is a local regular Dirichlet form on $L^2(K,\mu)$.*

PROOF. By Lemma 5.5, we have $g_1(x,y) \le \gamma(1)h(x,y)$ for any $x, y \in K$. This shows $\sup_{x\in K}\int_K g_1(x,y)\mu(dy) \le \gamma(1)h_\mu(\emptyset)$. Hence by Lemma 6.6, we see that

$$\gamma(1)h_\mu(\emptyset)\mathrm{Cap}(M) \ge \mu(M)$$

for any compact set $M \subseteq K$. So μ charges no set of 0 capacity. Moreover, $P_x(\tau_Y = 0) = 1$ for any $x \in K$, where Y is the quasisupport of μ. Using these facts and the general theory of Dirichlet forms in [**20**], we verify that $(\mathcal{E}, \mathcal{F}_\mu)$ is a local regular Dirichlet form on $L^2(K,\mu)$. See detailed discussion after [**11**, Lemma 2.5]. □

We use $(\{X_t^\mu\}_{t>0}, \{P_x^\mu\}_{x\in K})$ to denote the diffusion process associated with the Dirichlet form $(\mathcal{E}, \mathcal{F}_\mu)$ on $L^2(K,\mu)$ and E_x^μ to denote the corresponding expectation. Let U be an open subset of K. We define $\mathcal{F}_{U,\mu}$ be the closure of $\mathcal{D}_U$ with respect to the inner-product $\mathcal{E}(u,v) + \int_K uvd\mu$. Note that $\mathcal{F}_{U,\mu} \subseteq \mathcal{F}_\mu$ and that $u(x) = 0$ for μ-a.e. $x \in K\backslash U$. Hence $\mathcal{F}_{U,\mu}$ is regarded as a subspace of $L^2(U, \mu|_U)$. Define $\mathcal{E}_{U,\mu}(u,v) = \mathcal{E}(u,v)$ for any $u, v \in \mathcal{F}_{U,\mu}$. Using the results in [**20**, Section 4.4], we see that $(\mathcal{E}_{U,\mu}, \mathcal{F}_{U,\mu})$ is a local regular Dirichlet form on $L^2(U, \mu|_U)$. We denote the diffusion process associated with the Dirichlet form $(\mathcal{E}_{U,\mu}, \mathcal{F}_{U,\mu})$ by $(\{X_t^{U,\mu}\}_{t>0}, \{P_x^{U,\mu}\}_{x\in U})$ and the expectation by $\{E_x^{U,\mu}\}$.

The next theorem gives a sufficient condition for a measure μ to belong to $\mathcal{M}_P^{TC}(K)$.

THEOREM 6.9. *Let $\mu \in \mathcal{M}_P(K)$. If*

$$\begin{cases} \displaystyle\sum_{m=0}^{\infty} \inf_{s\in S} \frac{\mu(K_{\omega_1\dots\omega_m s})(r_*)^m}{h_\mu(\omega_1\dots\omega_m s)} = +\infty & \text{if } \alpha > 0, \\ \displaystyle\sum_{m=0}^{\infty} m \inf_{s\in S} \frac{\mu(K_{\omega_1\dots\omega_m s})}{h_\mu(\omega_1\dots\omega_m s)} = +\infty & \text{if } \alpha = 0 \end{cases} \tag{6.2}$$

for any $\omega = \omega_1\omega_2\dots \in \Sigma$, then $\mu \in \mathcal{M}_P^{TC}(K)$.

To prove the above theorem we need the following lemma, which is a consequence of Assumption 6.1.

LEMMA 6.10. *Let $x \in K$ and let $w \in W_m$. Suppose $x \in K_w$. Then there exists $s \in S$ such that $K_{ws} \cap V_{m+2}(x) = \emptyset$.*

PROOF. There exists $i \in S$ such that $x \in K_{wi}$. By Lemma 6.2, we find $s \in S$ which satisfies $K_{wi} \cap K_{ws} = \emptyset$. Then since $l \geq 4$, it follows that $K_{ws} \cap V_{m+2}(x) = \emptyset$. □

PROOF OF THEOREM 6.9. First we show that if A is a measurable set with $\mu(A) = 1$, then for any $x \in K$,

$$\begin{cases} \sum_{m\geq 1} (r_*)^m \mathrm{Cap}(A_m(x)) = +\infty & \text{if } \alpha > 0, \\ \sum_{m\geq 1} m\mathrm{Cap}(A_m(x)) = +\infty & \text{if } \alpha = 0, \end{cases} \tag{6.3}$$

where $A_m(x) = A \cap (V_{m-2}(x) \backslash V_m(x))$ for $m \geq 2$.

Assume that $\alpha > 0$. There exists a compact subset M of $A \cap K_w$ such that $\mu(M) \geq \mu(A \cap K_w)/2$. Note that $\mu(K_w) = \mu(K_w \cap A)$. Then

$$\frac{1}{\mu(M)} \int_M g_1(x, y)\mu(dy) \leq c(1)\frac{1}{\mu(M)} h_\mu(w) \leq 2c(1)\frac{h_\mu(w)}{\mu(K_w)}$$

By (6.1), we obtain

$$\mathrm{Cap}(K_w \cap A) \geq \mathrm{Cap}(M) \geq \frac{1}{2c(1)} \frac{\mu(K_w)}{h_\mu(w)}$$

for any $w \in W_*$.

Fix $\omega \in \Sigma$ which satisfies $\pi(\omega) = x$. By (6.2), we have either

$$\sum_{k=0}^{\infty} \inf_{s\in S} \frac{\mu(K_{\omega_1\ldots\omega_{2k+1}s})(r_*)^{2k+1}}{h_\mu(\omega_1 \ldots \omega_{2k+1}s)} = +\infty$$

or

$$\sum_{k=0}^{\infty} \inf_{s\in S} \frac{\mu(K_{\omega_1\ldots\omega_{2k}s})(r_*)^{2k}}{h_\mu(\omega_1 \ldots \omega_{2k}s)} = +\infty$$

Assume the latter. By Lemma 6.10, for any $k \geq 1$, there exists $i \in S$ such that $K_{\omega_1\ldots\omega_{2k-2}i} \subseteq V_{2k-2}(x)\backslash V_{2k}(x)$. Set $w(k) = \omega_1 \ldots \omega_{2k-2}$. Then

$$\begin{aligned} (r_*)^{2k}\mathrm{Cap}(A_{2k}(x)) \geq (r_*)^{2k}\mathrm{Cap}(A \cap K_{w(k)i}) &\geq \frac{1}{2c(1)} \frac{\mu(K_{w(k)i})(r_*)^{2n}}{h_\mu(w(k)i)} \\ &\geq \frac{1}{2c(1)} \inf_{s\in S} \frac{\mu(K_{w(k)s})(r_*)^{2k}}{h_\mu(w(k)s)} \end{aligned}$$

Summing these up from $k = 1$ to ∞, we obtain (6.3). The case $\alpha = 0$ can be shown by entirely the same arguments.

Now, (6.3) enable us to use the classical Wiener test argument and show that $P_x(\tau_Y) = 0$ for any $x \in K$, where Y is the quasi-support of μ. See detailed discussion after [**11**, Lemma 2.5] for reference. □

The rest of this section is devoted to finding more effective sufficient condition for (6.2). In the next lemma, we obtain an estimate of $h_\mu(w)$ by means of order of decay of $\mu(K_w)$ as $|w| \to \infty$.

LEMMA 6.11. *Let $w \in W_*$. Assume that there exists $f_w : \mathbb{N} \to (0,1)$ such that*

$$\mu(K_{wv}) \le f_w(|v|)\mu(K_w) \tag{6.4}$$

for any $v \in W_*$. If $\alpha > 0$, then

$$h_\mu(w) \le c_{6.5}\mu(K_w)(r_*)^{|w|}\sum_{k=0}^{\infty}(r_*)^k f_w(k). \tag{6.5}$$

If $\alpha = 0$, then

$$h_\mu(w) \le c_{6.6}\mu(K_w)\Big((|w|+1)\sum_{k=0}^{\infty} f_w(k) + \sum_{k=1}^{\infty} k f_w(k)\Big). \tag{6.6}$$

The constants $c_{6.5}$ and $c_{6.6}$ are independent of μ and w.

PROOF. Note that

$$|x-y| \ge l^{-m}. \tag{6.7}$$

for any $x \in K$, $m \ge 0$ and $y \notin V_m(x)$. Write $m = |w|$. Assume $\alpha > 0$. By (6.7), we have

$$\begin{aligned}\int_{K_w} h(x,y)\mu(dy) &= \sum_{k=0}^{\infty}\int_{K_w\cap V_{m+k}(x)\backslash V_{m+k+1}(x)} h(x,y)\mu(dy)\\ &\le \sum_{k=0}^{\infty} l^{\alpha(m+k+1)}\mu(V_{m+k}(x)\cap K_w) \le 4^n N\mu(K_w)\sum_{k=0}^{\infty} l^{\alpha(m+k+1)} f_w(k)\\ &\le 4^n N\mu(K_w)(r_*)^{|w|} l^{\alpha}\sum_{k=0}^{\infty} l^{\alpha k} f_w(k)\end{aligned}$$

for any $x \in K_w$. The constant $4^n N$ appears in the above inequality because $\{v|v \in W_{m+k+1}, K_v \subseteq V_{m+k}(x)\}$ contains at most $4^n N$ elements. If $\alpha = 0$,

$$\begin{aligned}\int_{K_w} h(x,y)\mu(dy) &= \sum_{k=0}^{\infty}\int_{K_w\cap V_{m+k}(x)\backslash V_{m+k+1}(x)} h(x,y)\mu(dy)\\ &\le \sum_{k=0}^{\infty}\big((m+k+1)\log l + \log\sqrt{n}e\big)\mu(V_{m+k}(x)\cap K_w)\\ &\le 4^n N\mu(K_w)\sum_{k=0}^{\infty}\big(m\log l + (k+1)\log l\big) f_w(k)\\ &\le 4^n N(\log l)\mu(K_w)\Big((m+1+\log\sqrt{n}e)\sum_{k=0}^{\infty} f_w(k) + \sum_{k=1}^{\infty} k f_w(k)\Big).\end{aligned}$$

□

The following lemma gives a simple sufficient condition for (6.2).

LEMMA 6.12. *Let $\mu \in \mathcal{M}_P(K)$. Let $\{\rho_m\}_{m\ge 0}$ satisfy*

$$\max_{w\in W_m, s\in S}\frac{\mu(K_{ws})}{\mu(K_w)} \le \rho_m$$

for any $m \geq 0$. *Set* $\delta_m = \rho_0\rho_1 \cdots \rho_{m-1}$ *for* $m \geq 1$. *If*

$$
\begin{cases} \displaystyle\sum_{k=1}^{\infty} k\delta_k < +\infty & \text{in case } \alpha = 0, \\ \displaystyle\sum_{k\geq 0} (r_*)^k \delta_k < +\infty & \text{in case } \alpha > 0, \end{cases} \tag{6.8}
$$

then (6.2) *is satisfied and hence* $\mu \in \mathcal{M}_P^{TC}(K)$. *Moreover, if* (6.8) *is satisfied, then*

$$
h_\mu(w) \leq \begin{cases} c_{6.6} \displaystyle\sum_{k\geq m} (k+1)\delta_k & \text{in case } \alpha = 0, \\ c_{6.5} \displaystyle\sum_{k\geq 0} (r_*)^{|w|+k} \delta_{|w|+k} & \text{in case } \alpha > 0 \end{cases} \tag{6.9}
$$

and

$$
h_{\mu_w}(\emptyset) \leq \begin{cases} c_{6.6} \dfrac{\sum_{k\geq 0}(k+1)\delta_{|w|+k}}{\delta_{|w|}} & \text{in case } \alpha = 0, \\ c_{6.5} \dfrac{1}{(r_*)^{|w|}\delta_{|w|}} \displaystyle\sum_{k=0}^{\infty} (r_*)^{|w|+k} \delta_{|w|+k} & \text{in case } \alpha > 0. \end{cases} \tag{6.10}
$$

As is shown in Example 8.7, if (6.8) is satisfied, then the resolvent operator associated with the time changed process is a compact operator on $L^\infty(K, \mu)$.

Proof. We present a proof for the case $\alpha = 0$. For the case $\alpha > 1$, the results follow by entirely analogous discussion using (6.5). If $\alpha = 0$, then

$$
\mu(K_{wv}) \leq \frac{\delta_{|w|+|v|}}{\delta_{|w|}} \mu(K_w) \tag{6.11}
$$

for any $w, v \in W_*$. By Lemma 6.11, if $m = |w|$, we have

$$
\begin{aligned} h_\mu(w) &\leq c_{6.6} \frac{(m+1)\sum_{k\geq 0} \delta_{m+k} + \sum_{k\geq 1} k\delta_{m+k}}{\delta_m} \mu(K_w) \\ &\leq c_{6.6} \frac{\sum_{k\geq m}(k+1)\delta_k}{\delta_m} \mu(K_w) \leq c_{6.6} \sum_{k\geq m} (k+1)\delta_k. \end{aligned} \tag{6.12}
$$

Since $\rho_m \geq 1/N$, it follows that

$$
\begin{aligned} \sum_{k\geq m} (k+1)\delta_k &= \sum_{k\geq m+1} k\delta_k + \sum_{k\geq m+1} \delta_k + (m+1)\delta_m \\ &\leq 2 \sum_{k\geq m+1} k\delta_k + (m+1)\delta_{m+1} N \leq (N+2) \sum_{k\geq m+1} k\delta_k \end{aligned}
$$

Using Lemma 6.13, we see that

$$
\frac{1}{N+2} \sum_{m\geq 1} \frac{m\delta_m}{\sum_{k\geq m+1} k\delta_k} \leq c_{6.6} \sum_{m\geq 1} m \min_{w\in W_m} \frac{h_\mu(w)}{\mu(K_w)} = +\infty.
$$

This yields (6.2). Hence Theorem 6.9 implies that $\mu \in \mathcal{M}_P^{TC}(K)$. Since $\mu_w(K_v) = \mu(K_{wv})/\mu(K_w)$, (6.11) implies

$$
\mu_w(K_v) \leq \frac{\delta_{|w|+|v|}}{\delta_{|v|}}.
$$

This and Lemma 6.11 show (6.10).

□

LEMMA 6.13. *If $\sum_{n\geq 1} a_n < +\infty$ for a positive sequence $\{a_n\}_{n\geq 1}$, then*

$$\sum_{m=1}^{\infty} \frac{a_m}{\sum_{k\geq m+1} a_k} = +\infty.$$

PROOF. Let $b_m = \frac{a_m}{\sum_{k\geq m+1} a_k}$ and let $A_m = \sum_{k=m}^{\infty} a_k$. Then

$$\sum_{i=1}^{m} b_k \geq \sum_{i=1}^{m} \log{(1+b_k)} = \log A_1 - \log A_m.$$

Since $A_m \downarrow 0$ as $m \to \infty$, we have $\sum_{k\geq 1} b_k = +\infty$. □

Making use of Lemma 6.12, we may observe how slow decay of $\mu(K_w)$ as $|w| \to \infty$ can be in order to make time change possible. Note that ρ_m can be chosen as $\max_{w\in W_m, i\in S} \mu(K_{wi})/\mu(K_w)$. The next example is an example of the application of Lemma 6.12.

EXAMPLE 6.14. We use the same notation as in Lemma 6.12. Assume $\alpha = 0$. Fix $\epsilon > 0$ and set

$$\delta_k = \frac{1}{k^{2+\epsilon}}.$$

Then $k\delta_k = 1/k^{1+\epsilon}$. Hence

$$\sum_{k\geq m+1} k\delta_k \asymp \frac{1}{m^{\epsilon}}$$

By Lemma 6.12, we have $\mu \in \mathcal{M}_P^{TC}(K)$. By (6.11), there exists $c^1_{6.13} > 0$ such that

$$h_\mu(w) \leq c^1_{6.13} \frac{1}{|w|^{\epsilon}} \tag{6.13}$$

for any $w \in W_*$. Moreover, by (6.12), there exists $c_{6.14} > 0$ such that

$$h_{\mu_w}(\emptyset) \leq c_{6.14}|w|^2. \tag{6.14}$$

for any $w \in W_*$.

In the above example, we only present the case $\alpha = 0$. If $\alpha > 0$, we set $\delta_k = (r_*)^{-k}\frac{1}{k^{1+\epsilon}}$. Then it follows that $h_\mu(w) \leq c/|w|^{\epsilon}$ and $h_{\mu_w}(\emptyset) \leq c|w|$.

7. Scaling of the Green function

If two domains in K are similar to each other, by scale and translation invariances of the Brownian motion, the Green functions of those domains are expected to have simple relation. In this section, we are going to rationalize such an intuition and give upper and lower estimates of integration of the Green function, which corresponds to average exit time of the time changed process from the domain, by means of $h_\mu(w)$'s.

We start with exact definition of the similarity of domains.

The following lemma is straightforward from the definition of $n(\Gamma_1, \Gamma_2)$ in Definition 4.7.

LEMMA 7.1. *For any equivalence class $\mathcal{C}$ under $\underset{B}{\sim}$, there exists $\Gamma_* \in \mathcal{C}$ such that $n(\Gamma_*, \Gamma) \geq 0$ for any $\Gamma \in \mathcal{C}$.*

DEFINITION 7.2. (1) For an independent finite subset of W_*, we denote the equivalence class of Γ under the equivalence relation $\underset{B}{\sim}$ by $[\Gamma]$.
(2) Let $\mathcal{C}$ be an equivalence class under $\underset{B}{\sim}$. An element $\Gamma_* \in \mathcal{C}$ is said to be maximal if $n(\Gamma_*, \Gamma) \geq 0$ for any $\Gamma \in \mathcal{C}$. Define $I_B(\mathcal{C}) = \max_{w \in \Gamma_*} |w|$, where Γ_* is a maximal element of $\mathcal{C}$.

REMARK. There can be several maximal elements in an equivalence class $\mathcal{C}$ under $\underset{B}{\sim}$.

Now we give relations between Dirichlet forms and the Green functions on B-similar domains.

LEMMA 7.3. *Assume that Γ_1 and Γ_2 are independent finite subsets of W_* and $\Gamma_1 \underset{B}{\sim} \Gamma_2$. Let $\psi : \Gamma_1 \to \Gamma_2$ be the B-isomorphism between Γ_1 and Γ_2 and let $\varphi : K(\Gamma_1) \to K(\Gamma_2)$ be the associated B-similitude.*
(1) *For any $u, v \in \mathcal{F}_{K^o(\Gamma_2)}$, $u \circ \varphi, v \circ \varphi \in \mathcal{F}_{K^o(\Gamma_1)}$ and*

$$\mathcal{E}_{K^o(\Gamma_1)}(u \circ \varphi, v \circ \varphi) = (r_*)^{n(\Gamma_1, \Gamma_2)} \mathcal{E}_{K^o(\Gamma_2)}(u, v) \tag{7.1}$$

(2)

$$g^{K^o(\Gamma_1)}(x, y) = (r_*)^{-n(\Gamma_1, \Gamma_2)} g^{K^o(\Gamma_2)}(\varphi(x), \varphi(y)) \tag{7.2}$$

for any $x, y \in K$.

PROOF. Set $U_i = K^o(\Gamma_i)$ for $i = 1, 2$. Note that $n(\Gamma_1, \Gamma_2) = |\psi(w)| - |w|$ for any $w \in \Gamma_1$.
(1) By (10.3),

$$\begin{aligned}
\mathcal{E}_{U_1}(u \circ \varphi, v \circ \varphi) &= \sum_{w \in \Gamma_1} \frac{1}{(r_*)^{|w|}} \mathcal{E}(u \circ \varphi \circ F_w, v \circ \varphi \circ F_w) \\
&= \sum_{w \in \Gamma_1} r^{|\psi(w)| - |w|} \frac{1}{(r_*)^{|\psi(w)|}} \mathcal{E}(u \circ F_{\psi(w)}, v \circ F_{\psi(w)}) \\
&= r^{n(\Gamma_1, \Gamma_2)} \mathcal{E}_{U_2}(u, v)
\end{aligned}$$

(2) Let G^i be the Green operator of U_i for $i = 1, 2$, i.e.

$$(G^i u)(x) = \int_{U_i} g^{U_i}(x, y) u(y) \nu_*(dy)$$

for any $u \in L^2(U, \nu_*|_{U_i})$. Recall that $G^i u \in \mathcal{F}_{U_i}$ is also characterized by

$$\mathcal{E}_{U_i}(G^i u, v) = (u, v)_{U_i}$$

for any $v \in \mathcal{F}_{U_i}$, where $(u, v)_{U_i} = \int_{U_i} u(x) v(x) \nu_*(dx)$. Let $u \in L^2(U_2, \nu_*|_{U_2})$. Then for any $v \in \mathcal{F}_{U_2}$, by (7.1) and the definition of G^i, we have

$$\begin{aligned}
(r_*)^{n(\Gamma_1, \Gamma_2)} \mathcal{E}_{U_2}((G^1(u \circ \varphi)) \circ \varphi^{-1}, v) &= \mathcal{E}_{U_1}(G^1(u \circ \varphi), v \circ \varphi) \\
&= (u \circ \varphi, v \circ \varphi)_{U_1} = N^{n(\Gamma_1, \Gamma_2)}(u, v)_{U_2}.
\end{aligned}$$

Hence $G^2u = (r_*/N)^{n(\Gamma_1,\Gamma_2)}(G^1(u\circ\varphi))\circ\varphi^{-1}$. Therefore

$$\int_{U_2} g^{U_2}(x,y)u(y)\nu_*(dy) = \Big(\frac{r_*}{N}\Big)^{n(\Gamma_1,\Gamma_2)}\int_{U_1} g^{U_1}(\varphi^{-1}(x),y)u(\varphi(y))\nu_*(dy)$$
$$= (r_*)^{n(\Gamma_1,\Gamma_2)}\int_{U_2} g^{U_1}(\varphi^{-1}(x),\varphi^{-1}(y))u(y)\nu_*(dy).$$

This immediately imply (7.2). $\square$

Next lemma shows an estimate of integration of the Green function by means of the sum of $h_\mu(w)$'s over Γ. The important point is that the constants in the estimates (7.3) and (7.4) only depend on the B-equivalence class of Γ.

Lemma 7.4. *Let $\mathcal{C}$ be an equivalence class under $\underset{B}{\sim}$. Assume that $\partial K(\Gamma)\neq\emptyset$ for $\Gamma\in\mathcal{C}$. Let Γ_* be a maximal element of $\mathcal{C}$.*
(1) *In case $\alpha>0$, if $\Gamma\in\mathcal{C}$, $\mu\in\mathcal{M}_P^{TC}(K)$ and $x\in K^o(\Gamma)$, then*

$$\int_{K^o(\Gamma)} g^{K^o(\Gamma)}(x,y)\mu(dy)\leq c_{5.5}(\Gamma_*)\sum_{w\in\Gamma}h_\mu(w) \tag{7.3}$$

(2) *In case $\alpha=0$, if $\Gamma\in\mathcal{C}$, $\mu\in\mathcal{M}_P^{TC}(K)$ and $x\in K^o(\Gamma)$, then*

$$\int_{K^o(\Gamma)} g^{K^o(\Gamma)}(x,y)\mu(dy)\leq c_{5.5}(\Gamma_*)\sum_{w\in\Gamma}(h_{\mu_w}(\emptyset)+(|w|-n(\Gamma_*,\Gamma))\log l)\mu(K_w) \tag{7.4}$$

Proof. Let $\Gamma\in\mathcal{C}$. Let $\psi:\Gamma_*\to\Gamma$ be the B-isomorphism and let $\varphi: K(\Gamma_*)\to K(\Gamma)$ be the associated B-similitude. Set $U_* = K^o(\Gamma_*)$, $U=K^o(\Gamma)$ and $m = n(\Gamma_*,\Gamma)$. Since Γ_* is maximal, it follows that $m\geq 0$ and hence $|\psi^{-1}(w)| = |w|-m$ for any $w\in\Gamma$. By (7.2) and (5.5),

$$g^U(x,y) = (r_*)^m g^{U_*}(\varphi^{-1}(x),\varphi^{-1}(y))\leq c_{5.5}(\Gamma_*)(r_*)^m h(\varphi^{-1}(x),\varphi^{-1}(y)) \tag{7.5}$$

If $\alpha>0$, then by Lemma 6.4 and (7.5), we have

$$g^U(x,y)\leq c_{5.5}(\Gamma_*)h(x,y).$$

Hence

$$\int_U g^U(x,y)\mu(dy)\leq c_{5.5}(\Gamma_*)\int_U h(x,y)\mu(dy)\leq c_{5.5}(\Gamma_*)\sum_{w\in\Gamma}h_\mu(w).$$

If $\alpha=0$, then we have

$$g^U(x,y)\leq c_{5.5}(\Gamma_*)(h(x,y)-m\log l).$$

Hence

$$\int_U g^U(x,y)\mu(dy) = c_{5.5}(\Gamma_*)\int_U(h(x,y)-m\log l)\mu(dy)$$
$$\leq c_{5.5}(\Gamma_*)\sum_{w\in\Gamma}(h_\mu(w)-m\log l\mu(K_w)).$$

By Lemma 6.5, we obtain (7.4). $\square$

Next we focus on special class of subsets $\{V_m(x)\}_{m\geq 0,x\in K}$, which constitutes a kind of standard system of neighborhoods of x. Note that $V_m(x)=\cup_{w\in\Gamma_m(x)}K_w$.

Lemma 7.5. $\{\Gamma_m(x)|x\in K, m\geq 1\}/\underset{B}{\sim}$ *is finite.*

Proof. As we have seen in Example 4.6-(1), the gauge function $\mathbf{g}_*(w) = l^{-|w|}$ is locally finite and elliptic. By (4.1), Theorem 4.9 yields the desired conclusion. $\square$

By the above lemma, we have an uniform upper estimate of integration of the Green function of $V_m(x)$.

Lemma 7.6. *There exist $c_{7.6}, c_{7.7} > 0$ such that*

$$\int_{V_m(x)} g^{V_m^o(x)}(y,z)\mu(dz) \le c_{7.6} \sum_{w \in \Gamma_m(x)} h_\mu(w) \tag{7.6}$$

and

$$\int_{V_m(x)} g^{V_m^o(x)}(y,z)\mu(dz) \le c_{7.7}\Big(\max_{w\in\Gamma_m(x)} h_{\mu_w}(\emptyset)\Big)(r_*)^m \mu(V_m(x)) \tag{7.7}$$

for any $x \in X$, any $m \ge 1$ and any $y \in V_m^o(x)$.

Proof. Lemma 7.5 implies that $\{\Gamma_m(x) | x \in K, m \ge 1\}/\underset{B}{\sim} = \{\mathcal{C}_1, \dots, \mathcal{C}_k\}$. Let $\Gamma_i \in \mathcal{C}_i$ be a maximal element of $\mathcal{C}_i$ and define $c_{7.6} = \max_{i=1,\dots,k} c_{5.5}(\Gamma_i)$. Then Lemma 7.4 immediately shows (7.6). For $\alpha > 0$, (7.7) is obvious by Lemma 6.5. Assume $\alpha = 0$. If $\Gamma_m(x) \in \mathcal{C}_i$, then $|w| - n(\Gamma_i, \Gamma_m(x)) \le I_B(\mathcal{C}_i)$. Note that $h_{\mu_w}(\emptyset) \ge \log l - 1$. Since there exists $c_1 > 0$ such that $x + \max_{i=1,\dots,k} I_B(\mathcal{C}_i)\log l \le c_1 x$ for any $x \ge \log l - 1$, (7.4) and Lemma 2.11 yield

$$\begin{aligned}\int_{V_m(x)} g^{V_m^o(x)}(y,z)\mu(dz) &\le c \sum_{w\in\Gamma_m(x)} h_{\mu_w}(\emptyset)(r_*)^m \mu(K_w)\\ &\le 2^n c \max_{w\in\Gamma_m(x)} h_{\mu_w}(\emptyset)\mu(V_m(x)),\end{aligned}$$

where $c > 0$ is independent of μ, x, y and m. $\square$

Finally, we obtain an uniform lower estimate as well.

Lemma 7.7. *There exists $c_{7.8} > 0$ such that if $x \in K$ and $V_m(x) \ne K$, then*

$$c_{7.8}(r_*)^m \le g^{V_m^o(x)}(x,y) \tag{7.8}$$

for any $y \in V_{m+1}(x)$.

Proof. Fix $\mathcal{C} \in \{\Gamma_m(x) | x \in K, m \ge 1\}/\underset{B}{\sim}$ and choose a maximal element Γ_* of $\mathcal{C}$. Then $\Gamma_* \subseteq W_{m_*}$ for some m_*. Set

$$\mathcal{U} = \{V_{m_*+1}(z) | z \in K, V_{m_*}(z) = K(\Gamma_*)\}.$$

Then $\mathcal{U}$ is a finite set. Hence

$$\inf_{Z\in\mathcal{U}}\Big(\inf_{x_1,x_2\in Z} g^{K^o(\Gamma_*)}(x_1,x_2)\Big) > 0$$

Define $L(\mathcal{C})$ as the above infimum.

Now assume that $\Gamma_m(x) \in \mathcal{C}$. Let $\psi : \Gamma_* \to \Gamma_m(x)$ be the B-isomorphism and $\varphi : K(\Gamma_*) \to V_m(x)$ be the associated B-similitude. Then $m_* = m - n(\Gamma_*, \Gamma)$, $K(\Gamma_*) = V_{m_*}(\varphi^{-1}(x))$ and $\Gamma_* = \Gamma_{m_*}(\varphi^{-1}(x))$. By (7.2),

$$\begin{aligned}\inf_{y\in V_{m+1}(x)} g^{V_m^o(x)}(x,y) &= (r_*)^{n(\Gamma_*,\Gamma)} \inf_{z\in V_{m_*+1}(\varphi^{-1}(x))} g^{K^o(\Gamma_*)}(\varphi^{-1}(x), z)\\ &\ge L(\mathcal{C})(r_*)^{m-m_*}\end{aligned}$$

Since $L(\mathcal{C})(r_*)^{-m_*}$ only depends on $\mathcal{C}$, Lemma 7.5 implies (7.8). $\square$

8. Resolvents

In this section, we study resolvents associated with time changed processes. The aim is to find a usable sufficient condition for the compactness of resolvent as an operator from $L^\infty(K,\mu)$ to itself. Throughout this section, we assume that $\mu \in \mathcal{M}_P^{TC}(K)$. Recall that by Theorem 6.8, this assumption holds if (6.2) is satisfied.

For simplicity, we write $\widetilde{P}_x = P_x^\mu, \widetilde{X}_t = X_t^\mu, \widetilde{P}_x^U = P_x^{U,\mu}, \widetilde{E}_x = E_x^\mu$ and $\widetilde{E}_x^U = E_x^{U,\mu}$.

By [**9**, Theorem 5.9], we immediately have the following lemma from the elliptic Harnack inequality.

LEMMA 8.1. *There exist $k \in \mathbb{N}$, $c_1 > 0$ and $\xi > 0$ such that*

$$|h(x) - h(y)| \le c_1 |x-y|^\xi l^{m\xi} \sup_{x \in V_m(x_0)} |h(x)|$$

for any $m \ge 0$, $x_0 \in K$, harmonic function h on $V_m(x_0)$ and $x, y \in V_{m+k}(x_0)$.

Next we define resolvent operators associated with the time changed processes $(\{\widetilde{X}_t\}_{t>0}, \{\widetilde{P}_x\}_{x\in K})$ and $(\{\widetilde{X}_t^U\}_{t>0}, \{\widetilde{P}_x^U\}_{x\in K})$.

DEFINITION 8.2. Let $\gamma \ge 0$ and let U be an open subset of K. Define

$$(G_\gamma^\mu f)(x) = \widetilde{E}_x\Big(\int_0^\infty e^{-\gamma t} f(\widetilde{X}_t)dt\Big) \text{ and } (G_\gamma^{U,\mu} f)(x) = \widetilde{E}_x\Big(\int_0^{\tau_U} e^{-\gamma t} f(\widetilde{X}_t)dt\Big)$$

for any bounded measurable function $f : K \to \mathbb{R}$ and any $x \in K$. If no confusion may occur, we use G_γ and G_γ^U to denote G_γ^μ and $G_\gamma^{U,\mu}$ respectively.

Note that $(G_\gamma f)(x)$ and $(G_\gamma^U f)(x)$ are defined for every $x \in K$.

PROPOSITION 8.3. *Let A be an open subset of K. Then, for any $\gamma > 0$,*

$$G_\gamma f(x) = G_\gamma^A(f)(x) + \widetilde{E}_x(e^{-\gamma\tau_A} G_\gamma f(\widetilde{X}_{\tau_A})).$$

PROOF.

$$\begin{aligned} G_\gamma f(x) &= \widetilde{E}_x\left(\int_0^\infty e^{-\gamma t} f(\widetilde{X}_t)dt\right) \\ &= \widetilde{E}_x\left(\int_0^{\tau_A} e^{-\gamma t} f(\widetilde{X}_t)dt\right) + \widetilde{E}_x\left(\int_{\tau_A}^\infty e^{-\gamma t} f(\widetilde{X}_t)dt\right) \\ &= \widetilde{E}_x\left(\int_0^{\tau_A} e^{-\gamma t} f(\widetilde{X}_t)dt\right) + \widetilde{E}_x\left(e^{-\gamma\tau_A}\int_0^\infty e^{-\gamma t} f(\widetilde{X}_{\tau_A+t})dt\right) \end{aligned}$$

Let $\mathcal{F}_{\tau_A}$ be the σ-algebra associated with τ_A. Since $e^{-\gamma\tau_A}$ is $\mathcal{F}_{\tau_A}$-measurable, we have

$$\begin{aligned} \widetilde{E}_x\left(e^{-\gamma\tau_A}\int_0^\infty e^{-\gamma t} f(\widetilde{X}_{\tau_A+t})dt\right) &= \widetilde{E}_x\left(\widetilde{E}_x\left(e^{-\gamma\tau_A}\int_0^\infty e^{-\gamma t} f(\widetilde{X}_{\tau_A+t})dt|\mathcal{F}_{\tau_A}\right)\right) \\ &= \widetilde{E}_x\left(e^{-\gamma\tau_A}\widetilde{E}_x\left(\int_0^\infty e^{-\gamma t} f(\widetilde{X}_{\tau_A+t})dt|\mathcal{F}_{\tau_A}\right)\right) \end{aligned}$$

(See [**12**, (1.12) Proposition] for example.) Using the strong Markov property, we may continue the above equality:

$$= \widetilde{E}_x\left(e^{-\gamma\tau_A}\widetilde{E}_{\widetilde{X}_{\tau_A}}\left(\int_0^\infty e^{-\gamma t} f(\widetilde{X}_t)\right)\right) = \widetilde{E}_x\left(e^{-\gamma\tau_A} G_\gamma f(\widetilde{X}_{\tau_A})\right).$$

∎

LEMMA 8.4. *There exists $c_2 > 0$ such that if $V_m(x) \neq K$, then*

$$\widetilde{E}_y(\tau_{V_m(x)}) \leq c_2 \max_{w \in \Gamma_m(x)} h_\mu(w)$$

for any $x \in K$, $m \geq 0$ and $y \in V_m(x)$.

PROOF. Using the fact that $\Gamma_m(x) \leq 4^n$, we obtain this lemma immediately from Lemma 7.6. □

Next lemma is the main result of this section.

LEMMA 8.5. *Assume that*

$$\lim_{m \to \infty} \max_{w \in W_m} h_\mu(w) = 0. \tag{8.1}$$

Then there exist $c_{8.2} > 0$ and a monotonically increasing continuous function F : $[0, \infty) \to [0, \infty)$ with $F(0) = 0$ such that for any $\gamma > 0$,

$$|(G_\gamma f)(x) - (G_\gamma f)(y)| \leq c_{8.2}(1 + \gamma^{-1}) F(|x - y|)||f||_\infty \tag{8.2}$$

for any bounded measurable function $f : K \to \mathbb{R}$ and $x, y \in K$, where $||f||_\infty = \sup_{x \in K} |f(x)|$.

By this lemma, if (8.1) holds, then G_γ maps bounded measurable functions to continuous functions and it is bounded as an operator from $L^\infty(K, \mu)$ to itself. Such a property is sometimes called strong Feller property of resolvents. Moreover, under the strong Feller property, if $\mathcal{U}$ is a bounded subset of $L^\infty(K, \mu)$, then the Arzelà-Ascoli theorem shows that $G_\gamma(\mathcal{U})$ contains a uniformly convergent subsequence. As a result, we see that G_γ can be thought of as a compact operator from $L^\infty(K, \mu)$ to itself.

PROOF. We adapt the discussion in the proof of [**14**, Proposition 3.3]. Fix k as in Lemma 8.1. Let $x, y \in K$ satisfy $|x - y| < l^{-(2k+2)}$. Define

$$m(x, y) = \max\{m | y \in V_m^o(x)\},$$

where $V_m^o(x) = K^o(\Gamma_m(x))$. (Recall that $V_m(x) = K(\Gamma_m(x))$.) Then there exists $c_3 > 0$ independent of x, y and $m(x, y)$ such that $l^{-(m+1)} \leq |x - y| \leq c_3 l^{-m}$, where $m = m(x, y)$. Note that $m(x, y) \geq 2k + 2$. Let $p = [m(x, y)/2]$. Then $p \geq k + 1$. Hence

$$m(x, y) \geq 2p \geq p + k + 1. \tag{8.3}$$

Proposition 8.3 yields

$$G_\gamma f(z) = G_\gamma^{V_p^o(x)} f(z) + \widetilde{E}_z\big((e^{-\tau\gamma} - 1) G_\gamma f(\widetilde{X}_\tau)\big) + \widetilde{E}_z\big(G_\gamma f(\widetilde{X}_\tau)\big).$$

for any $z \in V_p^o(x)$, where $\tau = \tau_{V_p^o(x)}$. Set $\lambda_m = \max_{w \in W_m} h_\mu(w)$. By Lemma 8.4,

$$|G_\gamma^{V_p^o(x)} f(z)| = \left| \widetilde{E}_z \left(\int_0^\tau e^{-\gamma t} f(\widetilde{X}_t) dt \right) \right| \leq \widetilde{E}_z(\tau)||f||_\infty \leq c_2 \lambda_p ||f||_\infty. \tag{8.4}$$

Again using Lemma 8.4, we have

$$\big|\widetilde{E}_z\big((e^{-\tau\gamma} - 1) G_\gamma f(\widetilde{X}_\tau)\big)\big| \leq \gamma \widetilde{E}_z(\tau)||G_\gamma f||_\infty \leq c_2 \lambda_p ||f||_\infty. \tag{8.5}$$

As a function of z, $\widetilde{E}_z(G_\gamma f(\widetilde{X}_\tau))$ is harmonic on $V_{p+1}(x)$ by [**20**, Theorem 4.6.5]. (8.3) shows that $y \in V_{p+1+k}(x)$. Therefore Lemma 8.1 implies

$$(8.6)\quad |\widetilde{E}_x(G_\gamma f(\widetilde{X}_\tau)) - \widetilde{E}_y(G_\gamma r(\widetilde{X}_\tau))| \leq c_1|x-y|^\xi l^{(p+1)\xi}||G_\gamma f||_\infty \leq \frac{c_1}{\gamma}|x-y|^\xi l^{(p+1)\xi}||f||_\infty.$$

Since $p = [m(x,y)/2]$, there exists c_4 and c_5 sucu that $c_4 l^{-p} \leq |x-y|^{1/2} \leq c_5 l^{-p}$. Note that there exists a continuous monotonically increasing function $F_1 : [0,\infty) \to [0,\infty)$ satisfying $F_1(0) = 0$ and $F_1((c_4)^2 l^{-2m}) = \lambda_m$ for any $m \geq 1$. Then by (8.4), (8.5) and (8.6), choosing a proper constant $C > 0$, we have

$$|G_\gamma f(x) - G_\gamma f(y)| \leq C\Big(1 + \frac{1}{\gamma}\Big)(F_1(|x-y|) + |x-y|^{\xi/2})||f||_\infty,$$

for any $x, y \in K$. Finally we set $F(t) = F_1(t) + t^{\xi/2}$. □

If $\mu(K_w)$ decays exponentially as $|w| \to \infty$, then the image $G_\mu f$ is Hölder continuous.

Corollary 8.6. *Let $\mu \in \mathcal{M}_P^{TC}(K)$. If there exist $c > 0$ and $\delta > \alpha$ such that*

$$\mu(K_w) \leq c l^{-|w|\delta}$$

for any $w \in W_*$. Then (8.1) is satisfied. In particular, (8.2) holds with $F(t) = t^{\min\{\xi/2, (\delta-\alpha)/2\}}$ if $\alpha > 0$. If $\alpha = 0$, then for any $\epsilon > 0$, (8.2) holds with $F(t) = t^{\min\{\xi/2, (\delta-\epsilon)/2\}}$.

Proof. Letting $c_w = c l^{-\delta|v|}/\mu(K_w)$, we see that

$$\mu(K_{wv}) \leq c_w l^{-\delta|v|}\mu(K_w).$$

for any $v \in W_*$. If $\alpha > 0$, then Lemma 6.11 yields that

$$h_\mu(K_w) \leq c' l^{-(\delta-\alpha)|w|}$$

for any $w \in W_*$. Hence in this case, $F_1(t) = ct^{(\delta-\alpha)/2}$ if $\alpha > 0$. The case $\alpha = 0$ is entirely the same. □

Example 8.7. Let $\mu \in \mathcal{M}_P(K)$ and let δ_k be the same as in Lemma 6.12. By Lemma 6.12, if (6.8) is satisfied, then $\mu \in \mathcal{M}_P^{TC}$. Moreover, in case $\alpha = 0$, by (6.12), we have

$$\max_{w \in W_m} h_\mu(w) \leq c_{6.6} \sum_{k \geq m+1} k\delta_k \to 0 \text{ as } m \to \infty.$$

Hence the assumption (8.1) of Lemma 8.5 holds.

In case $\alpha > 0$, (6.9) implies $\max_{w \in W_m} h_\mu(w) \leq c_{6.5} \sum_{k \geq 0}(r_*)^{k+m}\delta_{k+m} \to 0$ as $m \to \infty$. Hence (8.1) is satisfied in this case as well.

9. Poincaré inequality

In this section, we are going to show the Poincaré inequality (9.1) for the Dirichlet form $(\mathcal{E}, \mathcal{F}_\mu)$ associated with time change. Let H_μ be the non-negative self-adjoint operator on $L^2(K,\mu)$ associated with $(\mathcal{E}, \mathcal{F}_\mu)$. The Poincaré inequality essentially gives a lower bound of the second eigenvalue of H_μ. Note that the first eigenvalue of H_μ is 0 since H_μ is Neumann Laplacian. We will use the Poincaré inequality to derive Nash type inequality in Section 10.

As in the last section, we assume that $\mu \in \mathcal{M}_P^{TC}(K)$ throughout this section.

THEOREM 9.1. *There exists* $c_{9.1} > 0$ *such that if* $\mu \in \mathcal{M}_P^{TC}(K)$ *and* $u \in \mathcal{F}_\mu$, *then*

$$\mathcal{E}(u,u) \geq \frac{c_{9.1}}{h_\mu(\emptyset)^2}\int_K (u(y)-(u)_\mu)^2\mu(dy). \tag{9.1}$$

We will give a proof of the above theorem at the end of this section. As a step to prove Theorem 9.1, we first show a weak Poincaré inequality (9.5) by adapting the method developed in [**13**].

DEFINITION 9.2. For any $s \in S$, define $\Gamma(s) = \{s' | s' \in S, K_s \cap K_{s'} \neq \emptyset\}$.

By Assumption 6.1, $K_s \subseteq K^o(\Gamma(s)) \neq K$ for any $s \in S$. Write $K^o(s) = K^o(\Gamma(s))$. By Lemma 5.6, the Green function $g^{K^o(s)}(x,y)$, which is denoted by $g^s(x,y)$, is continuous on $\{(x,y)|x,y \in K, x \neq y\}$. The next three lemmas lead to the weak Poincaré inequality (9.5).

LEMMA 9.3. *Write* $g^s(x,y) = g^{K^o(s)}(x,y)$. *There exists* $c_{9.2} > 0$ *such that*

$$c_{9.2}h(x,y) \leq g^s(x,y) \tag{9.2}$$

for any $s \in S$ *and* $x, y \in K_s$.

PROOF. Note that if $x \in K_s$, then $B_*(x, l^{-1}/2) \subseteq K^o(s)$. Define

$$O_s = \{(x,y)|(x,y) \in K_s \times K_s, |x-y| < c_{5.6}^2/(2l)\}.$$

By Lemma 5.9, there exists $c > 0$ such that $ch(x,y) \leq g^s(x,y)$ for any $(x,y) \in O_s$. Since $h(x,y)$ and $g^s(x,y)$ are both positive and continuous on the compact set $(K_s)^2 \backslash O_s$, there exists $c' > 0$ such that $c'h(x,y) \leq g^s(x,y)$. Letting $c_{9.2} = \min\{c, c''\}$, we have (9.2). □

LEMMA 9.4. *There exists* $c_{9.3} > 0$ *such that if* $\mu \in \mathcal{M}_P^{TC}(K)$, *then*

$$g^s(x,y) - \frac{c_{9.3}}{h_{\mu,s}(\emptyset)}\int_K g^s(x,z)g^s(z,y)\mu(dz) \geq \frac{c_{9.2}}{2}h(x,y) \tag{9.3}$$

for any $s \in S$ *and* $x, y \in K_s$, *where* $h_{\mu,s}(\emptyset) = \sup_{x \in K_s}\int_K h(x,y)\mu(dy)$.

PROOF. Let $X_1 = \{z | z \in K, |x-z| \geq |x-y|/2\}$ and let $X_2 = \{z | z \in K, |x-z| < |x-y|/2\}$. Note that $|y-z| \geq |x-y|/2$ for any $z \in X_2$. Hence there exists a constant c_α depending only on α such that $h(x,z) \leq c_\alpha h(x,y)$ if $z \in X_1$ and $h(z,y) \leq c_\alpha h(x,y)$ if $z \in X_2$. Write $g(x,y) = g^s(x,y)$. By (5.5), we have

$$\begin{aligned}
\int_K g(x,z)g(z,y)\mu(dz) &= \int_{X_1} g(x,z)g(z,y)\mu(dz) + \int_{X_2} g(x,z)g(z,y)\mu(dz) \\
&\leq A^2\int_{X_1} h(x,z)h(z,y)\mu(dz) + A^2\int_{X_2} h(x,z)h(z,y)\mu(dz) \\
&\leq c_\alpha A^2\int_K h(x,y)h(z,y)\mu(dz) + c_\alpha A^2\int_K h(x,z)h(x,y)\mu(dz) \\
&\leq 2c_\alpha A^2 h_{\mu,s}(\emptyset)h(x,y),
\end{aligned}$$

where $A = \max_{s \in S} c_{5.5}(\Gamma(s))$. Choosing $c_{9.3}$ so that $2c_\alpha A^2 c_{9.3} = c_{9.2}/2$, we deduce the desired inequality by Lemma 9.3. □

LEMMA 9.5. *There exists* $c_{9.4} > 0$ *such that if* $\mu \in \mathcal{M}_P^{TC}(K)$, $s \in S$, $\gamma = c_{9.3}/h_{\mu,s}(\emptyset)$ *and* $u : K \to [0,\infty)$ *is a bounded measurable function, then*

$$(G_\gamma u)(x) \geq c_{9.4} \int_{K_s} u(y)\mu(dy) \tag{9.4}$$

for any $x \in K_s$.

PROOF. Let $u_* = \chi_{K_s} \cdot u$, where χ_{K_s} is the characteristic function of K_s, and let $U = K^o(s)$. By the resolvent equation and Lemma 9.4,

$$\begin{aligned}(G_\gamma u)(x) \geq (G_\gamma u_*)(x) \geq (G_\gamma^U u_*)(x) &= (G^U u_*)(x) - \gamma(G^U \circ G_\gamma^U u_*)(x) \\ &\geq (G^U u_*)(x) - \gamma(G^U \circ G^U u_*)(x) \\ &= \int_{K_s} \left(g(x,y) - \gamma \int_K g(x,z)g(z,y)\mu(dz) \right) u(y)\mu(dy) \\ &\geq \frac{c_{9.2}}{2} \int_{K_s} h(x,y)u(y)\mu(dy) \geq \frac{c_{9.2}}{2} \min_{x_1,x_2 \in K_s} h(x_1,x_2) \int_{K_s} u(y)\mu(dy).\end{aligned}$$

□

PROPOSITION 9.6. *There exists* $c_{9.5} > 0$ *such that if* $\mu \in \mathcal{M}_P^{TC}(K)$ *and* $f \in \mathcal{F} \cap C(K)$, *then*

$$\mathcal{E}(f,f) \geq c_{9.5} \frac{\mu(K_s)}{h_{\mu,s}(\emptyset)^2} \int_{K_s} (f(y) - (f)_{\mu,s})^2 \mu(dy) \tag{9.5}$$

for any $s \in S$, *where* $(f)_{\mu,s} = \int_{K_s} f(y)\mu(dy)/\mu(K_s)$.

The inequality (9.5) can be thought of as weak Poincaré inequality. The reason why it is "weak" is that the quantity $\mathcal{E}(f,f)$ reflects the values of f on the entire space K while the right-hand side of (9.5) depends only on information of μ and f on K_s.

PROOF. Write $\gamma = c_{9.3}/h_{\mu,s}(\emptyset)$. For any $f \in \mathcal{F} \cap C(K)$, let $u(y) = (f(y) - \gamma(G_\gamma f)(x))^2$. Then

$$\begin{aligned}\gamma(G_\gamma u)(x) &= \gamma(G_\gamma f^2)(x) - 2\gamma^2(G_\gamma f)(x)^2 + \gamma^3(G_\gamma f)(x)^2(G_\gamma 1)(x) \\ &= \gamma(G_\gamma f^2)(x) - \big(\gamma(G_\gamma f)(x)\big)^2.\end{aligned} \tag{9.6}$$

By (9.6) and Lemma 9.5,

$$\begin{aligned}\gamma(G_\gamma f^2)(x) - \big(\gamma(G_\gamma f)(x)\big)^2 &\geq \gamma c_{9.4} \int_{K_s} (f(y) - \gamma(G_\gamma f)(x))^2 \mu(dy) \\ &\geq \gamma c_{9.4} \int_{K_s} (f(y) - (f)_{\mu,s})^2 \mu(dy)\end{aligned} \tag{9.7}$$

for any $x \in K_s$.

Let H_μ be the non-negative self-adjoint operator associated with the Dirichlet form $(\mathcal{E}, \mathcal{F}_\mu)$ on $L^2(K,\mu)$ and let $\{Z_\theta\}_{\theta \geq 0}$ be the spectral resolution of H_μ. Then

$$\begin{aligned}&\int_K \Big(\gamma(G_\gamma f^2)(x) - \big(\gamma(G_\gamma f)(x)\big)^2\Big)\mu(dx) = ||f||_2^2 - ||\gamma G_\gamma f||_2^2 \\ &\quad = \int_0^\infty \left(1 - \Big(\frac{\gamma}{\gamma+\theta}\Big)^2\right) d\langle Z_\theta f, Z_\theta f\rangle \leq \frac{2}{\gamma}\int_0^\infty \theta d\langle Z_\theta f, Z_\theta f\rangle = \frac{2}{\gamma}\mathcal{E}(f,f).\end{aligned} \tag{9.8}$$

Note that $G_\gamma u(x) \geq 0$ for any $x \in K$. Combining (9.7) and (9.8), we obtain

$$\mathcal{E}(f,f) \geq \frac{1}{2}\gamma^2 c_{9.4}\mu(K_s)\int_{K_s}(f(y)-(f)_{\mu,s})^2\mu(dy).$$

□

To prove strong Poincaré inequality (9.1) from the weaker version (9.5), we make use of the self-similarities of the space K, the measure ν_* and the Dirichlet from $(\mathcal{E},\mathcal{F})$.

For $s \in \mathcal{Q}$, let $\Psi_s : H_0 \to H_s$ be the folding map introduced in [**10**, Definition 2.12], which is characterized by the following properties:

(FM1) $\Psi_s : H_0 \to H_s$ is continuous and piecewise affine. $\Psi_s(K) = K_s$.

(FM2) For each $s_0 \in \mathcal{Q}$, define $\pi_{s,s_0} = \Psi_s|_{H_{s_0}}$. Then π_{s,s_0} is an (affine) isometry from H_{s_0} to H_s, $\pi_{s,s}$ is an identity on K_s and $\pi_{s,s_0}(K_{s_0}) = K_s$ if $s, s_0 \in S$.

Proposition 9.7. *If $f \in \mathcal{F}$ and $s \in S$, then $f\circ F_s^{-1}\circ\Psi_s \in \mathcal{F}$.*

Proof. Since we have the heat kernel estimate (5.2), by [**26**, Theorem 4.2], it follows that $u \in \mathcal{F}$ if and only if

$$\sup_{0<r\leq 1}\frac{1}{r^{d_H+d_w}}\int_K\int_{B_*(x,r)}|u(x)-u(y)|^2\nu_*(dy)\nu_*(dx) < +\infty \tag{9.9}$$

Let $0 < r \leq 1/l$. Fix $j \in S$. If $x \in K_j$, then $B_*(x,r) \in \cup_{i\in\Gamma_1(j)}K_i$, where $\Gamma_1(j) = \{i|i \in S, K_j \cap K_i \neq \emptyset\}$. Define $\rho_{j,i} : H_j \cup H_i \to H_j \cup H_i$ as the reflection in $H_j \cap H_i$. If $B_*(x,r) \cap K_i \neq \emptyset$ for $i \in \Gamma_1(j)\backslash\{j\}$, then $\rho_{j,i}(B_*(x,r)\cap K_i) \subseteq B_*(x,r)\cap K_j$. Moreover, if $u = f\circ F_s^{-1}\circ\Psi_s$, then $u(y) = u(\rho_{j,i}(y))$ for any $y \in B_*(x,r)\cap K_i$ because $\Psi_s(y) = \Psi_s(\rho_{j,i}(y))$. Hence

$$\begin{aligned}\int_{B_*(x,r)\cap K_i}|u(x)-u(y)|^2\nu_*(dy) &= \int_{\rho_{j,i}(B_*(x,r)\cap K_i)}|u(x)-u(y)|^2\nu_*(dy)\\ &\leq \int_{B_*(x,r)\cap K_j}|u(x)-u(y)|^2\nu_*(dy)\end{aligned}$$

Therefore, since $\#(\Gamma_1(j)) \leq 3^n$, we have

$$\begin{aligned}\int_{B_*(x,r)}|u(x)-u(y)|^2\nu_*(dy) &\leq \#(\Gamma_1(j))\int_{B_*(x,r)\cap K_j}|u(x)-u(y)|^2\nu_*(dy)\\ &\leq 3^n\int_{B_*(x,r)\cap K_j}|u(x)-u(y)|^2\nu_*(dy)\end{aligned}$$

This implies

$$\begin{aligned}&\int_{K_j}\int_{B_*(x,r)}|u(x)-u(y)|^2\nu_*(dy)\nu_*(dx)\\ &\qquad\leq 3^n\int_{K_j}\int_{B_*(x,r)\cap K_j}|u(x)-u(y)|^2\nu_*(dy)\nu_*(dx)\\ &\qquad= 3^n\int_{K_s}\int_{B_*(x,r)\cap K_s}|u(x)-u(y)|^2\nu_*(dy)\nu_*(dx)\\ &\qquad= \frac{3^n}{l^2}\int_K\int_{B_*(x,lr)}|f(x)-f(y)|^2\nu_*(dy)\nu_*(dx).\end{aligned}$$

Summing this over $j \in S$, we obtain

$$(9.10)\quad \frac{1}{r^{d_H+d_w}}\int_K\int_{B_*(x,r)}|u(x)-u(y)|^2\nu_*(dy)\nu_*(dx) \\ \leq \frac{3^nN}{l^2r^{d_H+d_w}}\int_K\int_{B_*(x,lr)}|f(x)-f(y)|^2\nu_*(dy)\nu_*(dx)$$

Since $f \in \mathcal{F}$, (9.9) shows that the supremum of the right-hand side over $r \in [0, 1/l]$ is finite. Hence the supremum of the left-hand side over $r \in [0, 1/l]$ is finite as well. By the fact that

$$\int_K\int_{B_*(x,r)}|u(x)-u(y)|^2\nu_*(dy)\nu_*(dx) \\ \leq \int_K\int_K 2(|u(x)|^2+|u(y)|^2)\nu_*(dy)\nu_*(dx) < +\infty$$

we see that the supremum of the left-hand side of (9.10) over $[0, 1]$ is finite. Again by (9.9), we conclude that $u \in \mathcal{F}$. $\square$

Definition 9.8. For $s \in S$, we define $\Phi_s : \mathcal{F} \to \mathcal{F}$ by $\Phi_s(f) = f \circ (F_s)^{-1} \circ \Psi_s$.

Note that if $f \in \mathcal{F} \cap C(K)$, then $\Phi_s(f) \in \mathcal{F} \cap C(K)$.

Corollary 9.9. *Let* $s \in S$.
(1) $\mathcal{F} = \{u \circ F_s | u \in \mathcal{F}\}$.
(2) $\{u : K_s \to \mathbb{R} | u \circ \Psi_s \in \mathcal{F}\} = \{f \circ (F_s)^{-1} | f \in \mathcal{F}\}$.

Proof. (1) Theorem 5.1 shows $\{u \circ F_s | u \in \mathcal{F}\} \subseteq \mathcal{F}$. Since $\Phi_s(f) \circ F_s = f$, the converse is obvious.
(2) If $u : K_s \to \mathbb{R}$ and $u \circ \Psi_s \in \mathcal{F}$, then $u = u \circ \Psi_s \circ (F_s)^{-1}$. Conversely, if $f \in \mathcal{F}$, then $f \circ (F_s)^{-1} \circ \Psi_s = \Phi_s(f) \in \mathcal{F}$. $\square$

Remark. The set $\{u : K_s \to \mathbb{R} | u \circ \Psi_s \in \mathcal{F}\}$ is denoted by $\mathcal{F}^s$ in [**10**].

Lemma 9.10. *For any* $f \in \mathcal{F} \cap C(K)$,

$$\mathcal{E}(\Phi_s(f), \Phi_s(f)) = \frac{N}{r_*}\mathcal{E}(f, f).$$

Proof. By (5.1),

$$\mathcal{E}(\Phi_s(f), \Phi_s(f)) = \frac{1}{r_*}\sum_{i \in S}\mathcal{E}(\Phi_s(f) \circ F_i, \Phi_s(f) \circ F_i).$$

Note that $\Phi_s(f) \circ F_i = f \circ (F_s)^{-1} \circ \pi_{s,i} \circ F_i$. Since $(F_s)^{-1} \circ \pi_{s,i} \circ F_i$ is an isometry from K to itself, the invariance of $\mathcal{E}$ under isometries of K implies

$$\mathcal{E}(\Phi_s(f) \circ F_i, \Phi_s(f) \circ F_i) = \mathcal{E}(f, f).$$

$\square$

Finally, we are ready to prove the Poincaré inequality (9.1).

Proof of Theorem 9.1. For $\mu \in \mathcal{M}_P^{TC}(K)$, we define a Borel regular probability measure $\mu^{(\lambda,s)}$ for $\lambda \in (0, 1)$ and $s \in S$ as the Borel regular measure which satisfies

$$\int_K u(x)\mu^{(\lambda,s)}(dx) = \frac{(1-\lambda)N}{N-1}\int_{K\setminus K_s}u(x)\nu_*(dx) + \lambda\int_K u \circ F_s(x)\mu(dx)$$

for any $u \in C(K)$. It is easy to see that $\mu^{(\lambda,s)} \in \mathcal{M}_P^{TC}(K)$. First we assume that $f \in \mathcal{F} \cap C(K)$. It follows that $\mu^{(\lambda,s)}(K_s) = \lambda$ and

$$\int_{K_s} \Phi_s(f)(x)\mu^{(\lambda,s)}(dx) = \lambda \int_K f\circ(F_s)^{-1} \circ F_s(x)\mu(dx) = \lambda \int_K f(x)\mu(dx).$$

Hence we have $(\Phi_s(f))_{\mu^{(\lambda,s)},s} = (f)_\mu$. In the same way,

$$\int_{K_s} (\Phi_s(f)(x) - (\Phi_s(f))_{\mu^{(\lambda,s)},s})^2 \mu^{(\lambda,s)}(dx) = \lambda \int_K (f(x) - (f)_\mu)^2)\mu(dx).$$

Now applying Proposition 9.6 to $\mu^{(\lambda,s)}$ and $\Phi_s(f)$ and using Lemma 9.10, we see

$$\frac{N}{r_*}\mathcal{E}(f,f) \geq c_{9.5}\frac{\lambda^2}{h_{\mu^{(\lambda,s)},s}(\emptyset)^2}\int_K (f(x) - (f)_\mu)^2\mu(dx). \tag{9.11}$$

On the other hand, since

$$h(F_s(x), F_s(y)) \begin{cases} = l^\alpha h(x,y) & \text{if } \alpha > 0, \\ \leq (1 + \log l)h(x,y) & \text{if } \alpha = 0, \end{cases}$$

it follows that

$$\begin{aligned} &h_{\mu^{(\lambda,s)},s}(\emptyset) \\ &\quad\leq \frac{(1-\lambda)N}{N-1}\sup_{x\in K_s}\int_{K\setminus K_s} h(x,y)\nu_*(dy) + \lambda \sup_{x\in K_s}\int_K h(x, F_s(y))\mu(dy) \\ &\quad\quad = (1-\lambda)C_1 + \lambda \sup_{x\in K}\int_K h(F_s(x), F_s(y))\mu(dy) \\ &\quad\quad\quad \leq (1-\lambda)C_1 + \lambda C_2 h_\mu(\emptyset), \end{aligned}$$

where $C_1 = \frac{N}{N-1}\sup_{x\in K_s}\int_{K\setminus K_s} h(x,y)\nu_*(dy)$ and $C_2 = \max\{l^\alpha, 1 + \log l\}$. Therefore, (9.11) yields

$$\frac{N}{r_*}\mathcal{E}(f,f) \geq c_{9.5}\frac{\lambda^2}{\big((1-\lambda)C_1 + \lambda C_2 h_\mu(\emptyset)\big)^2}\int_K (f(x) - (f)_\mu)^2\mu(dx).$$

By letting $\lambda \to 1$, we obtain (9.1) for $f \in \mathcal{F} \cap C(K)$. Since the closure of $\mathcal{F} \cap C(K)$ is $\mathcal{F}_\mu$, we verify (9.1) for any $f \in \mathcal{F}_\mu$. □

10. Heat kernel, existence and continuity

In this section, we will study the existence and the continuity of heat kernels and introduce the notion of measures controlled by rate functions as a collection of measures for which time changes are possible, the associated heat kernels exist and are jointly continuous. As we will see in the following sections, this class is wide enough to contain self-similar measures, some class of random measures, Liouville measure on $[0,1]^2$ and measures having the volume doubling property. The main tool of this section is the Poincaré inequality shown in the last section.

We start with introducing a gauge function $\overline{\sigma}_\mu$ naturally associated with time change.

DEFINITION 10.1. Let μ be a Borel regular probability measure on K.

(1) Define $\sigma_\mu(w)$ and $\overline{\sigma}_\mu(w)$ by

$$\sigma_\mu(w) = (r_*)^{|w|}\mu(K_w)$$

and

$$\overline{\sigma}_\mu(w) = \sup_{v \in W_*} \sigma_\mu(wv) / \sup_{v \in W_*} \sigma_\mu(v).$$

(2) μ is called admissible if $\mu \in \mathcal{M}_P^{TC}(K)$ and (8.1) is satisfied.

Note that $\overline{\sigma}_\mu$ is a kind of normalized version of σ_μ. Intuitively, $\sigma_\mu(w)$ is proportional to the average exit time of the time changed process from K_w. In fact, this intuition will be justified (partially at least) in (12.4) and (12.5).

Proposition 10.2. *If μ is admissible, then $\overline{\sigma}_\mu$ is a gauge function.*

Proof. Since $|x-y| \leq \sqrt{n} l^{-|w|}$ for any $x, y \in K_w$, it follows that

$$h_\mu(w) \geq \begin{cases} n^{-\alpha/2}\sigma_\mu(w) & \text{if } \alpha > 0, \\ (|w|\log l + 1)\sigma_\mu(w) & \text{if } \alpha = 0. \end{cases}$$

Therefore, if (8.1) is satisfied, then $\overline{\sigma}_\mu$ is a gauge function. $\square$

Throughout the rest of this section, we assume that μ is admissible. As a consequence, $\mu \in \mathcal{M}_P^{TC}(K)$ and hence $(\mathcal{E}, \mathcal{F}_\mu)$ is a local regular Dirichlet form on $L^2(K, \mu)$. Recall that H_μ is the self-adjoint operator associated with $(\mathcal{E}, \mathcal{F}_\mu)$.

Definition 10.3. A function $f : [0, \infty) \to [0, \infty)$ is called doubling if and only if there exists $\gamma > 1$ and $c > 1$ such that $f(\gamma t) \leq c f(t)$ for any $t \geq 0$.

Now we define measures controlled by rate functions.

Definition 10.4. Let $\mu \in \mathcal{M}_P(K)$. μ is said to be controlled by rate functions $(\xi_\mu, \xi_\sigma, \xi_h)$ if and only if the following conditions (CRF1) and (CRF2) are satisfied:
(CRF1) ξ_μ and ξ_σ are monotonically non-decreasing doubling functions from $[0, \infty)$ to itself satisfying

$$\mu(K_{wi}) \geq \mu(K_w)\xi_\mu(\mu(K_w))$$

and

$$\mu(K_w) \geq \xi_\sigma(\overline{\sigma}_\mu(w))$$

for any $w \in W_*$ and $i \in S$.
(CRF2) ξ_h is a monotonically non-increasing continuous function from $(0, \infty)$ to itself and

$$h_{\mu_w}(\emptyset)^2 \leq \xi_h(\overline{\sigma}_\mu(w))$$

for any $w \in W_*$. There exist $c^1_{10.1} > 1$ and $c^2_{10.1} > 0$ such that $c^1_{10.1}c^2_{10.1} > 1$,

$$\xi_h(c^1_{10.1}t) \geq c^2_{10.1}\xi_h(t) \tag{10.1}$$

for any $t > 0$. Moreover $\xi_h(t)t$ is monotonically increasing, and $\lim_{t \downarrow 0} \xi_h(t)t = 0$.

Remark. If $\alpha = 0$, then $\overline{\sigma}_\mu(w) = \sigma_\mu(w) = \mu(K_w)$ and hence $\xi_\sigma(t) = t$.

If μ is elliptic, then ξ_μ can be chosen as a constant. In addition if

$$\sup_{w \in W_*, i \in S} \mu(K_{wi})/\mu(K_w) < \min\{1, 1/r_*\},$$

then (6.10) implies $\sup_{w \in W_*} h_{\mu_w}(\emptyset) < +\infty$. Hence in such a case, ξ_h can be chosen as a constant as well. In particular, if $\alpha = 0$ and μ is elliptic, then μ is controlled by rate functions (c_1, t, c_2), where $c_1, c_2 > 0$ are constants.

Notation. For a bounded linear operator $A : L^p(K, \mu) \to L^q(K, \mu)$, we define $||A||_{p \to q}$ as the operator norm $\sup_{f \in L^p(K,\mu), f \neq 0} ||Af||_p/||f||_q$.

The next theorem shows that the strong continuous semigroup associated with the Dirichlet form $(\mathcal{E}, \mathcal{F}_\mu)$ on $L^2(K,\mu)$ is ultracontractive as an operator from $L^1(K,\mu)$ to $L^\infty(K,\mu)$ if μ is controlled by rate functions. The Poincaré inequality is crucial in the proof.

THEOREM 10.5. *Assume that μ is admissible and controlled by rate functions $(\xi_\mu, \xi_\sigma, \xi_h)$. Set $T_t = e^{-H_\mu t}$. Define θ as the inverse of $t\xi_h(t)$. Then T_t maps $L^1(K,\mu)$ to $L^\infty(K,\mu)$ and there exists $c_{10.2} > 0$ such that*

$$||T_t||_{1\to\infty} \le c_{10.2} \max\{1, \xi(t)^{-1}\} \tag{10.2}$$

for any $t > 0$, where $\xi(t) = \xi_\sigma(\theta(t))\xi_\mu(\xi_\sigma(\theta(t)))$. In particular, $\{T_t\}_{t>0}$ is ultracontractive.

REMARK. If $\alpha = 0$, then $\xi(t) = \theta(t)\xi_\mu(\theta(t))$.

LEMMA 10.6. *Assume that μ is controlled by rate functions $(\xi_\mu, \xi_\sigma, \xi_h)$. Let $f(t) = t\xi_h(t)$. If θ is the inverse of f, then θ is doubling.*

PROOF. Let $c_1 = c_{10.1}^1$ and let $c_2 = c_{10.1}^2 c_{10.1}^1$. Then $c_1 > 0$ and $c_2 > 0$ and

$$f(c_1 t) \ge c_2 f(t)$$

for any $t > 0$. This implies $c_1\theta(f(t)) \ge \theta(c_2 f(t))$. $\square$

PROOF OF THEOREM 10.5. If Λ is a partition of Σ, then by using (5.1) and induction on the number of elements of Λ, we see that

$$\mathcal{E}(f,f) = \sum_{w\in\Lambda} \frac{1}{(r_*)^{|w|}} \mathcal{E}(f\circ F_w, f\circ F_w), \tag{10.3}$$

for any $f \in \mathcal{F}_\mu$. By Theorem 9.1, we have

(10.4)

$$\begin{aligned}
\frac{1}{(r_*)^{|w|}}\mathcal{E}(u\circ F_w, u\circ F_w) &\ge \frac{c_{9.1}}{h_{\mu_w}(\emptyset)^2}\frac{1}{(r_*)^{|w|}}\int_K (u\circ F_w(y) - (u\circ F_w)_{\mu_w})^2\mu_w(dy) \\
&\ge \frac{c_{9.1}}{h_{\mu_w}(\emptyset)^2}\frac{1}{(r_*)^{|w|}}\bigg(\int_K (u\circ F_w(y))^2\mu_w(dy) - \Big(\int_K u\circ F_w(y)\mu_w(dy)\Big)^2\bigg) \\
&\ge \frac{c_{9.1}}{\sigma_\mu(w)\xi_h(\sigma_\mu(w))}\bigg(\int_{K_w} u(y)^2\mu(dy) - \frac{1}{\mu(K_w)}\Big(\int_{K_w} u(y)\mu(dy)\Big)^2\bigg)
\end{aligned}$$

Write $\Lambda_\rho = \Lambda_\rho^{\overline{\sigma}_\mu}$. Let $w = w_1 \dots w_m \in \Lambda_\rho$. Set $w' = w_1\dots w_{m-1}$. If $C = \sup_{w\in W_*}\sigma_\mu(w)$, then

$$\overline{\sigma}_\mu(w') > \rho \ge \overline{\sigma}_\mu(w) \ge \frac{1}{C}\sigma_\mu(w),$$

Hence $\mu(K_{w'}) \ge \xi_\sigma(\overline{\sigma}_\mu(w')) \ge \xi_\sigma(\rho)$. This yields

$$\mu(K_w) \ge \mu(K_{w'})\xi_\mu(\mu(K_{w'})) \ge \xi_\sigma(\rho)\xi_\mu(\xi_\sigma(\rho)) \ge \xi_\sigma(\rho)\xi_\mu\circ\xi_\sigma(\rho). \tag{10.5}$$

Since $t\xi_h(t)$ is monotonically increasing, (10.4) implies

(10.6)

$$\frac{1}{(r_*)^{|w|}}\mathcal{E}(u\circ F_w, u\circ F_w) \ge \frac{c_{9.1}}{C\rho\xi_h(\rho)}\bigg(\int_{K_w} u(y)^2\mu(dy) - \frac{1}{\mu(K_w)}\Big(\int_{K_w} u(y)\mu(dy)\Big)^2\bigg),$$

for any $w \in \Lambda_\rho$ and $u \in \mathcal{F}_\mu$. Define $\Lambda_\rho(u) = \{w | w \in \Lambda_\rho, K_w \cap \mathrm{supp}(u) \neq \emptyset\}$. Then, by Lemma 2.11,

$$
(10.7) \quad \sum_{w\in\Lambda_\rho} \frac{1}{\mu(K_w)}\Big(\int_{K_w} u(y)\mu(dy)\Big)^2 = \sum_{w\in\Lambda_\rho(u)} \frac{1}{\mu(K_w)}\Big(\int_{K_w} u(y)\mu(dy)\Big)^2
$$
$$
\leq \frac{1}{\min_{w\in\Lambda_\rho(u)}\mu(K_w)}\bigg(\sum_{w\in\Lambda_\rho(u)}\int_{K_w}|u(y)|\mu(dy)\bigg)^2
$$
$$
\leq \frac{2^{2n}}{\min_{w\in\Lambda_\rho(u)}\mu(K_w)}||u||_{\mu,1}^2.
$$

Making use of (10.3), (10.6) and (10.7), we have

$$
(10.8) \qquad \mathcal{E}(u,u) + \frac{c_{9.1}2^{2n}C^{-1}}{\rho\xi_h(\rho)\min_{w\in\Lambda_\rho(u)}\mu(K_w)}||u||_{\mu,1}^2 \geq \frac{c_{9.1}}{\rho\xi_h(\rho)}||u||_{\mu,2}^2.
$$

for any $u \in \mathcal{F}_\mu$ and $\rho \in (0,1]$. Furthermore, by (10.5), this inequality implies

$$
\mathcal{E}(u,u) + \frac{c_{9.1}2^{2n}C^{-1}}{\rho\xi_h(\rho)\xi(\rho)}||u||_{\mu,1}^2 \geq \frac{c_{9.1}}{\rho\xi_h(\rho)}||u||_{\mu,2}^2
$$

for any $u \in \mathcal{F}_\mu$ and $\rho \in (0,1]$. Hence,

$$
(10.9) \qquad \mathcal{E}(u,u) + \frac{c_{9.1}2^{2n}C^{-1}}{t\xi(t)}||u||_{\mu,1}^2 \geq \frac{c_{9.1}}{t}||u||_{\mu,2}^2.
$$

In [**33**], (10.9) is called the homogeneous Nash inequality. Since θ, ξ_μ and ξ_σ are doubling by Lemma 10.6, ξ is doubling as well. Therefore by [**33**, Theorem 3.2], we obtain (10.2). □

Using the above theorem, we are about to show the existence and the continuity of heat kernel. Next lemma shows that the ultracontractivity of $\{T_t\}_{t>0}$ yields the fact that H_μ has compact resolvent.

LEMMA 10.7. *If a Borel regular probability measure μ on K is admissible and controlled by some rate functions, then H_μ has compact resolvent and any eigenfunction of H_μ is continuous. Furthermore, if $\{\varphi_i\}_{i\geq 1}$ is the complete orthonormal base of $L^2(K,\mu)$ consisting of the eigenfunctions of H_μ, $\{\lambda_i\}_{i\geq 1}$ is the corresponding eigenvalues, i.e. $H_\mu\varphi_i = \lambda_i\varphi_i$ for any $i \geq 1$, $0 \leq \lambda_1 \leq \lambda_2 \ldots$ and $\lim_{m\to\infty}\lambda_m = \infty$, then $\varphi_1 = 1$, $\lambda_1 = 0$ and $\lambda_2 > 0$.*

PROOF. By Lemma 8.5, G_γ is a compact operator from $L^\infty(K,\mu)$ to itself and $G_\gamma(L^\infty(K,\mu)) \subseteq C(K)$. Let $T_t = e^{-H_\mu t}$. Note that $\{T_t\}_{t>0}$ is ultracontractive by Theorem 10.5. Hence if $\{u_n\}_{n\geq 1}$ is a bounded sequence in $L^2(K,\mu)$, then $\{T_t u_n\}_{n\geq 0}$ is a bounded sequence in $L^\infty(K,\mu)$. This implies that $\{G_\gamma T_t u_n\}_{n\geq 1}$ contains a subsequence which converges in $L^\infty(K,\mu)$ and in $L^2(K,\mu)$ as well. Thus, if follows that $G_\gamma T_t$ is a compact operator from $L^2(K,\mu)$ to itself. Since $G_\gamma T_t$ is self-adjoint as well, there exist a complete orthonormal system $\{\varphi_i\}_{i\geq 1}$ of $L^2(K,\mu)$ and $\{a_i\}_{i\geq 1}$ such that $G_\gamma T_t\varphi_i = a_i\varphi_i$ for any $i \geq 1$, $a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$ and $\lim_{i\to\infty} a_i = 0$. Let λ_i be the unique real number satisfying

$$
\frac{e^{-\lambda_i t}}{\gamma + \lambda_i} = a_i.
$$

Then by the spectral resolution of H_μ, we see that $H_\mu \varphi_i = \lambda_i \varphi_i$. Furthermore, since every eigenfunction of H_μ is a finite linear combination of $\{\varphi_i\}_{i\geq 1}$, an eigenfunction of H_μ is continuous. Since $\mathcal{E}(1,1) = 0$, we see that $\lambda_1 = 1$ and $\varphi_1 = 1$. Note that φ_2 is orthogonal to $\varphi_1 = 1$. The Poincaré inequality (9.1) shows that $\mathcal{E}(\varphi_2, \varphi_2) > 0$. Hence $\lambda_2 > 0$. □

REMARK. Note that φ_i and λ_i depend on μ. In this sense, they should be written as φ_i^μ and λ_i^μ respectively. By using these exact notations, ψ_i and λ_i^* appearing in Proposition 5.2 are identified with $\varphi_i^{\nu_*}$ and $\lambda_i^{\nu_*}$ respectively. If no confusion may occur, however, we mainly use φ_i and λ_i.

In the rest of this section, we assume that μ is admissible and controlled by some rate functions. Then by the above lemmas, any eigenfunction is continuous, H_μ has compact resolvent and $||T_t||_{1\to\infty} < +\infty$ for any $t > 0$. In particular, there exists a sequence $\{(\lambda_i, \varphi_i)\}_{i\geq 1}$ of pairs of an eigenvalue and an eigenfunction such that $\lambda_1 = 0 < \lambda_2 \leq \lambda_3 \leq \ldots$ and $\{\varphi_i\}_{i\geq 1}$ is a complete orthonormal system of $L^2(K, \mu)$.

LEMMA 10.8. *Define*

$$p_n(t,x,y) = \sum_{i=1}^{n} e^{-\lambda_i t}\varphi_i(x)\varphi_i(y).$$

Then for any $x \in K$ and $t > 0$, it follows that $p_n(2t,x,x) \leq ||T_t||^2_{1\to\infty}$. In particular, $\sum_{i=1}^n e^{-2\lambda_i t} \leq ||T_t||^2_{1\to\infty}$ for any $n \geq 1$.

PROOF. Let $p_n^{t,x}(y) = p_n(t,x,y)$. Since $\varphi_i \in L^\infty(K,\mu)$,

$$||T_t p_n^{t,x}||_\infty \leq ||p_n^{t,x}||_1 ||T_t||_{1\to\infty} \leq ||p_n^{t,x}||_2 ||T_t||_{1\to\infty}$$

On the other hand,

$$(T_t p_n^{t,x})(y) = p_n(2t,x,y).$$

Therefore,

$$p_n(2t,x,x) \leq \sup_{y\in K} |p_n(2t,x,y)| \leq ||p_n^{t,x}||_2 ||T_t||_{1\to\infty}$$

Since $||p_n^{t,x}||_2^2 = \sum_{i=1}^n e^{-2\lambda_i t}\varphi_i(x)^2 = p_n(2t,x,x)$, it follows that

$$p_n(2t,x,x) \leq ||T_t||^2_{1\to\infty}.$$

□

LEMMA 10.9. *For any $L > 0$, the sum*

$$\sum_{i\geq 1} e^{-\lambda_i t}\varphi_i(x)\varphi_n(y)$$

converges absolutely and uniformly on $[L,\infty) \times K \times K$.

PROOF. Since ξ is doubling, there exist $c > 0$ and $a > 0$ such that $\xi(t) \geq ct^a$ for any $t \in (0,1]$. Hence by (10.2),

$$||T_t||_{2\to\infty} \leq ||T_t||_{1\to\infty} \leq c\max\{1, t^{-a}\}$$

for any $t > 0$. By the fact that $T_t\varphi_i = e^{-\lambda_i t}\varphi_i$, it follows $||\varphi_i||_\infty \leq e^{\lambda_i t}||T_t||_{2\to\infty}$. Letting $t = 1/\lambda_i$, we obtain

$$||\varphi_i||_\infty \leq c(\lambda_i)^a.$$

This yields

$$|e^{-\lambda_i t}\varphi_i(x)\varphi_i(y)| \le c(\lambda_i)^{2a}e^{-\lambda_i L}$$

for any $x, y \in K$ and $t \ge L$. Note that if $M = \sup_{i\ge1}(\lambda_i)^{2a}e^{-\lambda_i L/2}$, then $M < +\infty$. Hence by Lemma 10.8,

$$\sum_{i\ge1}(\lambda_i)^{2a}e^{-\lambda_i L} \le M\sum_{i\ge1}e^{-\lambda_i L/2} \le M||T_{L/4}||^2_{1\to\infty}.$$

Therefore by the Weierstrass majorant convergence theorem, i.e. M-test, we have the desired statement. □

Combining all the results together, we have the following theorem.

THEOREM 10.10. *Assume that μ is admissible and controlled by rate functions $(\xi_\mu, \xi_\sigma, \xi_h)$. Then there exists a jointly continuous heat kernel $p_\mu(t,x,y) > 0$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{F}_\mu)$ on $L^2(K,\mu)$, i.e. $p_\mu(t,x,y)$ is continuous and positive on $(0,\infty)\times K\times K$ and*

$$(T_t u)(x) = \int_K p_\mu(t,x,y)u(y)\mu(dy) \tag{10.10}$$

for any $u \in L^2(K,\mu)$, $t > 0$ and $x \in X$. Moreover,

$$\widetilde{E}_x(u(\widetilde{X}_t)) = \int_K p_\mu(t,x,y)u(y)\mu(dy) \tag{10.11}$$

for any bounded measurable function $u : K \to \mathbb{R}$, $x \in K$, and $t > 0$. Furthermore, H_μ has compact resolvent and there exist a complete orthonormal system $\{\varphi_i\}_{i\ge1}$ of $L^2(K,\mu)$ consisting of the eigenfunctions of H_μ and a sequence $\{\lambda_i\}_{i\ge1}$ such that $H_\mu\varphi_i = \lambda_i\varphi_i$ for any $i \ge 1$, $\lambda_1 = 0 < \lambda_2 \le \lambda_3 \le \ldots$, $\lim_{i\to\infty}\lambda_i = \infty$, $\varphi_1 = 1$, φ_i is continuous on K for any $i \ge 1$ and

$$p_\mu(t,x,y) = \sum_{i\ge1}e^{-\lambda_i t}\varphi_i(x)\varphi_i(y), \tag{10.12}$$

where the infinite sum converges uniformly and absolutely on $[L,\infty)\times K\times K$ for any $L > 0$.

PROOF. We have proved all the statements except the positivity of $p_\mu(t,x,y)$ and (10.11) in the course of the discussion in this section. Using the same argument as in the proof of [**32**, Proposition 5.1.10-(1)], we obtain the positivity of $p_\mu(t,x,y)$. About (10.11), in [**1**, Proof of Theorem 5.1-(i)], the authors have essentially shown that the strong Feller property of resolvents and the uniform convergence of (10.12) suffice for (10.11). Recall that G_γ has strong Feller property by Lemma 8.5 and that the uniform convergence of (10.12) has been shown in Lemma 10.9. Thus, we obtain (10.11). □

REMARK. If μ is controlled by rate functions $(\xi_\mu, \xi_\sigma, \xi_h)$, then (10.2) implies

$$p_\mu(t,x,y) \le c\max\{1, \xi(t)^{-1}\} \tag{10.13}$$

for any $t > 0$.

By (10.12), we also have expansions of the time derivatives of $p_\mu(t,x,y)$ as follows.

THEOREM 10.11. *Assume that μ is admissible and controlled by some rate functions as well. Under the same notations as in Theorem 10.10, for each $(x, y) \in K^2$, $p_\mu(t, x, y)$ belongs to $C^\infty(0, \infty)$ as a function of t and the derivatives $\frac{\partial^m}{\partial t^m} p_\mu(t, x, y)$'s for $m = 1, 2, \ldots$ are jointly continuous on $(0, \infty) \times K \times K$. In particular,*

$$\frac{\partial^m}{\partial t^m} p_\mu(t, x, y) = \sum_{i \geq 1} (\lambda_i)^m e^{-\lambda_i t} \varphi_i(x) \varphi_i(y),$$

where the right-hand side converges uniformly on $[L, \infty) \times K \times K$ for any $L > 0$. Moreover,

$$\left| \frac{\partial^m}{\partial t^m} p_\mu(t, x, y) \right| \leq \frac{1}{e} \Big(\frac{2m}{t} \Big)^m \sqrt{p_\mu(t/2, x, x) p_\mu(t/2, y, y)} \tag{10.14}$$

for any $(t, x, y) \in (0, \infty) \times K \times K$.

PROOF. Similar arguments as in the proof of Lemma 10.9 imply that

$$\sum_{n \geq 1} \lambda_i e^{-\lambda_n z} \varphi_n(x) \varphi_n(y)$$

converges compact uniformly on the right-half plane $H_R = \{z | \mathrm{Re}\, z > 0\} \subset \mathbb{C}$. Hence it is analytic on H_R and

$$\frac{\partial^m}{\partial z^m} p_\mu(z, x, y) = \sum_{n \geq 1} (-\lambda_n)^m e^{-\lambda_n z} \varphi_n(x) \varphi_n(y)$$

for any $z \in H_R$, where the right-hand side converges compact uniformly on H_R. Since $\max_{x \in \mathbb{R}} x^m e^{-x} = m^m/e$,

$$\left| \frac{\partial^m}{\partial t^m} p_\mu(t, x, x) \right| \leq \sum_{n \geq 1} (\lambda_n)^m e^{-\lambda_n t} \varphi_i(x)^2 \leq$$
$$\Big(\frac{2}{t} \Big)^m \sum_{n \geq 1} \Big(\frac{\lambda_n t}{2} \Big)^m e^{-\lambda_n t/2} e^{-\lambda_n t/2} \varphi_n(x)^2 \leq \Big(\frac{2}{t} \Big)^m \frac{m^m}{e} p_\mu(t, x, x).$$

Hence by the Schwartz inequality,

$$\left| \frac{\partial^m}{\partial t^m} p_\mu(t, x, y) \right| \leq \sum_{n \geq 1} (\lambda_n)^m e^{-\lambda_n t} |\varphi_n(x) \varphi_n(y)| \leq$$
$$\Big(\frac{\partial^m}{\partial t^m} p_\mu(t, x, x) \frac{\partial^m}{\partial t^m} p_\mu(t, y, y) \Big)^{1/2} \leq \frac{1}{e} \Big(\frac{2m}{t} \Big)^m \sqrt{p_\mu(t/2, x, x) p_\mu(t/2, y, y)}$$

□

11. Measures having weak exponential decay

In this section, we introduce a class of measures, called measures having weak exponential decay. Eventually this class will turn out to be a subclass of measures controlled by rate functions. The reason why we need this subclass is that the conditions for having weak exponential decay are much simpler and easier to verify than those for being controlled by rate functions. Naturally, if μ has weak exponential decay, then one has all the consequences in the last section. In particular, $\mu \in \mathcal{M}_P^{TC}(K)$ and the time changed process has a jointly continuous hear kernel $p_\mu(t, x, y)$. In Section 14, certain class of random measures is shown to have weak exponential decay for example. Moreover, if a measure has weak exponential decay, then the associated heat kernel is shown to satisfy a diagonal lower estimate in Section 12.

DEFINITION 11.1. A Borel regular probability measure μ is said to have weak exponential decay if and only if there exist positive constants $C_1, C_2, C_3, \lambda_1, \lambda_2$ such that $0 < \lambda_1 \leq \lambda_2 < 1/r_*$,

$$C_1(\lambda_1)^{|w|} \leq \mu(K_w) \leq C_2(\lambda_2)^{|w|} \tag{11.1}$$

for any $w \in W_*$, and

$$\mu(K_{wv}) \leq C_3(r_*)^{-|v|}\mu(K_w) \tag{11.2}$$

for any $w, v \in W_*$.

Note that if $\alpha = 0$, i.e. $r_* = 1$, then the condition (11.2) always holds.

The following proposition gives an equivalent condition for the condition (11.1) in terms of Euclidean balls.

PROPOSITION 11.2. *Let μ be a Borel regular probability measure on K. The condition* (11.1) *holds if and only if there exist positive constants $c_1, c_2, \alpha_1, \alpha_2$ such that $\alpha_1 \geq \alpha_2 > \alpha$ and*

$$c_1 r^{\alpha_1} \leq \mu(B_*(x, r)) \leq c_2 r^{\alpha_2} \tag{11.3}$$

for any $x \in K$ and any $r \in (0, 1]$. Furthermore, if (11.1) *holds, then $\lambda_1 = l^{-\alpha_1}$ and $\lambda_2 = l^{-\alpha_2}$. In particular, if $\alpha = 0$, i.e. $r_* = 1$, then μ has weak exponential decay if and only if it satisfies* (11.3).

REMARK. It follows by Proposition 11.6-(1) and (2) that

$$\lambda_1 \leq \frac{1}{N} \leq \lambda_2 \quad \text{and} \quad \alpha_1 \geq d_H \geq \alpha_2.$$

PROOF. For any $w \in W_*$, there exists $x \in K_w$ such that

$$B_*(x, l^{-(m+1)}) \subseteq K_w \subseteq B_*(x, \sqrt{n}l^{-m}).$$

This implies (11.1) from (11.3). Conversely the fact that

$$B_*(x, l^{-m}) \subseteq V_m(x) \subseteq B_*(x, 3\sqrt{n}l^{-m})$$

implies (11.3) from (11.1). □

EXAMPLE 11.3 (Liouville measure on the square). By [**21**, Theorem 2.2] and [**1**, Lemma 3.1], the condition (11.3) holds for Liouville measure on $[0, 1]^2$ and hence it has weak exponential decay.

Next we introduce a refined version of a measure having weak exponential decay.

DEFINITION 11.4. Let $\eta \geq 1, \mathbf{p} = (\overline{p}, \underline{p}) \in (0, (r_*)^{-1})^2$ and let $\kappa = (\overline{\kappa}, \underline{\kappa})$ be a pair of a monotonically non-decreasing function from $[0, \infty)$ to $[0, \infty)$. A Borel regular probability measure μ on K is said to have $(\eta, \mathbf{p}, \kappa)$-weak exponential decay if $\overline{\kappa}$ is doubling,

$$\sup_{m \in \mathbb{N}} \frac{\overline{\kappa}(m)}{m} < +\infty, \tag{11.4}$$

$$\mu(K_{wv}) \leq \eta\mu(K_w) \times \begin{cases} \overline{p}^{|v|} & \text{if } |v| \geq \overline{\kappa}(|w|), \\ (r_*)^{-|v|} & \text{otherwise,} \end{cases} \tag{11.5}$$

there exist positive constants $c^1_{11.6}$ and $c^2_{11.6}$ such that

$$\underline{\kappa}(x + c^1_{11.6}) \leq \underline{\kappa}(x) + c^2_{11.6} \tag{11.6}$$

for any $x \geq 0$, and

$$\mu(K_{wi}) \geq \frac{1}{\eta}\underline{p}^{\underline{\kappa}(|w|)}\mu(K_w) \tag{11.7}$$

for any $w \in W_*$ and $i \in S$ and

$$\mu(K_w) \geq \frac{1}{\eta}\underline{p}^{|w|} \tag{11.8}$$

for any $w \in W_*$. If both $\overline{\kappa}$ and $\underline{\kappa}$ are bounded, then μ is said to have uniform exponential decay.

Proposition 11.5. *Let μ be a Borel regular probability measure on K. μ has weak exponential decay if and only if μ has $(\eta, \mathbf{p}, \kappa)$-weak exponential decay for some $(\eta, \mathbf{p}, \kappa)$.*

Proof. Assume (11.1). Let $\lambda_1 = (\lambda_2)^{1+\gamma}$ and let $C = C_2/C_1$. Then

$$\mu(K_{wv}) \leq C_2(\lambda_2)^{|w|+|v|} \leq C(\lambda_2)^{-\gamma|w|}(\lambda_2)^{|v|}\mu(K_w).$$

Choose sufficiently small $\epsilon > 0$ so that $(\lambda_2)^{1-\epsilon} < 1/r_*$. Set $\overline{p} = (\lambda_2)^{1-\epsilon}$ and $\overline{\kappa}(x) = \gamma x/\epsilon$. Then we have $\mu(K_{wv}) \leq C\overline{p}^{|v|}\mu(K_w)$ for any $v \in W_*$ with $|v| \geq \overline{\kappa}(|w|)$. Combining this with (11.2), we obtain (11.4) and (11.5).

Next, let $\underline{p} = \min\{\lambda_1, \lambda_1/\lambda_2\}$ and define $\underline{\kappa}(x) = x$. Then

$$\mu(K_{wi}) \geq c_1(\lambda_1)^{|w|+1} \geq \frac{\lambda_1}{C}\underline{p}^{\underline{\kappa}(|w|)}\mu(K_w)$$

for any $w \in W_*$ and $i \in S$, and

$$\mu(K_w) \geq C_1\underline{p}^{|w|}$$

for any $w \in W_*$. Thus we have obtained (11.6), (11.7) and (11.8). (The constant η can be chosen properly.)

Conversely, if μ has $(\eta, \mathbf{p}, \kappa)$-weak exponential delay, we can deduce (11.1) from (11.5) and (11.7) by letting $w = \emptyset$. The condition (11.2) follows from (11.5). Thus μ has weak exponential decay. $\square$

Suppose that μ has (η, p, κ)-weak exponential decay. Note that the conditions on $\kappa : [0, \infty)^2 \to [0, \infty)$ only concern the values of κ on nonnegative integers. In other words, given values on $\mathbb{N} \cup \{0\}$, we may interpolate values between integers so that $\overline{\kappa}$ and $\underline{\kappa}$ are continuous monotone functions without losing the required properties. Moreover, adjusting the value of η, we may assume that

$$\overline{\kappa}(0) = \underline{\kappa}(0) = 0 \tag{11.9}$$

without loss of generality. Furthermore, due to (11.4), modifying $\overline{\kappa}$ without changing the order of increase, one may assume that

$$\lambda^{-x}\overline{\kappa}(x) \text{ is monotonically decreasing and } \lim_{x\to\infty} \lambda^{-x}\overline{\kappa}(x) = 0, \tag{11.10}$$

where $\lambda = r_*\underline{p}$. Thus whenever μ has $(\eta, \mathbf{p}, \kappa)$-weak exponential decay, then the conditions (11.9) and (11.10) are always assumed to be true hereafter.

The followings are basic facts on the conditions in Definition 11.4.

PROPOSITION 11.6. *Let* $\mu \in \mathcal{M}_P(K)$.
(1) *If* (11.5) *holds, then* $\overline{p} \geq 1/N$.
(2) *If* (11.7) *holds, then* $\underline{p} \leq 1/N$.
(3) (11.7) *holds and* $\underline{\kappa}$ *is bounded if and only if* μ *is elliptic.*
(4) μ *has uniform exponential decay if and only if there exist* $\eta > 1$, $\overline{p}, \underline{p} \in (0, (r_*)^{-1})$ *such that*

$$\frac{1}{\eta}\underline{p}^{|v|}\mu(K_w) \leq \mu(K_{wv}) \leq \eta\overline{p}^{|v|}\mu(K_w)$$

for any $w, v \in W_*$.
(5) *If* μ *is a self-similar measure on* K *with weight* $(\mu_i)_{i\in S}$. *Then* μ *has weak exponential decay if and only if* $\mu_i r_* < 1$ *for any* $i \in S$. *Moreover, if* $\mu_i r_* < 1$ *for any* $i \in S$, *then* μ *has uniform exponential decay.*

PROOF. (1) Choosing sufficiently large η, we have $\mu(K_w) \leq \eta\overline{p}^{|w|}$ for any $w \in W_*$. Hence

$$1 \leq \sum_{w\in W_m} \mu(K_w) \leq \eta(N\overline{p})^m.$$

This immediately implies $\overline{p} \geq 1/N$.
(2) By (11.8),

$$2^n \geq \sum_{w\in W_m} \mu(K_w) \geq \eta^{-1}(N\underline{p})^m.$$

Therefore, we obtain $\underline{p} \leq 1/N$.
(3) Assume that (11.7) is satisfied and $\underline{\kappa}$ is bounded. Choose $M \in \mathbb{N}$ so that $\sup_{x\geq 0} \underline{\kappa}(x) \leq M$. Then

$$\mu(K_{wi}) \geq \eta^{-1}\underline{p}^M\mu(K_w)$$

for any $w \in W_*$ and $i \in S$. Set $\gamma = \eta^{-1}\underline{p}^M$. Then it is straightforward to verify the condition (ELm) in [**34**, Theorem 1.2.4]. Hence by [**34**, Theorem 1.2.4 and its remark], we see that μ is elliptic. The converse direction is obvious.
(4) This is immediate from definitions.
(5) Let μ be a self-similar measure with weight $(\mu_i)_{i\in S}$. By [**34**, Theorem 1.2.7], it follows that $\mu(K_w) = \mu_{w_1}\cdots\mu_{w_m}$ for any $w = w_1 \dots w_m \in W_*$. Hence if (11.5) holds, then $\overline{p} \geq \max_{i\in S}\mu_i$. This yields $\mu_i r_* < 1$ for any $i \in S$. Conversely if $\mu_i r_* < 1$ for any $i \in S$, we let $\overline{p} = \max_{i\in S}\mu_i$ and obtain (11.5) with $\eta = 1$ and $\overline{\kappa}(x) = 0$ for any x. At the same time, we obtain (11.7) by letting $\underline{\kappa}(x) = 0$ for any $x \in X$ and $\underline{p} = \min_{i\in S}\mu_i$. □

The following proposition shows an upper estimate of $h_\mu(w)$ when μ has weak exponential decay. As a result, μ is shown to be admissible as well.

PROPOSITION 11.7. *Let* μ *have* $(\eta, \mathbf{p}, \kappa)$*-weak exponential decay. Define* $\lambda = r_*\overline{p}$. *If* $\alpha > 0$, *then*

$$h_\mu(w) \leq c_{6.5}\eta\Big(\overline{\kappa}(|w|) + \frac{1}{1-\lambda}\Big)\sigma_\mu(w) \tag{11.11}$$

for any $w \in W_*$. *If* $\alpha = 0$, *then*

$$h_\mu(w) \leq c_{6.6}\eta\left(|w|\Big(\overline{\kappa}(|w|) + \frac{1}{1-\lambda}\Big) + \overline{\kappa}(|w|)^2 + \frac{1}{(1-\lambda)^2}\right)\sigma_\mu(w) \tag{11.12}$$

for any $w \in W_*$. In particular, μ is admissible. Moreover,

$$\frac{1}{\eta}\sigma_\mu(w) \le \overline{\sigma}_\mu(w) \le \eta\sigma_\mu(w) \tag{11.13}$$

and

$$\overline{\sigma}_\mu(wv) \le \eta^3\overline{\sigma}_\mu(w) \times \begin{cases} \lambda^{|v|} & \text{if } |v| \ge \overline{\kappa}(|w|), \\ 1 & \text{otherwise.} \end{cases} \tag{11.14}$$

for any $w, v \in W_*$.

Proof. The estimates (11.11) and (11.12) are immediate by Lemma 6.11. Combining these with (11.4), we obtain (6.2). Hence $\mu \in \mathcal{M}_P^{TC}(K)$. By (11.5), there exists $\eta' > 0$ such that $\sigma_\mu(w) \le \eta'\lambda^{|w|}$ for any $w \in W_*$. This and (11.4) imply (8.1). (11.5) yields

$$\sigma_\mu(wv) \le \eta\sigma_\mu(w) \tag{11.15}$$

for any $w, v \in W_*$. Note that $1 \le \overline{\sigma}_\mu(\emptyset) \le \eta$. It follows by (11.15) that $\overline{\sigma}_\mu(w) \le \eta\sigma_\mu(w)$. Thus μ is admissible. At the same time we have (11.13). Combining (11.15) and (11.13), we obtain (11.14). □

The next proposition shows a simple equivalence condition for having uniform exponential decay.

Proposition 11.8. *Let $\mu \in \mathcal{M}_P(K)$. $\overline{\sigma}_\mu$ is an elliptic gauge function if and only if μ has uniform exponential decay.*

Remark. By [**34**, Theorem 1.2.4], if μ is elliptic, then $\mu(F_w(V_0)) = 0$ for any $w \in W_*$, where $V_0 = \partial H_0 \cap K$. This implies that $\mu(\partial K(\Gamma)) = 0$ for any $\Gamma \subseteq W_*$. Furthermore, if $\Gamma \subseteq W_*$ is independent, then

$$\int_{K(\Gamma)} f(x)\mu(dx) = \sum_{w \in \Gamma}\int_{K_w} f(x)\mu(dx) \tag{11.16}$$

for any $f \in L^1(K, \mu)$.

Proof. Assume that $\overline{\sigma}_\mu$ is an elliptic gauge function. Then, there exist $a > 0$ and $b \in (0, 1)$ such that

$$\overline{\sigma}_\mu(wv) \le ab^{|v|}\overline{\sigma}_\mu(w) \tag{11.17}$$

for any $w, v \in W_*$. If $M = \min\{m | ab^m \le 1\}$, then $\overline{\sigma}_\mu(w) = \max\{\sigma_\mu(wv) | |v| \le M\}$. Hence there exists $v_* \in W_*$ such that $|v_*| \le M$ and $\overline{\sigma}_\mu(w) = \sigma_\mu(wv_*) = (r_*)^{|v_*|}\mu(K_{wv_*})$. This implies that

$$\sigma_\mu(w) \le \overline{\sigma}_\mu(w) = (r_*)^{|v_*|}\mu(K_{wv_*}) \le (r_*)^M\sigma_\mu(w) \tag{11.18}$$

By (11.17) and (11.18), we have

$$\sigma_\mu(wv) \le \overline{\sigma}_\mu(wv) \le ab^{|v|}\overline{\sigma}_\mu(w) \le a(r_*)^M b^{|v|}\sigma_\mu(w).$$

This yields

$$\mu(K_{wv}) \le a(r_*)^M (b/r_*)^{|v|}\mu(K_w).$$

Let $\overline{\kappa}(x) = 0$ for any $x \ge 0$, $\eta = a(r_*)^M$ and $\overline{p} = b/r_*$. Then (11.5) holds. Since $\overline{\sigma}_\mu$ is elliptic, there exists $c > 0$ such that $\overline{\sigma}_\mu(wi) \ge c\overline{\sigma}(w)$ for any $w \in W_*$ and $i \in S$. This along with (11.18) shows that there exists $c' > 0$ such that $\sigma_\mu(wi) \ge c'\sigma_\mu(w)$

for any $w \in W_*$ and $i \in S$. Therefore, $\mu(K_{wi}) \geq c'(r_*)^{-1}\mu(K_w)$. Thus we have shown that μ has uniform exponential decay.

Conversely assume that μ has uniformly weak exponential decay. We have (11.14) by Proposition 11.7. By Proposition 11.6-(3), there exists $\gamma > 0$ such that $\mu(K_{wi}) \geq \gamma\mu(K_w)$ for any $w \in W_*$ and $i \in S$. Hence $\sigma_\mu(wi) \geq \gamma(r_*)^{-1}\sigma_\mu(w)$ for any $w \in W_*$ and $i \in S$. Using (11.13), we see that there exists $c'' > 0$ such that $\overline{\sigma}_\mu(wi) \geq c''\overline{\sigma}_\mu(w)$ for any $w \in W_*$ and $i \in S$. This and (11.14) show that $\overline{\sigma}_\mu$ is an elliptic gauge function. □

As is seen in Proposition 11.7, a measure having weak exponential decay is admissible. In the next theorem, it is also shown to be controlled by some rate functions. As a consequence, if a measure has weak exponential decay, then time change is possible and there exists a jointly continuous heat kernel with upper estimate (10.13).

Theorem 11.9. *If a Borel regular probability measure μ on K has weak exponential decay, then $\mu \in \mathcal{M}_P^{TC}(K)$ and it is controlled by some rate functions $(\xi_\mu^*, \xi_\sigma^*, \xi_h^*)$. More specifically, assume that μ has $(\eta, \mathbf{p}, \kappa)$-weak exponential decay. If $\gamma_1 = -1/\log(r_*\overline{p})$ and $\gamma_2 = 3\gamma_1 \log \eta$, then*

$$\xi_\mu^*(t) = \frac{1}{\eta}\underline{p}^{\underline{\kappa}(-\gamma_1 \log t + \gamma_2)},$$

$$\xi_\sigma^*(t) = \begin{cases} \underline{p}^{\gamma_2}\eta^{-1}t^{-\gamma_1 \log \underline{p}} & \text{if } \alpha > 0, \\ t & \text{if } \alpha = 0, \end{cases}$$

and

$$\xi_h^*(t) = \begin{cases} \gamma_3(\overline{\kappa}(-\gamma_1 \log t + \gamma_2)^2 + 1) & \text{if } \alpha > 0, \\ \gamma_3(\overline{\kappa}(-\gamma_1 \log t + \gamma_2)^4 + 1) & \text{if } \alpha = 0, \end{cases}$$

where γ_3 is a constant determined by $(\eta, \mathbf{p}, \kappa)$. In particular, if μ has uniform exponential decay, then ξ_h^* and ξ_μ^* are constants.

We will prove the above theorem later in this section. For the moment, we present a corollary on diagonal upper heat kernel estimate.

Corollary 11.10. *Let μ be a Borel regular probability measure on K. Assume that μ has $(\eta, \mathbf{p}, \kappa)$-exponential decay. If μ is controlled by rate functions $(\xi_\mu, \xi_\sigma, \xi_h)$ and $\lim_{x\to\infty} \underline{\kappa}(x)/x = 0$, then*

$$\limsup_{t\downarrow 0} -\frac{\log p_\mu(t,x,x)}{\log t} \leq \limsup_{s\downarrow 0} \frac{\log\big(\max\{\xi_\sigma(s), \xi_\sigma^*(s)\}\big)}{\log s} \tag{11.19}$$

for any $x \in K$. In particular, if $\alpha = 0$, then

$$\limsup_{t\downarrow 0} -\frac{\log p_\mu(t,x,x)}{\log t} \leq 1$$

for any $x \in K$.

Remark. If $\xi_\sigma^1(t) = \max\{\xi_\sigma(t), \xi_\sigma^*(t)\}$, then $\xi_\sigma^1(t)$ is better than both $\xi_\sigma(t)$ and $\xi_\sigma^*(t)$ as a rate function. In fact, $\mu(K_w) \geq \xi_\sigma^1(\overline{\sigma}_\mu(w)) \geq \xi_\sigma^*(\overline{\sigma}_\mu(w))$ for example.

Remark. If μ has uniform exponential decay, then $\underline{\kappa}$ is bounded and hence $\lim_{x\to\infty} \underline{\kappa}(x)/x = 0$. Thus we have (11.19).

PROOF OF COROLLARY 11.10. Define $\xi_\sigma^1(t) = \max\{\xi_\sigma(t), \xi_\sigma^*(t)\}$. Note that μ is controlled by rate functions $(\xi_\mu^*, \xi_\sigma^1, \xi_h^*)$. Hence by Theorems 10.5 and 10.10, we have

$$p_\mu(t,x,x) \le \frac{1}{\xi_\sigma^1(\theta(t))\xi_\mu^*(\xi_\sigma^1(\theta(t)))}$$

for any $x \in X$ and any $t \in (0,1]$. Since θ is the inverse of $t\xi_h^*(t)$,

$$\limsup_{t\downarrow} -\frac{\log p_\mu(t,x,x)}{\log t} \le \limsup_{s\downarrow 0} \frac{\log \xi_\sigma^1(s)\xi_\mu^*(\xi_\sigma^1(s))}{\log s\xi_h^*(s)} \tag{11.20}$$

By (11.4), for sufficiently small $t > 0$, we see that $1 \le \xi_h^*(s) \le c(\log s)^4$. Hence

$$\lim_{s\downarrow 0} \frac{\log \xi_h^*(s)}{\log s} = 0.$$

Furthermore, $\log \xi_\mu^*(\xi_\sigma^*(s)) = (\log \underline{p})\underline{\kappa}(-c_1 \log t + c_2) - \log \eta$, where $c_1 > 0$ and c_2 are constants. Since $\lim_{x\to\infty} \underline{\kappa}(x)/x = 0$ and $|\log \xi_\mu^*(\xi_\sigma^*(s))| \ge |\log \xi_\mu^*(\xi_\sigma^1(s))|$ for sufficiently small $s > 0$, it follows that

$$\lim_{s\downarrow 0} \frac{\log \xi_h^*(\xi_\sigma^1(s))}{\log s} = 0.$$

Hence, we obtain (11.19) from (11.20). □

We now begin to prove Theorem 11.9. First we prepare a lemma.

LEMMA 11.11. *Assume that $\overline{\kappa} : [0,\infty) \to [0,\infty)$ is a doubling non-decreasing function and satisfies (11.4). Fix $k \in \mathbb{N}$ and $c > 0$. Define $f(t) : (0,\infty) \to [0,\infty)$ by*

$$f(t) = \begin{cases} \overline{\kappa}(-c\log t)^k + 1 & \text{if } t \in (0,1], \\ 1 & \text{if } t > 1. \end{cases}$$

Then there exist $c_1 > 1$ and $c_2 > 0$ such that $c_1c_2 > 1$ and

$$f(c_1 t) \ge c_2 f(t)$$

for any $t > 0$.

PROOF. Since $\overline{\kappa}$ is doubling, there exist $\gamma_1, \gamma_2 \in (0,1)$ such that $\overline{\kappa}(\gamma_1 t) \ge \gamma_2\overline{\kappa}(t)$ for any $t > 0$. Let $x = -\log t$ for $t \in (0,1]$. Choose $s > 1$ so that $1 - 1/s = \gamma_1$. Let $A > 1$. Then if $x \ge s\log A$,

$$\frac{\overline{\kappa}(c(x-\log A))^k + 1}{\overline{\kappa}(cx)^k + 1} \ge \min\left\{1, \frac{\overline{\kappa}(c(x-\log A))^k}{\overline{\kappa}(cx)^k}\right\} \ge (\gamma_2)^k.$$

On the other hand, for $0 < x \le s\log A$, we see that

$$\frac{1}{\overline{\kappa}(cx)^k + 1} \ge \frac{1}{\overline{\kappa}(cs\log A)^k + 1}.$$

These inequalities imply that

$$\frac{f(At)}{f(t)} \ge \min\left\{(\gamma_2)^k, \frac{1}{\overline{\kappa}(cs\log A)^k + 1}\right\}$$

Define $F(A)$ as the right-hand side of this inequality. Then by (11.4), we see that $AF(A) \to \infty$ as $A \to \infty$. In particular, we may choose $A > 1$ so that $AF(A) > 1$. Letting $c_1 = A$ and $c_2 = F(A)$, we obtain the desired conclusion. □

Proof of Theorem 11.9. First we consider ξ_h^*. By (11.14) and (11.9), it follows that $\overline{\sigma}_\mu(w) \leq \eta^3 \lambda^{|w|}$ for any $w \in W_*$. Hence

$$-\gamma_1 \log \overline{\sigma}_\mu(w) + \gamma_2 \geq |w|. \tag{11.21}$$

Therefore by (11.11) and (11.12),

$$h_{\mu_w}(\emptyset)^2 \leq \gamma_3 \times \begin{cases} (\overline{\kappa}(-\gamma_1 \log \overline{\sigma}_\mu(w) + \gamma_2)^2 + 1) & \text{if } \alpha > 0, \\ (\overline{\kappa}(-\gamma_1 \log \overline{\sigma}_\mu(w) + \gamma_2)^4 + 1) & \text{if } \alpha = 0. \end{cases}$$

for some $\gamma_3 > 0$. Thus $h_\mu(\emptyset)^2 \leq \xi_h^*(\overline{\sigma}_\mu(w))$. Furthermore, by (11.10), $t\xi_h^*(t)$ is continuous, monotonically increasing and $\lim_{t\downarrow 0} t\xi_h^*(t) = 0$.

Next, if $\alpha = 0$, we may choose $\xi_\sigma^*(t) = t$. Assume $\alpha > 0$. By (11.8) and (11.21),

$$\mu(K_w) \geq \frac{1}{\eta}(\underline{p})^{|w|} \geq \frac{\underline{p}^{\gamma_2}}{\eta}(\overline{\sigma}_\mu(w))^{-\gamma_1 \log \underline{p}}$$

Therefore, $\mu(K_w) \geq \xi_\sigma^*(\overline{\sigma}_\mu(w))$. Obviously ξ_σ^* is doubling.

Finally about ξ_μ^*, by (11.7) and (11.21),

$$\mu(K_{wi}) \geq \xi_\mu^*(\overline{\sigma}_\mu(w))\mu(K_w)$$

By (11.6), ξ_μ^* is doubling. □

12. Protodistance and diagonal lower estimate of heat kernel

In this section, we will present a diagonal lower estimate of heat kernel (12.11) whose principal part is the volume of the "ball' with respect to "protodistance" δ_μ defined in this section. Note that we do not attempt to create a general notion of "protodistance" but we are going to call the nonnegative function $\delta_\mu : K \times K \to [0, \infty)$ by the name "protodistance", which is not even symmetric nor a quasimetric in general. Once μ has the volume doubling property with respect to d_*, however, our protodistance δ_μ is equivalent to some power of a distance under which sub-Gaussian heat kernel estimates (1.7) and (1.8) hold as we will see in Section 15.

After the introduction of δ_μ, assuming that μ has weak exponential decay, we study lower estimate of $p_\mu(t, x, x)$ as $t \downarrow 0$. Note that uniform upper estimate of $p_\mu(t, x, x)$ has been obtained in the previous section.

Throughout this section, we assume the following property:

$$\lim_{m\to\infty} (r_*)^m \mu(V_m(x)) = 0. \tag{12.1}$$

for any $x \in K$. If μ has weak exponential decay, then this assumption is satisfied.

Next we define our "protodistance" δ_μ.

Definition 12.1. For $m \geq 0$, $x \in K$, we define

$$\epsilon_\mu(m, x) = \max\{(r_*)^k \mu(V_k(x)) | k \geq m\},$$

$$\widetilde{m}_\mu(t, x) = \begin{cases} \max\{m | \epsilon_\mu(m, x) \geq t\} + 1 & \text{if } \epsilon_\mu(0, x) \geq t, \\ 0 & \text{if } \epsilon_\mu(0, x) < t \end{cases}$$

and

$$\delta_\mu(x, y) = \inf\{t | y \in V_{\widetilde{m}_\mu(t,x)}(x)\}.$$

We call δ_μ the protodistance associated with a measure μ. By the assumption (12.1), $\epsilon_\mu(m,x)$ is well-defined and $\lim_{m\to\infty}\epsilon_\mu(m,x) = 0$ for any $x \in K$. Consequently, $\widetilde{m}_\mu(t,x)$ and $\delta_\mu(x,y)$ are well-defined as well and $\delta_\mu(x,y) \geq 0$ and $\delta_\mu(x,y) = 0$ if and only if $x = y$. On the other hand the protodistance is not a (quasi)metric in general. For example, it is often the case that $\delta_\mu(x,y) \neq \delta_\mu(y,x)$ for $x \neq y$. Later in Section 19, we will show inequalities (19.1), (19.3) and (19.4) whose combination can be regarded as a kind of primitive counterpart of weakened triangle inequality;

$$d(x,y) \leq C(d(x,z) + d(z,y)),$$

where $C \geq 1$ is a fixed constant. Indeed, the combination of (19.1), (19.3) and (19.4) will be shown to yield weakened triangle inequality if μ has the volume doubling property.

If no confusion can occur, we write $\epsilon(m,x), \widetilde{m}(t,x)$ and $\delta(x,y)$ instead of $\epsilon_\mu(m,x), \widetilde{m}_\mu(t,x)$ and $\delta_\mu(x,y)$ respectively.

The protodistance δ_μ has another expression by means of the separation number $k(x,y)$ defined below.

Definition 12.2. Let $x, y \in K$. A sequence $(w(1),\ldots,w(j)) \in (W_*)^j$ is called a chain between x and y if and only if $x \in K_{w(1)}, y \in K_{w(j)}$ and $K_{w(i)} \cap K_{w(i+1)} \neq \emptyset$ for any $i = 1,\ldots,j-1$. Define

$$\begin{aligned}&\ell_m(x,y)\\ &\quad = \min\{k \mid \text{there exists a chain } (w(1),\ldots,w(k)) \in (W_m)^k \text{ between } x \text{ and } y\}\end{aligned}$$

and

$$k(x,y) = \max\{m \mid \ell_m(x,y) \leq 2\}.$$

The number $\ell_m(x,y)$ is the length of shortest walk in W_m between x and y. $k(x,y)$ represents the level at which two points x and y are separated. Obviously, $k(x,y) < +\infty$ if $x \neq y$. In case $x = y$, we think of $k(x,y) = +\infty$. The following lemma is straightforward from the above definition.

Lemma 12.3. *If $x \neq y \in K$, then $y \in V_{k(x,y)}(x) \backslash V_{k(x,y)+1}(x)$.*

Immediately by the above definitions, we obtain the next lemma.

Lemma 12.4. *Let $j \geq 1$. Then $\widetilde{m}_\mu(t,x) = j$ if and only if $\epsilon_\mu(j,x) < t \leq \epsilon_\mu(j-1,x)$.*

The above lemmas give the following alternative expression of δ_μ using $k(x,y)$.

Proposition 12.5. *For any $x, y \in K$,*

$$\delta_\mu(x,y) = \epsilon_\mu(k(x,y),x).$$

Proof. Let $k(x,y) = k$. Then by Lemmas 12.3 and 12.4,

$$\{t \mid y \in V_{\widetilde{m}(t,x)}(x)\} = \{t \mid l \geq \widetilde{m}(x,t)\} = \{t \mid \epsilon(k,x) < t\}.$$

Hence $\delta(x,y) = \epsilon(k(x,y),x)$. $\square$

A "ball" with respect to the protodistance δ_μ is identified with $V_m(x)$ as follows.

Proposition 12.6. *Define* $B_{\delta_\mu}(x,r) = \{y|\delta_\mu(x,y) < r\}$ *for* $x \in K$ *and* $r > 0$. *Then*

$$B_{\delta_\mu}(x,t) = V_{\widetilde{m}_\mu(t.x)}(x) \tag{12.2}$$

and

$$\mu(V_{\widetilde{m}_\mu(t,x)}(x)) \leq \frac{t}{(r_*)^{\widetilde{m}_\mu(t,x)}} \tag{12.3}$$

for any $x \in K$ *and* $t > 0$.

Proof. First assume that $\delta(x,y) < t$. Then by the definition of $\delta(\cdot,\cdot)$, if follows that $y \in V_{\widetilde{m}(t,x)}(x)$. Conversely, if $y \in V_{\widetilde{m}(t,x)}(x)$, then $k(x,y) \leq \widetilde{m}(t,x)$ and $\epsilon(m,x) < t \leq \epsilon(m-1,x)$, where $m = \widetilde{m}(t,x)$, by Lemma 12.4. This implies $\delta(x,y) = \epsilon(k(x,y),x) \leq \epsilon(m,x) < t$. Thus we have obtained (12.2).

By the definition of $\epsilon(m,x)$, it follows that $\epsilon(\widetilde{m}(t,x),x) < t$. Hence

$$\mu(V_{\widetilde{m}(t,x)}(x)) = \frac{(r_*)^{\widetilde{m}(t,x)}\mu(V_{\widetilde{m}(t,x)}(x))}{(r_*)^{\widetilde{m}(t,x)}} \leq \frac{\epsilon(\widetilde{m}(t,x),x)}{(r_*)^{\widetilde{m}(t,x)}} \leq \frac{t}{(r_*)^{\widetilde{m}(t,x)}}$$

□

The above proposition implies that δ_μ gives the same topology on K as d_*. More precisely, we have the following fact.

Corollary 12.7. *Define*

$$\mathcal{O}_{\delta_\mu} = \{O|O \subseteq K, \textit{for any } x \in O, \textit{ there exists } r > 0 \textit{ such that } B_{\delta_\mu}(x,r) \subseteq O\}.$$

Then $\mathcal{O}_{\delta_\mu}$ *coincides with the collection of open sets with respect to* d_*.

Now we start to study diagonal lower heat kernel estimate. In the rest of this section, $\mu \in \mathcal{M}_P(K)$ is assumed to have $(\eta, \mathbf{p}, \kappa)$-weak exponential decay. By the results of the last section, there exists a jointly continuous heat kernel $p_\mu(t,x,y)$.

To begin with, we have an upper estimate of exit time in terms of the volume of a neighborhood $V_m(x)$.

Lemma 12.8. *If* μ *has* $(\eta, \mathbf{p}, \kappa)$*-weak exponential decay, then there exists* $c_{12.4} > 0$ *such that*

$$\sup_{y \in V_m(x)} \widetilde{E}_y(\tau_{V_m(x)}) \leq c_{12.4}(r_*)^m \mu(V_m(x)) \times \begin{cases} \overline{\kappa}(m) + 1 & \textit{if } \alpha > 0, \\ \overline{\kappa}(m)^2 + 1 & \textit{if } \alpha = 0 \end{cases} \tag{12.4}$$

for any $x \in K$ *and any* $m \geq 1$.

Proof. Note that μ_w has $(\eta, \mathbf{p}, \kappa_{|w|})$-weak exponential decay, where $\overline{\kappa}_m(k) = \overline{\kappa}(k+m)$ and $\underline{\kappa}_m(k) = \underline{\kappa}(k+m)$. Hence by (11.11) and (11.12),

$$h_{\mu_w}(\emptyset) \leq \begin{cases} c_{6.5}\eta(\overline{\kappa}(|w|) + 1) & \text{if } \alpha > 0, \\ c_{6.6}\eta(\overline{\kappa}(|w|)^2 + 1) & \text{if } \alpha = 0. \end{cases}$$

Combining this with (7.7), we obtain (12.4). □

We also have a lower estimate of the exit time as follows.

Lemma 12.9.

$$\widetilde{E}_x(\tau_{V_m(x)}) \geq c_{7.8}(r_*)^m \mu(V_{m+1}(x)) \tag{12.5}$$

for any $x \in K$ *and* $m \geq 1$.

PROOF. This follows immediately by (7.8). □

Next we present three estimates concerning exit time and a heat kernel, which are know to hold in general setting of diffusion processes on metric measure spaces. The following fact has been obtained in the proof of [**28**, Lemma 3.12].

LEMMA 12.10. *Let U be an open subset of K. If $x \in U$, then*

$$\widetilde{E}_x(\tau_U) \leq t + \widetilde{P}_x(\tau_U > t) \sup_{y \in U} \widetilde{E}_y(\tau_U) \tag{12.6}$$

LEMMA 12.11. *Let U be an open subset of K. Then for any $x \in U$,*

$$\widetilde{P}_x(\tau_U > t)^2 \leq \mu(U) p_\mu(2t, x, x). \tag{12.7}$$

PROOF.

$$\begin{aligned}\widetilde{P}_x(\tau_U > t)^2 \leq \widetilde{P}_x(X_t \in U)^2 = \left(\int_U p_\mu(t, x, y) \mu(dy) \right)^2 \\ \leq \mu(U) \int_U p_\mu(t, x, y)^2 \mu(dy) = \mu(U) p_\mu(2t, x, x)\end{aligned}$$

□

By Lemmas 12.10 and 12.11, we have the following lower estimate of $p_\mu(2t, x, x)$.

LEMMA 12.12. *Let U be an open subset containing x. If $\widetilde{E}_x(\tau_U) > t$, then*

$$\left(\frac{\widetilde{E}_x(\tau_U) - t}{\sup_{y \in U} \widetilde{E}_x(\tau_U)} \right)^2 \frac{1}{\mu(U)} \leq p_\mu(2t, x, x). \tag{12.8}$$

REMARK. The inequality (12.8) holds without assuming that μ has weak exponential decay as long as $\mu \in \mathcal{M}_P^{TC}(K)$.

Combining the previous lemmas, we obtain the following lower estimate of $p_\mu(t, x, x)$ for a special $t = t_m$.

LEMMA 12.13. *Assume that μ has $(\eta, \mathbf{p}, \kappa)$-weak exponential decay. Define $t_m = \frac{1}{2} c_{7.8} (r_*)^m \mu(V_{m+1}(x))$ and set*

$$\overline{\kappa}^*(m) = \begin{cases} (\overline{\kappa}(m) + 1)^{-2} & \text{if } \alpha > 0, \\ (\overline{\kappa}(m)^2 + 1)^{-2} & \text{if } \alpha = 0. \end{cases}$$

Then

$$c_{12.9} \overline{\kappa}^*(m) \left(\frac{\mu(V_{m+1}(x))}{\mu(V_m(x))} \right)^2 \frac{1}{\mu(V_m(x))} \leq p(2t_m, x, x) \tag{12.9}$$

for any $m \geq 0$ and $x \in K$, where $c_{12.9} = \frac{1}{4}(c_{7.8}/c_{12.4})^2$.

PROOF. If $U = V_m(x)$, then (12.8) yields

$$\left(\frac{\widetilde{E}_x(\tau_{V_m(x)}) - t}{\sup_{y \in V_m(x)} \widetilde{E}_x(\tau_{V_m(x)})} \right)^2 \frac{1}{\mu(V_m(x))} \leq p_\mu(2t, x, x).$$

Setting $t = t_m$ and making use of (12.4) and (12.5), we obtain (12.9). □

Now we have diagonal lower estimate of the heat kernel $p_\mu(t, x, x)$.

THEOREM 12.14. *Assume that μ has $(\eta, \mathbf{p}, \kappa)$-weak exponential decay. Define* $\gamma_* = r_*/c_{7.8}$,

$$m_\mu(t,x) = \widetilde{m}_\mu(\gamma_* t, x) - 2, \tag{12.10}$$

and

$$C^*_\mu(t,x) = c_{12.9}\overline{\kappa}^*(m_\mu(t,x))\Big(\frac{\mu(V_{m_\mu(t,x)+1}(x))}{\mu(V_{m_\mu(t,x)}(x))}\Big)^3\Big(\frac{\mu(V_{m_\mu(t,x)+2}(x))}{\mu(V_{m_\mu(t,x)+1}(x))}\Big).$$

Then

$$C^*_\mu(t,x)\frac{(r_*)^{\widetilde{m}_\mu(\gamma_* t,x)}}{t} \le \frac{C^*_\mu(t,x)}{\mu(B_{\delta_\mu}(x,\gamma_* t))} \le p_\mu(t,x,y) \tag{12.11}$$

for any $t \in (0,1]$ and $x \in K$.

REMARK. If μ has weak exponential decay, then (12.1) is satisfied. Therefore, we may use the results on protodistances obtained before in the following proof.

PROOF. It follows that

$$m_\mu(t,x) = \max\{m | c_{7.8}(r_*)^m \mu(V_{m+1}(x)) \ge t\}.$$

By (12.9), the above equality yields

$$c_{12.9}\overline{\kappa}^*(m_\mu(t,x))\Big(\frac{\mu(V_{m_\mu(t,x)+1}(x))}{\mu(V_{m_\mu(t,x)}(x))}\Big)^2\frac{1}{\mu(V_{m_\mu(t,x)}(x))} \le p_\mu(t,x,x)$$

Since the left-hand side of this inequality equals $C^*_\mu(t,x)\mu(V_{m_\mu(t,x)+2}(x))^{-1}$, we obtain (12.11) by Proposition 12.6. $\Box$

As $t \downarrow 0$, the part $\mu(B_{\delta_\mu}(x,\gamma_* t))^{-1}$ is expected to be the principal part of the estimate (12.11). In fact, we are going to show that $\liminf_{t\downarrow 0} C^*_\mu(t,x)|\log t|^9 > 0$ for μ-a.e. $x \in K$ in Theorem 12.16. Furthermore, if μ has the volume doubling property and uniform exponential decay, then $C^*_\mu(t,x)$ is bounded from below by a constant independent of t and x.

The next lemma has essentially obtained by Andres and Kajino in [**1**]. They have used it to show a lower diagonal estimate of the heat kernels of Liouville Brownian motions. We modify their result in accordance with our framework.

LEMMA 12.15. *Let μ be a Borel regular probability measure on K and let $\{a_n\}_{n\ge 1}$ be a positive sequence. If $\sum_{n\ge 1} 1/a_n < +\infty$, then for μ-a.e. $x \in K$, there exists $n(x) \in \mathbb{N}$ such that $a_m\mu(V_m(x)) \ge \mu(V_{m-1}(x))$ for any $m \ge n(x)$.*

PROOF. For $w \in W_m$, set $V^0_m(w) = V^0_m(K_w)$ and $V^1_m(w) = V^1_m(K_w)$. Then $V^0_m(w) \subseteq V_m(x) \subseteq V^1_m(w)$ if $x \in K_w$. Note that $\#(\Gamma^1_m(K_w)) \le 5^n$. Define $\overline{w} = w_1 \dots w_{m-1}$ for any $w = w_1 \dots w_m$. Using Lemma 2.11, we obtain

$$\begin{aligned}\int_K \frac{\mu(V_{m-1}(x))}{\mu(V_m(x))}\mu(dx) \le \sum_{w\in W_m}\int_{K_w}\frac{\mu(V_{m-1}(x)))}{\mu(V_m(x))}\mu(dx) \le \\ \sum_{w\in W_m}\frac{\mu(V^1_{m-1}(\overline{w}))}{\mu(V^0_m(w))}\mu(K_w) \le \sum_{w\in W_m}\mu(V^1_{m-1}(\overline{w})) \\ \le 5^n N\sum_{w\in W_{m-1}}\mu(K_w) \le 10^n N\end{aligned} \tag{12.12}$$

Let

$$A_m = \{x | x \in K, a_m\mu(V_m(x)) \leq \mu(V_{m-1}(x))\}.$$

By (12.12),

$$a_m\mu(A_m) \leq 10^n N$$

and hence $\sum_{m\geq 1} \mu(A_m) < +\infty$. Now the Borel-Cantelli lemma implies the desired conclusion. □

THEOREM 12.16. *Assume that μ has $(\eta, \mathbf{p}, \kappa)$-weak exponential decay. Then there exist $c_{12.13} > 0$, $q \in [0, 9]$ and $\{T_x\}_{x\in K}$ such that $T_x > 0$ for μ-a.e. $x \in K$ and if $t \in (0, T_x]$, then*

$$\frac{c_{12.13}}{|\log t|^q} \leq C_\mu^*(t, x). \tag{12.13}$$

REMARK. By the following proof, one can see that if $\alpha > 0$, then $q = 6 + \epsilon$ for any $\epsilon > 0$ and if $\alpha = 0$, then $q = 8 + \epsilon$ for any $\epsilon > 0$.

PROOF. Since $(r_*)^{|w|}\mu(K_w) \leq c_1\lambda^{|w|}$ for any $w \in W_*$, we have $(r_*)^m V_{m+1}(x) \leq c_2\lambda^m$. By the definition of $m_\mu(t, x)$, it follows that $m_\mu(t, x) \leq c_3|\log t|$ for any $t \in (0, 1]$.

Let $a_m = (m - 1)^{1+\epsilon}$ for some $\epsilon > 0$. Then $\sum_{m\geq 2} \frac{1}{a_m} < +\infty$. Lemma 12.15 implies that for μ-a.e. $x \in K$, $m^{-(1+\epsilon)} \leq \mu(V_{m+1}(x))/\mu(V_m(x))$ for sufficiently large m. Hence,

$$\frac{c_4}{|\log t|^{1+\epsilon}} \leq \frac{V_{m_\mu(t,x)+1}(x)}{V_{m_\mu(t,x)}(x)} \tag{12.14}$$

for sufficiently small $t > 0$.

On the other hand, $\overline{\kappa}(m) \leq c_5 m$ for any $m \geq 1$. Hence if $\alpha > 0$, then $\overline{\kappa}^*(m) \geq c_6 m^{-2}$ for sufficiently large m. Moreover, it follows that $m^{-(1+\epsilon)}(m + 1)^{-(1+\epsilon)} \leq \mu(V_{m+2}(x))/\mu(V_m(x))$. Hence combining these with (12.14), we obtain (12.13). If $\alpha = 0$, then the arguments are entirely the same except that $\overline{\kappa}^*(m) \geq c_7 m^{-4}$. □

Since $r_* = 1$ if $\alpha = 0$, we immediately obtain the next corollary.

COROLLARY 12.17. *Assume that μ has $(\eta, \mathbf{p}, \kappa)$-weak exponential decay and that $\alpha = 0$. Then there exists $q \in [0, 9]$ such that for μ-a.e. $x \in K$,*

$$\frac{c_{12.13}}{|\log t|^q t} \leq p_\mu(t, x, y)$$

for sufficiently small $t > 0$. In particular,

$$1 \leq \liminf_{t\downarrow 0} -\frac{\log p_\mu(t, x, x)}{\log t}$$

for μ-a.e. $x \in K$. Furthermore, if $\lim_{x\to\infty} \underline{\kappa}(x)/x = 0$. then

$$\lim_{t\downarrow 0} -\frac{\log p_\mu(t, x, x)}{\log t} = 1$$

for μ-a.e. $x \in K$.

13. Proof of Theorem 1.1

In this section, we are going to give a proof of Theorem 13.1, which is an exact restatement of Theorem 1.1 when $K = [0,1]^2$. As in the previous sections, K is a generalized Sierpinski carpet GSC(n,l,S).

THEOREM 13.1. *Assume that $\alpha = 0$. Let μ be a Borel regular probability measure on K. Suppose that there exist positive constants $c_1, c_2, \alpha_1, \alpha_2$ such that $\alpha_1 \geq \alpha_2$ and*

$$c_1 r^{\alpha_1} \leq \mu(B_*(x,r)) \leq c_2 r^{\alpha_2}. \tag{13.1}$$

for any $x \in K$ and $r \in (0,1]$. Then μ has weak exponential decay, there exists a jointly continuous heat kernel $p_\mu(t,x,y)$ on $(0,\infty)\times K \times K$ associated with the time change of the Brownian motion with respect to μ and there exist $\gamma_* > 0$, $\{T_x\}_{x\in K} \subseteq [0,\infty)$ and $c_1 > 0$ such that $T_x > 0$ for μ-a.e. $x \in K$ and

$$\frac{c_1}{t|\log t|^9} \leq \frac{c_1}{\mu(B_{\delta_\mu}(x,\gamma_* t))|\log t|^9} \leq p_\mu(t,x,x) \tag{13.2}$$

for any $t \in (0, T_x]$. Furthermore, if there exists a monotonically non-increasing function $f : (0,\infty) \to [1,\infty)$ such that

$$\mu(B_*(x,2r)) \leq f(r)\mu(B_*(x,r)) \tag{13.3}$$

for any $x \in K$ and $r > 0$, and

$$\lim_{r\downarrow 0} \frac{\log f(r)}{\log r} = 0, \tag{13.4}$$

then

$$\lim_{t\downarrow 0} -\frac{\log p_\mu(t,x,x)}{\log t} = 1 \tag{13.5}$$

for any $x \in K$.

The condition (13.3) is a kind of weak volume doubling property. Note that the volume doubling property corresponds to the case when $f(r)$ is bounded. There is a slight difference between Corollary 12.17 and this theorem. Namely, in this theorem, (13.5) holds for any $x \in K$ while it holds only for μ-a.e. $x \in K$ in Corollary 12.17.

PROOF. By Proposition 11.2, μ has weak exponential decay. The existence of the heat kernel and (13.2) is immediately verified by Theorems 12.14, 12.16 and Corollary 12.17.

For any $r_1 > 0$, we define

$$k(r_1) = \min\{m | m \in \mathbb{N}\cup\{0\}, 2^m \geq r_1\} \quad \text{and} \quad f(r,r_1) = \prod_{i=1}^{k(r_1)} f(2^{i-1}r)$$

Then

$$\mu(B_*(x,r_1 r)) \leq f(r,r_1)\mu(B_*(x,r)) \tag{13.6}$$

and by (13.4)

$$\lim_{r\downarrow 0} \frac{\log f(r,r_1)}{\log r} = 0. \tag{13.7}$$

Choose $z \in K$ and $R > 0$ so that $\{y | y \in \mathbb{R}^n, |z - y| < R\} \subseteq [0,1]^n$. Then for any $w \in W_*$ we have $B_*(F_w(z), Rl^{-|w|}) \subseteq K_w$. Set $z_w = F_w(z)$. Note that

$$K_w \subseteq B_*(x, 2\sqrt{n}l^{-|w|}) \subseteq B_*(z_w, 3\sqrt{n}l^{-|w|})$$

for any $x \in K_w$. Let $f_1(r) = f(r, 2\sqrt{n}l/R)$. Then by (13.3)

$$\begin{aligned}\mu(K_{wi}) &\geq \mu(B_*(z_{wi}, Rl^{-|w|-1})) \\ &\geq \frac{\mu(B_*(z_{wi}, 2\sqrt{n}l^{-|w|}))}{f_1(Rl^{-|w|-1})} \geq \frac{\mu(K_w)}{f_1(Rl^{-|w|-1})}.\end{aligned} \tag{13.8}$$

Set $\eta_0 = f_1(Rl^{-1})$ and define

$$\underline{\kappa}(m) = \frac{1}{\log l}\Big(\log f_1(Rl^{-m-1}) - \log \eta_0\Big).$$

Then we see that

$$\mu(K_{wi}) \geq \frac{1}{\eta_0} l^{-\underline{\kappa}(|w|)} \mu(K_w)$$

and by (13.7) we also have

$$\lim_{m\to\infty} \frac{\underline{\kappa}(m)}{m} = 0. \tag{13.9}$$

By Corollary 11.10 and (13.9),

$$\limsup_{t\downarrow 0} -\frac{\log p_\mu(t,x,x)}{\log t} \leq 1. \tag{13.10}$$

for any $x \in K$. Next note that

$$B_*(x, l^{-m}) \subseteq V_m(x) \subseteq B_*(x, 3\sqrt{n}l^{-m}).$$

Define $f_2(r) = f(r, 3\sqrt{n}l)$. Then

$$\begin{aligned}\mu(V_m(x)) &\leq \mu(B_*(x, 3\sqrt{n}l^{-m})) \\ &\leq f_2(l^{-m-1})\mu(B_*(x, l^{-m-1})) \leq f_2(l^{-m-1})\mu(V_{m+1}(x))\end{aligned}$$

By the definition of $C^*_\mu(t,x)$ given in Theorem 12.14,

$$C^*_\mu(t,x) \geq c\frac{\overline{\kappa}^*(m)}{f_2(l^{-m-1})^3 f_2(l^{-m-2})}, \tag{13.11}$$

where $m = m_\mu(t,x)$ and c is independent of t and x. Recalling the proof of Theorem 12.16, we see that $m_\mu(t,x) \leq c_3|\log t|$. This fact along with (13.7) shows that

$$\frac{\log f_2(l^{-m_\mu(t,x)-1})}{|\log t|} \leq \frac{\log f_2(l^{-m_\mu(t,x)-1})}{m_\mu(t,x)} \frac{m_\mu(t,x)}{|\log t|} \to 0$$

as $t \downarrow 0$. Again by the proof of Theorem 12.16, it follows that $\overline{\kappa}^*(m) \geq c_6 m^{-4}$ for sufficiently large m. Making use of (13.11), we obtain

$$\lim_{t\downarrow 0} -\frac{\log C^*_\mu(t,x)}{\log t} = 0$$

and hence by (12.11) it follows that

$$\liminf_{t\downarrow 0} -\frac{\log p_\mu(t,x,x)}{\log t} \geq 1$$

for any $x \in K$. This, together with (13.10), completes the proof. □

14. Random measures having weak exponential decay

In this section, we study a class of random measures $\{P^\nu_\omega\}_{\omega\in\Omega}$ and prove that they almost surely have weak exponential decay. Our random measure P^ν_ω can be though of as a random self-similar measure where a weight $(\mu_i)_{i\in S}$ is randomly chosen in every step according to a probability measure ν on the space of weights Δ_N.

Definition 14.1. Define $\Delta_N \subseteq \mathbb{R}^N$ by

$$\Big\{(x_1,\ldots,x_N)\Big|\sum_{i=1}^N x_i = 1, x_i\in[0,1] \text{ for any } i\in\{1,\ldots,N\}\Big\}$$

Let $\mathcal{B}$ be the collection of Borel sets of Δ_N. For a Borel regular probability measure ν on Δ_N, let $\{(\Delta_{N,w},\mathcal{B}_w,\nu_w)\}_{w\in W_*}$ be a collection of independent copies of $(\Delta_N,\mathcal{B},\nu)$ and define $(\Omega,\mathcal{F},\mathbb{P}_\nu)$ as the direct product $\prod_{w\in W_*}(\Delta_w,\mathcal{B}_w,\nu_w)$. For any $\omega=\{\omega_w\}_{w\in W_*}\in\Omega$, we define a probability measure $\widetilde{P}^\nu_\omega$ on Σ as the unique probability measure satisfying

$$\widetilde{P}^\nu_\omega(\Sigma_{w_1\ldots w_m}) = \omega_\emptyset(w_1)\omega_{w_1}(w_2)\omega_{w_1w_2}(w_3)\cdots\omega_{w_1\ldots w_{m-1}}(w_m)$$

for any $w_1\ldots w_m\in W_*$, where $\omega_w=(\omega_w(1),\ldots,\omega_w(N))\in\Delta_N$.

Note that the measures $\{\widetilde{P}^\nu_\omega\}_{\omega\in\Omega}$ are measures on the Cantor set Σ. Using the canonical map $\pi:\Sigma\to K$, we obtain induced measures $\{\widetilde{P}^\nu_\omega\circ\pi^{-1}\}_{\omega\in\Omega}$ on the generalized Sierpinski carpet K.

Such random measures have been considered by Falconer [**18**]. In his case, however, spaces are also randomized, i.e. there is randomness in contraction ratios of the collection of similitudes which characterizes the space. We remark that wider classes of random self-similar measures have been studied by many authors, for example, [**23, 38, 2**].

Throughout this section, we fix a Borel regular probability measure ν on Δ_N satisfying the following assumption.

Assumption 14.2. $\nu(\Delta_N\cap(0,1/r_*)^N)=1$ and there exists $q>0$ such that

$$\int_{\Delta_N}(x_i)^{-q}d\nu<+\infty$$

for any $i=1,\ldots,N$.

Recall that $\pi:\Sigma\to K$ is the natural surjective map given by $\{\pi(i_1i_2\ldots)\}=\cap_{j\geq1}K_{i_1\ldots i_j}$.

Definition 14.3. For any $\omega\in\Omega$, define a probability measure P^ν_ω by $P^\nu_\omega(A)=\widetilde{P}^\nu_\omega(\pi^{-1}(A))$ for any Borel set $A\subseteq K$.

P^ν_ω is the random measure which we are going to study. We use E_ν and $\mathbb{E}_\nu$ to denote the expectation with respect to ν and $\mathbb{P}_\nu$ respectively. If no confusion can occur, we use $\mathbb{P},\widetilde{P}_\omega,P_\omega$ and $\mathbb{E}$ in place of $\mathbb{P}_\nu,\widetilde{P}^\nu_\omega,P^\nu_\omega$ and $\mathbb{E}_\nu$ respectively.

By Lemma 2.7, π is one to one on $\pi^{-1}(K\backslash V_*)$. The next proposition shows that for $\mathbb{P}_\nu$-a.e. ω, $\widetilde{P}^\nu_\omega(A)=P^\nu_\omega(\pi(A))$ for any Borel set $A\subseteq\Sigma$. In other words, we may identify two probability spaced $(\Sigma,\widetilde{P}^\nu_\omega)$ and (K,P^ν_ω) in the measurable sense.

PROPOSITION 14.4. *Under Assumption 14.2, for $\mathbb{P}_\nu$-a.e. ω, $P_\omega^\nu(V_*) = 0$. In particular, for $\mathbb{P}_\nu$-a.e. ω,*

$$P_\omega^\nu(K_w) = \widetilde{P}_\omega^\nu(\Sigma_w) \tag{14.1}$$

for any $w \in W_*$.

To prove the above proposition, we use the following lemma.

LEMMA 14.5. *Let $J \subseteq \{1, \dots, N\}$. If $E_\nu(\sum_{j\in J} x_j) < 1$, then, for $\mathbb{P}_\nu$-a.e. ω,*

$$\widetilde{P}_\omega^\nu(wJ^{\mathbb{N}}) = 0$$

for any $w \in W_*$, where $wJ^{\mathbb{N}} = \{wj_1j_2\dots | j_i \in J$ for any $i \in \mathbb{N}\}$.

PROOF. Set $Z = E_\nu(\sum_{j\in J} x_j)$. Define

$$F_m(\omega) = \frac{1}{Z^m} \sum_{v\in J^m} P_\omega(\Sigma_{wv}).$$

Then

$$F_m(\omega) = \frac{1}{Z^m} P_\nu(\Sigma_w) \sum_{v_1\dots v_m\in J^m} \omega_w(v_1)\omega_{wv_1}(v_2)\cdots\omega_{wv_1\dots v_{m-1}}(v_m).$$

Define $\mathcal{B}^m$ as the Borel σ-algebra of $\prod_{w\in W_m, |w|\le m} \Delta_{N,w}$ and $\mathcal{F}_m = \{A | A \subseteq \Omega, A = B \times \prod_{w\in W_*, |w|>m} \Delta_{N,w}, B \in \mathcal{B}^m\}$. Then $\{F_m\}_{m\ge0}$ is a $\mathbb{P}_\nu$-martingale with respect to the filtration $\{\mathcal{F}_m\}_{m\ge0}$. By the martingale convergence theorem, for $\mathbb{P}_\nu$-a.e. ω, $F(\omega) = \lim_{m\to\infty} F_m(\omega)$ exists and is finite. Then

$$P_\omega(wJ^{\mathbb{N}}) \le Z^m F_m(\omega) \to 0$$

as $m \to \infty$. □

PROOF OF PROPOSITION 14.4. By Proposition 2.6 and Lemma 2.7, we see that

$$V_* = \bigcup_{w\in W_*} \bigcup_{i=1,\dots,n, j=1,2} F_w(B_{i,j}) \quad \text{and} \quad \pi^{-1}(V_*) = \bigcup_{w\in W_*} \bigcup_{i=1,\dots,n, j=1,2} w(S_{i,j})^{\mathbb{N}}.$$

By Assumption 14.2, it follows that $E_\nu(\sum_{k\in S_{i,j}} x_k) < 1$. Using the above lemma, we see that for $\mathbb{P}_\nu$-a.e. ω, $\widetilde{P}_\nu(w(S_{i,j})^{\mathbb{N}}) = 0$ for any $w \in W_*$ and i, j. Therefore, $\widetilde{P}_\omega^\nu(\pi^{-1}(V_*)) = 0$ and hence $P_\omega^\nu(V_*) = 0$. □

Next we show that P_ω^ν has weak exponential decay with linear $\overline{\kappa}$ and $\underline{\kappa}$ almost surely.

THEOREM 14.6. *Under Assumption 14.2, for $\mathbb{P}_\nu$-a.e. ω, there exist $\eta_\omega \ge 1$, $\mathbf{p} = (\overline{p}, \underline{p}) \in (0, 1/r_*)^2$ and $a_\omega, b_\omega > 0$ such that if $\overline{\kappa}_\omega(s) = a_\omega s$ and $\underline{\kappa}_\omega(s) = b_\omega s$ for any $s \ge 0$, then P_ω^ν has $(\eta_\omega, \mathbf{p}, \kappa_\omega)$-weak exponential decay, where $\kappa_\omega = (\overline{\kappa}_\omega, \underline{\kappa}_\omega)$.*

To prove this theorem, we need the next two lemmas.

LEMMA 14.7. *Let $(r_1, \dots, r_N) \in (0, 1)^N$. Define $r_w = r_{w_1}\cdots r_{w_m}$ for any $w = w_1 \dots w_m \in W_*$. If*

$$\sum_{j=1}^{N} \frac{E_\nu((x_i)^q)}{(r_i)^q} < 1 \tag{14.2}$$

for some $q \geq 1$, then for $\mathbb{P}_\nu$-a.e. $\omega \in \Omega$, there exist $\eta_\omega \geq 1$ and $a_\omega > 0$ such that if $|v| \geq a_\omega |w|$, then

$$P^\nu_\omega(K_{wv}) \leq \eta_\omega r_v P^\nu_\omega(K_w).$$

Proof. We may assume that (14.1) holds for any $w \in W_*$. Set $f_{\omega,w}(v) = P^\nu_\omega(\Sigma_{wv})/P^\nu_\omega(\Sigma_w)$. For any $v = v_1 \dots v_k$,

$$f_{\omega,w}(v) = \prod_{i=1,\dots,k} \omega_{wv_1\dots v_{i-1}}(v_i)$$

Hence we have

$$E_\nu((f_{\omega,w}(v))^q) = \prod_{i=1}^k E_\nu((x_{v_i})^q) = \prod_{i=1}^k \int_{\Delta_N} (x_{v_i})^q \nu(dx). \tag{14.3}$$

Define $A_k(w) = \{\omega | \omega \in \Omega, f_{\omega,w}(v) > r_v \text{ for some } v \in W_k\}$. Using Chebyshev's inequality and (14.3), we obtain

$$\begin{aligned}\mathbb{P}(A_k(w)) &\leq \sum_{v \in W_k} \mathbb{P}(f_{\omega,w}(v) \geq r_v) \\ &\leq \sum_{v \in W_k} \frac{\mathbb{E}(f_{\omega,w}(v)^q)}{(r_v)^q} = \sum_{v \in W_k} \prod_{i=1}^k \Big(\frac{E_\nu((x_{v_i})^q)}{(r_{v_i})^q}\Big) = \left(\sum_{j=1}^N \frac{E_\nu((x_i)^q)}{(r_i)^q}\right)^k.\end{aligned}$$

Hence if (14.2) holds, then the Borel-Cantelli lemma implies that there exists $m \in \mathbb{N}$ such that $f_{\omega,w}(v) < r_v$ if $|v| \geq m$. Define $M(\omega, w)$ as the minimum of such m. Then $\{\omega | M(\omega, w) > k\} = \cup_{i \geq k} A_i(w)$. Hence

$$\begin{aligned}\mathbb{P}(M(\omega,w) \geq k \text{ for some } w \in W_m) &\leq \sum_{w \in W_m} \mathbb{P}(M(\omega,w) \geq k) \\ &= \sum_{w \in W_m} \mathbb{P}(\cup_{i \geq k} A_i(w)) \leq \lambda^k \frac{N^m}{1-\lambda},\end{aligned}$$

where $\lambda = \sum_{j=1}^N \frac{E_\nu((x_i)^q)}{(r_i)^q}$. Choose $L \in \mathbb{N}$ so that $\lambda^L N < 1$. Then the above inequality implies

$$\sum_{m \geq 0} \mathbb{P}(M(\omega,w) \geq Lm \text{ for some } w \in W_m) \leq \sum_{i=0}^\infty \frac{(\lambda^L N)^m}{1-\lambda} < +\infty.$$

Again by the Borel-Cantelli lemma, for $\mathbb{P}$-a.e. ω, there exists $k \in \mathbb{N}$ such that if $|w| \geq k$, then $M(\omega, w) \leq L|w|$. Hence if $a_\omega = \max_{w \in W_* \setminus W_0} M(\omega,w)/|w|$ and $\eta_\omega = \sup_{v \in W_*} (r_v)^{-1} P^\nu_\omega(K_v)$, then a_ω and η_ω are finite and we have the desired statement. $\square$

Lemma 14.8. *Let $(r_1, \dots, r_N) \in (0,1)^N$. If*

$$\sum_{i=1}^N E_\nu((x_i)^{-q})(r_i)^q < 1 \tag{14.4}$$

for some $q > 0$, then for $\mathbb{P}_\nu$-a.e. $\omega \in \Omega$, there exist $\eta_\omega \geq 1$ and $\beta_\omega \in \mathbb{N}$ such that if $|v| \geq b_\omega |w|$, then

$$P^\nu_\omega(\Sigma_{wv}) \geq \frac{1}{\eta_\omega} r_v P^\nu_\omega(\Sigma_w)$$

for any $w \in W_*$.

Proof. We use the same notation as in the proof of Lemma 14.7. Define $B_k(w) = \{\omega | \omega \in \Delta_N, f_{\omega,w}(v) < r_v \text{ for some } v \in W_k\}$. Using Chebyshev's inequality and (14.3), we obtain

$$\mathbb{P}(B_k(w)) \leq \sum_{v\in W_k} \mathbb{P}(f_{\omega,w}(v)^{-q} \geq (r_v)^{-q}) \leq \sum_{v\in W_k} \mathbb{E}(f_{\omega,w}(v)^{-q})(r_v)^q$$
$$= \sum_{v\in W_k} \prod_{i=1}^{N} E_\nu((x_{w_i})^{-q})(r_{w_i})^q = \left(\sum_{j=1}^{N} E_\nu((x_i)^{-q})(r_i)^q\right)^k.$$

The rest is entirely analogous to the counterpart of the proof of Lemma 14.7. □

Proof of Theorem 14.6. By Assumption 14.2,

$$P^\nu_\omega(K_{wv}) \leq (r_*)^{-|v|} P^\nu_\omega(K_w)$$

for any $w, v \in W_*$. Again by Assumption 14.2, for any $i \in \{1, \ldots, N\}$,

$$E_\nu((r_* x_i)^q) \to 0$$

as $q \to \infty$. Hence for sufficiently large q, we may choose $\overline{p} \in (0, 1/r_*)$ so that

$$\sum_{i=1}^{N} \frac{E_\nu((x_i)^q)}{\overline{p}^q} < 1.$$

By Lemma 14.7, for $\mathbb{P}_\nu$-a.e. ω, if $\overline{\kappa}_\omega(x) = a_\omega x$, then we have (11.4) and (11.5).

By Assumption 14.2, there exists $\underline{p} \in (0, 1/r_*)$ such that

$$\sum_{i=1}^{N} E_\nu((x_i)^q)\underline{p}^q < 1.$$

Hence by Lemma 14.8, we have (11.6), (11.7) and (11.8). □

By Theorem 14.6, P^ν_ω has weak exponential decay with $\underline{\kappa}(x) = b_\omega x$. If ν decays rapidly near the boundary of Δ_N, we have better $\underline{\kappa}$.

Theorem 14.9. *Define $F_\nu(t) = \nu(\Delta_M \cap [0, t]^N)$. Let $r \in (0, 1)$ and let $\underline{\kappa} : [0, \infty) \to [0, \infty)$ be monotonically nondecreasing. If*

$$\sum_{m=0}^{\infty} N^m F_\nu(r^{\underline{\kappa}(m)}) < +\infty, \tag{14.5}$$

then for $\mathbb{P}_\nu$-a.e. ω, there exists $c_\omega > 0$ such that

$$\nu(K_{wi}) \geq c_\omega r^{\underline{\kappa}(|w|)} \nu(K_w) \tag{14.6}$$

for any $w \in W_*$ and $i \in S$.

Proof. Set

$$Y_m = \{\omega | \text{there exist } w \in W_m, i \in S \text{ such that } \omega_w(i) < r^{\underline{\kappa}(m)}\}.$$

Then

$$\mathbb{P}_\nu(Y_m) \leq \sum_{w\in W_m, i\in S} \mathbb{P}_\nu(\omega_w(i) < r^{\underline{\kappa}(m)}) \leq N^{m+1} F_\nu(r^{\underline{\kappa}(m)}).$$

Since we have (14.5), the Borel-Cantelli lemma shows that for $\mathbb{P}_\nu$-a.e. ω, there exists $k \in \mathbb{N}$ such that $P^\nu_\omega(K_{wi}) \geq r^{\underline{\kappa}(|w|)} P^\nu_\omega(K_w)$ if $|w| \geq k$. Choosing sufficiently small $c_\omega > 0$, we verify (14.6). □

For example, if $\nu([0, t_*]^n) = 0$ for some $t_* > 0$, then we we may choose $\underline{\kappa}(x)$ as a constant. In case $\alpha = 0$, Corollary 12.17 implies the following assertion.

COROLLARY 14.10. *Assume that $\alpha = 0$. Let $r \in (0, 1)$ and let $\underline{\kappa} : [0, \infty) \to [0, \infty)$ is nondecreasing. If $\underline{\kappa}(x)/x \to 0$ as $x \to \infty$ and (14.5) is satisfied, then for $\mathbb{P}_\nu$-a.e. ω, P_ν^ω has weak exponential decay and*

$$\lim_{t \downarrow 0} -\frac{\log p_{P_\nu^\omega}(t, x, x)}{\log t} = 1$$

for P_ν^ω-a.e. $x \in K$.

15. Volume doubling measure and sub-Gaussian heat kernel estimate

Throughout the rest of this paper, we consider the case when μ has the volume doubling property with respect to d_*. The volume doubling property is known to be a necessary condition for sub-Gaussian heat kernel estimates. See [**27, 33, 28**] for example. So, if the volume doubling property holds, at least one may hope to show sub-Gaussian heat kernel estimate as (5.2). At the same time, however, by Theorem 15.3, it turns out to be hopeless to have sub-Gaussian heat kernel estimates with respect to d_* unless μ is absolutely continuous with respect to the normalized d_H-dimensional Hausdorff measure ν_*. Consequently we must have a metric that is different from d_* in our sub-Gaussian heat kernel estimates, if such a heat kernel estimate does hold at all. One candidate of such a metric is the "protodistance" δ_μ introduced in Section 12. Indeed, although δ_μ itself may not be a metric, with the volume doubling property it is going to produce a family of intrinsic metrics appearing in sub-Gaussian heat kernel estimates obtained in Theorem 15.7.

Throughout this section, we always assume that $\mu \in \mathcal{M}_P(K)$.

DEFINITION 15.1. Let $\mu \in \mathcal{M}_P(K)$.
(1) Let d be a metric on K. μ is said to have the volume doubling property with respect to d if and only if there exists $c > 0$ such that

$$\mu(B_d(x, 2r)) \leq c\mu(B_d(x, r))$$

for any $x \in K$ and $r > 0$.
(2) We say that μ has upper uniform exponential decay if and only if there exist $\eta \geq 1$ and $r \in (0, 1/(r_*))$ such that

$$\mu(K_{wv}) \leq \eta r^{|v|} \mu(K_w)$$

for any $w, v \in W_*$.

Immediately by the above definition, it follows that μ has upper exponential decay if and only if there exist $\eta \geq 1$ and $\lambda \in (0, 1)$ such that $\sigma_\mu(wv) \leq \eta \lambda^{|v|} \sigma_\mu(w)$. This fact yields the following proposition.

PROPOSITION 15.2. *Let $\mu \in \mathcal{M}_P(K)$. If μ has upper uniform exponential decay and the volume doubling property with respect to d_*, then μ has uniform exponential decay. In particular, in case $\alpha = 0$, if μ has the volume doubling property with respect to d_*, then it has uniform exponential decay.*

By this proposition, if μ has upper uniform exponential decay and the volume doubling property with respect to d_*, then $\mu \in \mathcal{M}_P^{TC}(K)$ and we have jointly continuous heat kernel $p_\mu(t, x, y)$.

PROOF. Since μ has upper uniform exponential decay, we have (11.5) with a bounded $\overline{\kappa}$. By [**34**, Theorem 1.3.5], the volume doubling property implies that μ is elliptic. Using Proposition 11.6-(3), we see that (11.7) holds with a bounded $\underline{\kappa}$. Thus μ has uniform exponential decay. $\square$

Now, we show that the volume doubling property and upper sub-Gaussian heat kernel estimate with respect to d_* imply the absolute continuity of μ with respect to the normalized Hausdorff measure ν_*.

THEOREM 15.3. *Let $\mu \in \mathcal{M}_P(K)$. Assume that μ has upper uniform exponential decay and the volume doubling property with respect to d_*. If there exist $\beta > 1, c^1_{15.1} > 0$ and $c^2_{15.1} > 0$ such that*

$$p_\mu(t,x,y) \le \frac{c^1_{15.1}}{\mu(B_*(x,t^{1/\beta}))} \exp\left(- c^2_{15.1}\Big(\frac{d_*(x,y)^\beta}{t}\Big)^{\frac{1}{\beta-1}}\right) \tag{15.1}$$

for any $x, y \in K$ and $t \in (0,1]$, then $\beta = d_w$ and there exist $c^1_{15.2}, c^2_{15.2} > 0$ such that

$$c^1_{15.2}\nu_*(A) \le \mu(A) \le c^2_{15.2}\nu_*(A) \tag{15.2}$$

for any Borel set $A \subseteq K$.

PROOF. By Proposition 15.2, μ has uniform exponential decay. Hence $\overline{\kappa}(x)$ is bounded. By (12.4) and (12.5), there exist $c_3 > 0$ and $c_4 > 0$ such that

$$c_3(r_*)^m\mu(V_{m+1}(x)) \le \widetilde{E}_x(\tau_{V_m(x)}) \le c_4(r_*)^m\mu(V_m(x))$$

for any $x \in K$ and $m \ge 0$. Note that $B_*(x, l^{-m}) \subseteq V_m(x) \subseteq B_*(x, 3\sqrt{n}l^{-m})$. The volume doubling property along with this fact and the above inequality imply that there exist $c_5 > 0$ and $c_6 > 0$ such that

$$c_5 r^{-\alpha}\mu(B_*(x,r)) \le \widetilde{E}_x(\tau_{B_*(x,r)}) \le c_6 r^{-a}\mu(B_*(x,r)) \tag{15.3}$$

for any $x \in K$ and $r \in (0,1]$.

On the other hand, applying [**33**, Theorem 2.10] and using the volume doubling property and (15.1), we see that there exist $c_7 > 0$ and $c_8 > 0$ such that

$$c_7 r^\beta \le \widetilde{E}_x(\tau_{B_*(x,r)}) \le c_8 r^\beta \tag{15.4}$$

for any $x \in K$ and $r \in (0,1]$. By (15.3) and (15.4), there exist $c_9 > 0$ and $c_{10} > 0$ such that

$$c_9 r^{\beta+\alpha} \le \mu(B_*(x,r)) \le c_{10} r^{\alpha+\beta} \tag{15.5}$$

for any $x \in K$ and $r \in (0,1]$. Since there exist $c_{11} > 0$, $c_{12} > 0$ and $\{x_w\}_{w\in W_*} \subseteq K$ such that $B_*(x_w, c_{11}l^{-|w|}) \subseteq K_w \subseteq B_*(x_w, c_{12}l^{-|w|})$ and $B_*(x_w, c_{11}l^{-|w|}) \cap \partial K_w = \emptyset$ for any $w \in W_*$, (15.5) imply

$$\begin{aligned} c_{11}^{\beta+\alpha}N^m l^{-m(\alpha+\beta)} &\le \mu(\cup_{w\in W_m} B_*(x_w, c_{11}l^{-m})) \le 1 \\ &\le \sum_{w\in W_m}\mu(B_*(x_w, c_{12}l^{-m})) \le c_{12}^{\alpha+\beta}N^m l^{-m(\alpha+\beta)}. \end{aligned}$$

for any $m \ge 0$. This yields $\alpha + \beta = d_H$ and hence $\beta = d_w$. Moreover,

$$(c_{11})^{d_H}\nu_*(K_w) \le \mu(K_w) \le (c_{12})^{d_H}\nu_*(K_w)$$

for any $w \in W_*$. Using [**32**, Theorem 1.4.10], we obtain (15.2). $\square$

To state our main theorem of this section, we need several notions. The first one is quasisymmetry which has been introduced by Tukia and Väisälä in [**41**] as a generalization of quasiconformal mappings in the complex plane.

Definition 15.4. Let d_1 and d_2 be metrics on K giving the same topology as d_*. d_1 is said to be quasisymmetric to d_2 if and only if there exists a homeomorphism h from $[0, +\infty)$ to itself such that $h(0) = 0$ and, for any $t > 0$, $d_1(x, z) < h(t)d_1(x, y)$ whenever $d_2(x, z) < td_2(x, y)$. If d_1 is quasisymmetric to d_2, we write $d_1 \underset{QS}{\sim} d_2$.

The relation $\underset{QS}{\sim}$ on metrics on K has been shown to be an equivalence relation in [**41**]. See also [**35**, Section 12]. Since quasisymmetric deformation of metrics distorts the balls in uniformly bounded fashion, it preserves the volume doubling property and the elliptic Harnack inequality.

Next, we introduce the notions of quasimetric and bi-Lipschitz equivalence. For the sake of later use, we give definitions of those notions on general set X.

Definition 15.5. Let X be a set.

(1) Let $\varphi : X \times X \to [0, \infty)$ and let $C > 0$. φ is called a C-quasimetric if and only if $\varphi(x, y) = \varphi(y, x) > 0$ for any $x \neq y \in K$, $\varphi(x, x) = 0$ for any $x \in X$ and

$$\varphi(x, z) \leq C(\varphi(x, y) + \varphi(y, z)). \tag{15.6}$$

for any $x, y, z \in X$. Moreover, φ is called a quasimetric if and only if φ is a C-quasimetric for some $C > 0$. For a quasimetric φ, we define C_φ as

$$C_\varphi = \sup_{x,y,z \in X, x \neq z} \frac{\varphi(x, z)}{\varphi(x, y) + \varphi(y, z)}. \tag{15.7}$$

(2) Let φ_1 and φ_2 be non-negative valued function on $X \times X$. We say that φ_1 is bi-Lipschitz equivalent to φ_2 if and only if there exist $c_1, c_2 > 0$ such that

$$c_1\varphi_1(x, y) \leq \varphi_2(x, y) \leq c_2\varphi_1(x, y)$$

for any $x, y \in X$. We write $\varphi_1 \underset{\mathrm{BL}}{\sim} \varphi_2$ if and only if φ_1 is bi-Lipschitz equivalent to φ_2.

The inequality (15.6) is called the extended (or weakened) triangle inequality. The quantity C_φ is the minimal value of $\{C|\varphi \text{ is } C\text{-quasimetric}\}$. Note that $C \geq 1$ in (15.6) because $\varphi(x, z) \leq C(\varphi(x, z) + \varphi(z, z)) = C\varphi(x, z)$.

Remark. There seems an ambiguity in the usage of the word "quasimetric" in mathematical community. Our definition is based on the book by Heinonen[**29**]. The same notion is called "near-metric" and the word "quasimetric" has different definition in Deza & Deza[**17**].

Now we come back to our assumption that X is generalized Sierpinski carpet K. The quasisymmetric equivalence $\underset{QS}{\sim}$ is weaker than the bi-Lipschitz equivalence $\underset{\mathrm{BL}}{\sim}$, i.e. if d and ρ are metrics on K and $d \underset{\mathrm{BL}}{\sim} \rho$, then $d \underset{QS}{\sim} \rho$.

We need one more definition to state our main theorem.

Definition 15.6. For a Borel regular probability measure μ on K, define

$$\mathfrak{B}_\mu = \{\beta|\beta > 0, \text{there exists a metric } d \text{ on } K \text{ giving the same topology as } d_* \text{ such that } d^\beta \underset{\mathrm{BL}}{\sim} \delta_\mu.\}$$

THEOREM 15.7. *Assume that μ has upper uniform exponential decay and the volume doubling property with respect to d_*. Then $\mathfrak{B}_\mu \neq \emptyset$ and*

$$\mathfrak{B}_\mu = \bigcup_{\varphi:\text{quasimetric}, \varphi \underset{\mathrm{BL}}{\sim} \delta_\mu} \Big[1 + \frac{\log C_\varphi}{\log 2}, \infty\Big) \subseteq [2, \infty). \tag{15.8}$$

Furthermore, for any $\beta \in \mathfrak{B}_\mu$, if d is a metric on K and $d^\beta \underset{\mathrm{BL}}{\sim} \delta_\mu$, then d is quasisymmetric to d_* and there exist positive constants $c^1_{15.9}, c^2_{15.9}, c^1_{15.10}, c^2_{15.10}$ and $c_{15.11}$ such that

$$p_\mu(t, x, y) \leq \frac{c^1_{15.9}}{\mu(B_d(x, t^{1/\beta}))} \exp\left(- c^2_{15.9} \Big(\frac{d(x, y)^\beta}{t}\Big)^{\frac{1}{\beta - 1}} \right). \tag{15.9}$$

for any $x, y \in K$ and $t \in (0, \infty)$,

$$\frac{c^1_{15.10}}{\mu(B_d(x, t^{1/\beta}))} \leq p_\mu(t, x, y) \tag{15.10}$$

if $d(x, y)^\beta \leq c^2_{15.10} t$, and

$$p_\mu(t, x, x) \leq c_{15.11} p_\mu(2t, x, x) \tag{15.11}$$

for any $x \in K$ and $t > 0$. In particular, if $r_* = 1$, then $\mu(B_d(x, t^{1/\beta}))$ in (15.9) *and* (15.10) *can be replaced by t.*

The estimate (15.9) is called off-diagonal upper sub-Gaussian estimate and the estimate (15.10) is called near diagonal lower estimate. We are going to prove this theorem step by step in the subsequent sections starting from Section 17.

REMARK. By Proposition 15.2, the assumption of Theorem 15.7 is equivalent to that μ has uniform exponential decay and the volume doubling property.

REMARK. The protodistance δ_μ is not even symmetric in general. Under the assumption of Theorem 15.7, however, it is bi-Lipschitz equivalent to the symmetrized version v_μ defined by $v_\mu(x, y) = \delta_\mu(x, y) + \delta_\mu(y, x)$. In fact, we show that v_μ is a quasimetric under the assumption of Theorem 15.7. See Propositions 19.3 and 19.7.

REMARK. If d is a metric on K and $d^\beta \underset{\mathrm{BL}}{\sim} \delta_\mu$, then by Corollary 12.7 the metric d induces the same topology on K as d_*.

Observing (15.9) and (15.10), one notices that the protodistance δ_μ plays the essential roll. Namely, we can replace $d(x, y)^\beta$ by $\delta_\mu(x, y)$ and obtain the following corollary.

COROLLARY 15.8. *Assume that μ has upper uniform exponential decay and the volume doubling property with respect to d_*. If $\beta \in \mathfrak{B}_\mu$, then there exist positive constants $c^1_{15.12}, c^2_{15.12}, c^1_{15.13}$ and $c^2_{15.13}$ such that $c^1_{15.12}$ and $c^2_{15.12}$ depend on β while $c^1_{15.13}$ and $c^2_{15.13}$ do not,*

$$p_\mu(t, x, y) \leq \frac{c^1_{15.12}}{\mu(B_{\delta_\mu}(x, t))} \exp\left(- c^2_{15.12} \Big(\frac{\delta_\mu(x, y)}{t}\Big)^{\frac{1}{\beta - 1}} \right). \tag{15.12}$$

for any $(x, y, t) \in K^2 \times (0, \infty)$, and if $\delta_\mu(x, y) \leq c^2_{15.13} t$, then

$$\frac{c^1_{15.13}}{\mu(B_{\delta_\mu}(x, t))} \leq p_\mu(t, x, y). \tag{15.13}$$

In view of (15.12), it is interesting to know what happens if we lower the value of β towards $\inf \mathfrak{B}_\mu$. In the special case $\mu = \nu_*$, which is the normalized Hausdorff measure of K with respect to d_*, we see that $\mathfrak{B}_{\nu_*} = [d_w, \infty)$ and the metric d with $d \underset{\text{BL}}{\sim} (\delta_\mu)^{1/d_w}$ equals d_*, the restriction of the Euclidean metric. In particular, $d_w = \inf \mathfrak{B}_{\nu_*}$. This means that $\inf \mathfrak{B}_{\nu_*}$ is a characterization of the walk dimension d_w in this case. This may help us to understand how to characterize the notion of walk dimension in general. Pursuing this direction, however, we need to solve the following problem first.

Open Problem Let $\beta_* = \inf \mathfrak{B}_\mu$. Then $\beta_* \in \mathfrak{B}_\mu$ or not? If $\beta_* \in \mathfrak{B}_\mu$ and d is a metric giving the same topology on K as d_* and $d^{\beta_*} \underset{\text{BL}}{\sim} \delta_\mu$, then does d satisfy the chain condition?

A metric space (X, d) is said to satisfy the chain condition if and only if there exists $C > 0$ such that, for any $x, y \in X$ and $m \in \mathbb{N}$, there exists a sequence $\{x_i\}_{i=1,\ldots,m+1} \subseteq X$ such that $x_1 = x$, $x_{m+1} = y$ and

$$d(x_i, x_{i+1}) \leq C\frac{d(x,y)}{m}$$

for any $i = 1, \ldots, m$. It is known that if the chain condition is satisfied, then we can deduce the off-diagonal lower sub-Gaussian estimate

$$\frac{c^1_{15.14}}{\mu(B_d(x, t^{1/\beta_*}))} \exp\left(- c^2_{15.14}\Big(\frac{d(x,y)^{\beta_*}}{t}\Big)^{\frac{1}{\beta_*-1}}\right) \leq p_\mu(t, x, y) \tag{15.14}$$

from (15.9) and (15.10) with $\beta = \beta_*$. See [**28**] for example. If this is the case, then the metric d can be regarded as the best intrinsic metric for the heat kernel $p_\mu(t, x, y)$ and the infimum β_* may be called the "walk dimension".

Now we know that the protodistance may be a good candidate for an intrinsic metric to time change. Even with the alternative expression in Proposition 12.5, however, the definition of δ_μ is rather complicated and difficult to see what it is intuitively. So it is nice to have a simpler version.

Definition 15.9. Define $Q_\mu(x, y)$ for each $x, y \in K$ by

$$Q_\mu(x, y) = d_*(x, y)^{d_w - d_H} \mu(B_*(x, d_*(x, y))).$$

and

$$\psi_\mu(x, y) = \frac{Q_\mu(x, y) + Q_\mu(y, x)}{2}.$$

The function ψ_μ is the symmetrized version of Q_μ.

The next proposition tells that if μ has the volume doubling property, then Q_μ is a good substitute of δ_μ.

Proposition 15.10. *Assume that μ has upper uniform exponential decay and the volume doubling property with respect to d_*. Then $\delta_\mu \underset{\text{BL}}{\sim} Q_\mu \underset{\text{BL}}{\sim} \psi_\mu$.*

This proposition will be proven in Section 19.

By this proposition, we may replace δ_μ in Theorem 15.7 and Corollary 15.8 by Q_μ or ψ_μ. As a consequence, we obtain the following statement: under the same assumptions as in Corollary 15.8, if $\beta \in \mathfrak{B}_\mu$, then there exist positive constants

$c^1_{15.15}, c^2_{15.15}, c^1_{15.16}$ and $c^2_{15.16}$ such that

$$p_\mu(t,x,y) \le \frac{c^1_{15.15}}{\mu(B_{\psi_\mu}(x,t))} \exp\left(-c^2_{15.15}\Big(\frac{\psi_\mu(x,y)}{t}\Big)^{\frac{1}{\beta-1}}\right). \tag{15.15}$$

for any $(x,y,t) \in K^2 \times (0,\infty)$, and if $\psi_\mu(x,y) \le c^2_{15.16}t$, then

$$\frac{c^1_{15.16}}{\mu(B_{\psi_\mu}(x,t))} \le p_\mu(t,x,y). \tag{15.16}$$

The next theorem is a version of Theorem 15.7 without using any expression related to self-similarity of K. In other words, it is written in the "conventional" language.

THEOREM 15.11. *Let $\mu \in \mathcal{M}_P(K)$. Assume that there exist $c, \epsilon > 0$ such that*

$$\mu(B_*(x,ar)) \le ca^{\alpha+\epsilon}\mu(B_*(x,r)) \tag{15.17}$$

for any $r \in (0,1]$ and $a \in (0,1]$. Then μ has the volume doubling property with respect to d_* if and only if the following conditions (TC1), (TC2) *and* (TC3) *are satisfied:*

(TC1) *Let $\mathcal{D} = \mathcal{F} \cap C(K)$. Then $(\mathcal{E}|_{\mathcal{D}\times\mathcal{D}}, \mathcal{D})$ is closable on $L^2(K,\mu)$ and its closure $(\mathcal{E}_\mu, \mathcal{F}_\mu)$ is a strong local regular Dirichlet form on $L^2(K,\mu)$.*

(TC2) *There exist a diffusion process $(\{\widetilde{X}_t\}_{t>0}, \{\widetilde{P}_x\}_{x\in K})$ that is associated with the Dirichlet form $(\mathcal{E}_\mu, \mathcal{F}_\mu)$ on $L^2(K,\mu)$ and a continuous function $p_\mu(t,x,y)$: $(0,\infty)\times K \times K \to (0,\infty)$ such that*

$$\widetilde{E}_x(f(\widetilde{X}_t)) = \int_K p_\mu(t,x,y)f(y)\mu(dy)$$

for any bounded measurable function $f : K \to \mathbb{R}$, $x \in K$ and $t > 0$.

(TC3) *There exist a metric d on K which is quasisymmetric to d_* and positive constants $\beta, c^1_{15.9}, c^1_{15.9}, c^2_{15.10}, c^2_{15.10}$ and $c_{15.11}$ such that $\beta \ge 2$,* (15.9) *holds for any $t > 0$ and $x, y \in K$,* (15.10) *holds if $d(x,y)^\beta \le c^2_{15.10}t$ and* (15.11) *holds for any $t > 0$ and $x \in K$.*

By the definition of $Q_\mu(x,y)$, the condition (1.6) is equivalent to the condition (15.17).

In the rest of this section, we show that Theorem 15.7 implies Theorem 15.11.

LEMMA 15.12. *Let $\mu \in \mathcal{M}_P(K)$. Assume that there exist $c, \epsilon > 0$ such that* (15.17) *is satisfied for any $r \in (0,1]$ and $a \in (0,1]$ and that μ has the volume doubling property with respect to d_*. Then μ has upper uniform exponential decay .*

PROOF. For each $w \in W_*$, define $\{W^m(w)\}_{m\ge0}$ and $\{K^m(w)\}_{m\ge0}$ inductively by $W^0(w) = \{w\}$, $K^m(w) = K(W^m(w))$ and $W^{m+1}(w) = \Gamma^0_{|w|}(W^m(w))$. Choose $x \in K_{wv}$ so that $K_{wv} \subseteq B_*(x, \sqrt{n}l^{-|wv|})$. Let $M = [\sqrt{n}] + 1$. Then $K_w \subseteq B_*(x, \sqrt{n}l^{-|w|}) \subseteq K^M(w)$. Since μ has the volume doubling property with respect to d_*, it follows that μ is elliptic and $\mathbf{g}_* \underset{\mathrm{GE}}{\sim} \mu$ by [**34**, Theorem 1.3.5]. (The definition of the relation $\underset{\mathrm{GE}}{\sim}$ is give in Definition 17.1.) Hence there exists $c_0 > 0$ such that $\mu(K_{w'}) \le c_0\mu(K_w)$ for any $w \in W_*$ and $w' \in W^M(w)$. Since $\#(W^M(w)) \le (2M)^n$, we see that

$$\mu(B_*(x, \sqrt{n}l^{-|w|})) \le c_0(2M)^n\mu(K_w).$$

Therefore,

$$\mu(K_{wv}) \le \mu(B_*(x, \sqrt{n}l^{-|wv|}) \le c(l^{-|v|})^{\alpha+\epsilon}\mu(B_*(x, \sqrt{n}l^{-|w|})$$
$$\le c \cdot c_0(2M)^n(l^{-|v|})^{\alpha+\epsilon}\mu(K_w) = c \cdot c_0(2M)^n(r_*)^{-|v|}(l^{\epsilon})^{-|v|}\mu(K_w).$$

This implies

$$\sigma_\mu(wv) = (r_*)^{|wv|}\mu(K_{wv}) \le c \cdot c_0(2M)^n(l^{\epsilon})^{-|v|}(r_*)^{|w|}\mu(K_w) = c_1\lambda^{|v|}\sigma_\mu(w),$$

where $\lambda = l^{-\epsilon} \in (0, 1)$ and $c_1 = c \cdot c_0(2M)^n$. Thus we see that μ has upper uniform exponential decay. □

Proof of Theorem 15.11. Assume (15.17). By Lemma 15.12, if μ has the volume doubling property with respect to d_*, then it has upper uniform exponential decay. Making use of Proposition 15.2, we see that μ has uniform exponential decay. Now, Proposition 11.7 implies that $\mu \in \mathcal{M}_P^{TC}(K)$ and Theorem 11.9 shows that μ is controlled by some rate functions. Theorems 6.8, 10.10 and 15.7 yield (TC1), (TC2) and (TC3). Conversely, assume that (TC1), (TC2) and (TC3). By (15.9) and (15.10),

$$\frac{c_3}{\mu(B_d(x, t^{1/\beta}))} \le p_\mu(t, x, x) \le \frac{c_1}{\mu(B_d(x, t^{1/\beta}))}$$

for any $t > 0$ and $x \in K$. This along with (15.11) implies the volume doubling property of μ with respect to d. Since d is quasisymmetric to d_*, μ has the volume doubling property with respect to d_*. □

16. Examples

In this section, we will present two classes of measures which satisfy the two conditions, the volume doubling property and upper uniform exponential decay, in Theorem 15.7. The first class consists of self-similar measures and the second class consists of measures which are absolutely continuous with respect to the normalize Hausdorff measure ν_*.

It is known that not all the self-similar measures have the volume doubling property with respect to d_*. We are going to apply results in [**34**] to our case to obtain simple criterion.

Definition 16.1. Let $i \in \{1, \ldots, n\}$. For $Q_1 \in S_{i,0}$ and $Q_2 \in S_{i,1}$, we write $Q_1 \underset{i}{\sim} Q_2$ if and only if Q_1 and Q_2 are symmetric with respect to the reflection in the hyper-plane $x_i = 1/2$.

Theorem 16.2. *Let μ be a self-similar measure on K with weight $(\mu_i)_{i\in S}$.*
(1) *μ has the volume doubling property with respect to d_* if and only if $\mu_{Q_1} = \mu_{Q_2}$ whenever $Q_1 \in S_{i,0}$ and $Q_2 \in S_{i,1}$ for some $i \in \{1, \ldots, n\}$ and $Q_1 \underset{i}{\sim} Q_2$.*
(2) *μ has the upper uniform exponential decay if and only if $\mu_i r_* < 1$ for any $i \in S$.*

Proof. (1) Let φ_i be the reflection in the hyper-plane $x_i = 1/2$. Note that if $Q \in S_{i0}$, then $\varphi_i(Q) \in S_{i1}$. Therefore we may regard φ_i as a map from S_{i0} to S_{i1}. For $Q_1 \in S_{i0}$ and $Q_2 \in S_{i1}$, it follows that $Q_1 \underset{i}{\sim} Q_2$ if and only if $\varphi_i(Q_1) = Q_2$. If

$$\mathcal{R} = \{(S_{i0}, S_{i1}, \varphi_i, s_1, s_2)|i \in \{1, \ldots, n\}, s_1, s_2 \in S, F_{s_1}(B_{i0}) = F_{s_2}(B_{i1})\},$$

then, by [**34**, Proposition 3.4.3], $\mathcal{L} = (K, S, \{F_Q\}_{Q\in S})$ is a rationally ramified self-similar structure with a relation set $\mathcal{R}$. Since the gauge functions μ and $\mathbf{g}_*$ are elliptic and $\mathbf{g}_*$ is locally finite, by [**34**, Theorem 1.3.5], μ has the volume doubling

property with respect to d_* if and only if $\mu \underset{\text{GE}}{\sim} \mathbf{g}_*$, where $\underset{\text{GE}}{\sim}$ is defined in Definition 17.1-(1). Applying [**34**, Theorem 1.6.6], we see that $\mu \underset{\text{GE}}{\sim} \mathbf{g}_*$ if and only if $\mu_{Q_1} = \mu_{Q_2}$ for any pair $(Q_1, Q_2) \in S_{i0} \times S_{i1}$ satisfying $\varphi_i(Q_1) = Q_2$. Thus we have obtained the desired equivalence.
(2) Set $\lambda = \max_{i \in S} \mu_i r_*$. If $\lambda < 1$, then

$$\sigma_\mu(wv) \leq \lambda^{|w|}\sigma_\mu(w)$$

for any $w, v \in W_*$. Hence μ has upper uniform exponential decay. The converse is immediate. $\square$

The second example is a measure given as $\mu(dx) = c|x - x_*|^{-\delta}\nu_*(dx)$, where $x \in K$, $0 < \delta$ and c is a normalizing constant. If $0 < \delta < d_H$, then $\int_K |x - x_*|^{-\delta}\nu_*(dx) < +\infty$ and the normalizing constant c is given by the reciprocal of this integral.

THEOREM 16.3. *Let $x_* \in K$ and let $0 < \delta < d_H$. Define*

$$\mu_{x_*,\delta}(A) = \int_A |x - x_*|^{-\delta}\nu_*(dx) \Big/ \int_K |x - x_*|^{-\delta}\nu_*(dx).$$

for any Borel set $A \subseteq K$. Then $\mu_{x_*,\delta}$ has upper uniform exponential decay if and only if $0 < \delta < d_w$. Moreover, if $0 < \delta < d_w$, then μ has the volume doubling property with respect to d_*.

The rest of this section is devoted to proving this theorem.

For simplicity, we only consider the case where $x_* = 0$. We define

$$\mu_*(A) = \int_A |x|^{-\delta}\nu_*(dx)$$

for any Borel set $A \subseteq K$. Note that $\mu_* = \big(\int_K |x|^{-\delta}\nu_*(dx)\big)\mu_{0,\delta}$. Therefore, to show the upper uniform exponential decay and the volume doubling property for $\mu_{0,\delta}$, it is enough to show the corresponding properties for μ_*.

LEMMA 16.4. *μ_* has upper uniform exponential decay if and only if $0 < \delta < d_w$.*

PROOF. Let $w \in W_m$ and let $I \in S$. Set $I_* = [0, 1/l] \times \cdots [0, 1/l]$ and write $K_m = K_{(I_*)^m}$. Then

$$\mu_*(K_w) = \int_{K_m} |x + a_w|^{-\delta}\nu_*(dx),$$

where $a_w = F_w(0)$, and

$$\begin{aligned}
\mu_*(K_{wI}) &= \int_{K_{m+1}} |x + a_w + F_I(0)/l^{m+1}|^{-\delta}\nu_*(dx) \\
&= \frac{1}{N}\int_{K_m} |x/l + a_w + F_I(0)/l^{m+1}|^{-\delta}\nu_*(dx) \\
&= \frac{l^\delta}{N}\int_{K_m} |x + a_w l + F_I(0)/l^m|^{-\delta}\nu_*(dx)
\end{aligned}$$

Since a_w and $F_I(0)$ are nonnegative vector,

$$|x + a_w| \leq |x + a_w l + F_I(0)/l^m|$$

for any $x \in K_m$. Hence

$$\mu_*(K_{wI}) \le \frac{l^\delta}{N}\mu_*(K_w).$$

Note that if $w = (I_*)^m$ and $I = I_*$, equality holds in the above inequality. By the definition of d_w, we see that

$$\frac{r_*}{N}l^\delta < 1$$

if and only if $\delta < d_w$. Thus μ_* has upper uniform exponential decay if and only if $\delta < d_w$. □

LEMMA 16.5. *There exists $c > 0$ such that $\mu(K_{wi}) \le c\mu(K_{wj})$ for any $w \in W_*$ and $i, j \in S$.*

PROOF. Note that

$$\mu_*(K_{wi}) = \int_{K_{m+1}} |x + a_{wi}|^{-\delta}\nu_*(dx)$$

for any $w \in W_*$ and $i \in S$. Set $I_0 = [1 - 1/l, 1]^n$. Since $|x + a_w I_*| \le |x + a_{wi}| \le |x + a_{wI_0}|$ for any $x \in K_{m+1}$ and $i \in S$, we see that $\mu_*(K_{wI_0}) \le \mu_*(K_{wi}) \le \mu_*(K_{wI_*})$ for any $i \in S$. Assume that $w \ne (I_*)^m$. Then

$$\mu_*(K_{wI_0}) \ge (|a_w| + \sqrt{n}l^{-1})^{-\delta}N^{-(m+1)} \quad \text{and} \quad \mu(K_{wI_*}) \le |a_w|^{-\delta}N^{-(m+1)}.$$

Set $c_1 = (\sqrt{n} + 1)^\delta$. Since $w \ne (I_*)^m$, we have $|a_w| \ge 1/l$ and this implies

$$c_1\mu_*(K_{wL_0}) \ge c_1(|a_w| + \sqrt{n}l^{-1})^{-\delta}N^{-(m+1)} \ge |a_w|^{-\delta}N^{-(m+1)} \ge \mu(K_{wI_*}).$$

Next, let $w = (I_*)^m$. Then

$$\mu(K_{wI_*}) = \int_{K_{m+1}} |x|^{-\delta}\nu_*(dx) = \frac{l^{m\delta}}{N^m}\int_{(I_*)^n} |y|^{-\delta}\nu_*(dy)$$

$$\mu(K_{wI_0}) = \int_{[l^{-m}(1-1/l),l^{-m}]^n} |x|^{-\delta}\nu_*(dx) = \frac{l^{m\delta}}{N^m}\int_{[1-1/l,1]^n} |y|^{-\delta}\nu_*(dy).$$

Hence there exists $c_2 > 0$ such that $c_2\mu(K_{wI_0}) \ge \mu(K_{wI_*})$ for any $m \ge 0$. Finally define $c = \max\{c_1, c_2\}$. Then $\mu(K_{wi}) \le c\mu(K_{wj})$ for any $w \in W_*$ and $i, j \in S$. □

LEMMA 16.6. *μ_* is elliptic.*

PROOF. For any $w \in W_*$, there exists $i \in S$ such that $\mu_*(K_{wi}) \ge \mu_*(K_w)/N$. By Lemma 16.5, for any $j \in S$,

$$c\mu_*(K_{wj}) \ge \mu_*(K_{wi}) \ge \mu_*(K_w)/N.$$

Combining this with Lemma 16.4, we see that μ_* is elliptic. □

LEMMA 16.7. *Define $e_k = (\delta_{1k}, \dots, \delta_{nk}) \in \mathbb{R}^n$, where δ_{ij} is Kronecker's δ. Then there exists $c_3 > 0$ such that*

$$\mu_*(K_w) \le c_3\mu_*(K_v)$$

if $w, v \in W_*$, $|w| = |v|$, $a_v = a_w + e_k/l^{|w|}$ for some $k \in \{1, \dots, n\}$.

PROOF. Let $|w| = m$. Note that

$$\mu_*(K_w) = \int_{K_m} |x + a_w|^{-\delta} \nu_*(dx) \text{ and } \mu_*(K_v) = \int_{K_m} |x + a_w + e_k/l^m|^{-\delta} \nu_*(dx).$$

In case $w \neq (I_*)^m$, then since $x + a_w - e_k/l^m$ is a nonnegative vector,

$$|x + a_w| \leq |x + a_w + \epsilon_k/l^m| \leq |x + a_w + e_k/l^m + (x + a_w - e_k/l^m| = 2|x + a_w|.$$

Hence $2^{-\delta}\mu_*(K_w) \leq \mu_*(K_v) \leq \mu_*(K_w)$.

If $w = (I_*)^m$, then

$$\mu_*(K_w) = \frac{l^{\delta m}}{N^m} \int_K |x|^{-\delta} \nu_*(dx) \quad \text{and} \quad \mu_*(K_v) = \frac{l^{\delta m}}{N^m} \int_K |x + e_k|^{-\delta} \nu_*(dx)$$

Hence there exists $c' > 0$ is independent of m and k such that $\mu_*(K_w) \leq c'\mu_*(K_v)$. Thus we have shown the lemma. $\square$

PROOF OF THEOREM 16.3. By Lemma 16.4, μ_* has upper uniform exponential decay if and only if $0 < \delta < d_w$. Now we show that μ has the volume doubling property if $0 < \delta < d_w$. By Lemma 16.6, μ_* is elliptic. Moreover, Lemma 16.7 shows that $\mu \underset{\text{GE}}{\sim} \nu_*$. (The definition of $\underset{\text{GE}}{\sim}$ is given in Definition 17.1.) Then by [**34**, Theorem 1.3.5], μ has the volume doubling property with respect to d_*. $\square$

17. Construction of metrics from gauge function

From this section, we start preparations to prove Theorem 15.7. In this section, we briefly review the theory of gauge functions and metrics developed in [**34, 36**] and modify it for our purpose.

DEFINITION 17.1. (1) A gauge function $\mathbf{g}_1$ on W_* is said to be gentle with respect to a gauge function $\mathbf{g}_2$ on W_* if and only if there exists $c > 0$ such that $\mathbf{g}_1(w) \leq c\mathbf{g}_1(v)$ whenever $w, v \in \Lambda_\rho^{\mathbf{g}_2}$ and $K_w \cap K_v \neq \emptyset$ for some $\rho \in (0, 1]$. We write $\mathbf{g}_1 \underset{\text{GE}}{\sim} \mathbf{g}_2$ if $\mathbf{g}_1$ is gentle with respect to $\mathbf{g}_2$.
(2) For $\gamma > 0$, we define $\mathbf{g}^\gamma : W_* \to (0, 1]$ as $\mathbf{g}^\gamma(w) = \mathbf{g}(w)^\gamma$ for any $w \in W_*$.

Note that $\mathbf{g}^\gamma$ is again a gauge function and if $\mathbf{g}$ is elliptic (resp. locally finite), then so is $\mathbf{g}^\gamma$.

PROPOSITION 17.2 ([**34**, Theorem 1.4.3]). (1) *The relation* $\underset{\text{GE}}{\sim}$ *is an equivalent relation on the collection of elliptic gauge functions.*
(2) *Let* $\mathbf{g}_1$ *and* $\mathbf{g}_2$ *be elliptic gauge functions on* W_*. *If* $\mathbf{g}_1$ *is locally finite and* $\mathbf{g}_1 \underset{\text{GE}}{\sim} \mathbf{g}_2$, *then* $\mathbf{g}_2$ *is locally finite.*

Recall that $U^{\mathbf{g}}(x, r)$ defined in Definition 4.4 is the "ball" with center x and radius r associated with a gauge function $\mathbf{g}$.

DEFINITION 17.3. Let $\mathbf{g}$ be a gauge function on K. A metric d on K is said to be 1-adapted to $\mathbf{g}$ if and only if there exist $c_1, c_2 > 0$ such that

$$B_d(x, c_1 r) \subseteq U^{\mathbf{g}}(x, r) \subseteq B_d(x, c_2 r)$$

for any $x \in K$ and $r \in (0, 1]$.

This definition enable us to regard $U^{\mathbf{g}}(x, r)$ as a real ball with respect to the metric d if d is 1-adapted to $\mathbf{g}$.

Next we propose a natural way to construct a metric from a gauge function.

Definition 17.4. Let $\mathbf{g}$ be a gauge function on W_*. For any $x, y \in K$, define

$$D_{\mathbf{g}}(x,y) = \inf\Big\{\sum_{i=1}^{m} \mathbf{g}(w(i))\Big| m \geq 1, w(1), \ldots, w(m) \in W_*, x \in K_{w(1)},$$
$$K_{w(i)} \cap K_{w(i+1)} \neq \emptyset \text{ for any } i = 1, \ldots, m-1 \text{ and } y \in K_{w(m)}\Big\}$$

It is easy to see that $D_{\mathbf{g}}$ is a pseudo distance, i.e. $D_{\mathbf{g}}(x,y) = D_{\mathbf{g}}(y,x) \geq 0$, $D_{\mathbf{g}}(x,x) = 0$, $D_{\mathbf{g}}(x,y) \leq D_{\mathbf{g}}(x,z) + D_{\mathbf{g}}(z,y)$. Unfortunately, $D_{\mathbf{g}}(x,y)$ may be 0 even if $x \neq y$ in general.

Example 17.5. The restriction of the Euclidean metric d_* is 1-adapted to the gauge function $\mathbf{g}_*$ defined in Example 4.6. Moreover, $D_{\mathbf{g}_*}$ is a metric which is bi-Lipschitz equivalent to d_*.

The following theorem suggests that two relations $\underset{\mathrm{GE}}{\sim}$ and $\underset{QS}{\sim}$ are closely related.

Theorem 17.6. *Let $\mathbf{g}$ be an elliptic gauge function on W_*. Assume that $\mathbf{g} \underset{\mathrm{GE}}{\sim} \mathbf{g}_*$.*
(1) *If d is a metric on K which is 1-adapted to $\mathbf{g}^\epsilon$ for some $\epsilon > 0$, then d is quasisymmetric to d_*.*
(2) *There exists $\epsilon \in (0,1]$ such that $D_{\mathbf{g}^\epsilon}$ is a metric which is 1-adapted to $\mathbf{g}^\epsilon$ and quasisymmetric to d_*.*

Proof. (1) This is a direct consequence of [**36**, Theorem 3.4].
(2) Since $\mathbf{g}_*$ is locally finite, $\mathbf{g}$ is locally finite by Proposition 17.2-(2). Note that the self-similar structure associated with generalized Sierpinski carpet is rationally ramified. Combining [**34**, Theorem 2.3.11] and [**34**, Corollary 2.3.15], we see that $D_{\mathbf{g}^\epsilon}$ is a metric on K which is 1-adapted to $\mathbf{g}^\epsilon$ for some $\epsilon \in (0,1]$. Hence by (1), $D_{\mathbf{g}^\epsilon}$ is quasisymmetric to d_*. □

The next theorem is one of the key ingredients of the proof of Theorem 15.7.

Theorem 17.7. *Let μ has uniform exponential decay. Then the following three conditions are equivalent:*
(1) *μ is elliptic and gentle with respect to $\mathbf{g}_*$.*
(2) *μ has the volume doubling property with respect to d_*.*
(3) *$\overline{\sigma}_\mu$ is elliptic and gentle with respect to $\mathbf{g}_*$.*
Furthermore, if any of the above conditions holds, then there exists $\epsilon \in (0,1]$ such that $D_{(\overline{\sigma}_\mu)^\epsilon}$ is a metric on K which is 1-adapted to $(\overline{\sigma}_\mu)^\epsilon$ and quasisymmetric to d_*.

Proof. (1) $\Leftrightarrow$ (2): Since d_* is adapted to $\mathbf{g}_*$ and $\mathbf{g}_*$ is locally finite, this follows from [**34**, Theorem 1.3.5].
(1) $\Rightarrow$ (3): Proposition 11.8 yields that $\overline{\sigma}_\mu$ is elliptic. Since $\mu \underset{\mathrm{GE}}{\sim} \mathbf{g}_*$, there exists $c > 0$ such that if $w, v \in \Lambda_\rho^{\mathbf{g}_*}$ and $K_w \cap K_v \neq \emptyset$, then $\mu(K_w) \leq c\mu(K_v)$. Note that $|w| = |v|$ if $w, v \in \Lambda_\rho^{\mathbf{g}_*}$. Hence $\sigma_\mu(w) \leq c\sigma_\mu(v)$ if $w, v \in \Lambda_\rho^{\mathbf{g}_*}$ and $K_w \cap K_v \neq \emptyset$. By (11.13), we see that $\overline{\sigma}_\mu$ is gentle with respect to $\mathbf{g}_*$.
(3) $\Rightarrow$ (1): Proposition 11.8 yields that μ is elliptic. Since $\overline{\sigma}_\mu \underset{\mathrm{GE}}{\sim} \mathbf{g}_*$, there exists $c > 0$ such that if $w, v \in \Lambda_\rho^{\mathbf{g}_*}$ and $K_w \cap K_v \neq \emptyset$, then $\overline{\sigma}_\mu(w) \leq c\overline{\sigma}_\mu(v)$. By (11.13), there exists $c' > 0$ such that $\sigma_\mu(w) \leq c'\sigma_\mu(v)$. Note that $|w| = |v|$ if $w, v \in \Lambda_\rho^{\mathbf{g}_*}$. This implies $\mu(K_w) \leq c'\mu(K_v)$ if $w, v \in \Lambda_\rho^{\mathbf{g}_*}$ and $K_w \cap K_v \neq \emptyset$. Hence $\mu \underset{\mathrm{GE}}{\sim} \mathbf{g}_*$.

The rest of the statement is immediate by Theorem 17.6. □

18. Metrics and quasimetrics

In this section, we prepare another ingredient of the proof of Theorem 15.7. The main subject is the construction of metrics from powers of quasimetrics. First we give basic definitions. Since the results below may be useful in general framework, our space in this section is a general set X instead of a generalized Sierpinski carpet K.

DEFINITION 18.1. Let X be a set. Let $q : X \times X \to [0, \infty)$.
(1) q is called symmetric if $q(x, y) = q(y, x)$ for any $x, y \in X$
(2) q is called predistance if $q(x, x) = 0$ and $q(x, y) > 0$ for any $x \neq y$.
(3) Define $\rho_q(x, y)$ by

$$\rho_q(x,y) = \inf\Big\{\sum_{i=1}^{k} q(x_i, x_{i+1})\Big| k \geq 1, x_1, \ldots, x_{k+1} \in X, x_1 = x, x_{k+1} = y\Big\}$$

for $x, y \in X$.
(4) Let $\kappa > 0$. q is called κ-quasiultrametric on X if and only if q is a symmetric predistance and

$$q(x, z) \leq \kappa \max\{q(x, y), q(y, z)\} \tag{18.1}$$

for any $x, y, z \in X$.

REMARK. $\rho_q(x, y) \leq q(x, y)$ for any $x, y \in X$.

In the above definition, if X contains more than two points, we have $\kappa \geq 1$ in (18.1).

Let us recall the definition of quasimetric in Definition 15.5. If q is a quasimetric, then its power is also a quasimetric as is seen in the next proposition.

PROPOSITION 18.2. *Let $C \geq 1$. If $q(x, y)$ is a C-quasimetric, then for any $\epsilon > 0$,*

$$q(x, y)^\epsilon \leq C^\epsilon 2^{\max\{\epsilon - 1, 0\}}(q(x, z)^\epsilon + q(y, z)^\epsilon).$$

One can prove the above proposition by routine calculus. Next we discuss when a predistance is equivalent to a metric.

PROPOSITION 18.3. *Let $q : X \times X \to [0, \infty)$ be a symmetric predistance. The following three statements are equivalent:*
(A) *There exists a metric on X which is bi-Lipschitz equivalent to q.*
(B) *There exists $c > 0$ such that $q(x, y) \leq c\rho_q(x, y)$ for any $x, y \in X$.*
(C) *ρ_q is a metric on K and $\rho_q \underset{\mathrm{BL}}{\sim} q$.*

PROOF. (A) ⇒ (B): Let d be a metric on X which is bi-Lipschitz equivalent to q. Then there exist $c_1, c_2 > 0$ such that

$$c_1 d(x, y) \leq q(x, y) \leq c_2 d(x, y).$$

Hence if $x_1 = x$ and $x_{n+1} = y$, then

$$c_1 d(x, y) \leq c_1 \sum_{i=1}^{n} d(x_i, x_{i+1}) \leq \sum_{i=1}^{n} q(x_i, x_{i+1}).$$

This shows that $c_1 q(x,y)/c_2 \leq c_1 d(x,y) \leq \rho_q(x,y)$. Thus we have shown that (A) implies (B).
It is straightforward to show the statements (B) $\Rightarrow$ (C) and (C) $\Rightarrow$ (A). $\square$

A metric is always 2-quasiultrametric. How about the converse of this statement? The following classical theorem gives a kind of answer to this question. It shows that 2-quasiultrametric may not be a metric but it is always bi-Lipschitz equivalent to a metric.

THEOREM 18.4 (Frink[**19**]). *Assume that $q(x,y)$ is a κ-quasiultrametric. If $\kappa \leq 2$, then $q(x,y)$ is bi-Lipschitz equivalent to a metric. More precisely, if $\kappa \leq 2$, then*

$$\rho_q(x,y) \leq q(x,y) \leq 2\kappa \rho_q(x,y) \tag{18.2}$$

for any $x, y \in X$.

See also [**39**] for a proof of Theorem 18.4.

COROLLARY 18.5. *Let q be a C-quasimetric. If $(2C)^\epsilon \leq 2$, then ρ_{q^ϵ} is a metric. More precisely, if $\epsilon \leq \dfrac{\log 2}{\log 2 + \log C}$, then*

$$\rho_{q^\epsilon}(x,y) \leq q(x,y)^\epsilon \leq 4\rho_{q^\epsilon}(x,y)$$

for any $x, y \in X$.

This corollary is a quantitative version of [**29**, Proposition 14.5], where the condition has been $(2C)^\epsilon \leq \sqrt{2}$ instead of $(2C)^\epsilon \leq 2$ in the above corollary. (The condition $(2C)^\epsilon \leq \sqrt{2}$ has not explicitly written in the statement of [**29**, Proposition 14.5]. One can extract, however, this condition from its proof.) This improvement is crucial to obtain Theorem 18.7.

PROOF OF COROLLARY 18.5. For any $x, y, z \in K$, we have

$$q(x,y) \leq C(q(x,z) + q(z,y)) \leq 2C \max\{q(x,z), q(z,y)\}.$$

Thus we see that

$$q(x,y)^\epsilon \leq (2C)^\epsilon \max\{q(x,z)^\epsilon, q(z,y)^\epsilon\}.$$

Using Theorem 18.4, we conclude our proof of this corollary. $\square$

DEFINITION 18.6. For a quasimetric q, define

$$\mathcal{A}_q = \{\epsilon | q(x,y)^\epsilon \text{ is bi-Lipschitz equivalent to a metric}\}.$$

The following theorem gives a characterization of $\mathcal{A}_q$. Recall (15.7) for the definition of the quantity C_φ.

THEOREM 18.7. *If q is a quasimetric of X, then*

$$\mathcal{A}_q \cap (0,1] = \bigcup_{\varphi:\ \text{quasimetric},\ \varphi \underset{\mathrm{BL}}{\sim} q} \left(0, \frac{\log 2}{\log 2 + \log C_\varphi}\right]. \tag{18.3}$$

PROOF. Choose any $\epsilon \in \mathcal{A}_q \cap (0,1]$. Then ρ_{q^ϵ} is a metric on K which is bi-Lipschitz equivalent to q^ϵ. Set $\varphi = (\rho_{q^\epsilon})^{1/\epsilon}$. By Proposition 18.2,

$$\varphi(x,y) \leq 2^{1/\epsilon - 1}(\varphi(x,z) + \varphi(z,y))$$

for any $x, y, z \in X$. This implies $C_\varphi \leq 2^{\epsilon-1}$ and hence $\epsilon \leq \frac{\log 2}{\log 2 + \log C_\varphi}$.

Next, if φ is a quasimetric which is bi-Lipschitz equivalent to q, then φ is C_φ-quasimetric. Using Corollary 18.5, we see that $(0, \frac{\log 2}{\log 2 + \log C_\varphi}] \subseteq \mathcal{A}_q$.

□

19. Protodistance and the volume doubling property

In this section, we study properties of the protodistance δ_μ with or without the volume doubling property of μ. Although δ_μ is not symmetric and does not fulfill extended triangle inequality (15.6) in general, it satisfies primitive counterparts given in Lemma 19.2 and Proposition 19.5. In fact, if μ has the volume doubling property, then the combination of Lemma 19.2 and Proposition 19.5 implies that δ_μ is equivalent to a quasimetric. See Proposition 19.7 for the exact statement.

In this section, we always assume that $\mu \in \mathcal{M}_P(K)$ and (12.1) holds. Note that this assumption is satisfied by all measures having weak exponential decay.

First we consider how far δ_μ is apart from being symmetric.

Definition 19.1. Define

$$j_\mu(m, x) = \min\{k - m | k \geq m, (r_*)^k \mu(V_k(x)) = \epsilon_\mu(m, x)\}.$$

Lemma 19.2. *For any $x, y \in K$,*

$$\delta_\mu(x, y) \leq (r_*)^{j_\mu(k(x,y),x)} \frac{\mu(V_{k(x,y)-1}(y))}{\mu(V_{k(x,y)}(y))} \delta_\mu(y, x). \tag{19.1}$$

Proof. Since $V_{k(x,y)}(x) \subseteq V_{k(x,y)-1}(y)$, we see that

$$\begin{aligned}(r_*)^{k(x,y)} \mu(V_{k(x,y)}(x)) &\leq \frac{\mu(V_{k(x,y)-1}(y))}{\mu(V_{k(x,y)}(y))} (r_*)^{k(x,y)} \mu(V_{k(x,y)}(y)) \\ &\leq \frac{\mu(V_{k(x,y)-1}(y))}{\mu(V_{k(x,y)}(y))} \epsilon_\mu(k(x,y), y) = \frac{\mu(V_{k(x,y)-1}(y))}{\mu(V_{k(x,y)}(y))} \delta_\mu(y, x).\end{aligned}$$

Hence

$$\begin{aligned}\delta_\mu(x, y) = \epsilon_\mu(k(x,y), x) &= (r_*)^{j_\mu(k(x,y),x)+k(x,y)} \mu(V_{j_\mu(k(x,y),x)+k(x,y)}(x)) \\ &\leq (r_*)^{j_\mu(k(x,y),x)} (r_*)^{k(x,y)} \mu(V_{k(x,y)}(x)) \\ &\leq (r_*)^{j_\mu(k(x,y),x)} \frac{\mu(V_{k(x,y)-1}(y))}{\mu(V_{k(x,y)}(y))} \delta_\mu(y, x).\end{aligned}$$

□

Under the volume doubling property, (19.1) leads to the fact that $\delta_\mu(x, y)$ and $\delta_\mu(y, x)$ are comparable as follows.

Proposition 19.3. *Assume that $\sup_{m \geq 0, x \in K} j_\mu(m, x) < +\infty$. Then there exists $c_{19.2} > 0$ such that*

$$\delta_\mu(x, y) \leq c_{19.2} \delta_\mu(y, x) \tag{19.2}$$

for any $x, y \in K$ if and only if μ has the volume doubling property with respect to d_*.

Proof. Let $M = \sup_{m \geq 0, x \in K} j_\mu(m, x)$. If μ has the volume doubling property, then there exists $c_1 > 0$ such that

$$\mu(V_m(x)) \leq c_1 \mu(V_{m+1}(x))$$

for any $m \geq 0$ and $x \in K$. By Lemma 19.2, it follows that

$$\delta_\mu(x,y) \leq (r_*)^M c_1 \delta_\mu(y,x).$$

Conversely, for any $m \geq 0$ and $x \in K$, choose $y \in V_m(x)\backslash V_{m+1}(x)$. Then there exists $k \leq M$ such that

$$\begin{aligned}(r_*)^m \mu(V_m(y)) \leq \delta_\mu(y,x) \leq c_{19.2}\delta_\mu(x,y)&\\ = c_{19.2}(r_*)^{m+k}\mu(V_{m+k}(x)) &\leq c_{19.2}(r_*)^{m+M}\mu(V_m(x)).\end{aligned}$$

Hence if $c_2 = c_{19.2}(r_*)^M$, then

$$\mu(V_m(y)) \leq c_2 \mu(V_m(x))$$

for any $x, y \in K$ with $\ell_m(x,y) \leq 2$. Using this inductively, we see that if $\ell_m(x,y) \leq k$, then

$$\mu(V_m(y)) \leq (c_2)^{k-1}\mu(V_m(x)).$$

On the other hand,

$$V_{m-1}(x) \subseteq \bigcup_{w \in \Gamma_m^{2l-1}(x)} K_w.$$

Choosing $y_w \in K_w$ for each $w \in \Gamma_m^{2l-1}(x)$, we have

$$\begin{aligned}\mu(V_{m-1}(x)) \leq &\sum_{w \in \Gamma_m^{2l-1}(x)} \mu(V_m(y_w))\\ &\leq \#(\Gamma_m^{2l-1}(x))(c_2)^{2l-1}\mu(V_m(x)) \leq (4l-1)^n (c_2)^{2l-1}\mu(V_m(x)).\end{aligned}$$

Set $c_3 = (4l-1)^n(c_2)^{2l-1}$. Inductively, we obtain

$$\mu(V_m(x)) \leq (c_3)^k \mu(V_{m+k}(x)).$$

Note that $B_*(x, l^{-m}) \subseteq V_m(x) \subseteq B_*(x, 3\sqrt{n}l^{-m})$. Let $k = \min\{i \in \mathbb{N} | l^i > 3\sqrt{n}\}$. Then

$$\mu(B_*(x, l^{-m}) \leq (c_3)^k \mu(B_*(x, (3\sqrt{n}l^{-k})l^{-m})$$

for any $m \geq 0$ and $x \in K$. Since $3\sqrt{n}l^{-k} < 1$, μ has the volume doubling property with respect to d_*. $\square$

LEMMA 19.4. *For any $x, y, z \in K$.*

$$\min\{k(x,y), k(y,z)\} - 1 \leq k(x,z)$$

PROOF. Set $m = \min\{k(x,y), k(y,z)\}$. Then $\ell_m(x,z) \leq 3$. Hence $\ell_{m-1}(x,z) \leq 2$. This immediately implies $m - 1 \leq k(x,z)$. $\square$

Next we have a primitive version of extended (or weakened) triangle inequality (15.6), although it is difficult to find resemblance at a glance.

PROPOSITION 19.5. *For any $x, y, z \in K$, either*

$$\delta_\mu(x,z) \leq \max\Big\{\frac{\mu(V_{k(x,y)-1}(x))}{r_*\mu(V_{k(x,y)}(x))}, 1\Big\}\delta_\mu(x,y) \tag{19.3}$$

or

$$\delta_\mu(z,x) \leq \max\Big\{\frac{\mu(V_{k(z,y)-1}(x))}{r_*\mu(V_{k(z,y)}(x))}, 1\Big\}\delta_\mu(z,y) \tag{19.4}$$

holds.

PROOF. We have two cases as follows.
Case I $\min\{k(x,y), k(y,z)\} \leq k(x,z)$: In this case,

$$\delta_\mu(x,z) \leq \delta_\mu(x,y) \text{ if } k(x,y) \leq k(x,z),$$
$$\delta_\mu(z,x) \leq \delta_\mu(z,y) \text{ if } k(y,z) \leq k(x,z).$$

Case II $\min\{k(x,y), k(y,z)\} > k(x,z)$: Lemma 19.4 shows that

$$k(x,z) = \min\{k(x,y), k(y,z)\} - 1.$$

Suppose $k(x,z) = k(x,y) - 1$. If $j_\mu(k(x,y) - 1, x) \geq 1$, then $\epsilon(k(x,y) - 1, x) = \epsilon(k(x,y), x)$. This implies $\delta_\mu(x,z) = \epsilon(k(x,z), x) = \epsilon(k(x,y), x) = \delta_\mu(x,y)$. If $j_\mu(k(x,y) - 1, x) = 1$, then

$$\begin{aligned}\delta_\mu(x,z) &= \epsilon(k(x,y) - 1, x) = (r_*)^{k(x,y)-1}\mu(V_{k(x,y)-1}(x)) \\ &\leq (r_*)^{-1}\frac{\mu(V_{k(x,y)-1}(x))}{\mu(V_{k(x,y)}(x))}\epsilon(k(x,y), x) = (r_*)^{-1}\frac{\mu(V_{k(x,y)-1}(x))}{\mu(V_{k(x,y)}(x))}\delta_\mu(x,y)\end{aligned}$$

Hence if $k(x,z) = k(x,y)-1$, we have (19.3). In case $k(z,x) = k(z,y)-1$, switching x and z, we obtain (19.4). □

LEMMA 19.6. *If μ has upper uniform exponential decay, then*

$$\sup_{x \in K, m \geq 0} j_\mu(m,x) < +\infty.$$

Moreover, there exists $c_{19.5} \geq 1$ such that

$$(r_*)^{k(x,y)}\mu(V_{k(x,y)}(x)) \leq \delta_\mu(x,y) \leq c_{19.5}(r_*)^{k(x,y)}\mu(V_{k(x,y)}(x)) \tag{19.5}$$

for any $x, y \in K$.

PROOF. By the definition, μ has upper uniform exponential decay if and only if there exist $\eta \geq 1$ and $\lambda \in (0,1)$ such that $\sigma_\mu(wv) \leq \eta\lambda^{|v|}\sigma_\mu(w)$ for any $w, v \in W_*$. Since $v_1 \dots v_m \in \Gamma_m(x)$ for any $v = v_1 \dots v_{m+k} \in \Gamma_{m+k}(x)$, we have

$$\begin{aligned}(r_*)^{m+k}\mu(V_{m+k}(x)) &= \sum_{v \in \Gamma_{m+k}(x)} \sigma_\mu(v) \\ &\leq \#(\Gamma_{m+k}(x))\eta\lambda^k \max_{w \in \Gamma_m(x)} \sigma_\mu(w) \leq 4^n\eta\lambda^k(r_*)^m\mu(V_m(x)).\end{aligned}$$

Hence choosing k so that $4^n\eta\lambda^k \leq 1$, we have $j_\mu(m,x) \leq k$. At the same time, if $c_{19.5} = (r_*)^k$, then (19.5) holds. □

PROOF OF PROPOSITION 15.10. For any $x, y \in K$,

$$B_*(x, l^{-k(x,y)}) \subseteq V_{k(x,y)}(x) \subseteq B_*(x, 3\sqrt{n}l^{-k(x,y)})$$

and

$$l^{-k(x,y)-1} \leq d_*(x,y) \leq 2l^{-k(x,y)}.$$

These inequalities with the volume doubling property show that there exist $c_1, c_2 > 0$ such that

$$c_1\mu(B_*(x, d_*(x,y))) \leq \mu(V_{k(x,y)}(x)) \leq c_2\mu(B_*(x, d_*(x,y))) \tag{19.6}$$

for any $x, y \in K$. Since $(r_*)^{k(x,y)} = (l^{-k(x,y)})^{d_w - d_H}$, the inequalities (19.5) and (19.6) imply that $\delta_\mu \underset{\text{BL}}{\sim} Q_\mu$. By Proposition 19.3, it follows that $\delta_\mu \underset{\text{BL}}{\sim} \psi_\mu$ as well. □

Now assuming the volume doubling property, we are going to deduce the extended triangle inequality from (19.3) and (19.4) as promised.

PROPOSITION 19.7. *Assume that μ has upper uniform exponential decay and the volume doubling property with respect to d_*. If $v_\mu(x,y) = \delta_\mu(x,y) + \delta_\mu(y,x)$, then v_μ is a quasimetric.*

PROOF. By Proposition 19.5 and the volume doubling property, there exists $c_1 > 0$ such that $\delta_\mu(x,z) \le c_1\delta_\mu(x,y)$ or $\delta_\mu(z,x) \le c_1\delta_\mu(z,y)$ for any $x,y,z \in K$. On the other hand, by Proposition 19.3 and Lemma 19.6, $\delta_\mu(x,y) \le c_{19.2}\delta_\mu(y,x)$ for any $x,y \in K$. Hence

$$\begin{aligned}\delta_\mu(x,z) &\le c_1 \min\{\delta_\mu(x,y), c_{19.2}\delta_\mu(z,y)\}\\ &\le c_1 \min\{\delta_\mu(x,y), c_{19.2}^2\delta_\mu(y,z)\} \le c_1c_{19.2}^2(\delta_\mu(x,y)+\delta_\mu(y,z)).\end{aligned}$$

Replacing (x,y,z) with (z,y,x) and then adding two inequalities, we obtain

$$v_\mu(x,z) \le c_1c_{19.2}^2(v_\mu(x,y)+v_\mu(y,z)).$$

□

If $v_\mu(x,y)$ is a quasimetric, then by [**29**, Proposition 14.5], there exists $\epsilon_0 > 0$ such that, for any $\epsilon \in (0,\epsilon_0]$, $(v_\mu)^\epsilon \underset{\mathrm{BL}}{\sim} d_\epsilon$. By (19.2), we have $(\delta_\mu)^\epsilon \underset{\mathrm{BL}}{\sim} d_\epsilon$. In fact, the metric $d_\epsilon \underset{\mathrm{BL}}{\sim} D_{\overline{\sigma}_\mu^\epsilon}$ as follows.

THEOREM 19.8. *Assume that μ has upper uniform exponential decay and the volume doubling property with respect to d_*. Then $\mathfrak{B}_\mu \neq \emptyset$. Let $\beta \in \mathfrak{B}_\mu$ and let d be a metric on K satisfying $d^\beta \underset{\mathrm{BL}}{\sim} \delta_\mu$. Then d and $D_{\overline{\sigma}_\mu^{1/\beta}}$ are 1-adapted to $(\overline{\sigma}_\mu)^{1/\beta}$, $d \underset{\mathrm{BL}}{\sim} D_{\overline{\sigma}_\mu^{1/\beta}}$ and $d \underset{QS}{\sim} d_*$. In particular, $\delta_\mu \underset{\mathrm{BL}}{\sim} (D_{\overline{\sigma}_\mu^{1/\beta}})^\beta$.*

To prove the above theorem, we need several lemmas.

LEMMA 19.9. *Under the same assumption as Theorem 19.8, there exist $c_{19.7}^1 > 0$ and $c_{19.7}^2 > 0$ such that*

$$c_{19.7}^1\overline{\sigma}_\mu(w) \le \delta_\mu(x,y) \le c_{19.7}^2\overline{\sigma}_\mu(w) \tag{19.7}$$

for any $x,y \in K$ and $w \in \Gamma_{k(x,y)}(x)$.

PROOF. By Theorem 17.7, $\overline{\sigma}_\mu$ and μ are elliptic and gentle with respect to $\mathbf{g}_*$. Since $W_m = \Lambda_{l-m}^*$ and $\Gamma_m(x) = \Lambda_{l-m,1}^*(x)$, there exists $c_1 > 0$ such that $\mu(K_v) \le c_1\mu(K_w)$ for any $x \in K$, $m \ge 0$ and $w,v \in \Gamma_m(x)$. This implies

$$(r_*)^m\mu(K_w) \le (r_*)^m\mu(V_m(x)) \le c_14^n(r_*)^m\mu(K_w) \tag{19.8}$$

for any $x \in K$, $m \ge 0$ and $w \in \Gamma_m(x)$. By Proposition 15.2, μ has uniform exponential decay and hence by (11.14), there exist $c_2, c_3 > 0$ such that

$$c_2\overline{\sigma}_\mu(w) \le (r_*)^m\mu(V_m(x)) \le c_3\overline{\sigma}_\mu(w)$$

for any $x \in K$, $m \ge 0$ and $w \in \Gamma_m(x)$. This immediately implies (19.7). □

DEFINITION 19.10. Define

$$\widetilde{\delta}_\mu(x,y) = \inf\{s|y \in U^{\overline{\sigma}_\mu}(x,s)\}.$$

and $B_{\widetilde{\delta}_\mu}(x,r) = \{y|\widetilde{\delta}_\mu(x,y) < r\}$.

By the above definition, it is easy to see that $\widetilde{\delta}_\mu(x,y)$ is a predistance and

$$B_{\widetilde{\delta}_\mu}(x,r) \subseteq U^{\overline{\sigma}_\mu}(x,r) \subseteq B_{\widetilde{\delta}_\mu}(x,\gamma r) \tag{19.9}$$

for any $x \in K$, $r > 0$ and $\gamma > 1$. However, $\widetilde{\delta}_\mu$ does not satisfy the (extended) triangle inequality in general.

LEMMA 19.11. *Under the same assumption as Theorem 19.8, $\delta_\mu \underset{\mathrm{BL}}{\sim} \widetilde{\delta}_\mu$.*

PROOF. We write $\widetilde{\delta}(x,y) = \widetilde{\delta}_\mu(x,y)$ in this proof if no confusion may occur. If $\max_{w\in\Gamma_{k(x,y)}(x)} \overline{\sigma}_\mu(w) \le s$, then for any $w \in \Gamma_{k(x,y)}(x)$, we have $w = vu$ for some $v \in \Lambda^{\overline{\sigma}_\mu}_{s,1}(x)$ and $u \in W_*$. Therefore, $y \in U^{\overline{\sigma}_\mu}(x,s)$ and hence $\widetilde{\delta}(x,y) \le \max_{w\in\Gamma_{k(x,y)}(x)} \overline{\sigma}_\mu(w) \le (c^1_{19.7})^{-1}\delta_\mu(x,y)$.

Since $\overline{\sigma}_\mu$ is elliptic, there exists $\gamma \in (0,1)$ such that $\Lambda^{\overline{\sigma}_\mu}_\rho \cap \Lambda^{\overline{\sigma}_\mu}_{\gamma\rho} = \emptyset$. Hence if $\gamma \min_{w\in\Gamma_{k(x,y)}(x)} \overline{\sigma}_\mu(w) > s$, then for any $w \in \Lambda^{\overline{\sigma}_\mu}_{s,1}(x)$, there exists $v \in \Gamma_{k(x,y)}(x)$ such that $w = vu$ for some $u \in W_*$ and $v \in W_*\backslash W_0$. If $y \in U^{\overline{\sigma}_\mu}(x,s)$, then there exist $w, w' \in \Lambda^{\overline{\sigma}_\mu}_{s,1}(x)$ such that $x \in K_w$, $y \in K_{w'}$ and $K_w \cap K_{w'} \neq \emptyset$. Since $|w| \ge k(x,y)+1$ and $w' \ge k(x,y)+1$, it follows that $\ell_{k(x,y)+1}(x,y) \le 2$. This contradiction yields $y \notin U^{\overline{\sigma}_\mu}(x,s)$ and hence $\widetilde{\delta}(x,y) \ge \gamma \min_{w\in\Gamma_{k(x,y)}(x)} \overline{\sigma}_\mu(w) \ge \gamma(c^2_{19.7})^{-1}\delta_\mu(x,y)$. □

PROOF OF THEOREM 19.8. By Proposition 19.7 and Lemma 19.11, it follows that $\widetilde{\delta}_\mu$ is a quasimetric. Using [**29**, Proposition 14.5] (or equivalently [**34**, Proposition 2.3.3]), we obtain $\epsilon_0 > 0$ and a metric d_ϵ for each $\epsilon \in (0,\epsilon_0]$ satisfying $d_\epsilon \underset{\mathrm{BL}}{\sim} (\widetilde{\delta}_\mu)^\epsilon \underset{\mathrm{BL}}{\sim} (\delta_\mu)^\epsilon$. Hence $\mathfrak{B}_\mu \neq \emptyset$. Let $\beta \in \mathfrak{B}_\mu$ and let d be a metric giving the same topology on K as d_* and satisfying $d^\beta \underset{\mathrm{BL}}{\sim} \delta_\mu$. The fact that $d^\beta \underset{\mathrm{BL}}{\sim} \widetilde{\delta}_\mu$ along with (19.9) implies that d is 1-adapted to $(\overline{\sigma}_\mu)^{1/\beta}$. By [**34**, Lemma 2.3.10], we see that $d \underset{\mathrm{BL}}{\sim} D_{(\overline{\sigma}_\mu)^{1/\beta}}$. Since $(\overline{\sigma}_\mu)^{1/\beta}$ is elliptic and gentle with respect to $\mathbf{g}_*$ by Theorem 17.7, it follows from Theorem 17.6-(1) that d and $D_{(\overline{\sigma}_\mu)^{1/\beta}}$ are quasisymmetric to d_*. □

20. Upper estimate of $p_\mu(t,x,y)$

In this section, we are going to give the first half of our proof of Theorem 15.7. Throughout this section, we assume that μ has upper uniform exponential decay and that μ has the volume doubling property with respect to d_*. Hence by Proposition 15.2, μ has uniform exponential decay.

For simplicity, we write $\Lambda_\rho = \Lambda^{\overline{\sigma}_\mu}_\rho$, $\Lambda_\rho(x) = \Lambda^{\overline{\sigma}_\mu}_\rho(x)$, $K(x,\rho) = K^{\overline{\sigma}_\mu}(x,\rho)$, $\Lambda_{\rho.1}(x) = \Lambda^{\overline{\sigma}_\mu}_{\rho,1}(x)$ and $U(x,\rho) = U^{\overline{\sigma}_\mu}(x,\rho)$ as far as no confusion may occur.

By Theorem 17.7, μ and $\overline{\sigma}_\mu$ are elliptic and $\mu \underset{\mathrm{GE}}{\sim} \overline{\sigma}_\mu \underset{\mathrm{GE}}{\sim} \mathbf{g}_*$. Hence by Proposition 17.2, μ and $\overline{\sigma}_\mu$ are locally finite. In particular, there exists $c_{20.1} > 0$ such that

$$\mu(K_{wi}) \ge c_{20.1}\mu(K_w) \tag{20.1}$$

for any $w \in W_*$ and $i \in S$.

LEMMA 20.1. *There exists $m_{20.2} > 0$ such that if $w, v \in \Lambda_\rho$ and $K_w \cap K_v \neq \emptyset$, then*

$$||w| - |v|| \leq m_{20.2}. \tag{20.2}$$

PROOF. Since $\overline{\sigma}_\mu \underset{\text{GE}}{\sim} \mathbf{g}_*$, there exists $c > 0$ such that if $w, v \in \Lambda_\rho$ and $K_w \cap K_v \neq \emptyset$, then $\mathbf{g}_*(w) = 2^{-|w|} \leq c\mathbf{g}_*(v) = c2^{-|v|}$. This immediately implies the desired statement. $\square$

LEMMA 20.2. *There exist $c^1_{20.3} > 0$ and $c^2_{20.3} > 0$ such that*

$$c^1_{20.3}\rho \leq \sigma_\mu(w) \leq c^2_{20.3}\rho \tag{20.3}$$

for any $\rho \in (0, 1]$ and $w \in \Lambda_\rho$.

PROOF. Since $\overline{\sigma}_\mu$ is elliptic, there exist positive constants c_1 and c_2 such that

$$c_1\rho \leq \overline{\sigma}_\mu(w) \leq c_2\rho$$

for any $\rho \in (0, 1]$ and $w \in \Lambda_\rho$. This along with (11.13) suffices. $\square$

LEMMA 20.3. *There exist $\rho_1 \in (0, 1]$ and $c^1_{20.4}, c^2_{20.4} > 0$ such that*

$$c^1_{20.4}\rho \leq \widetilde{E}_x(\tau_{U(x,\rho)}) \leq c^2_{20.4}\rho \tag{20.4}$$

for any $\rho \in (0, \rho_1]$ and $x \in K$.

PROOF. Choose $w \in \Lambda_\rho(x)$ so that $|w| = \max\{|v| : v \in \Lambda_\rho(x)\}$. Lemma 20.1 implies that $|v| \geq |w| + m_{20.2}$ for any $v \in \Lambda_{\rho,1}(x)$. Hence we have $V_{|w|+m_{20.2}}(x) \subseteq U(x, \rho)$. By Lemma 7.7, if $M = m_{20.2} + 1$, then

$$c_{7.8}(r_*)^{|w|+m_{20.2}}\mu(V_{|w|+M}(x)) \leq \int_{U(x,\rho)} g^{U(x,\rho)}(x, y)\mu(dy) = \widetilde{E}_x(\tau_{U(x,\rho)}). \tag{20.5}$$

Since $w \in \Lambda_\rho(x)$, there exists $v \in W_M$ such that $x \in K_{wv} \subseteq V_{|w|+M}(x)$. By (20.1), $\mu(K_{wv}) \geq (c_{20.1})^M\mu(K_w)$. By (20.5),

$$c_{7.8}(r_*)^{m_{20.2}}(c_{20.1})^M\sigma_\mu(w) \leq \widetilde{E}_x(\tau_{U(x,\rho)}).$$

Using Lemma 20.2, we obtain $c^1_{20.3}c_{7.8}(r_*)^{m_{20.2}}(c_{20.1})^M\rho \leq \widetilde{E}_x(\tau_{U(x,\rho)})$.

Next we show the upper estimate. Since $\overline{\sigma}_\mu$ is locally finite and elliptic, Theorem 4.9 implies that the number of equivalence classes of $\{\Lambda_{\rho,x}\}_{x\in K,\rho\in(0,1]}$ under $\underset{B}{\sim}$ is finite. Let $\{\Gamma_1, \ldots, \Gamma_k\}$ be the collection of equivalence classes of $\{\Lambda_{\rho,x}\}_{x\in K,\rho\in(0,1]}$ under $\underset{B}{\sim}$. Set $C = \max_{i=1,\ldots,k} c_{7.3}([\Gamma_i], \eta, \lambda)$. Note that there exists $\rho_1 \in (0, 1)$ such that $\partial U(x, \rho) \neq \emptyset$ for any $(x, \rho) \in K \times (0, \rho_1]$. By Lemma 7.3, if $\rho \in (0, \rho_1]$, then

$$\widetilde{E}_x(\tau_{U(x,\rho)}) = \int_{U(x,\rho)} g^{U(x,\rho)}(x, y)\mu(dy) \leq C \sum_{w\in\Lambda_{\rho,1}(x)} \sigma_\mu(w) \tag{20.6}$$

Since $\overline{\sigma}_\mu$ is locally finite, it follows that $L = \sup_{x\in K,\rho\in(0,1]} \#(\Lambda_{\rho,1}(x)) < +\infty$. Combining this fact with Lemma 20.2 and (20.6), we obtain

$$\widetilde{E}_x(\tau_{U(x,\rho)}) \leq c^2_{20.3}C\rho.$$

$\square$

LEMMA 20.4. *There exists $c_{20.7} > 0$ such that*

$$c_{20.7}\mu(K_w) \geq \mu(U(x, \rho)) \tag{20.7}$$

for any $x \in K$, $\rho \in (0, 1]$ and $w \in \Lambda_\rho(x)$.

PROOF. By the fact that $\mu \underset{\mathrm{GE}}{\sim} \overline{\sigma}_\mu$, there exists $c_1 > 0$ such that

$$\mu(K_w) \geq c_1 \mu(K_v)$$

whenever $x \in K$ and $w, v \in \Lambda_{\rho,1}(x)$. Hence if $w \in \Lambda_\rho(x)$, then

$$\mu(U(x,\rho)) = \sum_{v \in \Lambda_{\rho,1}(x)} \mu(K_v) \leq \frac{1}{c_1} \sum_{v \in \Lambda_{\rho,1}(x)} \mu(K_w) \leq \frac{L}{c_1} \mu(K_w),$$

where $L = \sup_{x \in K, \rho \in (0,1]} \#(\Lambda_{\rho,1}(x))$ appearing in the proof of Lemma 20.3. $\square$

First part of proof of Theorem 15.7. By Theorem 19.8, $\mathfrak{B}_\mu$ is not empty. Let $\beta \in \mathfrak{B}_\mu$ and let d be a metric on K satisfying $d^\beta \underset{\mathrm{BL}}{\sim} \delta_\mu$. Again by Theorem 19.8, $d \underset{QS}{\sim} d_*$ and d is 1-adapted to $(\overline{\sigma}_\mu)^{1/\beta}$. Consequently μ has the volume doubling property with respect to d. Moreover, since $U^{(\overline{\sigma}_\mu)^{1/\beta}}(x, r) = U(x, r^\beta)$, Lemma 20.3 implies that there exist $c^1_{20.8}, c^2_{20.8} > 0$ and $R > 0$ such that

$$c^1_{20.8} r^\beta \leq \widetilde{E}_x(\tau_{B_d(x,r)}) \leq c^2_{20.8} r^\beta \tag{20.8}$$

for any $x \in K$ and $r \in (0, R]$. By [**33**, Lemma 4.4], it follows that $\beta > 1$.

For a compact set $A \subseteq K$ and a gauge function $\mathbf{g}$, define $\Lambda^{\mathbf{g}}_\rho(A) = \{w | w \in \Lambda^{\mathbf{g}}_\rho, A \cap K_w \neq \emptyset\}$. Then by Lemma 20.4,

$$c_{20.7} \inf_{w \in \Lambda^{\overline{\sigma}_\mu}_\rho(A)} \mu(K_w) \geq \inf_{x \in A} \mu(U(x,\rho))$$

for any $\rho \in (0, 1]$. Let $r = \rho^{1/\beta}$. Since d is 1-adapted to d and μ has the volume doubling property with respect to d, we have

$$\begin{aligned}\inf_{w \in \Lambda^{\overline{\sigma}_\mu}_\rho(A)} \mu(K_w) &\geq c_1 \inf_{x \in A} \mu(U(x,\rho)) \\ &\geq c_2 \inf_{x \in A} \mu(B_d(x, c_3 r)) \geq c_4 \inf_{x \in A} \mu(B_d(x, r)),\end{aligned}$$

where the constants c_1, c_2, c_3 and c_4 are independent of A and r. This and (10.8) yield

$$\mathcal{E}(f,f) + \frac{c_5}{r^\beta \inf_{x \in \mathrm{supp}(f)} \mu(B_d(x,r))} ||f||^2_{\mu,1} \geq \frac{c_6}{r^\beta} ||f||^2_{\mu,2} \tag{20.9}$$

for any $r \in (0, 1]$ and any $f \in \mathcal{F}$. This inequality (20.9) is called the local Nash inequality in [**33**]. Recall that μ has the volume doubling property with respect to d. Combining this fact with (20.8) and (20.9), we have (15.9) by [**33**, Theorem 2.10]. Now Theorem 22.2 shows that $\beta \geq 2$. Thus $\mathfrak{B}_\mu \subseteq [2, \infty)$. Since v_μ is a quasimetric by Proposition 19.7 and $\delta_\mu \underset{\mathrm{BL}}{\sim} \widetilde{\delta}_\mu$ by Lemma 19.11, we see that $\widetilde{\delta}_\mu$ is a quasimetric. Again by the fact that $\delta_\mu \underset{\mathrm{BL}}{\sim} \widetilde{\delta}_\mu$, we obtain

$$\mathfrak{B}_\mu = \{1/\epsilon | \epsilon \in \mathcal{A}_{\widetilde{\delta}_\mu}\}$$

from Definition 18.6. Since $\mathfrak{B}_\mu \subseteq [2, \infty)$, (18.3) implies (15.8). $\square$

21. Lower estimate of $p_\mu(t,x,y)$

This section is devoted to giving the second half of the proof of Theorem 15.7. The ideas of the proof in this section are essentially due to [**28**, Section 5]. We adapt their arguments to our situation where the space is compact. As in the last section, we assume that $\mu \in \mathcal{M}_P(K)$ has uniform exponential decay and the volume doubling property with respect to d_*. Let $\beta \in \mathfrak{B}_\mu$ and let d be a metric on K satisfying $d^\beta \underset{\mathrm{BL}}{\sim} \delta_\mu$. Then, by the results in the last section, d is quasisymmetric to d_* and

$$p_\mu(t,x,y) \leq \frac{c_1}{\mu(B_d(x,t^{1/\beta}))} \exp\left(-c_2\Big(\frac{d(x,y)^\beta}{t}\Big)^{\frac{1}{\beta-1}}\right). \tag{21.1}$$

Since d is quasisymmetric to d_*, $(\mathcal{E},\mathcal{F})$ satisfies the elliptic Harnack inequality (5.3) with respect to d as well.

Let $\{(\lambda_i,\varphi_i)\}_{i\geq 1}$ be the collection of pairs of an eigenvalue and an eigenfunction given in Lemma 10.7. Define

$$u^{t,x}(y) = \sum_{i\geq 1}(\lambda_i+\gamma)e^{-\lambda_i t}\varphi_i(x)\varphi_i(y). \tag{21.2}$$

Using the same discussion as in the proofs of Lemma 10.9 and Theorem 10.11, we see that the above infinite sum converges uniformly on $[T,\infty)\times K\times K$ for any $T>0$, and that

$$G_\gamma u^{t,x}(y) = p_\mu(t,x,y) \tag{21.3}$$

and

$$u^{t,x}(y) = \gamma p_\mu(t,x,y) - \frac{\partial}{\partial t}p_\mu(t,x,y) \tag{21.4}$$

for any $(t,x,y)\in(0,\infty)\times K\times K$.

The next lemma is well-known consequence of the elliptic Harnack inequality. The present statement is a slight modification of [**28**, Lemma 5.2].

Lemma 21.1. *There exist $c>0$ and $\theta>0$ such that for any $x\in K$, $r>0$, bounded harmonic function f on $B_d(x,r)$ and $y\in B_d(x,r)$,*

$$|f(x)-f(y)| \leq c\left(\frac{d(x,y)}{r}\right)^\theta ||f||_{\infty,B_d(x,r)},$$

where $||f||_{\infty,A} = \sup_{x\in A}|f(x)|$.

Lemma 21.2. *For any $f\in C(K)$, $\gamma>0$ and $x,y\in K$, if $r>d(x,y)$, then*

$$|G_\gamma f(x) - G_\gamma f(y)| \leq 2\sup_{x\in B}\widetilde{E}_x(\tau_B)(||f||_{\infty,B}+\gamma||G_\gamma f||_{\infty,B}) + c\left(\frac{d(x,y)}{r}\right)^\theta ||G_\gamma f||_{\infty,B}, \tag{21.5}$$

where c and θ are the same constants as in Lemma 21.1 and $B=B_d(x,r)$.

Proof. By Proposition 8.3, we have

$$G_\gamma f(z) = G_\gamma^B f(z) + \widetilde{E}_z\big((e^{-\gamma\tau_B}-1)G_\gamma f(\widetilde{X}_{\tau_B})\big) + \widetilde{E}_z\big(G_\gamma f(\widetilde{X}_{\tau_B})\big). \tag{21.6}$$

We are going to give an estimate of each of the tree terms in the right-hand side of (21.6). For the first term, it follows

$$|G_\gamma^B f(z)| = \left|\widetilde{E}_z\left(\int_0^{\tau_B} e^{-\gamma s} p_\mu f(\widetilde{X}_s)ds\right)\right|$$
$$\le \widetilde{E}(\tau_B)||f||_{\infty,B} \le \sup_{z\in B} \widetilde{E}(\tau_B)||f||_{\infty,B}.$$

For the second term,

$$|\widetilde{E}_z\big((e^{-\gamma\tau_B}-1)G_\gamma f(\widetilde{X}_{\tau_B})\big)| \le \gamma\widetilde{E}_z(\tau_B)||G_\gamma f||_{\infty,B}.$$

By [**20**, Theorem 4.6.5], the last term $\widetilde{E}_z\big(G_\gamma f(\widetilde{X}_{\tau_B})\big)$ is a harmonic function on B whose boundary value at ∂B is $G_\gamma f$. Hence by Lemma 21.1,

$$|\widetilde{E}_x\big(G_\gamma f(\widetilde{X}_{\tau_B})\big) - \widetilde{E}_y\big(G_\gamma f(\widetilde{X}_{\tau_B})\big)| \le c\left(\frac{d(x,y)}{r}\right)^\theta ||G_\gamma f||_{\infty,B}.$$

Combining all three terms, we have (21.5). □

LEMMA 21.3. *There exists $C_1>0$ such that*

$$\left|\frac{\partial}{\partial t}p_\mu(t,x,y)\right| \le \frac{C_1}{t\mu(B_d(x,t^{1/\beta}))}$$

if $d(x,y)^\beta \le t$.

PROOF. By (10.14) and (21.1),

$$\left|\frac{\partial}{\partial t}p_\mu(t,x,y)\right| \le \frac{1}{t}\sqrt{p_\mu(t/2,x,x)p_\mu(t/2,y,y)} \le$$
$$\frac{c_1}{t}\frac{1}{\sqrt{\mu(B_d(x,(t/2)^{1/\beta}))\mu(B_d(y,(t/2)^{1/\beta}))}}$$
$$\le \frac{c_1}{t}\frac{1}{\sqrt{\mu(B_d(x,t^{1/\beta}))\mu(B_d(y,t^{1/\beta}))}}.$$

By the volume doubling property, there exists $c>0$ such that

$$\mu(B_d(x,r)) \le \mu(B_d(y,2r)) \le c\mu(B_d(y,r))$$

whenever $d(x,y)\le r$. Hence

$$\left|\frac{\partial}{\partial t}p_\mu(t,x,y)\right| \le \frac{c_1}{t}\frac{1}{\sqrt{\mu(B_d(x,t^{1/\beta}))\mu(B_d(y,t^{1/\beta}))}} \le \frac{c_1c}{t}\frac{1}{\mu(B_d(x,t^{1/\beta}))}$$

if $d(x,y)^\beta \le t$. □

LEMMA 21.4. *For any $A>0$ and $T>0$, there exists $C>0$ such that*

$$|p_\mu(t,x,x)-p_\mu(t,x,y)| \le \frac{A}{\mu(B_d(x,t^{1/\beta}))}$$

whenever $t\in(0,T]$ and $d(x,y)^\beta \le Ct$.

PROOF. Let $f=u^{t,x}$ in (21.5). Assume that $d(x,y)^\beta \le t$. Then by (21.1), (21.3), (21.4) and Lemma 21.3, there exist c_3, c_4 and c_5 such that

$$(21.7)\quad |p_\mu(t,x,x)-p_\mu(t,x,y)| \le \left(r^\beta\left(\frac{c_3}{t}+c_4\right)+c_5\left(\frac{d(x,y)}{r}\right)^\theta\right)\frac{1}{\mu\big(B_d(x,t^{1/\beta})\big)}$$

if $d(x,y) \leq r$. Set $c_6 = \max\{1, (2c_5/A)^{1/\theta}\}$. Define $R = c_6 d(x,y)$. We have $d(x,y) \leq 2R$ and $c_5\left(\dfrac{d(x,y)}{2R}\right)^{\theta} \leq A/2$. Next, note that if $t \in (0,T]$,

$$\frac{c_3}{t} + c_4 \leq \frac{c_3 + c_4 T}{t}.$$

Let $C = \min\left\{1, \frac{A}{2^{1+\beta}(c_3+c_4T)(c_6)^\beta}\right\}$. Then

$$(2R)^\beta\left(\frac{c_3}{t} + c_4\right) \leq \frac{A}{2}$$

if $d(x,y)^\beta \leq Ct$. Thus, letting $r = 2R$ in (21.7), we verify the desired inequality. $\square$

LEMMA 21.5. *For any* $T > 0$,

$$0 < \inf_{x,y\in K, t\geq T} p_\mu(t,x,y) \tag{21.8}$$

PROOF. By (10.12), if $F(t,x,y) = \sum_{i\geq 2} e^{-(\lambda_i-\lambda_2)t}\varphi_i(x)\varphi_i(y)$, then

$$p_\mu(t,x,y) = 1 + e^{-\lambda_2 t}F(t,x,y).$$

Since $F(t,x,y)$ is bounded on $[1,\infty)\times K\times K$ and $\lambda_2 > 0$, there exists $T_* > 0$ such that $e^{-\lambda_2}tF(t,x,y) \leq 1/2$ for any $(t,x,y) \in [T,\infty)\times K\times K$. It is enough to consider the case when $T < T_*$. Since $p_\mu(t,x,y)$ is positive, $0 < \inf_{x,y\in K,t\in[T,T_*]} p_\mu(t,x,y)$. This immediately implies (21.8). $\square$

PROOF OF (15.10). Since μ has uniform exponential decay, $\overline{\kappa}$ and $\underline{\kappa}$ can be chosen as constants. Moreover, by the volume doubling property of μ with respect to d_*, it follows that $C^*_\mu(t,x)$ defined in Theorem 12.14 is uniformly bounded from below. Hence by (12.11), there exists $c_1 > 0$ such that

$$\frac{c_1}{\mu(B_{\delta_\mu}(x,\gamma_* t))} \leq p_\mu(t,x,x)$$

for any $x \in K$ and $t \in (0,1]$. Note that $\delta_\mu \underset{\mathrm{BL}}{\sim} d^\beta$ and that μ has the volume doubling property with respect to d. So, there exists $c_{21.9} > 0$ such that

$$\frac{c_{21.9}}{\mu(B_d(x,t^{1/\beta}))} \leq p_\mu(t,x,x) \tag{21.9}$$

for any $x \in K$ and $t \in (0,1]$. Using Lemma 21.5 and changing the value of $c_{21.9}$ if necessary, we verify that (21.9) holds for any $x \in K$ and $t > 0$. Set $T = \mathrm{diam}(K,d)^\beta$. Define $D = \inf_{x,y\in K,t\geq T} p_\mu(t,x,y)$, which is positive by Lemma 21.5. Then

$$\frac{D}{\mu(B_d(x,t^{1/\beta}))} = D \leq p_\mu(t,x,y) \tag{21.10}$$

for any $(t,x,y) \in [T,\infty)\times K\times K$. Let $A = c_{21.9}/2$. Applying Lemma 21.4 and using (21.9), we have

$$\frac{1}{2}\frac{c_{21.9}}{\mu(B_d(x,t^{1/\beta}))} \leq p_\mu(t,x,y) \tag{21.11}$$

if $d(x,y)^\beta \leq Ct$ and $t \in (0,T]$. Combining (21.10) and (21.11), we obtain (15.10). $\square$

Proof of (15.11). By the upper and lower diagonal estimates obtained above, it follows that

$$\frac{c_1}{\mu(B_d(x, t^{1/\beta}))} \leq p_\mu(t, x, x) \leq \frac{c_2}{\mu(B_d(x, t^{1/\beta}))}$$

for any $t > 0$ and $x \in K$. The volume doubling property of μ with respect to d immediately yields (15.11). □

22. Non existence of super-Gaussian heat kernel behavior

In this section, we will give a proof of the fact that if the heat kernel estimate (15.9) holds, then $\beta \geq 2$. This means that there is no super-Gaussian heat kernel behavior. If μ is Ahlfors regular, i.e. $\mu(B_d(x, r)) \asymp r^a$, and both the lower and upper off-diagonal heat kernel estimate (15.9) and (15.14) hold, the inequality $\beta \geq 2$ has been shown in [**3, 26, 25**]. In the general framework of local and conservative Dirichlet spaces, it has been shown in Hino-Ramirez [**31**, Section 3] by using their version of extended Varahdan's formula. Moreover, in Hino [**30**, comments after Theorem 4.1], this fact has been shown without assuming the local property. Here we present an alternative proof using Theorem 22.3, which characterizes the domain of the Dirichlet form under (15.9).

Throughout this section, we assume that (X, d) is a locally compact metric space, that μ is a Radon measure on (X, d) and that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(X, \mu)$. We set $B_d(x, r) = \{y | y \in X, d(x, y) < r\}$ and $V_d(x, r) = \mu(B_d(x, r))$ for any $x \in X$ and $r \geq 0$.

The following is an abstract definition of a heat kernel.

Definition 22.1. $p(t, x, y)$ is said to be a heat kernel associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, \mu)$ if and only if

(1) For any $t > 0$, $p(t, x, y)$ is non-negative measurable function on $X \times X$.

(2) For any $t > 0$, $p(t, x, y) = p(t, y, x)$ for any $x, y \in X$.

(3) For $t > 0$ and $x \in X$, define $p^{t,x}(y) = p(t, x, y)$. Then $p^{t,x} \in L^1(X, \mu) \cap L^2(X, \mu)$ for any $t > 0$ and $x \in X$.

(4) For any $f \in L^2(X, \mu)$, $(T_t f)(x) = \int_X p(t, x, y) f(y) \mu(dy)$ for μ-a.e. $x \in X$, where $\{T_t\}_{t>0}$ is the strongly continuous semigroup associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, \mu)$.

Now we state the main theorem of this section.

Theorem 22.2. *Assume that μ has the volume doubling property with respect to d and that there exists a heat kernel $p(t, x, y)$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, \mu)$ which is stochastically complete, i.e.*

$$\int_X p(t, x, y) \mu(dy) = 1$$

for any $t > 0$ and μ-a.e. $x \in X$. If there exist a monotonically non-increasing function $\Phi : [0, \infty) \to \mathbb{R}$ and $\beta \geq 1$ such that

$$\int_0^\infty s^{\beta + \delta - 1} \Phi(s) ds < +\infty \tag{22.1}$$

and

$$p(t, x, y) \leq \frac{C}{V_d(x, t^{1/\beta})} \Phi\left(\frac{d(x, y)}{t^{1/\beta}}\right) \tag{22.2}$$

for any $t \in (0, 1]$ and $x, y \in X$, then $\beta \geq 2$.

The key step to prove the above theorem is the following fact. We define $\mathcal{E}(u,u) = +\infty$ if $u \in L^2(X,\mu)$ and $u \notin \mathcal{F}$.

THEOREM 22.3. *Under assumptions of Theorem 22.2, there exists $C > 0$ such that*

$$\mathcal{E}(u,u) \le C \varlimsup_{r\downarrow 0} \frac{1}{r^\beta} \int_X \frac{1}{V_d(x,r)} \Big(\int_{B(x,r)} |u(x) - u(y)|^2 \mu(dy) \Big) \mu(dx) \tag{22.3}$$

for any $u \in L^2(X,\mu)$. In particular, $u \in \mathcal{F}$ if the right-hand side of (22.3) *is finite.*

This theorem is essentially due to [**26**] if μ is Ahlfors regular. The generalization under the volume doubling condition has been given by Sturm-Kumagai in [**40**].

LEMMA 22.4. *Let (X,d) be a locally compact metric space. Define*

$$C_0(X) = \{f | f \in C(X), \mathrm{supp}(f) \text{ is compact.}\}$$
$$C_0^L(X) = \{f | f \in C_0(X), f \text{ is Lipschitz continuous}\}.$$

Then for any $f \in C_0(X)$, there exist a compact set $K \subseteq X$ and $\{h_n\}_{n\ge1} \subset C_0^L(X)$ such that $||f - h_n||_\infty \to 0$ as $n \to \infty$ and $\mathrm{supp}(h_n) \subseteq K$ for any $n \ge 1$.

PROOF. If X is compact, then this is immediate from the Stone-Weierstrass theorem. (See [**42**], for example, for the Stone-Weierstrass theorem.) Assume that X is not compact. Let $f : X \to [0,\infty)$ belong to $C_0(X)$. Let $F = \mathrm{supp}(f)$. We may choose an open set $U \subseteq X$ so that $F \subseteq U$ and $\overline{U}$ is compact. Using the result for the compact case, we may choose $\{f_n\}_{n\ge0} \subseteq C_0^L(\overline{U})$ such that $||f_n - f||_{\infty,\overline{U}} \to 0$ as $n \to \infty$. Without loss of generality, we may assume that $||f_n - f||_{\infty,\overline{U}\setminus K} \le 2^{-n}$. Define $h_n(x) = \max\{f_n(x) - 2^{-n}, 0\}$ on $\overline{U}$ and $h_n(x) = 0$ on the compliment of $\overline{U}$. Since $\inf_{x\in F, y\notin\overline{U}} d(x,y) > 0$, it follows that $h_n \in C_0^L(X)$ and $||f - h_n||_{\infty,X} \to 0$ as $n \to \infty$. Moreover, $\mathrm{supp}(h_n) \subseteq \overline{U}$. Thus we see that $\{h_n\}_{n\ge1}$ is the desired sequence if $f \ge 0$. This suffices for general case by considering the positive and the negative parts of $f \in C_0(X)$. □

PROOF OF THEOREM 22.2. Assume that $\beta < 2$. Let $u \in C_0^L(X)$ and let L be the Lipschitz constant of u, i.e.

$$L = \sup_{x,y\in X, x\ne y} \frac{|u(x) - u(y)|}{d(x,y)}.$$

Denote the support of u by F. There exists an open set $U \subseteq X$ such that $F \subseteq U$ and $\overline{U}$ is compact. Let $R = \inf_{x\in F, y\in X\setminus U} d(x,y)$. Then $R > 0$ and, for any $r \in (0,R)$,

$$\frac{1}{V(x,r)} \int_{B_d(x,r)} |u(x) - u(y)|^2 \mu(dy) \begin{cases} \le L^2 r^2 & \text{if } x \in U, \\ = 0 & \text{otherwise.} \end{cases}$$

Hence by Lemma 22.4,

$$\mathcal{E}(u,u) \le \varlimsup_{r\downarrow 0} L^2 r^{2-\beta} \mu(U) = 0$$

Consequently $u \in \mathcal{F}$ and $\mathcal{E}(u,u) = 0$. This immediately implies $\mathcal{E}(u,v) = 0$ for any $v \in \mathcal{F}$. Hence letting H be the non-negative self-adjoint operator associated with the Dirichlet from $(\mathcal{E},\mathcal{F})$ on $L^2(X,\mu)$, we see

$$\int_X uHv = 0$$

for any $v \in \mathrm{Dom}(H)$. By Lemma 22.4, if $v \in \mathrm{Dom}(H)$, then

$$\int_X uHv = 0$$

for any $u \in C_0(X)$. By the regularity of $(\mathcal{E}, \mathcal{F})$, $C_0(X)$ is dense in $L^2(X, \mu)$. Hence $Hv = 0$ for any $v \in \mathrm{Dom}(H)$. This implies that $H = 0$ and $T_t f = f$ for any $f \in L^2(X, \mu)$ and $t > 0$. This contradicts to the existence of the integral density $p(t, x, y)$ of T_t. $\square$

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List of Notations

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