Random fractal dendrites

David Croydon

St. Cross College

University of Oxford

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Abstract

Dendrites are tree-like topological spaces, and in this thesis, the physical characteristics of various random fractal versions of this type of set are investigated. This work will contribute to the development of analysis on fractals, an area which has grown considerably over the last twenty years.

First, a collection of random self-similar dendrites is constructed, and their Hausdorff dimension is calculated. Previous results determining this quantity for random self-similar structures have often relied on the scaling factors being bounded uniformly away from zero. However, using a percolative argument, and taking advantage of the tree-like structure of the sets considered here, it is shown that this condition is not necessary; a simple condition on the tail of the distribution of the scaling factors at zero is all that is assumed. The scaling factors of these recursively defined structures form what is known as a multiplicative cascade, and results about the height of this random object are also obtained.

With important physical and probabilistic applications, the heat equation has justifiably received a substantial amount of attention in a variety of settings. For certain types of fractals, it has become clear that a key factor in estimating the heat kernel is the volume growth with respect to the resistance metric on the space. In particular, uniform polynomial volume growth, which occurs for many deterministic self-similar fractals, immediately implies uniform (on-diagonal) heat kernel behaviour. However, in the random fractal setting, this is frequently not the case, and volume fluctuations are often observed. Motivated by this, an analysis of how volume fluctuations lead to corresponding heat kernel fluctuations for measure-metric spaces equipped with a resistance form is conducted here. These results apply to the aforementioned random self-similar dendrites, amongst other examples.

The continuum random tree (CRT) of Aldous is an important random example of a measure-metric space, and fits naturally into the framework of the previous paragraph. In this thesis, quenched (almost-sure) volume growth asymptotics for the CRT are deduced, which show that the behaviour in almost-every realisation is not uniform. Applying the results introduced above, these yield heat kernel bounds for the CRT, demonstrating that heat kernel fluctuations occur almost-surely. Finally, a new representation of the CRT as a random self-similar dendrite is presented.

Contents

Introduction

This thesis contains a study of the geometrical and analytical properties of a range of random fractal dendrites. Throughout, we use the definition that a *dendrite* is a path-connected Hausdorff space containing no subset homeomorphic to the circle. Thus a dendrite is most easily thought of as the topological analogue of a graph tree. Although we assign no precise meaning to the word fractal, the term is included in the title to allude to the fact that the more interesting examples to which the results of this work apply all have some kind of fine structure, precluding them from being investigated by some of the standard tools of classical analysis. Finally, the majority of the objects of interest will be built as random variables on some underlying probability space. As we shall demonstrate in several cases, this can lead to a qualitative distinction between the properties of the structures considered here and those of the associated deterministic structures.

The nature of the thesis is such that the three chapters, although being interlinked by a common theme, are mathematically and notationally independent. As such, it seems more sensible to introduce the background material and notation at the start of each chapter separately, rather than collecting it all here. Instead, we shall simply summarise the main contribution of each chapter in words, leaving the presentation of the precise mathematical statements until the preparations are in place.

In Chapter 1, the focus of study is a class of self-similar dendrites. The particularly high level of symmetry of the so-called post-critically finite self-similar fractals, of which these are a subset, has allowed through analytical and probabilistic methods a mathematical development of an analysis on fractals. At the centre of this work has been the solution of the heat equation on fractals and calculation of related spectral properties. For these problems, possibly the most powerful approach currently available is that of [39], in which the first step is to build an intrinsic self-similar Dirichlet form on the relevant sets. It is the randomisation of this construction that we undertake, resulting in a Dirichlet form which is only statistically self-similar. It should be noted that such an approach has been considered on specific fractals before, see [32] for example. However, the simplicity of the structures on which we are working has allowed a weakening of the conditions on the random scaling factors involved. In particular, we are able to avoid uniform bounds, and only require simple tail estimates to complete our arguments. Furthermore, we are also able to avoid having to choose the scaling factors to be consistent at each stage by introducing random "resistance perturbations", which take into account the tail fluctuations in the construction.

Perhaps establishing the existence of a random Dirichlet form on a fixed set seems a trifle contrived. However, there is a more natural way to view the results we prove. In fact, the Dirichlet form we build also satisfies the definition of a resistance form, see [38]. In this article, it was shown that on dendrites there is a one-to-one correspondence between these quadratic forms and resistance metrics, where, in this setting, such metrics can be viewed simply as shortest path (additive along paths) metrics. Thus it is also an accurate description to say that we have constructed a statistically self-similar dendrite and the naturally associated Dirichlet form. In the course of doing this, we are required to prove some results of interest in their own right about the height of multiplicative cascades, which are probabilistic objects appearing in a number of diverse settings. In the latter half of the chapter, we investigate some of the geometrical properties of the random dendrites. Specifically, we calculate the Hausdorff dimension of the fractals, and also present some measure bounds for the innate statistically self-similar measure on the sets. Finally, we discuss some examples to illustrate the conclusions of the chapter.

Chapter 2 sees a more general approach to the study of the heat equation on fractals, which is motivated by results arising in the random fractal setting. It has been seen that for measure-metric spaces equipped with a resistance form one major factor in obtaining heat kernel estimates for the associated Laplacian is knowledge about the volume growth of the space with respect to the resistance metric. In particular, if uniform volume doubling occurs, then it is possible to obtain sharp bounds, at least for the on-diagonal part of the heat kernel, see [41]. For volume doubling, it is required that the volume of a ball of a certain radius is controlled uniformly by the volume of a ball, centred on the same point, with only half of its radius. However, although many deterministic sets satisfy this condition, it has been demonstrated that certain random fractals do not and display fluctuations in the volume about a leading order doubling term, see [33]. In a specific case, Hambly and Kumagai showed that this leads to fluctuations in the heat kernel, see [34]. In Chapter 2, we prove similar results in a much wider setting, making no assumptions on the structure of the space, and only relatively weak assumptions on the volume fluctuations. In this setting, we are able to deduce global and local (point-wise) bounds which confirm that non-trivial volume fluctuations will always lead to non-trivial heat kernel fluctuations. Although the results proved here are not specifically targeted at dendrites, the condition that is necessary to apply the off-diagonal bounds holds most naturally in this case. The chapter is concluded by a brief presentation of the effect of small polynomial or logarithmic volume fluctuations about a leading order polynomial term. These cases both arise naturally for random fractals, including the dendrites of Chapter 1.

The final chapter is nothing more than an, albeit significant, example. In it, we deduce volume growth asymptotics for the continuum random tree of Aldous, see [1]. Since the continuum random tree is a random dendrite, the results of the previous chapter are readily applicable and so we are immediately able to deduce from these heat kernel asymptotics for this set. We provide global and local quenched (almost-sure) versions of these results, as well as annealed (expected) bounds at the root. To obtain the volume bounds, we conduct a sample path analysis of the normalised Brownian excursion, which is the contour process of the continuum random tree. The chapter also contains a construction of the Brownian motion on the continuum random tree that is substantially more concise than the construction appearing in the literature, see [40].

In fact, it transpires that the continuum random tree is also an example of a statistically self-similar dendrite, as discussed in Chapter 1. In Appendix A, we use inductively the observation of Aldous, [5], that the continuum random tree has a random self-similarity to show that it may be constructed via a random change of metric on a fixed subset of $\mathbb{R}^2$. This model for the continuum random tree gives us an extremely clear picture of the structure and symmetry of the set and measure, which is not obvious from the graph tree descriptions that are available.

Throughout this thesis, we use numbered constants of the form $c_{1.1}$ and t_1 to represent (possibly random) constants whose precise value is unimportant to our study. Exponents of the form $\theta_{\cdot}$ are always deterministic, and we will provide some bounds for these in certain results.

Chapter 1

Random self-similar dendrites

In this chapter we randomise the construction presented in [39] of a Dirichlet form on a post-critically finite self-similar (p.c.f.s.s.) set, when this set is a dendrite. We then calculate the Hausdorff dimension in the resistance metric of the random p.c.f.s.s. dendrite to be almost-surely equal to the solution of a stochastic version of the equation which gives the Hausdorff dimension in the deterministic case. This result is analogous to the expression derived for the Hausdorff dimension of the random recursive constructions in [23] and [47]. We provide measure bounds for a class of these sets, where the measure that we consider is the stochastically self-similar measure naturally associated with the random set. Finally, we present three examples to which we are able to apply the results of this chapter.

1.1 Background and notation

We start by outlining briefly the procedure used in [39] to build a Dirichlet form on a p.c.f.s.s. set, which will provide a template for our random construction, and also allow us to introduce much of the notation that will be used throughout the chapter. It should be noted that a similar treatment can also be found in [9]. To define a self-similar set it suffices to define a finite collection of contractions on (X, d), a complete metric space. For the arguments in later sections, it will be useful to restrict to continuous injections. Hence we fix a finite index set S, define $N := |S|$, and let $(F_i)_{i\in S}$ be a set of continuous injections on (X, d), with contraction ratios strictly less than 1. Throughout, we shall assume that $N \geq 2$ to exclude the trivial case that arises when $N = 1$. For $A \subseteq X$, define

$$F(A) := \bigcup_{i\in S} F_i(A). \tag{1.1}$$

Our *self-similar set*, T, is the non-empty compact fixed point of the equation $F(A) = A$. The existence and uniqueness of T is guaranteed by an extension of the usual contraction principle for complete metric spaces, see [39], Theorem 1.1.4.

An important idea in the understanding of the topological structure of self-similar sets is the relation with what is known as the *shift space*, which is made up of infinite sequences of elements of S. We denote this by $\Sigma := S^{\mathbb{N}}$. The corresponding finite sequences we write as, for $n \geq 0$,

$$\Sigma_n := S^n, \qquad \Sigma_* := \bigcup_{m \geq 0} \Sigma_m, \tag{1.2}$$

where $\Sigma_0 := \{\emptyset\}$. These spaces also serve as useful address spaces for labelling various objects in the discussion and we now introduce some related notation. For $i \in \Sigma_m, j \in \Sigma_n, k \in \Sigma$, write $ij = i_1 \dots i_m j_1 \dots j_n$, and $ik = i_1 \dots i_m k_1 k_2 \dots$. For $i \in \Sigma_*$, denote by $|i|$ the integer n such that $i \in \Sigma_n$ and call this the *length* of i. For $i \in \Sigma_n \cup \Sigma$, $n \geq m$, the *truncation* of i to length m is written as $i|m := i_1 \dots i_m$. For $i \in \Sigma_n$ and $A \subseteq T$, we define A_i to be equal to $F_i(A)$, where $F_i := F_{i_1} \circ \dots \circ F_{i_n}$ and for a function $f : T \to \mathbb{R}$, let $f_i := f \circ F_i$. The following theorem provides the connection between the shift space Σ and the self-similar set T.

Theorem 1.1.1 *([39], Theorem 1.2.3) For any $i \in \Sigma$, the set $\cap_{m \geq 1} T_{i|m}$ contains only one point. If we define $\pi : \Sigma \to T$ by $\{\pi(i)\} = \cap_{m \geq 1} T_{i|m}$, then π is a continuous surjective map. Moreover, if we define for $i \in S$ the map $\sigma_i : \Sigma \to \Sigma$ by $\sigma_i(j) = ij_1 j_2 \dots$, then $\pi \circ \sigma_i = F_i \circ \pi$.*

This result gives us that $(T, S, (F_i)_{i \in S})$ is a *self-similar structure* in the sense of [39], Definition 1.3.1. In the analysis of a self-similar structure, a particularly useful condition for the structure to satisfy is post-critical finiteness, which makes precise the idea that the intersections of the sets $(T_i)_{i \in S}$ should not be too large. If we write the union of the pairwise intersections of sets in $(T_i)_{i \in S}$ as

$$\mathcal{C}' := \bigcup_{\substack{i,j \in S \\ i \neq j}} T_i \cap T_j,$$

then the *critical set* is the pre-image under π of this set, $\mathcal{C} := \pi^{-1}(\mathcal{C}')$, and the *post-critical set* is defined to be

$$\mathcal{P} := \bigcup_{n \geq 1} \sigma^n(\mathcal{C}),$$

where $\sigma : \Sigma \to \Sigma$ is the *shift map*, characterised by $\sigma(i) = i_2 i_3 \dots$ for $i \in \Sigma$. The self-similar structure $(T, S, (F_i)_{i\in S})$ is said to be *post-critically finite* if $|\mathcal{P}| < \infty$. The main obstacles to $|\mathcal{P}|$ being finite are when T is not finitely ramified, as in the case of the Sierpinski carpet (see [39], Example 1.3.17), or when there is too much overlap of the sets $(T_i)_{i\in S}$. Since we will be focussing on dendrites, which are certainly finitely ramified, the first of these problems will not arise and it is the second problem we want to rule out. Henceforth, we assume that $(T, S, (F_i)_{i\in S})$ is a p.c.f.s.s. set.

The Dirichlet form on T of [39] is constructed as the limit of a sequence of Dirichlet forms on the approximating finite subsets of T that we now introduce. First, let $V^0 := \pi(\mathcal{P})$, which may be thought of as the boundary of T, and define

$$V^n := \bigcup_{i\in\Sigma_n} V_i^0.$$

The sequence $(V^n)_{n\geq 0}$ satisfies $V^n \subseteq V^{n+1}$ and it is also a fact that $V^* := \bigcup_{n\geq 0} V^n$ is dense in T with respect to the metric d whenever $V^0 \neq \emptyset$, ([39], Lemma 1.3.11). We exclude the trivial case $V^0 = \emptyset$ in all of what follows. A result that holds for p.c.f.s.s. sets that we will apply repeatedly is, ([39], Proposition 1.3.5), for $i, j \in \Sigma_n$, $i \neq j$,

$$T_i \cap T_j = V_i^0 \cap V_j^0. \tag{1.3}$$

Now, consider a Dirichlet form on the finite set V^0 defined by

$$D(f,f) := \frac{1}{2} \sum_{\substack{x,y\in V^0:\\ x\neq y}} H_{xy}((f(x) - f(y))^2, \qquad \forall f \in C(V^0),$$

where for a countable set, A, we denote $C(A) := \{f : A \to \mathbb{R}\}$. To make this a Dirichlet form we require $H_{xy} \geq 0$, $\forall x, y \in V^0$, $x \neq y$, and if $H_{xx} := -\sum_{y\in V^0,\, y\neq x} H_{xy}$, then we also require the matrix $H := (H_{xy})_{x,y\in V^0}$ to be non-positive definite. Furthermore, we make the assumption of irreducibility, so that $Hf = 0$ if and only if $f \in C(V^0)$ is constant. Given this form and a set of scaling factors $\mathbf{r} := (r_i)_{i\in S}$ with $r_i > 0$ for each $i \in S$, we can use D to define a Dirichlet form on each of the V^n by setting, for $n \geq 0$,

$$\mathcal{E}^n(f,f) := \sum_{i\in\Sigma_n} \frac{1}{r_i} D(f_i, f_i), \qquad \forall f \in C(V^n), \tag{1.4}$$

where $r_i := r_{i_1} \dots r_{i_n}$ for $i \in \Sigma_n$ and $r_\emptyset := 1$. Whilst each $\mathcal{E}^n$ is a Dirichlet form, to establish the existence of a non-trivial limit as $n \to \infty$, we need to place some restrictions on the choice of $(D, \mathbf{r})$ so that the sequence $\{(V^n, \mathcal{E}^n)\}_{n\geq 0}$ is compatible in a sense that we shall now define. First, we introduce the *trace* operator, which gives a natural restriction of a Dirichlet form, $\mathcal{E}$, from a set A to a finite set $B \subseteq A$ and is defined by

$$\mathrm{Tr}(\mathcal{E}|B)(f,f) := \inf\{\mathcal{E}(g,g) : \ g \in \mathcal{F}, \ g|_B = f\}, \qquad \forall f \in C(B), \tag{1.5}$$

where $\mathcal{F}$ is the domain of $\mathcal{E}$. The domain of $\mathrm{Tr}(\mathcal{E}|B)$ is defined to be the set of functions for which the above infimum is finite. The sequence $\{(V^n, \mathcal{E}^n)\}_{n\geq 0}$ is said to be *compatible* if $\mathcal{E}^n = \mathrm{Tr}(\mathcal{E}^{n+1}|V^n)$ for each n, and in this case $(D, \mathbf{r})$ is said to be a *harmonic structure*. A harmonic structure $(D, \mathbf{r})$ is said to be *regular* if $0 < r_i < 1$, $\forall i \in S$.

The general problem of finding a harmonic structure is known ([39], Proposition 3.1.3) to be equivalent to finding a Dirichlet form D on V^0 such that, when we define $\mathcal{E}^1$ by (1.4), we have $\mathrm{Tr}(\mathcal{E}^1|V^0) = D$. Proving the existence of a solution to this renormalisation problem is not trivial and has not been achieved for p.c.f.s.s sets in general. A significant step towards answering this question was taken by Sabot, [53], who provided conditions for the existence and uniqueness of such a Dirichlet form. One particular application of this work is that for nested fractals, which are a special class of p.c.f.s.s. sets, there is precisely one solution to the renormalisation problem (up to multiplicative constants) associated with equal weights ($r_i = r_j$, for all $i, j \in S$).

Suppose now that $(D, \mathbf{r})$ is a regular harmonic structure, so that the sequence $\{(V^n, \mathcal{E}^n)\}_{n\geq 0}$ is compatible. Before taking limits, we introduce the notion of a resistance form. A non-negative symmetric quadratic form $(\mathcal{E}, \mathcal{F})$ is called a *resistance form* on a set X if it satisfies the following conditions:

- $\mathcal{F}$ is a linear subspace containing constants. $\mathcal{E}(f,f) = 0$ if and only if f is constant.
- Let $f \sim g$ if and only if $f - g$ is constant on X. Then $(\mathcal{F}/\sim, \mathcal{E})$ is a Hilbert space.
- If V is a finite subset of X and $f \in C(V)$, then there exists $g \in \mathcal{F}$ such that $g|_V = f$.
- For any $x, y \in X$,

$$\sup\left\{\frac{|f(x) - f(y)|^2}{\mathcal{E}(f,f)} : \ f \in \mathcal{F}, \ \mathcal{E}(f,f) > 0\right\} < \infty.$$

- If $f \in \mathcal{F}$ and $\overline{f} := (0 \vee f) \wedge 1$, then $\overline{f} \in \mathcal{F}$ and $\mathcal{E}(\overline{f}, \overline{f}) \leq \mathcal{E}(f,f)$.

In fact, the quadratic forms $\{(\mathcal{E}^n, C(V^n))\}_{n\geq 0}$ are also resistance forms. Working in a greater generality than the p.c.f.s.s. case, Kigami shows that if we define

$$\mathcal{E}'(f,f) := \lim_{n\to\infty} \mathcal{E}^n(f,f), \qquad \forall f \in \mathcal{F}', \tag{1.6}$$

where

$$\mathcal{F}' := \{f \in C(V^*) : \lim_{n\to\infty} \mathcal{E}^n(f,f) < \infty\}, \tag{1.7}$$

then $(\mathcal{E}', \mathcal{F}')$ is a resistance form on V^*. Note that we have abused notation slightly by using the convention that if a form $\mathcal{E}$ is defined for functions on a set A and f is a function defined on $B \supseteq A$, then we write $\mathcal{E}(f,f)$ to mean $\mathcal{E}(f|_A, f|_A)$. There are now two steps remaining: we first need to extend $(\mathcal{E}', \mathcal{F}')$ from V^* to T, and finally we need to check that it satisfies the definition of a Dirichlet form.

Naturally associated with a resistance form $(\mathcal{E}, \mathcal{F})$ on a set X is a *resistance metric*, R, which as the name suggests is a metric on X and may be defined by, for $x, y \in X$, $x \neq y$,

$$R(x,y)^{-1} := \inf\{\mathcal{E}(f,f) :\ f \in \mathcal{F},\ f(x) = 1,\ f(y) = 0\}, \tag{1.8}$$

and $R(x,x) = 0$. We can define such a function R' on $V^* \times V^*$ from our resistance form $(\mathcal{E}', \mathcal{F}')$, and because we are considering a p.c.f.s.s. set with a regular harmonic structure we have that (T, R) is the completion of the metric space (V^*, R'), where R is the natural extension of R' to T. Moreover, the topology of (T, R) is compatible with the original topology of (T, d), see [39], Theorem 3.3.4. Furthermore, if we define

$$\mathcal{E}(f,f) := \mathcal{E}'(f,f), \qquad \forall f \in \mathcal{F}, \tag{1.9}$$

where

$$\mathcal{F} := \{f \in C(T) :\ f|_{V^*} \in \mathcal{F}'\}, \tag{1.10}$$

and we use $C(T)$ to represent the continuous functions on T (with respect to d or R), then $(\mathcal{E}, \mathcal{F})$ is a resistance form on on T and has associated resistance metric R. To complete the construction of a Dirichlet form we need a measure on our p.c.f.s.s. set, and we shall assume that μ is a Borel probability measure on T that charges every non-empty open set. Under this assumption, it follows from results in Chapter 2 of [39] that $(\mathcal{E}, \mathcal{F})$ is actually an irreducible, conservative, local, regular Dirichlet form on $L^2(T, \mu)$.

One of the main goals of this chapter is to calculate the Hausdorff dimension of a random p.c.f.s.s. dendrite. For comparison, we note that the Hausdorff dimension of the fixed metric space (T, R) is the unique positive α that satisfies

$$\sum_{i \in S} r_i^{\alpha} = 1, \tag{1.11}$$

see [9], Corollary 8.10. This is analogous to the result proved by Moran in 1946 for the Hausdorff dimension of Euclidean self-similar sets satisfying an open set condition, [48].

Finally, we summarise the concept of a *self-similar measure* on T. If $p := (p_i)_{i \in S}$ is a set of weights satisfying $\sum_{i \in S} p_i = 1$, $0 < p_i < 1$ for $i \in S$, then there exists a Borel probability measure, μ on T that satisfies the following self-similarity relation

$$\mu(A) = \sum_{i \in S} p_i \mu(F_i^{-1}(A)),$$

for any Borel set $A \subseteq T$. For this measure, it is possible to show that

$$\mu(T_i) = p_i, \qquad \forall i \in \Sigma_*, \tag{1.12}$$

where $p_i := p_{i_1} \dots p_{i_n}$ for $i \in \Sigma_n$, and $p_\emptyset := 1$. In particular, when $p_i := r_i^{\alpha}$, with α defined as at (1.11), the measure μ may be used in the computation of the Hausdorff dimension of (T, R). In the case of T being a random p.c.f.s.s. dendrite, we will use a stochastically self-similar measure that satisfies a randomised version of (1.12) to prove the corresponding dimension result.

1.2 Geometry of p.c.f.s.s. dendrites

We saw in the previous section the standard method of approximating a p.c.f.s.s. set T by the finite subsets $(V^n)_{n \geq 0}$. Henceforth, we restrict our attention to the case when we have a p.c.f.s.s. structure, $(T, S, \{F_i\}_{i \in S})$, with T a dendrite, as defined in the introduction. For a graph on V^n, the natural edge set is

$$E^n := \{\{x, y\} : \ x, y \in T_i, \text{ for some } i \in \Sigma_n\}.$$

However, for our purposes, the graphs (V^n, E^n) are not the best way to approximate T. The main problem is that, even though T is a dendrite and contains no loops, the graphs (V^n, E^n) do not in general reflect this and may contain cycles. For example, this is the case for the well-known Vicsek set and Hata's tree-like set (see Examples 1.2 and 1.3). In this section, we introduce the graphs that we will use to approximate T and present a discussion of some of their simpler properties. In particular, we will show that they are graph trees, which has as an advantage that the resistance between vertices will simply be the sum of edge resistances along the path between them, making much of the analysis in subsequent sections more tractable.

We do not disregard the idea of (V^n, E^n) completely. We shall use the idea of refinement to obtain a sequence of vertex sets based on $(V^n)_{n\geq 0}$, and then choose edge sets more closely related to the underlying dendrite T, so that the resulting graph sequence has the properties that we would like. First though, we introduce some further notation. It is a consequence of the definition of a dendrite that, for any $x, y \in T$ there exists a unique path connecting x and y. Precisely, there exists a continuous injection $\gamma : [0,1] \to T$ such that $\gamma(0) = x$ and $\gamma(1) = y$. We shall denote a function with these properties γ_{xy}. The *path* connecting x and y can be defined to be the image of such a map, and we shall write it as G_{xy}, i.e. $G_{xy} := \gamma_{xy}([0,1])$. For a finite subset $V \subseteq T$, define the *direct neighbours* of $x \in V$ by

$$V(x) := \{x' : \; x' \in V, \; x' \neq x, \; G_{xx'} \cap V = \{x, x'\}\}.$$

We say V is *fine* if and only if $G_{xx_1} \cap G_{xx_2} = \{x\}$ for all $x \in V$ and distinct $x_1, x_2 \in V(x)$. A fine subset U containing V is called a *refinement* of V. The lemma we now state guarantees we can always find a finite refinement for a finite subset of T.

Lemma 1.2.1 *([39], Lemma 5.3). Let T be a dendrite. For any finite subset $V \subseteq T$, there exists a finite set $U \subseteq T$ which is a refinement of V.*

In the proof of the above lemma, the following refinement of V is introduced:

$$U := V \cup \{b(x, x_1, x_2) : \; x \in V, \; x_1, x_2 \in V(x), \; x_1 \neq x_2\}, \tag{1.13}$$

where $b = b(x, x_1, x_2)$ is defined to be the unique point in T such that $G_{xx_1} \cap G_{xx_2} = G_{xb}$. The function b picks out the *branch point* of the three vertices in its arguments, and so U is simply the set V with its branch points added. This is the *minimal refinement* in the sense that $U \subseteq U'$ for every refinement U' of V. The following lemma allows us to write down the minimal refinement in a more concise way.

Lemma 1.2.2 *Let V be a finite subset of T and U be defined by (1.13). Then $b(x_1, x_2, x_3) \in U$ for any $x_1, x_2, x_3 \in V$.*

Proof: Throughout this proof, write $b = b(x_1, x_2, x_3)$. First, suppose $x_1 = x_2$, then $G_{x_1x_2} \cap G_{x_1x_3} = G_{x_1x_1}$, and so $b = x_1 \in V$. Similarly for $x_1 = x_3$. If $x_2 = x_3$, then $G_{x_1x_2} \cap G_{x_1x_3} = G_{x_1x_2}$, and so $b = x_2 \in V$. Hence we can assume that x_1, x_2, x_3 are distinct. Clearly, if $b \in V$, then we are done. Suppose $b \notin V$. Let

$$t_i = \inf\{t \geq 0 : \; \gamma_{bx_i}(t) \in V\},$$

which takes values in $(0,1]$, because $\gamma_{bx_i}(1) \in V$, for $i = 1,2,3$. Furthermore, define $x'_i = \gamma_{bx_i}(t_i) \in V$. Now, by definition, $b \in G_{x_1x_2}$, and so applying the path uniqueness property of a dendrite, we find that $G_{bx_1} \cap G_{bx_2} = \{b\}$. Similarly, $G_{bx_1} \cap G_{bx_3} = \{b\}$. Suppose $x' \in G_{bx_2} \cap G_{bx_3}$, then clearly $x' \in G_{x_1x_2} \cap G_{x_1x_3} = G_{x_1b}$. Consequently, we also have $x' \in G_{bx_1} \cap G_{bx_2} = \{b\}$, and so $G_{bx_2} \cap G_{bx_3} = \{b\}$. Noting that $G_{bx'_i} \subseteq G_{bx_i}$, $i = 1,2,3$, it follows that

$$G_{bx'_1} \cap G_{bx'_2} = G_{bx'_1} \cap G_{bx'_3} = G_{bx'_2} \cap G_{bx'_3} = \{b\}.$$

Using these formulae, it is elementary to check that $b = b(x'_1, x'_2, x'_3)$ and $x'_1 \in V$, $x'_2, x'_3 \in V(x'_1)$, $x'_2 \neq x'_3$. Thus $b \in U$. $\square$

For a finite subset $V \subseteq T$, we shall denote

$$\mathcal{R}(V) := \{b(x_1,x_2,x_3) : \; x_1,x_2,x_3 \in V\},$$

which, by the previous lemma, is simply another way of representing the minimal refinement of V. It is clear from the minimal fineness of this set that, if V is fine, then $\mathcal{R}(V) = V$. We are now ready to define our alternative sequence of finite subsets of T. Let $\tilde{V}^0 := \mathcal{R}(V^0)$, and define

$$\tilde{V}^n := \bigcup_{i \in \Sigma_n} \tilde{V}^0_i.$$

By Lemma 1.2.1, $\tilde{V}^0$ is a finite set and consequently, so is $\tilde{V}^n$ for all $n \geq 0$. Analogous to the definition of V^*, we also define $\tilde{V}^* := \cup_{n\geq 0}\tilde{V}^n$.

Since $\tilde{V}^0$ is a non-empty compact set, $\tilde{V}^n \to T$ with respect to the Hausdorff metric on (T,d), ([39], Theorem 1.1.7). From this fact it follows that T is the closure of $\tilde{V}^*$. Note that is only the closure with respect to the metric d, which we are only interested in for the construction of T. We shall show later that, as in the deterministic case when we had a regular harmonic form, the topology induced by the random resistance metric that we construct in Section 1.4 is the same as that of d, (Proposition 1.4.8). This means that the closure of $\tilde{V}^*$ with respect to the resistance metric is also equal to T, and so $(\tilde{V}^n)_{n\geq 0}$ is a reasonable sequence to approximate T by.

As a corollary of the next three lemmas we have that $(\tilde{V}^n)_{n\geq 0}$ is an increasing sequence of finite subsets of T. This is important for the construction of the Dirichlet form on T. We start the series by showing, in Lemma 1.2.3, that $\mathcal{R}$ preserves order of finite subsets of T. Next, in Lemma 1.2.4, we demonstrate that $\tilde{V}^n$ is fine. From this, it is straightforward to show that $\mathcal{R}$ and F^n commute on V^0, where F is the function defined at (1.1). To clarify this statement, we note that Lemma 1.2.5 may be presented in the following alternative notation $\mathcal{R}(F^n(V^0)) = F^n(\mathcal{R}(V^0))$.

Lemma 1.2.3 *Let V and V' be two finite subsets of T. If $V \subseteq V'$, then $\mathcal{R}(V) \subseteq \mathcal{R}(V')$.*

Proof: Applying the definition of $\mathcal{R}(\cdot)$ we obtain, for $V \subseteq V'$,

$$\mathcal{R}(V) = \{b(x_1, x_2, x_3) : x_1, x_2, x_3 \in V\} \subseteq \{b(x_1, x_2, x_3) : x_1, x_2, x_3 \in V'\} = \mathcal{R}(V').$$

$\square$

Lemma 1.2.4

(a) For $x, y \in T$ and $f : T \to T$ a continuous injection, $f(G_{xy}) = G_{f(x)f(y)}$.

(b) Let V be a finite subset of T such that $V^0 \subseteq V$, and define $V' = F(V)$. Then, for $x \in V'$, $x' \in V'(x)$, we have $G_{xx'} \subseteq T_i$ for some $i \in S$.

(c) Let V be a fine finite subset of T such that $V^0 \subseteq V$, then $V' = F(V)$ is a fine finite subset of T with $V^0 \subseteq V'$.

(d) $\tilde{V}^n$ is fine.

Proof: Let $x, y \in T$ and suppose $f : T \to T$ is a continuous injection. By definition, we have that γ_{xy} is a continuous injection with $\gamma(0) = x$ and $\gamma(1) = y$. Hence $f \circ \gamma_{xy}$ is a continuous injection with $(f \circ \gamma_{xy})(0) = f(x)$ and $(f \circ \gamma_{xy})(1) = f(y)$, and so

$$(f \circ \gamma_{xy})([0,1]) = G_{f(x)f(y)}.$$

We also have that $\gamma_{xy}([0,1]) = G_{xy}$, which means that

$$(f \circ \gamma_{xy})([0,1]) = f(\gamma_{xy}([0,1])) = f(G_{xy}).$$

Comparing the two expressions for $(f \circ \gamma_{xy})([0,1])$ yields part (a).

Let V be a finite subset of T such that $V^0 \subseteq V$, define $V' = F(V)$. If (b) does not hold, then we can find $x \in V'$ and $x' \in V'(x)$ such that there exists $t_0 \in (0,1]$, $i, j \in S$, $i \neq j$ with $\gamma_{xx'}(0) \in T_i \backslash T_j$ and $\gamma_{xx'}(t_0) \in T_j \backslash T_i$. Let $t_1 = \inf\{t : \gamma_{xx'}(t) \notin T_i\}$. Clearly, t_1 is well-defined and not greater than t_0. Furthermore, by the continuity of $\gamma_{xx'}$ and the compactness of the sets $T_{i'}$, $i' \in S$, we must have that $t_1 \in (0,1)$ and

$$\gamma_{xx'}(t_1) \in T_i \cap T_k, \qquad \text{for some } i, k \in S, i \neq k.$$

By (1.3), this means that $\gamma_{xx'}(t_1) \in V_i^0 \cap V_k^0 \subseteq F(V^0) \subseteq V'$. However, this contradicts that $x' \in V'(x)$ and so (b) must hold.

Now assume that V is fine. Fix $x \in V'$ and let x_1 and x_2 be distinct points of $V'(x)$. By part (b) we know that $G_{xx_1} \subseteq T_i$, $G_{xx_2} \subseteq T_j$, for some $i, j \in S$. First suppose $i = j$. We can write $x = F_i(x')$, $x_1 = F_i(x_1')$ and $x_2 = F_i(x_2')$, where x', x_1' and x_2' are distinct points of V. Now if $y \in G_{x'x_1'} \cap V$ then, by (a) this would imply that $F_i(y) \in G_{xx_1} \cap V'$, and so $F_i(y) \in \{x, x_1\}$, because x and x_1 are direct neighbours in V'. Hence $y \in \{x', x_1'\}$ and so $x_1' \in V(x')$. Similarly, $x_2' \in V(x')$. Thus, because V is fine we have that $G_{x'x_1'} \cap G_{x'x_2'} = \{x'\}$. By (a), applying F_i to both sides of this equation yields $G_{xx_1} \cap G_{xx_2} = \{x\}$ and so V' is fine.

Now suppose $i \neq j$. This means that $G_{xx_1} \cap G_{xx_2} \subseteq T_i \cap T_j \subseteq F(V^0) \subseteq V'$. However, $G_{xx_1} \cap V' = \{x, x_1\}$ and $G_{xx_2} \cap V' = \{x, x_2\}$. Thus

$$G_{xx_1} \cap G_{xx_2} = G_{xx_1} \cap G_{xx_2} \cap V' = \{x, x_1\} \cap \{x, x_2\} = \{x\}.$$

This completes the proof that V' is fine. The last part of (c) is trivial on noting that $V^0 \subseteq F(V^0)$.

Part (d) is obtained by applying part (c) repeatedly to $\tilde{V}^0$, which is fine by definition. $\square$

Lemma 1.2.5 *For $n \geq 0$, $\mathcal{R}(V^n) = \tilde{V}^n$.*

Proof: By definition, $V^0 \subseteq \tilde{V}^0$. Applying F^n to this we obtain $V^n \subseteq \tilde{V}^n$. Thus, by Lemma 1.2.3, we have $\mathcal{R}(V^n) \subseteq \mathcal{R}(\tilde{V}^n) = \tilde{V}^n$, where the equality is a result of Lemma 1.2.4(d).

It remains to show that $\tilde{V}^n \subseteq \mathcal{R}(V^n)$. Let $x \in \tilde{V}^n$, then $x = F_i(x')$ for some $x' \in \tilde{V}^0$ and $i \in \Sigma_n$. Since $\tilde{V}^0 = \mathcal{R}(V^0)$ we must have $x' = b(x_1, x_2, x_3)$ for some $x_1, x_2, x_3 \in V^0$. This means that x' is the unique point in T such that $G_{x_1x_2} \cap G_{x_1x_3} = G_{x_1x'}$. Applying F_i to this equation, and using Lemma 1.2.4(a) yields $G_{F_i(x_1)F_i(x_2)} \cap G_{F_i(x_1)F_i(x_3)} = G_{F_i(x_1)x}$. Thus $x = b(F_i(x_1), F_i(x_2), F_i(x_3)) \in \mathcal{R}(V^n)$. $\square$

Corollary 1.2.6 *For $n \geq 0$, $\tilde{V}^n \subseteq \tilde{V}^{n+1}$.*

Proof: From the previous lemma, we know that $\mathcal{R}(V^n) = \tilde{V}^n$. Since $V^n \subseteq V^{n+1}$, Lemma 1.2.3 implies the claim. $\square$

To complete this section, we shall define a sequence of graphs on the nested sequence of vertex sets, $(\tilde{V}^n)_{n\geq 0}$. We shall take the natural choice of edges on $\tilde{V}^n$ given by pairs of direct neighbours. Precisely, we define the edge set by

$$\tilde{E}^n := \{\{x, y\} : x \in \tilde{V}^n,\ y \in \tilde{V}^n(x)\}.$$

The next proposition gives use that the graphs $(\tilde{V}^n, \tilde{E}^n)$ form a sequence of graph trees. This result is followed by a presentation of some other properties of the graphs that we will apply in later sections.

Proposition 1.2.7 $(\tilde{V}^n, \tilde{E}^n)$ *is a graph tree for each* $n \geq 0$.
Proof: Let $x, y \in \tilde{V}^n$, and set $t_0 = 0$. Define

$$t_{n+1} := \inf\{t > t_n : \gamma_{xy}(t) \in \tilde{V}^n\},$$

$x_i := \gamma_{xy}(t_i)$ and $M := \inf\{n : x_n = y\}$. By the injectivity of γ_{xy}, we must have that $M \leq |\tilde{V}^n| < \infty$. Using elementary arguments, it is possible to check that $x = x_0, \dots, x_M = y$ is a path from x to y with $\{x_{m-1}, x_m\} \in \tilde{E}^n$ for every $m = 1, \dots, M$. Hence $(\tilde{V}^n, \tilde{E}^n)$ is connected.

It remains to show that $(\tilde{V}^n, \tilde{E}^n)$ is acyclic and we shall do this using a proof by contradiction. Suppose $x_0, \dots, x_M = x_0$ is a cycle in $(\tilde{V}^n, \tilde{E}^n)$, necessarily we have $3 \leq M < \infty$ and $x_0, \dots, x_{M-1}$ distinct. We first note that, since $\tilde{V}^n$ is fine, we must have $G_{x_0x_1} \cap G_{x_{M-1}x_M} = \{x_0\}$, and so by adjoining the two paths end-to-end we have $G_{x_1x_{M-1}} = G_{x_0x_1} \cup G_{x_{M-1}x_M}$. Furthermore, it is immediate from our assumptions that $G_{x_1x_2} \cup \dots \cup G_{x_{M-2}x_{M-1}}$ is a path-connected subspace of T containing the points x_1 and x_{M-1}. By the uniqueness of paths on T, it follows that $G_{x_1x_{M-1}}$ is a subset of this union. Combining these facts we find that $x_0 \in G_{x_1x_2} \cup \dots \cup G_{x_{M-2}x_{M-1}}$, and in particular $x_0 \in G_{x_mx_{m+1}}$ for some $m \in \{1, \dots, M-2\}$. However, by the definition of the edges as direct neighbours we have that $G_{x_mx_{m+1}} \cap \tilde{V}^n = \{x_m, x_{m+1}\}$. Thus $x_0 = x_m$ for some $m \in \{1, \dots, M-1\}$, which is a contradiction and so no such cycle can exist. $\square$

The following lemma gives us an alternative representation for edges in $\tilde{E}^n$. In the proof, we will use the obvious notation $G_e := G_{xy}$ for an edge $e = \{x, y\}$.

Lemma 1.2.8 *For every edge* $e \in \tilde{E}^n$*, there exists a unique* $e' \in \tilde{E}^0$ *and* $i \in \Sigma_n$ *such that* $e = F_i(e')$.
Proof: We first prove existence. Let $e = \{x, y\} \in \tilde{E}^n$. Applying the obvious generalisation of Lemma 1.2.4(b), we immediately have $\{x, y\} \subseteq T_i$ for some $i \in \Sigma_n$ and hence $x = F_i(x')$, $y = F_i(y')$ for some $x', y' \in T$. We are required to show that $\{x', y'\} \in \tilde{E}^0$. Suppose there exists a $j \neq i$ such that $x = F_j(x'')$. Then $x \in T_i \cap T_j = V_i^0 \cap V_j^0$, by (1.3). It follows from this and the injectivity of F_i that $x' \in V^0 \subseteq \tilde{V}^0$. If no such j exists then $x \in \tilde{V}^n \cap (\cup_{j \in \Sigma_n, j \neq i} T_j^c) \subseteq \tilde{V}_i^0$. Again, by the injectivity of F_i, this implies that $x' \in \tilde{V}^0$. Similarly, $y' \in \tilde{V}^0$. Suppose now $z' \in G_{x'y'} \cap \tilde{V}^0$, then

$F_i(z') \in G_{xy} \cap \tilde{V}^n = \{x, y\}$, where we use the fineness property of the set $\tilde{V}^n$ for the equality. Thus $z' \in \{x', y'\}$ by injectivity, and so x' and y' are direct neighbours in $\tilde{V}^0$. This means that $e = F_i(e')$, where $e' = \{x', y'\} \in \tilde{E}^0$, and it remains to show this expression is unique.

Suppose $e = F_i(e') = F_j(e'')$ for some $i, j \in \Sigma_n$, $e', e'' \in \tilde{E}^0$. First note that by Lemma 1.2.4(a), $G_e = F_i(G_{e'})$ and also $G_e = F_j(G_{e''})$. Thus, if $i \neq j$, we have $G_e \subseteq T_i \cap T_j \subseteq V_i^0 \cap V_j^0$. This is a contradiction, because G_e is an uncountable set, whereas $V_i^0 \cap V_j^0$ is a finite set. Hence $i = j$. In this case, we have $F_i(e') = F_i(e'')$ and it follows from injectivity that $e' = e''$ and the proof is complete. $\square$

We now prove the converse result.

Lemma 1.2.9 *If $e \in \tilde{E}^0$, $i \in \Sigma_n$, then $F_i(e) \in \tilde{E}^n$.*

Proof: Suppose $e = \{x, y\} \in \tilde{E}^0$, $i \in \Sigma_n$. Clearly we have $F_i(x), F_i(y) \in \tilde{V}^n$ and also $G_{F_i(x)F_i(y)} \subseteq T_i$ by Lemma 1.2.4(a). Now, $\tilde{V}^n \cap T_i = \bigcup_{j \in \Sigma_n} \tilde{V}_j^0 \cap \tilde{V}_i^0 = \tilde{V}_i^0$. Hence

$$\begin{aligned} G_{F_i(x)F_i(y)} \cap \tilde{V}^n &= G_{F_i(x)F_i(y)} \cap \tilde{V}_i^0 \\ &= F_i(G_{xy} \cap \tilde{V}^0) \\ &= F_i(\{x, y\}) \\ &= \{F_i(x), F_i(y)\}, \end{aligned}$$

where the third equality holds because $e \in \tilde{E}^0$. This means that $F_i(x)$ and $F_i(y)$ are direct neighbours in $\tilde{V}^n$, which proves the lemma. $\square$

The identity we prove now will be particularly useful in proving the compatibility of the sequence of resistance forms introduced in Section 1.4.

Lemma 1.2.10 *Let $e \in \tilde{E}^k$, $e' \in \tilde{E}^l$, $i \in \Sigma_m$, $j \in \Sigma_n$, then*

$$\mathbf{1}_{\{F_{ij}(G_{e'}) \subseteq G_e\}} = \sum_{e'' \in \tilde{E}^0} \mathbf{1}_{\{F_j(G_{e'}) \subseteq G_{e''}\}} \mathbf{1}_{\{F_i(G_{e''}) \subseteq G_e\}}. \tag{1.14}$$

Proof: We prove only the result when $k = l = 0$ as the full result is easily deduced from this case. Fix $e, e' \in \tilde{E}^0$, $i \in \Sigma_m$, $j \in \Sigma_n$. First, note that if e^1, e^2 are distinct edges in $\tilde{E}^0$ then, by the fineness of $\tilde{V}^0$, we necessarily have $\#(G_{e^1} \cap G_{e^2}) \leq 1$. Hence the uncountably infinite set $F_j(G_{e'})$ can not be contained in $G_{e^1} \cap G_{e^2}$ and so the right-hand side of (1.14) is less than or equal to 1. Thus it will be sufficient to show that $F_{ij}(G_{e'}) \subseteq G_e$ if and only if

$$F_j(G_{e'}) \subseteq G_{e''} \text{ and } F_i(G_{e''}) \subseteq G_e \text{ for some } e'' \in \tilde{E}^0.$$

Since the $\Leftarrow$ implication is obvious, it remains to show $\Rightarrow$. To do this, write $e = \{e_+, e_-\}$ and consider the path from e_+ to e_- in $\tilde{V}^m$ constructed similarly to the path in the proof of Proposition 1.2.7. Denote this by $e_+ = x_0, \ldots, x_{m'} = e_-$. Let $e_+ = y_0, \ldots, y_{n'} = e_-$ be the corresponding path in $\tilde{V}^{m+n}$. Since $\tilde{V}^m \subseteq \tilde{V}^{m+n}$, this construction immediately implies that for every $l \in \{1, \ldots, n'\}$, $G_{y_{l-1}y_l} \subseteq G_{x_{l'-1}x_{l'}}$ for some $l' \in \{1, \ldots, m'\}$. Thus, by the uniqueness of paths in T, if $e^n \in \tilde{E}^{m+n}$ and $G_{e^n} \subseteq G_e$, then there exists an $e^m \in \tilde{E}^m$ such that $G_{e^n} \subseteq G_{e^m} \subseteq G_e$.

Now suppose $F_{ij}(G_{e'}) \subseteq G_e$. By the previous paragraph and the two previous lemmas, there exists $e'' \in \tilde{E}^0$, $i' \in \Sigma_m$ such that $F_{ij}(G_{e'}) \subseteq F_{i'}(G_{e''}) \subseteq G_e$. If $i \neq i'$, then $F_{ij}(G_{e'}) \subseteq V_i^0 \cap V_{i'}^0$, which cannot be true, and so $i' = i$. Hence $F_i(G_{e''}) \subseteq G_e$ and by injectivity, we also have that $F_j(G_{e'}) \subseteq G_{e''}$, which completes the proof. $\square$

Finally for this section, we prove a result about the paths on vertices of $(\tilde{V}^n)_{n\geq 0}$.

Lemma 1.2.11 *If $x \in \tilde{V}^0$, $y \in \tilde{V}^n$, then we can find a sequence $x_0, \ldots, x_m$, with $x_0 = x$, $x_m = y$, $\{x_{l-1}, x_l\} \in \cup_{n'\geq 0}\tilde{E}^{n'}$ for $l \in \{1, \ldots m\}$, and such that:*
(1) for $n' > n$, $\{x_{l-1}, x_l\} \notin \tilde{E}^{n'}$ for any $l \in \{1, \ldots m\}$,
(2) for $n' \leq n$, $\{x_{l-1}, x_l\} \in \tilde{E}^{n'}$ for at most $N|\tilde{E}^0|$ of the $l \in \{1, \ldots m\}$.
Proof: We shall proceed by induction on n. Clearly the assertion is true for $n = 0$. Hence assume that $n \geq 1$ and the conclusion holds for elements of $\tilde{V}^{n-1}$. If $y \in \tilde{V}^n$, then $y = F_i(y')$ for some $y' \in \tilde{V}^1$, $i \in \Sigma_{n-1}$. Now choose $y'' \in \tilde{V}^0$. Note that by Corollary 1.2.6, $y'' \in \tilde{V}^1$ and so there exists a path $y' = y_0, \ldots, y_{m'} = y''$ with length $m' \leq |\tilde{E}^1|$ and edges $\{y_{l-1}, y_l\} \in \tilde{E}^1$. Note that, Lemma 1.2.8 implies that $|\tilde{E}^1| \leq N|\tilde{E}^0|$. Hence $F_i(y_0), \ldots, F_i(y_{m'})$ is a path of length $m' \leq N|\tilde{E}^0|$. Also by Lemma 1.2.8, for each l, we must have $\{y_{l-1}, y_l\} = \{F_j(e_-), F_j(e_+)\}$ for some $j \in S$, $e \in \tilde{E}^0$, and so $\{F_i(y_{l-1}), F_i(y_l)\} = \{F_{ij}(e_-), F_{ij}(e_+)\} \in \tilde{E}^n$, by Lemma 1.2.9. This path starts at $F_i(y') = y$ and ends at $F_i(y'') \in \tilde{V}^{n-1}$. Applying the inductive hypothesis, the proof is complete. $\square$

1.3 Height of a multiplicative cascade

The results that we obtain in this section about the height of a multiplicative cascade will be useful in establishing various properties of the random fractal dendrite introduced in the next section, but they are also of interest in their own right. Following Theorem 1.3.6, we discuss the connection between our results and the classical results about the extinction time of a Galton-Watson branching process.

We will use the alphabet S and address spaces Σ_n, Σ_* and Σ defined at (1.2). We define a *multiplicative cascade* to be a collection of random variables, $(w(i))_{i\in\Sigma_*\backslash\{\emptyset\}}$, which take values in $(0,1]$ and satisfy the following assumption. The N-tuples

$$(w(ij))_{j\in S}, \qquad i\in\Sigma_*\backslash\{\emptyset\}, \tag{1.15}$$

are independent copies of $(w(j))_{j\in S}$. The multiplicative cascade has a naturally associated filtration, $(\mathcal{F}_n)_{n\geq 0}$, defined by

$$\mathcal{F}_n := \sigma(w(i) : |i| \leq n). \tag{1.16}$$

Furthermore, let

$$l(i) := \begin{cases} \prod_{n=1}^{|i|} w(i|n) & \text{if } i\in\Sigma_*\backslash\{\emptyset\}, \\ 1 & \text{if } i=\emptyset. \end{cases} \tag{1.17}$$

Models of this type have been studied extensively. In the case when $(w(i))_{i\in S}$ are independent we have what is known as Mandelbrot's multiplicative cascade, see [46]. For further examples of work on such multiplicative cascades, see [24], [44], [45]. Unlike the focus of this section, much of the previous work on multiplicative cascades has been targeted at determining properties of the limit of the martingale $Z^\theta(n)$, which is introduced below, rather than the height of the cascade.

Throughout the arguments, we use the following function

$$\phi(\theta) := \mathbf{E}\left(\sum_{i\in S} w(i)^\theta\right), \qquad \theta > 0. \tag{1.18}$$

It is useful because it allows estimates to be made over moments of all the random variables $(w(i))_{i\in S}$ simultaneously. Note that ϕ is a decreasing, continuous function. A simple result that is important in what follows is

$$\phi(\theta) \to \sum_{i\in S} \mathbf{P}(w(i)=1), \qquad \text{as } \theta\to\infty. \tag{1.19}$$

The function ϕ is also used as a scaling function in the following definition of the so-called *tree-martingale* (this term was introduced in [24]) associated with our multiplicative cascade. Let

$$Z^\theta(n) := \frac{\sum_{i\in\Sigma_n} l(i)^\theta}{\phi(\theta)^n}. \tag{1.20}$$

It is straightforward to check that, for each $\theta > 0$, $(Z^\theta(n))_{n\geq 0}$ is an $(\mathcal{F}_n)_{n\geq 0}$ martingale. In particular, $\mathbf{E}\left(Z^\theta(n)\right) = \mathbf{E}\left(Z^\theta(0)\right) = 1$.

To generalise our cascade model, we introduce random perturbations of the $l(i)$, denoted by $(X_i)_{i\in\Sigma_*}$. We assume that the X_i are identically distributed non-negative random variables satisfying the following conditions

$$\mathbf{E}(X_i^d) < \infty, \qquad \forall d > 0, \tag{1.21}$$

$$X_i \perp \mathcal{F}_{|i|}, \qquad \forall i \in \Sigma_*, \tag{1.22}$$

where $\perp$ is taken to mean "is independent of". The reason for the introduction of the factors $(X_i)_{i\in\Sigma_*}$ will become apparent in later sections where perturbations with these properties arise naturally in the construction of our random self-similar dendrite.

We can consider the cascade model as a weighted graph tree, rooted at $\emptyset$ with vertex set Σ_* and edge set $\{\{i, i|(|i|-1)\} : i \in \Sigma_*\backslash\{\emptyset\}\}$; where the edge $\{i, i|(|i|-1)\}$ has length $l(i)X_i$. For two vertices in Σ_*, we define the distance between them to be the sum of edge lengths along the shortest path in the graph. This is indeed a distance and allows the *height*, H, of the tree (the supremum of distances to the root) to be written as

$$H = \sup_{i\in\Sigma} \sum_{n=0}^{\infty} l(i|n)X_{i|n}.$$

Strictly, the sum should be from $n = 1$, but this definition will be more convenient for our purposes.

The main result of this section is Theorem 1.3.4, a corollary of which gives a relatively weak sufficient condition for the expected value of the height of our tree to be finite. In Theorem 1.3.6, we deal with the unperturbed cascade and show that the condition is necessary in this case. We start by estimating how fast $l(i)X_i$ decays as $|i| \to \infty$.

Lemma 1.3.1 *Suppose* $\sum_{i\in S} \mathbf{P}(w(i) = 1) < 1$ *and* $d \geq 1$, *then*
(i) there exist constants $c_{1.1}, c_{1.2}$ *such that*

$$\mathbf{E}\left(\left(\sup_{i\in\Sigma_n} l(i)X_i\right)^d\right) \leq c_{1.1}e^{-c_{1.2}n}, \qquad \forall n \geq 0.$$

(ii) $\mathbf{P}$*-a.s., there exist constants* $c_{1.3}, c_{1.4}$ *such that*

$$\sup_{i\in\Sigma_n} l(i)X_i \leq c_{1.3}e^{-c_{1.4}n}, \qquad \forall n \geq 0.$$

Proof: To prove (i) we first look for bounds on the tail of the distribution of

$\sup_{i\in\Sigma_n} l(i)X_i$. Using the definition of $Z^\theta(n)$, Markov's inequality and the independence assumption of the X_is we obtain, for $\theta > 0$,

$$\begin{aligned}
\mathbf{P}\left(\sup_{i\in\Sigma_n} l(i)X_i \geq \lambda\right) &\leq \mathbf{P}\left(\sum_{i\in\Sigma_n} l(i)^\theta X_i^\theta \geq \lambda^\theta\right) \\
&\leq \lambda^{-\theta}\mathbf{E}\left(X_\emptyset^\theta\right)\mathbf{E}\left(Z^\theta(n)\right)\phi(\theta)^n \\
&= \lambda^{-\theta}\mathbf{E}\left(X_\emptyset^\theta\right)\phi(\theta)^n. \qquad (1.23)
\end{aligned}$$

The condition $\sum_{i\in S}\mathbf{P}(w(i)=1) < 1$ and (1.19) imply that we can find $\theta_0 > d$ large enough so that $\phi(\theta_0) < 1$. Set $x := \|X_\emptyset\|_{\theta_0}$ ($<\infty$ by assumption) and define

$$\lambda_n := x\phi(\theta_0)^{\frac{n}{\theta_0}},$$

which is less than 1 for $n \geq n_0$ for some $n_0 \geq 0$. Assume for now that $n \geq n_0$ and $\theta = \theta_0$. For $\lambda \geq \lambda_n$, the upper bound at (1.23) is ≤ 1 and so is non-trivial. For $\lambda < \lambda_n$, we merely use the fact that we are trying to bound a probability, i.e. $\mathbf{P}(\sup_{i\in\Sigma_n} l(i)X_i \geq \lambda) \leq 1$. We apply these estimates to bound the moments of $\sup_{i\in\Sigma_n} l(i)X_i$ as follows:

$$\begin{aligned}
\mathbf{E}\left(\left(\sup_{i\in\Sigma_n} l(i)X_i\right)^d\right) &= d\int_0^\infty \lambda^{d-1}\mathbf{P}\left(\sup_{i\in\Sigma_n} l(i)X_i \geq \lambda\right)d\lambda \\
&\leq d\int_0^{\lambda_n}\lambda^{d-1}d\lambda + d\int_{\lambda_n}^\infty \lambda^{d-1-\theta_0}x^{\theta_0}\phi(\theta_0)^n d\lambda \\
&= \frac{x^d\theta_0}{\theta_0 - d}\phi(\theta_0)^{\frac{dn}{\theta_0}}
\end{aligned}$$

Hence taking $c_{1.2} = -\frac{d}{\theta_0}\ln\phi(\theta_0)$ and $c_{1.1}$ suitably large gives us part (i) of the lemma.

To prove (ii) we again look to bound the tail probability of $\sup_{i\in\Sigma_n} l(i)X_i$. We proceed as above to obtain the bound

$$\mathbf{P}\left(\sup_{i\in\Sigma_n} l(i)X_i \geq \lambda^n\right) \leq \lambda^{-n\theta}\mathbf{E}\left(X_\emptyset^\theta\right)\phi(\theta)^n.$$

If we fix $\theta = \theta_0$ we can find a $\lambda_0 \in (0,1)$ such that $\lambda_0^{-\theta_0}\phi(\theta_0) < 1$, and so

$$\sum_{n=0}^\infty \mathbf{P}\left(\sup_{i\in\Sigma_n} l(i)X_i \geq \lambda_0^n\right) \leq \mathbf{E}\left(X_\emptyset^{\theta_0}\right)\sum_{n=0}^\infty\left(\lambda_0^{-\theta_0}\phi(\theta_0)\right)^n < \infty.$$

An application of the Borel-Cantelli lemma then gives us part (ii) of the lemma. □

In the lines of the argument leading up to (1.23) we use the fact that $l(i)^\theta X_i^\theta \leq \sum_{j\in\Sigma_n} l(j)^\theta X_j^\theta$, which may seem rather crude. However, we show in the following proposition that if $\mathbf{P}(w(i) = 0) = 0$, for each i, then the $l(i)$ cannot decay quicker than exponentially.

Proposition 1.3.2 *Suppose* $\sum_{i\in S}\mathbf{P}(w(i)=1)<1$ *and that* $\mathbf{P}(w(i)=0)=0$*, for every* $i\in S$*. There exist constants* $\theta_{1.1},\theta_{1.2}$ *such that,* $\mathbf{P}$*-a.s., given* $\varepsilon>0$*, there exist constants* $c_{1.5},\dots,c_{1.8}$ *satisfying*

$$c_{1.5}e^{-(\theta_{1.1}+\varepsilon)n}\leq\inf_{i\in\Sigma_n}l(i)\leq c_{1.6}e^{-(\theta_{1.1}-\varepsilon)n},\qquad\forall n\geq 0,$$

and

$$c_{1.7}e^{-(\theta_{1.2}+\varepsilon)n}\leq\sup_{i\in\Sigma_n}l(i)\leq c_{1.8}e^{-(\theta_{1.2}-\varepsilon)n},\qquad\forall n\geq 0.$$

Proof: If we define $z_i^{(n)}:=\ln l(i)$, $i\in\Sigma_n$ to be the positions of a collection of particles at time n, then the $z_i^{(n)}$ form a branching random walk. Similarly, we may define a branching walk by setting $z_i^{(n)}:=-\ln l(i)$. The results are an immediate application of the results deduced in [15] to these branching random walks. $\square$

Remark 1.1 *In general we find that* $\theta_{1.1}>\theta_{1.2}$*. When this occurs, this result tells us that the variation of the relative branch size grows exponentially with the level of the tree.*

In the proof of the main result of this section, Theorem 1.3.4, we apply the following elementary lemma, which we state without proof.

Lemma 1.3.3 *Let* $(x_n)_{n\geq 0}$ *be a sequence of non-negative real numbers, then*

$$\left(\sum_{n=0}^{\infty}x_n\right)^d\leq K_d\sum_{n=0}^{\infty}x_n^d(1+n)^d,\qquad\forall d\geq 1,$$

where

$$K_d:=\begin{cases}\left(\sum_{n=0}^{\infty}(1+n)^{-\frac{d}{d-1}}\right)^{d-1} & \text{if } d>1\\ 1 & \text{if } d=1.\end{cases}$$

Theorem 1.3.4 *Let* $\sum_{i\in S}\mathbf{P}(w(i)=1)<1$ *and* $d\geq 0$*, then*

$$\mathbf{E}\left(\left(\sum_{n=0}^{\infty}\sup_{i\in\Sigma_n}l(i)X_i\right)^d\right)<\infty.$$

Proof: For $d\geq 1$, applying Lemmas 1.3.1 and 1.3.3 yields

$$\begin{aligned}\mathbf{E}\left(\left(\sum_{n=0}^{\infty}\sup_{i\in\Sigma_n}l(i)X_i\right)^d\right) &\leq K_d\mathbf{E}\left(\sum_{n=0}^{\infty}\left(\sup_{i\in\Sigma_n}l(i)X_i\right)^d(1+n)^d\right)\\ &\leq c_{1.1}K_d\sum_{n=0}^{\infty}(1+n)^de^{-c_{1.2}n},\end{aligned}$$

which is clearly finite. The result for $d \in [0,1)$ follows from the case when $d = 1$ by applying the inequality $x^d \leq 1 + x$, which holds for all $x \geq 0$. $\square$

Corollary 1.3.5 *Let* $\sum_{i \in S} \mathbf{P}(w(i) = 1) < 1$ *and* $d \geq 0$, *then* $\mathbf{E}(H^d) < \infty$.

Proof: This follows from Theorem 1.3.4 on noting that

$$H = \sup_{i \in \Sigma} \sum_{n=0}^{\infty} l(i|n) X_{i|n} \leq \sum_{n=0}^{\infty} \sup_{i \in \Sigma_n} l(i) X_i.$$

$\square$

In the final result of this section, we show how for unperturbed cascades the condition $\sum_{i \in S} \mathbf{P}(w(i) = 1) < 1$ is also necessary for finite moments of H.

Theorem 1.3.6 *Assume* $X_i \equiv 1$, $\forall i \in \Sigma_*$, *then*
(a) if $\sum_{i \in S} \mathbf{P}(w(i) = 1) < 1$, $\mathbf{E} H^d < \infty$, $\forall d \in \mathbb{R}$.
(b) if $\sum_{i \in S} \mathbf{P}(w(i) = 1) = 1$, $\mathbf{E} H = \infty$.
(c) if $\sum_{i \in S} \mathbf{P}(w(i) = 1) > 1$, $\mathbf{P}(H = \infty) > 0$.

Proof: Assume $\sum_{i \in S} \mathbf{P}(w(i) = 1) < 1$. Clearly, if $X_i \equiv 1$, $\forall i \in \Sigma_*$, then $(X_i)_{i \in \Sigma_*}$ satisfies the conditions that enable us to apply the previous results of the section. Hence, if $d \geq 0$, $\mathbf{E}(H^d) < \infty$ follows from Corollary 1.3.5. We note that, because $l(\emptyset) = 1$, $H \geq 1$. Hence, for $d < 0$, $\mathbf{E}(H^d) \leq 1 < \infty$. Thus part (a) holds.

To prove the remaining parts of the theorem, we use a Galton-Watson branching process related to our tree, which we now define. Given $(w(i))_{i \in \Sigma_* \setminus \{\emptyset\}}$, let

$$\tilde{w}(i) := \begin{cases} 1 & \text{if } w(i) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\tilde{l}(i) := \prod_{n=1}^{|i|} \tilde{w}(i|n).$$

It is then easy to check that if

$$Z_n := \sum_{i \in \Sigma_n} \tilde{l}(i), \qquad n \geq 0$$

then $(Z_n)_{n \geq 0}$ is a Galton-Watson process. Importantly, if $Z_n > 0$ then we must have $\tilde{l}(i) = 1$ for some $i \in \Sigma_n$, and so $l(i|m) = 1$ for $1 \leq m \leq n$. Consequently, $H \geq n$. This means we can use known results about the extinction of the Galton-Watson process to infer results about H.

A simple calculation yields that the expected number of offspring for this Galton-Watson process is equal to $\mathbf{E}Z_1 = \sum_{i\in S} \mathbf{P}(w(i) = 1)$. Hence, if $\sum_{i\in S} \mathbf{P}(w(i) = 1)$ is strictly greater than one, then the branching process is supercritical and survives with positive probability (see [8], Theorem I-5.1). This implies that $\mathbf{P}(H = \infty) > 0$, which proves (c).

Assume now $\mathbf{E}Z_1 = \sum_{i\in S} \mathbf{P}(w(i) = 1) = 1$. From [8], Theorem I-9.1, it follows that, if $\varepsilon > 0$, then

$$\mathbf{P}\left(Z_n > 0\right) \geq \frac{1}{n\left(\frac{\sigma^2}{2} + \varepsilon\right)} \tag{1.24}$$

for $n \geq n_0$ for some $n_0 \geq 0$, and where $\sigma^2 = \mathrm{Var}Z_1$. Note that $\sigma^2 \leq N < \infty$. If X is the extinction time of our Galton-Watson process then we have $H \geq X$ and also $\{X > n\} = \{Z_n > 0\}$ and so

$$\mathbf{E}H \geq \mathbf{E}X = \sum_{n=0}^{\infty} \mathbf{P}\left(X > n\right) = \sum_{n=0}^{\infty} \mathbf{P}\left(Z_n > 0\right) \geq \sum_{n=n_0}^{\infty} \frac{1}{n\left(\frac{\sigma^2}{2} + \varepsilon\right)}.$$

The proof of (b) is completed on noting that this sum is infinite. □

Remark 1.2 *By defining an associated multiplicative cascade, we may recover the result that a sub-critical Galton-Watson branching process with bounded offspring distribution becomes extinct,* $\mathbf{P}$*-a.s. Let* $(\tilde{Z}_n)_{n\geq 0}$ *be a Galton-Watson process with offspring distribution* $\tilde{Z}_1$ *satisfying* $\mathbf{E}\tilde{Z}_1 < 1$*. Assume* $\tilde{Z}_1 \leq N$, $\mathbf{P}$*-a.s., and define* $p_n := \mathbf{P}(Z_1 = n)$ *and* $S := \{1, \ldots, N\}$*. We can then define a multiplicative cascade such that*

$$\mathbf{P}(w(1) = \cdots = w(m) = 1, w(m+1) = \cdots = w(N) = 0) = p_m,$$

and also $(Z_n)_{n\geq 0}$ *to be a Galton-Watson process as in the proof of the above theorem. It is clear that* $(Z_n)_{n\geq 0} \overset{d}{=} (\tilde{Z}_n)_{n\geq 0}$*. We also have*

$$\sum_{i\in S} \mathbf{P}(w(i) = 1) = \sum_{n=1}^{N} \sum_{m=n}^{N} p_m = \sum_{n=0}^{N} np_n = \mathbf{E}\tilde{Z}_1.$$

Since $\mathbf{E}\tilde{Z}_1 < 1$*, we have* $\sum_{i\in S} \mathbf{P}(w(i) = 1) < 1$*, and so* $\mathbf{E}H < \infty$*. It follows that the process* $(\tilde{Z}_n)_{n\geq 0}$ *eventually becomes extinct,* $\mathbf{P}$*-a.s.*

Remark 1.3 *At criticality, the Galton-Watson process exhibits two distinct types of behaviour. First, there is the trivial case when* $N = 1$, $\mathbf{P}$*-a.s., and in this* *case, the process survives. This corresponds to a multiplicative cascade that satisfies $\sum_{i\in S}\mathbf{P}(w(i)=1)=1$ and $\mathbf{P}(\sup_{i\in S}w(i)=1)=1$. In this case $H=\infty$, $\mathbf{P}$-a.s. In the non-trivial case, $\mathbf{P}(N=1)<1$, the Galton-Watson process becomes extinct with probability 1. Hence, using the construction in Remark 1.2, we can find a multiplicative cascade with $\sum_{i\in S}\mathbf{P}(w(i)=1)=1$ and $H<\infty$, $\mathbf{P}$-a.s. The problem of whether $H<\infty$ in the general non-trivial case is left open.*

Remark 1.4 *If $N=\infty$, there exist random variables $(w(i))_{i\in S}$ that satisfy the conditions $\sum_{i\in S}\mathbf{P}(w(i)=1)<1$ and also $\sup_{i\in S}w(i)=1$, $\mathbf{P}$-a.s. In this case we have $H=\infty$, and so the conclusion of the theorem does not hold in general when $N=\infty$.*

Remark 1.5 *The condition $w(i)\in[0,1]$, $\mathbf{P}$-a.s., is not essential. It is an elementary probability exercise to show that, if $(w(i))_{i\in S}$ are independent, identically-distributed uniform $[0,x]$ random variables and $|S|\geq 2$, then to first order in $|S|^{-1}$ we have*

$$1+\frac{1}{|S|e}\leq \sup\{x:\ \mathbf{P}(H<\infty)=1\}\leq 1+\frac{1}{|S|}.$$

1.4 Random p.c.f.s.s. dendrite construction

In this section we construct a random resistance metric and Dirichlet form on the p.c.f.s.s. dendrite T. We take as a starting point the existence of a Dirichlet form on $(\tilde{V}^0,\tilde{E}^0)$ which is invariant under a renormalisation analogous to that used in the definition of a harmonic structure in Section 1.1. In particular, define D by

$$D(f,f):=\sum_{e\in\tilde{E}_0}H_e(f(e_+)-f(e_-))^2,\qquad \forall f\in C(\tilde{V}^0),\tag{1.25}$$

where $H_e>0$ for every $e\in\tilde{E}_0$, and we write $e=\{e_+,e_-\}$. Let $\mathbf{r}:=(r_i)_{i\in S}$, $r_i>0$, be a set of scaling factors. Using the terminology of Section 1.1, we shall say $(D,\mathbf{r})$ is a *harmonic structure* if

$$H_e^{-1}=\sum_{i\in S}\sum_{e'\in\tilde{E}^0}\frac{r_i\mathbf{1}_{\{F_i(G_{e'})\subseteq G_e\}}}{H_{e'}},\qquad \forall e\in\tilde{E}^0,\tag{1.26}$$

and that it is *regular* if $0<r_i<1$ for every $i\in S$. This will be consistent with our previous understanding of the term regular harmonic structure in that it will allow the construction of a compatible sequence of Dirichlet forms on $(\tilde{V}^n)_{n\geq 0}$ using a similar procedure to the non-random case. From now on, we assume a regular harmonic structure $(D, \mathbf{r})$ exists on T.

In our Dirichlet form construction, rather than rescaling simply using the deterministic scale factors r_i, we use random scaling factors $(w(i))_{i \in \Sigma_* \setminus \{\emptyset\}}$ which are assumed to satisfy the properties of a multiplicative cascade, as introduced in Section 1.3. Specifically, this means that the $w(i)$ are $(0,1]$ random variables satisfying the distributional assumption stated at (1.15). Whilst these are the only *a priori* assumptions we make on the $(w(i))_{i \in \Sigma_* \setminus \{\emptyset\}}$, we shall also have cause to use the following assumptions. For clarity, we will explicitly state when we require these further restrictions.

Assumption (W1): *The expected values of the random scaling factors are equal to the deterministic scaling factors of the regular harmonic structure* $(D, \mathbf{r})$*. i.e.* $\mathbf{E} w(i) = r_i$ *for all* $i \in S$.

Assumption (W2): *The random variables* $(w(i))_{i \in S}$ *have a distribution which does not place too much mass at one. In particular,* $\sum_{i \in S} \mathbf{P}(w(i) = 1) < 1$.

Assumption (W3a): *The random variables* $(w(i))_{i \in S}$ *are bounded away from zero. Specifically, there exists an* $\varepsilon > 0$ *such that* $\mathbf{P}(w(i) > \varepsilon) = 1$*, for all* $i \in S$.

Assumption (W3b): *The random variables* $(w(i))_{i \in S}$ *are independent and their distributions satisfy the following tail condition. If* $p \in (0,1)$*, there exists a constant* $\varepsilon > 0$*, such that*

$$\mathbf{P}(w(i) \leq \varepsilon x |\, w(i) \leq x) \leq p, \qquad \forall x \in (0,1], i \in S.$$

Assumption (W4): *The random variables* $(w(i))_{i \in S}$ *are bounded away from one. Specifically, there exists an* $\eta < 1$ *such that* $\mathbf{P}(w(i) < \eta) = 1$*, for all* $i \in S$.

Although we would like to simply replace the scaling factors r_i with the random variables $w(i)$ in a formula similar to (1.4), a sequence of forms defined in this way will not be compatible in general and taking limits will no longer be straightforward. To deal with the this problem, we introduce the random variables

$$R_i^e := \lim_{n \to \infty} \sum_{j \in \Sigma_n} \sum_{e' \in \tilde{E}^0} \frac{l(ij) H_e \mathbf{1}_{\{F_j(G_{e'}) \subseteq G_e\}}}{l(i) H_{e'}}, \qquad \forall e \in \tilde{E}^0, i \in \Sigma_*, \tag{1.27}$$

which we shall term *resistance perturbations.* The questions of convergence and distribution of $(R_i^e)_{e \in \tilde{E}^0, i \in \Sigma_*}$ may be answered in terms of the random scaling factors

$(w(i))_{i\in S}$ by means of multiplicative cascade or multi-type branching random walk arguments, but we postpone the discussion of this until Section 1.6. For now, we simply state the properties that we will need to prove the results of this and subsequent sections.

Assumption (R1): *$\mathbf{P}$-a.s., the limit at (1.27) exists and moreover, $R_i^e \in (0,\infty)$ for all $e \in \tilde{E}^0$, $i \in \Sigma_*$.*

Assumption (R2): *The resistance perturbations have finite positive moments of all orders. i.e. $\mathbf{E}\left((R_i^e)^d\right) < \infty$ for all $e \in \tilde{E}^0$, $i \in \Sigma_*$, $d \geq 0$.*

Assumption (R3): *The resistance perturbations have finite negative moments of some order. i.e. There exists a $\beta > 0$ such that $\mathbf{E}\left((R_i^e)^{-\beta}\right) < \infty$ for all $e \in \tilde{E}^0$, $i \in \Sigma_*$.*

In the following lemma we derive a decomposition for the random variables R_i^e which will be useful in proving the compatibility of the sequence of resistance forms that we introduce shortly. We also show that in the deterministic case $w(i) \equiv r_{i_{|i|}}$ for all $i \in \Sigma_*\backslash\{\emptyset\}$, the assumption at (1.26) implies that $R_i^e \equiv 1$, and so the assumptions (R1), (R2) and (R3) clearly hold.

Lemma 1.4.1
(a) Assume (R1). $\mathbf{P}$-a.s., for each $e \in \tilde{E}^0$, $i \in \Sigma_*$, the resistance perturbations satisfy

$$R_i^e = \sum_{j\in\Sigma_n}\sum_{e'\in\tilde{E}_0} \frac{l(ij)H_e \mathbf{1}_{\{F_j(G_{e'})\subseteq G_e\}} R_{ij}^{e'}}{l(i)H_{e'}}.$$

(b) In the case $w(i) \equiv r_{i_{|i|}}$, for all $i \in \Sigma_*\backslash\{\emptyset\}$, we have $R_i^e \equiv 1$, for every $e \in \tilde{E}^0$, $i \in \Sigma_*$.

Proof: For $n \geq 0$, define the random variables

$$R_i^e(n) = \sum_{j\in\Sigma_n}\sum_{e'\in\tilde{E}^0} \frac{l(ij)H_e \mathbf{1}_{\{F_j(G_{e'})\subseteq G_e\}}}{l(i)H_{e'}}, \qquad \forall e \in \tilde{E}^0, i \in \Sigma_*, \tag{1.28}$$

so that $R_i^e(n) \to R_i^e$ as $n \to \infty$, $\mathbf{P}$-a.s. Applying this definition and the identity of Lemma 1.2.10, we obtain

$$\begin{aligned} R_i^e(n+m) &= \sum_{j\in\Sigma_n}\sum_{k\in\Sigma_m}\sum_{e'\in\tilde{E}^0} \frac{l(ijk)H_e}{l(i)H_{e'}} \sum_{e''\in\tilde{E}^0} \mathbf{1}_{\{F_k(G_{e'})\subseteq G_{e''}\}}\mathbf{1}_{\{F_j(G_{e''})\subseteq G_e\}} \\ &= \sum_{j\in\Sigma_n}\sum_{e''\in\tilde{E}^0} \frac{l(ij)H_e \mathbf{1}_{\{F_j(G_{e''})\subseteq G_e\}}}{l(i)H_{e''}} \sum_{k\in\Sigma_m}\sum_{e'\in\tilde{E}^0} \frac{l(ijk)H_{e''}\mathbf{1}_{\{F_k(G_{e'})\subseteq G_{e''}\}}}{l(ij)H_{e'}} \qquad (1.29) \\ &= \sum_{j\in\Sigma_n}\sum_{e''\in\tilde{E}^0} \frac{l(ij)H_e \mathbf{1}_{\{F_j(G_{e''})\subseteq G_e\}} R_{ij}^{e''}(m)}{l(i)H_{e''}}. \end{aligned}$$

Letting $m \to \infty$ we get part (a) of the lemma. Now assume that $w(i) \equiv r_{i_{|i|}}$ for all $i \in \Sigma_* \backslash \{\emptyset\}$. Letting $m = 1$ in (1.29), we find that

$$R_i^e(n+1) = \sum_{j \in \Sigma_n} \sum_{e'' \in \tilde{E}^0} \frac{l(ij) H_e \mathbf{1}_{\{F_j(G_{e''}) \subseteq G_e\}}}{l(i) H_{e''}} \sum_{k \in S} \sum_{e' \in \tilde{E}^0} \frac{r_k H_{e''} \mathbf{1}_{\{F_k(G_{e'}) \subseteq G_{e''}\}}}{H_{e'}}.$$

However, the renormalisation property (1.26) implies that the two innermost sums (those over k and e') contribute exactly 1 for each $j \in \Sigma_n$, $e'' \in \tilde{E}^0$. Thus we observe that $R_i^e(n+1) = R_i^e(n)$ for every $n \geq 0$. Since $R_i^e(0) = 1$, the result follows. $\square$

Given the random scaling factors, $(w(i))_{i \in \Sigma_* \backslash \{\emptyset\}}$, and resistance perturbations satisfying (R1), we can define a sequence of resistance forms on the sequence of vertex sets, $(\tilde{V}^n)_{n \geq 0}$. Similarly to (1.4), let

$$\mathcal{E}^n(f,f) := \sum_{j \in \Sigma_n} \frac{1}{l(j)} D_j(f_j, f_j), \qquad \forall f \in C(\tilde{V}^n),$$

where $l(j)$ is defined by (1.17), and D_j is the perturbed version of D given by

$$D_j(f,f) := \sum_{e \in \tilde{E}_0} \frac{H_e}{R_j^e} (f(e_+) - f(e_-))^2.$$

In the next lemma we show that this definition gives us a compatible sequence of resistance forms, and the only assumption we need for the proof of this is (R1).

Theorem 1.4.2 *Under the assumption (R1), the sequence $\{(\tilde{V}^n, \mathcal{E}^n)\}_{n \geq 0}$ is a compatible sequence, $\mathbf{P}$-a.s.*

Proof: Since (R1) holds, we can assume that $R_i^e \in (0, \infty)$ for every $i \in \Sigma_*$, $e \in \tilde{E}^0$. Now, by Lemma 1.2.9, for $j \in \Sigma_n$, $e' \in \tilde{E}^0$, we have $F_j(e') \in \tilde{E}^n$, and furthermore, Lemma 1.2.8 tells us this representation is unique. Thus we have $\sum_{e \in \tilde{E}^n} \mathbf{1}_{\{F_j(G_{e'}) = G_e\}} = 1$. Consequently, we can write

$$\mathcal{E}^n(f,f) = \sum_{e \in \tilde{E}^n} \left(\sum_{j \in \Sigma_n} \sum_{e' \in \tilde{E}^0} \frac{H_{e'} \mathbf{1}_{\{F_j(G_{e'}) \subseteq G_e\}}}{l(j) R_j^{e'}} \right) (f(e_+) - f(e_-))^2, \tag{1.30}$$

and moreover, for each $e \in \tilde{E}^n$, we know exactly one term in the internal double sum has a strictly positive contribution. Hence the related electrical network on $\tilde{V}^n$ can only have strictly positive conductances on the edges of $\tilde{E}^n$. Now, fix $e \in \tilde{E}^n$. Using the fact that $(\tilde{V}^n, \tilde{E}^n)$ is a graph tree (Lemma 1.2.7), if $e = F_j(e')$ for $e' \in \tilde{E}^0$, $j \in \Sigma_n$, then it also follows from (1.30) that

$$R^{(n)}(e_+, e_-) = \frac{l(j) R_j^{e'}}{H_{e'}}, \tag{1.31}$$

where $R^{(n)}$ is the resistance metric associated with $\mathcal{E}^n$ by (1.8). Similarly, define $R^{(n+1)}$ to be the resistance metric associated with $\mathcal{E}^{n+1}$. Since $(\tilde{V}^{n+1}, \tilde{E}^{n+1})$ is a graph tree, $R^{(n+1)}$ is additive along paths. In particular, we have

$$\begin{aligned} R^{(n+1)}(e_+, e_-) &= \sum_{e' \in \tilde{E}^{n+1}} R^{(n+1)}(e'_+, e'_-) \mathbf{1}_{\{G_{e'} \subseteq G_e\}} \\ &= \sum_{j \in \Sigma_{n+1}} \sum_{e' \in \tilde{E}^0} \frac{l(j) R_j^{e'} \mathbf{1}_{\{F_j(G_{e'}) \subseteq G_e\}}}{H_{e'}}. \end{aligned}$$

From this and the identity of Lemma 1.2.10, it follows that

$$\begin{aligned} R^{(n+1)}(e_+, e_-) &= \sum_{j \in \Sigma_n} \sum_{k \in S} \sum_{e' \in \tilde{E}^0} \sum_{e'' \in \tilde{E}^0} \frac{l(jk) R_{jk}^{e'} \mathbf{1}_{\{F_k(G_{e'}) \subseteq G_{e''}\}} \mathbf{1}_{\{F_j(G_{e''}) \subseteq G_e\}}}{H_{e'}} \\ &= \sum_{j \in \Sigma_n} \sum_{e' \in \tilde{E}^0} \frac{l(j) R_j^{e'} \mathbf{1}_{\{F_j(G_{e'}) \subseteq G_e\}}}{H_{e'}}, \end{aligned}$$

which may be seen to be equal to $R^{(n)}(e_+, e_-)$ by applying again the uniqueness of the representation of edges in $\tilde{E}^n$. Note also that to obtain the second equality we have used the resistance perturbation decomposition of Lemma 1.4.1(a). Hence $R^{(n+1)} = R^{(n)}$ on $\tilde{V}^n$. By [39], Corollary 2.1.13, this is sufficient for the compatibility of $\mathcal{E}^n$ and $\mathcal{E}^{n+1}$, and so the proof is complete. □

We can now write down a resistance form on the countable set $\tilde{V}^*$ as the limit of the compatible sequence of resistance forms on $\tilde{V}^n$. When the sequence $(\mathcal{E}^n)_{n \geq 0}$ is defined, set

$$\mathcal{E}'(f, f) := \lim_{n \to \infty} \mathcal{E}^n(f, f), \qquad \forall f \in \mathcal{F}',$$

where

$$\mathcal{F}' := \{f \in C(\tilde{V}^*) : \lim_{n \to \infty} \mathcal{E}^n(f, f) < \infty\}.$$

Note that, by the compatibility of the sequence of resistance forms, the sequence $(\mathcal{E}^n(f, f))_{n \geq 0}$ is increasing in n and so it always has a limit in $[0, \infty]$.

Lemma 1.4.3 *Under the assumption (R1), $(\mathcal{E}', \mathcal{F}')$ is a resistance form on $\tilde{V}^*$,* **P**-*a.s.*

Proof: From the previous lemma we have that $\{(\tilde{V}^n, \mathcal{E}^n)\}_{n \geq 0}$ is compatible, **P**-a.s. This allows us to apply [39], Theorem 2.2.6, to deduce the result. □

As described at (1.8), naturally associated with a resistance form is a resistance metric. We shall denote by R' the resistance metric associated with $(\mathcal{E}', \mathcal{F}')$ when this is defined. To extend our resistance form to a resistance form on the whole of T it will be necessary to show that T is the completion of $\tilde{V}^*$ with respect to the metric R'. Before demonstrating that this is the case, we prove some preliminary results about the diameter of sets of the form $(\tilde{V}_i^*)_{i\in\Sigma_*}$ in the metric R'. Throughout the remainder of the chapter, for a subset A of a metric space (X,d), we shall denote the *diameter* of A by

$$\mathrm{diam}_d A := \sup\{d(x,y):\ x,y\in A\}.$$

In the proof of the next result, we will also use the notation

$$R_i := \sum_{e\in\tilde{E}^0} R_i^e, \qquad \forall i\in\Sigma_*.$$

Note that the random variables $(R_i)_{i\in\Sigma_*}$ are identically distributed.

Lemma 1.4.4 *If we assume (W2), (R1) and (R2), then* $\mathbf{E}((\mathrm{diam}_{R'}\tilde{V}^*)^d)<\infty$*, for all* $d>0$*.*

Proof: Let $x\in\tilde{V}^*$ and $y\in\tilde{V}^0$. Necessarily, $x\in\tilde{V}^n$ for some $n\geq 0$. Using the description of paths in $\tilde{V}^*$ that was proved in Lemma 1.2.11, we find that

$$R'(x,y)\leq N|\tilde{E}^0|\sum_{m=0}^{n}\sup_{e\in\tilde{E}^m}R'(e_+,e_-).$$

From the expression that was given at (1.31) for the resistance along an edge in $\tilde{E}^m$, we obtain from this that

$$\mathrm{diam}_{R'}\tilde{V}^*\leq\frac{2N|\tilde{E}^0|}{H_*}\sum_{m=0}^{\infty}\sup_{i\in\Sigma_m}l(i)R_i, \tag{1.32}$$

where $H_*:=\min\{H_e:\,e\in\tilde{E}^0\}$. Note that, if $\mathcal{F}_n$ is the filtration introduced at (1.16), then it is clear from the definition that, for each $i\in\Sigma_*$, R_i^e is independent of $\mathcal{F}_{|i|}$ for each e. Hence so is R_i. By assumption (R2), R_i has finite positive moments of all orders. Thus the expression on the right-hand side of (1.32) is the multiplicative cascade quantity considered in Theorem 1.3.4, where the multiplicative cascade is defined by $(w(i))_{i\in\Sigma_*\setminus\{\emptyset\}}$, and the perturbations, $(R_i)_{i\in\Sigma_*}$, satisfy the assumptions of (1.21) and (1.22). Since (W2) holds, Theorem 1.3.4 tells us that this has finite positive moments of all orders. □

We now introduce random variables $(W_i)_{i\in\Sigma_*}$ to represent the normalised diameters of the sets $(\tilde{V}_i^*)_{i\in\Sigma_*}$. Set

$$W_i:=\frac{\mathrm{diam}_{R'}\tilde{V}_i^*}{l(i)}, \qquad i\in\Sigma_*,$$

whenever the resistance metric R' is defined. In the next lemma we show that these random variables satisfy the assumptions placed on the perturbations of a multiplicative cascade in Section 1.3.

Lemma 1.4.5 *Assume (W2), (R1) and (R2). The random variables $(W_i)_{i\in\Sigma_*}$ are identically distributed, non-negative and satisfy, for all $i \in \Sigma_*$, $d > 0$,*

$$\mathbf{E}(W_i^d) < \infty, \quad \textit{and} \quad W_i \perp \mathcal{F}_{|i|}.$$

Proof: We have

$$\begin{aligned} l(i)W_i &= \sup_n \sup_{x,y\in\tilde{V}_i^n} R'(x,y) \\ &= \sup_n \sup_{x,y\in\tilde{V}_i^n} \sum_{j\in\Sigma_{n+|i|}} \sum_{e\in\tilde{E}^0} \frac{l(j)R_j^e \mathbf{1}_{\{F_j(G_e)\subseteq G_{xy}\}}}{H_e}, \end{aligned}$$

where we have used the description of edges in $\tilde{E}^{n+|i|}$ from Lemma 1.2.8, and the expression for the resistance along an edge from (1.31). If $x, y \in \tilde{V}_i^n$, then $x = F_i(x')$, $y = F_i(y')$ for some $x', y' \in \tilde{V}^n$, and $G_{xy} = F_i(G_{x'y'})$, by Lemma 1.2.4(a). Consequently, for $j \in \Sigma_{n+|i|}$, $F_j(G_e) \subseteq G_{xy}$ if and only if $j||i| = i$. From the injectivity of F_i, it follows that

$$W_i = \sup_n \sup_{x,y\in\tilde{V}^n} \sum_{j\in\Sigma_n} \sum_{e\in\tilde{E}^0} \frac{l(ij)R_{ij}^e \mathbf{1}_{\{F_j(G_e)\subseteq G_{xy}\}}}{l(i)H_e}.$$

As noted in the proof of Lemma 1.4.4, R_{ij} is independent of $\mathcal{F}_{|ij|}$, and so we certainly have that R_{ij} is independent of $\mathcal{F}_{|i|}$, because $\mathcal{F}_{|i|} \subseteq \mathcal{F}_{|ij|}$. Also, $l(ij)/l(i)$ is clearly independent of $\mathcal{F}_{|i|}$. Thus $W_i \perp \mathcal{F}_{|i|}$.

Using the similarity of the distributions of $(w(ij))_{j\in S}$, for $i \in \Sigma_*$, it may be deduced from the above expression for W_i that

$$W_i \stackrel{d}{=} \sup_n \sup_{x,y\in\tilde{V}^n} \sum_{j\in\Sigma_n} \sum_{e\in\tilde{E}^0} \frac{l(j)R_j^e \mathbf{1}_{\{F_j(G_e)\subseteq G_{xy}\}}}{H_e} = W_\emptyset = \mathrm{diam}_{R'} \tilde{V}^*.$$

Hence the $(W_i)_{i\in\Sigma_*}$ are identically distributed, non-negative random variables and, by Lemma 1.4.4, have finite positive moments of all orders. □

It is now easy to show that the diameters of the sets $(\tilde{V}_i)_{i\in\Sigma_*}$ decrease to 0 uniformly as $|i| \to \infty$.

Lemma 1.4.6 *Under the assumptions (W2), (R1) and (R2), we have*

$$\sup_{i\in\Sigma_n} \mathrm{diam}_{R'}\tilde{V}_i^* \to 0, \qquad \mathbf{P}\text{-}a.s.$$

Proof: This result is an immediate corollary of Lemma 1.4.5 and the corresponding multiplicative cascade result, Lemma 1.3.1(ii). □

This uniform decay of the diameter of the sets $(T_i)_{i\in\Sigma_*}$ allows us to extend the definition of R' to the whole of T. Recall that T is the closure of $\tilde{V}^*$ with respect to the original metric, d. Hence, for any $x, y \in T$, there exist sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in $\tilde{V}^*$ with $d(x_n, x) \to 0$ and $d(y_n, y) \to 0$. Define

$$R(x, y) = \lim_{n\to\infty} R'(x_n, y_n).$$

Before proceeding with the proof of Proposition 1.4.8, in which we show that this is a sensible definition for R, we introduce some further notation. For $x \in T$,

$$T_n(x) := \bigcup\{T_i :\ i \in \Sigma_n,\ x \in T_i\}, \tag{1.33}$$

$$\tilde{T}_n(x) := \bigcup\{T_i :\ i \in \Sigma_n,\ T_i \cap T_n(x) \neq \emptyset\}. \tag{1.34}$$

We will also apply the following result of [39], Proposition 1.3.6.

Lemma 1.4.7 $(T_n(x))_{n\in\mathbb{N}}$ *forms a base of neighbourhoods of x, with respect to the original metric, d.*

Proposition 1.4.8 *Assume (W2), (R1) and (R2). R is a well-defined metric on T, topologically equivalent to the original metric,* $\mathbf{P}$*-a.s.*

Proof: Under the assumptions of the lemma, the argument that we give holds $\mathbf{P}$-a.s. Let $x, y \in T$ and suppose there exist, for $m = 1, 2$, sequences (x_n^m) and (y_n^m) in $\tilde{V}^*$ such that $d(x_n^m, x) \to 0$ and $d(y_n^m, y) \to 0$. Fix $\varepsilon > 0$. By Lemma 1.4.6, we can choose $n_0 \geq 0$ such that $\sup_{i\in\Sigma_n} \mathrm{diam}_{R'}\tilde{V}_i^* < \varepsilon/4$, for every $n \geq n_0$. Furthermore, we can use the convergence of the sequences and Lemma 1.4.7 to show that there exists $n_1 \geq n_0$ such that $x_n^m \in T_{n_0}(x)$, $y_n^m \in T_{n_0}(y)$, for $m = 1, 2$ and $n \geq n_1$. Thus

$$|R'(x_n^1, y_n^1) - R'(x_{n'}^m, y_{n'}^m)| \leq R'(x_n^1, x_{n'}^m) + R'(y_n^1, y_{n'}^m) < \varepsilon,$$

for $m = 1, 2$ and $n, n' \geq n_1$. Taking $m = 1$, this implies that $(R(x_n^1, y_n^1))_{n\geq 0}$ is a Cauchy sequence and has a limit. Taking $m = 2$, $n' = n$, this implies that the limit is unique and so the function R is well-defined on $T \times T$. It follows immediately from the fact that R' is a metric on $\tilde{V}^*$ that R is positive, symmetric and satisfies the triangle inequality. To prove R is a metric on T, it remains to show that $R(x,y)=0$ implies that $x=y$. We shall prove the stronger claim that $R(x_n,y)\to 0$ implies that $d(x_n,y)\to 0$.

Suppose $(x_n)_{n\geq 0}$ is a sequence in T with $R(x_n,y)\to 0$ for some $y\in T$. Let $\delta\in(0,1)$ be the largest of the contraction ratios of $(F_i)_{i\in S}$, define $c_{1.9}:=\mathrm{diam}_d T$, fix $\varepsilon>0$, and choose n_0 such that $2c_{1.9}\delta^{n_0}<\varepsilon$. For $z\notin\tilde{T}_{n_0}(y)$, we must have that $z\in T_i$, $y\in T_j$ for some $i,j\in\Sigma_{n_0}$ with $T_i\cap T_j=\emptyset$. For any $z'\in\tilde{V}_i^*$, $y'\in\tilde{V}_j^*$, using the additivity along paths of the metric R' and the fact that the sets $(T_k)_{k\in\Sigma_{n_0}}$ only intersect at vertices of $\tilde{V}^{n_0}$, it is possible to show that

$$R(z',y')=R'(z',y')\geq\inf_{e\in\tilde{E}^{n_0}}R'(e_+,e_-)=:c_{1.10}.$$

It follows that $R(z,y)\geq c_{1.10}$. Since $c_{1.10}>0$ and $R(x_n,y)\to 0$, there exists an n_1 such that $R(x_n,y)<c_{1.10}$, for all $n\geq n_1$. Consequently, $x_n\in\tilde{T}_{n_0}(y)$ for $n\geq n_1$. By our choice of n_0, this implies that $d(x_n,y)\leq 2c_{1.9}\delta^{n_0}<\varepsilon$, for $n\geq n_1$. Hence $d(x_n,y)\to 0$.

To prove the equivalence of the metrics, it remains to show that, for all sequences $(x_n)_{n\in\mathbb{N}}$ in T with $d(x_n,x)\to 0$ for some $x\in T$, we have $R(x_n,x)\to 0$. We note that if $y\in T_i$, then there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in $\tilde{V}_i^*$ with $d(y_n,y)\to 0$. Consequently,

$$\mathrm{diam}_R T_i=\sup_{x,y\in T_i}R(x,y)=\sup_{x,y\in\tilde{V}_i^*}R'(x,y)=\mathrm{diam}_{R'}\tilde{V}_i^*. \tag{1.35}$$

Applying this fact and Lemma 1.4.6, we have that, given $\varepsilon>0$, there exists an n_0 such that $\sup_{i\in\Sigma_{n_0}}\mathrm{diam}_R T_i<\varepsilon$. By Lemma 1.4.7, we have that $x_n\in T_{n_0}(x)$, $\forall n\geq n_1$, for some n_1. It follows that $R(x_n,x)<\varepsilon$, for all $n\geq n_1$, and so $R(x_n,x)\to 0$ as desired. □

The following result is a simple but important consequence of this proposition.

Proposition 1.4.9 *Assume (W2), (R1) and (R2). The metric space (T,R) is the completion of $(\tilde{V}^*,R')$, $\mathbf{P}$-a.s.*

Proof: Recall from Section 1.2 that $\tilde{V}^*$ is dense in (T,d). By the previous result, the topologies induced by R and d are the same, $\mathbf{P}$-a.s. Hence $\tilde{V}^*$ is dense in (T,R) and trivially, we also have that $(\tilde{V}^*,R')$ is isometric to $(\tilde{V}^*,R)$, $\mathbf{P}$-a.s. The result follows. □

This result allows us to extend the resistance form $(\mathcal{E}', \mathcal{F}')$ to the whole of T. Define $\mathcal{E}$ and $\mathcal{F}$ from $(\mathcal{E}', \mathcal{F}')$, when they are defined, by the formulae (1.9) and (1.10), exactly as in the deterministic case.

Theorem 1.4.10 *Assume (W2), (R1) and (R2).* $(\mathcal{E}, \mathcal{F})$ *is a resistance form on* T *and has associated resistance metric* R, $\mathbf{P}$*-a.s.*

Proof: The fact that (T, R) is the completion of $(\tilde{V}^*, R')$, $\mathbf{P}$-a.s. allows us to apply [39], Theorem 2.3.10, to obtain the result. □

We have now constructed a random self-similar dendrite (T, R). Furthermore, because T is a dendrite, it is elementary to check that R must be a *shortest path metric* (additive along paths). To complete this section we show that the quadratic form $(\mathcal{E}, \mathcal{F})$ is actually Dirichlet form for any Borel measure on (T, R) which charges non-empty open sets. To do so, we will apply the following inequality, which is easily verified from the definition of the resistance metric,

$$|f(x) - f(y)|^2 \leq R(x, y)\mathcal{E}(f, f), \qquad \forall x, y \in T, \quad \forall f \in \mathcal{F}. \tag{1.36}$$

Before proceeding with the final theorem of this section, we state a result which was proved by Kigami.

Theorem 1.4.11 *([38], Theorem 5.4). Let* K *be a dendrite and let* d *be a shortest path metric on* K*. Suppose* (K, d) *is locally compact and complete, then* $(\mathcal{E}, \mathcal{F} \cap L^2(K, \nu))$*, where* $(\mathcal{E}, \mathcal{F})$ *is the finite resistance form associated with* (K, d)*, is an irreducible, conservative, local, regular Dirichlet form on* $L^2(K, \nu)$ *for any* σ*-finite Borel measure* ν *on* K *that charges every non-empty open set* $A \subseteq K$*.*

Theorem 1.4.12 *Assume (W2), (R1) and (R2). If* μ *is a (possibly random) finite Borel measure on* (T, R) *which charges every non-empty open subset of* T, $\mathbf{P}$*-a.s., then* $(\mathcal{E}, \mathcal{F})$ *is an irreducible, conservative, local, regular Dirichlet form on* $L^2(T, \mu)$, $\mathbf{P}$*-a.s.*

Proof: This result will clearly follow from Theorems 1.4.10 and 1.4.11 if we can show that $\mathcal{F} \subseteq L^2(T, \mu)$, $\mathbf{P}$-a.s. When T is compact and μ is a finite Borel measure, which is the case $\mathbf{P}$-a.s., we have that $C(T) \subseteq L^2(T, \mu)$. Since the inequality at (1.36) implies that $\mathcal{F} \subseteq C(T)$, $\mathbf{P}$-a.s., the proof is complete. □

1.5 Random p.c.f.s.s. dendrite properties

We collect in this section several properties of the random resistance metric and Dirichlet form we have constructed. Some of the results are of interest in their own right, whereas others will be useful in proving Hausdorff dimension and measure results for T. To prove an upper bound for the Hausdorff dimension of (T,R), we will require knowledge of the diameter of (T,R) and subsets of the form T_i, $i \in \Sigma_*$. We start by presenting some preliminary results regarding these random variables that are simple extensions of the corresponding results for $\mathrm{diam}_{R'}\tilde{V}^*$ proved in the previous section.

Lemma 1.5.1 *If we assume (W2), (R1) and (R2), then*
(a) $W_i = l(i)^{-1}\mathrm{diam}_R T_i$*, for all* $i \in \Sigma_*$*,* $\mathbf{P}$*-a.s.*
(b) $\mathbf{E}((\mathrm{diam}_R T)^d) < \infty$*, for all* $d > 0$*.*
(c) $\sup_{i\in\Sigma_n} \mathrm{diam}_R T_i \to 0$*,* $\mathbf{P}$*-a.s.*
Proof: Observe from (1.35) that under the assumptions of this lemma, $\mathrm{diam}_R T_i = \mathrm{diam}_{R'}\tilde{V}_i^*$, $\mathbf{P}$-a.s. The assertions follow immediately from this fact and the definition of $(W_i)_{i\in\Sigma_*}$, Lemma 1.4.4 and Lemma 1.4.6, respectively. □

On a deterministic p.c.f.s.s. set, the Dirichlet form that is constructed has a strict self-similarity, see [39], Proposition 3.3.1. In the next proposition we show that the random Dirichlet form we have constructed satisfies a similar stochastic self-similarity.

Proposition 1.5.2 *Assume (W2), (R1) and (R2).* $\mathbf{P}$*-a.s., for* $n \geq 0$*,*

$$\mathcal{E}(f,f) = \sum_{i\in\Sigma_n} \frac{1}{l(i)}\, \mathcal{E}_i(f_i, f_i), \qquad \forall f \in \mathcal{F}, \tag{1.37}$$

where $\mathcal{E}_i$*,* $i \in \Sigma_n$*, are independent copies of* $\mathcal{E}$*, and are also independent of* $\mathcal{F}_n$*.*
Proof: Define for $i \in \Sigma_*$, $n \geq 0$,

$$\mathcal{E}_i^n(f,f) := \sum_{j\in\Sigma_n} \frac{l(i)}{l(ij)} D_{ij}(f_j, f_j), \qquad \forall f \in C(\tilde{V}^n).$$

By relabelling, we can repeat the arguments of Section 1.4 to obtain a resistance form $(\mathcal{E}_i, \mathcal{F}_i)$ which is a limit of this sequence, $\mathbf{P}$-a.s. By definition, for $f \in C(\tilde{V}^{n+m})$,

$$\begin{aligned}
\mathcal{E}^{n+m}(f,f) &= \sum_{i\in\Sigma_n}\sum_{j\in\Sigma_m} \frac{1}{l(ij)} \sum_{e\in\tilde{E}^0} \frac{H_e}{R^e_{ij}} (f_{ij}(e_+) - f_{ij}(e_-))^2 \\
&= \sum_{i\in\Sigma_n} \frac{1}{l(i)} \sum_{j\in\Sigma_m} \frac{l(i)}{l(ij)} D_{ij}(f_{ij}, f_{ij}) \\
&= \sum_{i\in\Sigma_n} \frac{1}{l(i)}\, \mathcal{E}_i^m(f_i, f_i).
\end{aligned}$$

Taking the limit $m \to \infty$ implies the identity at (1.37). It is easy to check that $\mathcal{E}_i^m$ is statistically identical to $\mathcal{E}^m$ for each i, and that the sequences $(\mathcal{E}_i^m)_{m \geq 0}$, $i \in \Sigma_n$, are defined from collections of random variables that are independent of each other and of $\mathcal{F}_{|i|}$, which completes the proof. $\square$

In the final result of this section we prove two analytic properties of the form $(\mathcal{E}, \mathcal{F})$. Part (a) is the statement that the form we have constructed satisfies a *Poincaré inequality.* Part (b) implies that the Markov process naturally associated with $(\mathcal{E}, \mathcal{F})$ hits points of the dendrite, **P**-a.s.

Proposition 1.5.3 *If we assume (W2), (R1) and (R2), then*
(a) **P***-a.s., there exists a constant* $c_{1.11}$ *such that, if* μ *is a Borel probability measure on* (T, R)*, then*

$$\mathrm{Var}_\mu f := \int_T f^2 d\mu - \left(\int_T f d\mu \right)^2 \leq c_{1.11} \mathcal{E}(f, f), \qquad \forall f \in \mathcal{F}.$$

(b) **P***-a.s., all non-empty subsets of* T *have strictly positive capacity with respect to the Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *on* $L^2(T, \mu)$*, where* μ *is a Borel probability measure on* (T, R) *which charges every non-empty open set.*

Proof: Part (a) may be proved in exactly the same way as Proposition 7.16 of [9]. We only need to use the fact that $\mathrm{diam}_R T < \infty$, **P**-a.s., which follows from Lemma 1.5.1(b).

Suppose that we are able to construct the Dirichlet form, $(\mathcal{E}, \mathcal{F})$. By Lemma 3.2.2 of [27], if ν is a positive Radon measure on T with finite energy integral, i.e.,

$$\left(\int_T |f| d\nu \right)^2 \leq c_{1.12} \left(\mathcal{E}(f, f) + \int_T f^2 d\mu \right), \qquad \forall f \in \mathcal{F},$$

for some finite constant $c_{1.12}$, then ν charges no set of zero capacity. Hence, by monotonicity, it is sufficient to show that δ_x has finite energy integral for any $x \in T$, where δ_x is the probability measure putting all its mass at the point x. For any $x, y \in T$,

$$\left(\int_T |f| d\delta_x \right)^2 = f(x)^2 \leq 2(f(x) - f(y))^2 + 2f(y)^2. \tag{1.38}$$

Applying the bound at (1.36), and integrating (1.38) with respect to y gives

$$\left(\int_T |f| d\delta_x \right)^2 \leq 2 \, \mathrm{diam}_R T \mathcal{E}(f, f) + 2 \int_T f^2 d\mu,$$

which completes the proof of (b), because $\mathrm{diam}_R T$ is finite, **P**-a.s. $\square$

Remark 1.6 *One reason for establishing a Poincaré inequality for the form* $(\mathcal{E}, \mathcal{F})$ *is that it provides a lower bound for the associated spectral gap,* λ_* *say. One way of defining this quantity is as the infimum of* $\mathcal{E}(f,f)/\mathrm{Var}_\mu f$ *over functions in* $\mathcal{F}$ *with non-zero* μ*-variance. In our case, it is possible to take* $c_{1.11} = \frac{1}{2}\mathrm{diam}_R T$ *in part (a) of the previous lemma. Hence the above result implies that,* $\mathbf{P}$*-a.s.,*

$$\lambda_* \geq \frac{2}{\mathrm{diam}_R T}.$$

1.6 Resistance perturbations and scaling factors

Although we state many of the results in this chapter in terms of the conditions on the resistance perturbations, it is preferable to reduce these to conditions on the scaling factor distributions alone, and that is the aim of the discussion in this section.

We start by presenting a condition which eliminates the need for resistance perturbations by fixing them to be identically equal to 1. Define the $|\tilde{E}^0| \times |\tilde{E}^0|$ random matrix $M = (m_{ee'})_{e,e' \in \tilde{E}^0}$ by

$$m_{ee'} := \sum_{i \in S} \frac{w(i) H_e \mathbf{1}_{\{F_i(G_{e'}) \subseteq G_e\}}}{H_{e'}}, \tag{1.39}$$

and consider the following assumption on M, which implies that the resistance perturbations are non-random, see Lemma 1.6.1.

Assumption (M): $\mathbf{P}$*-a.s., the matrix* M *is stochastic.*

Lemma 1.6.1 *Under the assumption (M),* $R_i^e \equiv 1$*, for every* $e \in \tilde{E}^0$*,* $i \in \Sigma_*$*.*
Proof: This may be proved using an argument similar to the deterministic case which was part (b) of Lemma 1.4.1. □

This has the following obvious corollary.

Corollary 1.6.2 *If assumption (M) holds, then so does (R1), (R2) and (R3).*

What Lemma 1.6.1 means is that if each N-tuple of scaling factors, $(w(ij))_{j \in S}$, is for each $i \in \Sigma_*$ selected randomly from the space

$$\{\mathbf{r} : (D, \mathbf{r}) \text{ is a harmonic structure. } 0 < r_i \leq 1,\ i \in S\}, \tag{1.40}$$

where D is the quadratic form defined by $(H_e)_{e\in\tilde{E}^0}$ at (1.25), then there is an in-built consistency to the inductively defined sequence $(\mathcal{E}^n)_{n\geq 0}$, and we do not need to introduce random resistance perturbations to achieve compatibility. In Section 1.11, we give examples of when the collection of $\mathbf{r}$ contained in the set at (1.40) consists of more than one element (and indeed may be multi-dimensional) and show that random Dirichlet forms constructed from scaling factors satisfying the assumption (M) actually do exist.

The most complete results about the resistance perturbations that we are able to prove are when $|V^0| = 2$, or equivalently, when there is exactly one edge in $\tilde{E}^0$. In this case, the resistance perturbations are precisely limits of tree-martingales, about which much is known. By applying the results of this area of probability theory, we are able to obtain relatively mild sufficient conditions for the assumptions (R1), (R2) and (R3) to hold. We start by showing that (W1) implies the conditions (R1) and (R2) in this case. Since there is only one $e \in \tilde{E}^0$, we will drop the superscripts from the R_i^e for the proofs of results with $|V^0| = 2$.

Proposition 1.6.3 *Suppose $|V^0| = 2$. If assumption (W1) holds, then so does (R1) and (R2).*

Proof: We first check that the limit defining R_i exists. Let $i \in \Sigma_*$ and define $R_i(n)$ as at (1.28). Under (W1), it is straightforward to use the renormalisation property of an harmonic form, (1.26), to deduce that

$$\mathbf{E}(R_i(n+1)|\mathcal{F}_{|i|+n}) = R_i(n), \qquad \mathbf{P}\text{-a.s.},$$

where $(\mathcal{F}_n)_{n\geq 0}$ is the filtration defined at (1.16). Hence $(R_i(n))_{n\geq 0}$ is an $(\mathcal{F}_{|i|+n})_{n\geq 0}$ martingale and $\mathbf{E}R_i(n) = 1$. Thus $R_i := \lim_{n\to\infty} R_i(n)$ exists $\mathbf{P}$-a.s. by the almost-sure martingale convergence theorem and, moreover, $\mathbf{E}R_i \in [0,1]$. To prove that (R1) holds, it remains to demonstrate that R_i is non-zero, $\mathbf{P}$-a.s.

Applying the decomposition of R_i from Lemma 1.4.1(a) and noting that the scaling factors are non-zero by assumption, we find that

$$\begin{aligned}
\mathbf{P}(R_i = 0) &= \mathbf{P}\left(\sum_{j\in S} w(ij)\mathbf{1}_{\{F_j(G_e)\subseteq G_e\}} R_{ij} = 0\right) \\
&= \mathbf{P}\left(\mathbf{1}_{\{F_j(G_e)\subseteq G_e\}} R_{ij} = 0,\ \forall j \in S\right) \\
&= \mathbf{P}(R_i = 0)^{\#\{j:\, F_j(G_e)\subseteq G_e\}},
\end{aligned}$$

where we have used the fact that $(R_{ij})_{j\in S}$ are independent copies of R_i, and e is the only edge in $\tilde{E}^0$. Now, $e_+, e_- \in \tilde{V}^1$ and so decomposing the path from e_+ to e_- in T as in the proof of Lemma 1.2.7, we can show that $G_e = G_{e_1} \cup \cdots \cup G_{e_n}$, where $e_1, \dots, e_n$ are distinct edges $\tilde{E}^1$. Suppose that there is only one such edge, so that $G_e = G_{e_1}$. By Lemma 1.2.8, $e_1 = F_j(e)$ for some $j \in S$. Hence we must have $e = F_j(e)$. This implies that the contraction ratio of F_j is greater than or equal to 1, which is not true by assumption. Consequently, $\#\{j : F_j(G_e) \subseteq G_e\} > 1$, and we must have $\mathbf{P}(R_i = 0) \in \{0, 1\}$.

The proof that (R1) holds may be completed by checking the conditions of [44], Theorem 2.0, to show that $R_i(n)$ actually converges in mean to R_i and so $\mathbf{E}R_i = 1$. Hence $\mathbf{P}(R_i = 0) = 0$. That R_i has finite positive moments of all orders, i.e. (R2) holds, is also a consequence of [44], Theorem 2.0, under the assumptions that we have made on the scaling factors. $\square$

The right tail of the distribution of a tree-martingale limit has been considered by various authors, including Liu, who proves in [44], Theorem 2.1, a widely applicable result demonstrating exponential tails. However, in proving that the resistance perturbations have negative moments, (R3), we need some information about the tail of the distribution at zero, which, to the author's knowledge, has not been studied previously when the scaling factors are not bounded away from zero. Under the assumption that the scaling factors are independent and have finite negative moments of some order, we are able to show that the distribution of the tree-martingale limit also has exponential tails at zero, see Lemma 1.6.5. To prove this result we apply a result of Barlow and Bass, [10], which allows a polynomial estimate for a distribution function to be improved to an exponential one under suitable conditions. We start by proving the polynomial bound we need to apply this result.

Lemma 1.6.4 *Suppose $|V^0| = 2$ and (W1) holds. Fix $\beta > 0$, $\varepsilon \in (0,1)$. Then there exists a constant $c_{1.13}$ such that*

$$\mathbf{P}(R_\emptyset \le x) \le \varepsilon + c_{1.13} x^\beta, \qquad \forall x \ge 0.$$

Proof: Fix $\beta > 0, \varepsilon \in (0,1)$. By Proposition 1.6.3, the resistance perturbations satisfy (R1). Hence $\mathbf{P}(R_\emptyset \le x) \to 0$ as $x \to 0$. In particular, there is an $x_0 > 0$ such that $\mathbf{P}(R_\emptyset \le x_0) \le \varepsilon$. Thus, for $x \ge 0$, $\mathbf{P}(R_\emptyset \le x) \le \varepsilon + x_0^{-\beta} x^\beta$, which proves the result. $\square$

Lemma 1.6.5 *Suppose $|V^0| = 2$ and (W1) holds. Moreover, assume that $(w(i))_{i\in S}$ are independent and there exists a $\beta > 0$ such that*

$$\sup_{i\in S} \mathbf{E}(w(i)^{-\beta}) < \infty, \tag{1.41}$$

then there exist constants $c_{1.14}, \theta_{1.3}$ such that

$$\mathbf{P}(R_i \leq x) \leq e^{-c_{1.14}x^{-\theta_{1.3}}}, \qquad \forall x \geq 0.$$

Proof: Let $\beta > 0$ be a constant for which (1.41) holds and fix $\varepsilon \in (0,1)$. By the previous lemma, we can find a $c_{1.13}$ such that $\mathbf{P}(R_\emptyset \leq x) \leq \varepsilon + c_{1.13}x^\beta$, for all $x \geq 0$. Applying the relevant independence assumptions and the fact that $R_i \stackrel{d}{=} R_\emptyset$, we can deduce from this that, for all $x \geq 0$, $i \in \Sigma_n$,

$$\begin{aligned}
\mathbf{P}(l(i)R_i \leq x) &= \mathbf{E}\left(\mathbf{P}\left(l(i)R_i \leq x \,|\, \mathcal{F}_n\right)\right) \\
&\leq \mathbf{E}\left(\varepsilon + c_{1.13}\frac{x^\beta}{l(i)^\beta}\right) \\
&\leq \varepsilon + c_{1.13}c_{1.15}^n x^\beta,
\end{aligned}$$

where $c_{1.15} := \sup_{i\in S}\mathbf{E}(w(i)^{-\beta}) \vee (\tilde{N}+1)$ and $\tilde{N} := \#\{j : F_j(G_e) \subseteq G_e\}$. That $\tilde{N} \geq 2$ is demonstrated in the proof of Proposition 1.6.3.

By writing $R_\emptyset = \sum_{i\in\Sigma_n} l(i)R_i\mathbf{1}_{\{F_i(G_e)\subseteq G_e\}}$, one may easily check that the conditions of [10], Lemma 1.1 hold. It is also possible to show that the number of non-zero summands is $\tilde{N}^n$. Consequently, we obtain the following estimate for the left tail of the distribution of $R_\emptyset$,

$$\mathbf{P}(R_\emptyset \leq x) \leq \exp\left(c_{1.16}(c_{1.15}\tilde{N})^{n/2}x^{\beta/2} + \tilde{N}^n \ln\varepsilon\right), \qquad \forall x \geq 0, \tag{1.42}$$

for some constant $c_{1.16}$. We now look to choose n in a way that will give us the control we require over this bound. Define $n_0 = n_0(x)$ to be the unique solution to

$$\left(\frac{c_{1.15}}{\tilde{N}}\right)^{n_0/2} = \frac{-\ln\varepsilon}{x^{\beta/2}c_{1.16}},$$

and then set $n = \lfloor n_0 - 1\rfloor$. We have $c_{1.15} > \tilde{N}$ and so we can find an $c_{1.17} \in (0,1)$ such that $\tilde{N}{c_{1.15}}^{-1} \leq (1-c_{1.17})^2$. Consequently, because $n - n_0 \in (-2,-1]$, we have

$$(c_{1.15}\tilde{N})^{(n-n_0)/2} - \tilde{N}^{n-n_0} = \tilde{N}^{n-n_0}\left(\left(c_{1.15}\tilde{N}^{-1}\right)^{(n-n_0)/2} - 1\right) \leq -c_{1.17}\tilde{N}^{-2}.$$

By the choice of n_0, our upper bound, (1.42), now becomes

$$\begin{aligned}\ln \mathbf{P}(R_\emptyset \le x) &\le c_{1.16}(c_{1.15}\tilde{N})^{n_0/2}x^{\beta/2}(c_{1.15}\tilde{N})^{(n-n_0)/2} + \tilde{N}^{n_0}\tilde{N}^{n-n_0}\ln\varepsilon \\ &= -\tilde{N}^{n_0}\left((c_{1.15}\tilde{N})^{(n-n_0)/2} - \tilde{N}^{n-n_0}\right)\ln\varepsilon \\ &\le c_{1.17}\tilde{N}^{n_0-2}\ln\varepsilon \\ &= -c_{1.18}x^{-\frac{\beta\ln\tilde{N}}{\ln c_{1.15}-\ln\tilde{N}}},\end{aligned}$$

which proves the result. □

We now prove an alternative characterisation of the tail inequality of assumption (W3b) that will prove useful in applying the previous result.

Lemma 1.6.6 *Let X be a $(0,1]$ valued random variable with distribution function Φ, then the following statements are equivalent:*
(a) If $p \in (0,1)$, then there exists a constant $\varepsilon \in (0,1)$ such that

$$\Phi(\varepsilon x) \le p\Phi(x), \qquad \forall x \in (0,1]. \tag{1.43}$$

(b) There exist constants $\varepsilon \in (0,1)$ and $\beta > 0$ such that

$$\mathbf{E}\left(\left(1 - \frac{x^\beta \varepsilon}{X^\beta}\right)\mathbf{1}_{\{X\le x\}}\right) \ge 0, \qquad \forall x \in (0,1].$$

Proof: Assume (a) holds and fix $p \in (0,1)$. Choose ε so that (1.43) holds, and $\beta > 0$ so that $p < \varepsilon^\beta$. Integration by parts yields

$$\mathbf{E}(x^\beta X^{-\beta}\mathbf{1}_{\{X\le x\}}) = \lim_{\delta\to 0}\left\{\left[x^\beta y^{-\beta}\Phi(y)\right]_{y=\delta}^x + \beta\int_\delta^x x^\beta y^{-\beta-1}\Phi(y)dy\right\}.$$

Now, $\Phi(\varepsilon^n) \le p^n$, and so, for $y \in (\varepsilon^{n+1}, \varepsilon^n]$, we have $y^{-\beta}\Phi(y) \le \varepsilon^{-\beta(n+1)}p^n$. It follows that, because $p\varepsilon^{-\beta} < 1$,

$$\lim_{\delta\to 0}\left[x^\beta y^{-\beta}\Phi(y)\right]_{y=\delta}^x = \Phi(x). \tag{1.44}$$

Also,

$$\begin{aligned}\lim_{\delta\to 0}\beta\int_\delta^x x^\beta y^{-\beta-1}\Phi(y)dx &= \sum_{n=0}^\infty \beta x^\beta \int_{x\varepsilon^{n+1}}^{x\varepsilon^n} y^{-\beta-1}\Phi(y)dy \\ &\le \beta x^\beta \sum_{n=0}^\infty \int_{x\varepsilon^{n+1}}^{x\varepsilon^n}(x\varepsilon^{n+1})^{-\beta-1}p^n\Phi(x)dy \\ &\le \beta\Phi(x)\varepsilon^{-\beta-1}\sum_{n=0}^\infty (p\varepsilon^{-\beta})^n \\ &= \frac{\beta\Phi(x)}{\varepsilon^{\beta+1}(1-p\varepsilon^{-\beta})}.\end{aligned} \tag{1.45}$$

The results at (1.44) and (1.45) imply that (b) holds.

Conversely, suppose that (b) holds for some $\beta \in (0,1)$ and $\varepsilon > 0$. Fix $p \in (0,1)$ and define $\varepsilon' := (p\varepsilon)^{1/\beta}$. For $x \in (0,1]$, we obtain

$$\Phi(\varepsilon' x) \leq \mathbf{E}\left(\mathbf{1}_{\{X \leq \varepsilon' x\}} \frac{\varepsilon'^{\beta} x^{\beta}}{X^{\beta}}\right) \leq p\mathbf{E}\left(\mathbf{1}_{\{X \leq x\}} \frac{\varepsilon x^{\beta}}{X^{\beta}}\right) \leq p\Phi(x),$$

which is statement (a). $\square$

We can now write down the sufficient conditions for (R3) in terms of the scaling factor assumptions introduced in Section 1.4.

Corollary 1.6.7 *Suppose $|V^0| = 2$. If (W1) and (W3b) hold, then so does (R3).*

Proof: Under (W3b), the scaling factors $(w(i))_{i \in S}$ are independent and by Lemma 1.6.6, they have finite negative moments of some order. Hence we may apply Lemma 1.6.5 to show that $\mathbf{P}(R_i \leq x) \leq e^{-c_{1.14} x^{-\theta_{1.3}}}$. The result follows. $\square$

When $|V^0| \geq 3$, there is a class of p.c.f.s.s. sets for which the resistance perturbations fit naturally into the multi-type branching random walk setting. In particular, this is the case when

$$\#\{e' \in \tilde{E}^0 : F_j(G_{e'}) \subseteq G_e\} \leq 1, \tag{1.46}$$

for all $e \in \tilde{E}^0$, $j \in S$. We now explain the connection and why this condition is useful.

Consider that the zeroth generation of a branching process is made up of a single particle, labelled (i,e), where $i \in \Sigma_*$ and $e \in \tilde{E}^0$ is the type of the particle. We assume that the particle is positioned at the origin in $\mathbb{R}$. At time 1, this particle dies and leaves offspring $\{(ij, e') : j \in S, e' \in \tilde{E}^0, F_j(G_{e'}) \subseteq G_e\}$, where a particle (ij, e') is born at a position

$$-\ln w(ij) - \ln H_e + \ln H_{e'},$$

relative to its parent. Each particle in the first generation reproduces in a similar fashion, and so on. The condition (1.46) means that there cannot be a pair of particles of the form (ij, e') and (ij, e'') for $e' \neq e''$ born to (i,e). As a consequence, this means that the particles of the first generation reproduce independently of one another. Hence the process describes a multi-type branching random walk, with the nth generation being the particles in $\mathrm{Gen}(n) := \{(ik, e') : k \in \Sigma_n, , e' \in \tilde{E}^0, : F_k(G_{e'}) \subseteq G_e\}$, and the set of particle types being $\tilde{E}^0$.

An important object in the analysis of such a process is the matrix made up of Laplace transforms of the offspring point process. Define the $|\tilde{E}^0| \times |\tilde{E}^0|$ random matrix $M(\theta) = (m_{ee'}(\theta))_{e,e'\in\tilde{E}^0}$ by

$$m_{ee'}(\theta) := \sum_{i\in S}\left(\frac{w(i)H_e}{H_{e'}}\right)^{\theta}\mathbf{1}_{\{F_i(G_{e'})\subseteq G_e\}}, \qquad \forall\theta > 0.$$

Note that the M defined at (1.39) is simply equal to $M(1)$. The relevant matrix of Laplace transforms is then given by $\overline{M}(\theta) := \mathbf{E}M(\theta)$.

From the description of this branching process, we can check that $R_i^e(n)$, as defined at (1.28), may be written

$$R_i^e(n) = \sum_{(ik,e')\in\mathrm{Gen}(n)} e^{-\pi(ik,e')},$$

where $\pi(ik, e')$ is the position of the particle (ik, e'). Furthermore, under the condition (W1), the renormalisation property, (1.26), implies that $\overline{M}(1)$ is stochastic. Consequently, $R_i^e := \lim_{n\to\infty} R_i^e(n)$ is a martingale limit of a type which has received a substantial amount of attention in probabilistic literature. Hence we can immediately apply known results to give conditions on the scaling factors which lead to non-degeneracy and mean convergence (see [42], Theorem 1), and the finiteness of positive moments (the argument of [54], Theorem 2.1(ii) may easily be adapted to our situation) of the resistance perturbations. However, since stating a general result of this form would involve little more than verifying the conditions and regurgitating the results of these references, we shall omit to do so here. Instead, we will highlight some of the more important considerations for a particular p.c.f.s.s. dendrite, see Example 1.3. See also Appendix B.

1.7 Hausdorff dimension upper bound

In this section, we prove an upper bound for the Hausdorff dimension of (T, R), which holds under relatively weak assumptions. We start by explaining how the Hausdorff dimension is defined. For further background, see [25].

Let (X, d) be a metric space. For $\delta > 0$, call a finite or countable family of sets $(A_i)_{i=1}^{\infty}$ a *δ-cover* of $A \subseteq X$ if $\mathrm{diam}_d A_i \leq \delta$ for all i and $A \subseteq \bigcup_i A_i$. For $0 \leq s < \infty$, define

$$\mathcal{H}_\delta^s(A) := \inf\left\{\sum_{i=1}^{\infty}(\mathrm{diam}_d A_i)^s :\ (A_i)_{i=1}^{\infty} \text{ is a } \delta\text{-cover of } A\right\}.$$

This is non-decreasing as $\delta \searrow 0$ and allows the following definition of what is a metric outer measure, called the *Hausdorff s-dimensional measure*, $\mathcal{H}^s(A) := \lim_{\delta\to 0}\mathcal{H}_\delta^s(A)$. For any set $A \subseteq X$ it is straightforward to show that there exists an s such that, if

$t < s$, then $\mathcal{H}^t(A) = \infty$, and if $t > s$, then $\mathcal{H}^t(A) = 0$. This s is termed the *Hausdorff dimension* of A, and is denoted

$$\dim_H(A) = \inf\{s\colon\ \mathcal{H}^s(A) < \infty\} = \sup\{s\colon\ \mathcal{H}^s(A) > 0\}. \tag{1.47}$$

Recall from (1.18) the definition of $\phi(\theta)$. Since $w(i) > 0$ for every $i \in S$, $\mathbf{P}$-a.s., then $\phi(0) = |S| \geq 2$. Also, if we assume (W2), then $\phi(\theta)$ is strictly decreasing and is strictly less than 1 for large θ. Hence there is a unique positive solution to

$$\phi(\theta) = 1, \tag{1.48}$$

which is the stochastic version of (1.11). We shall denote this solution α and show that, $\mathbf{P}$-a.s., this provides an upper bound for the Hausdorff dimension of T with respect to the metric R, which we will write as $\dim_H(T)$ throughout the remainder of this chapter.

Deducing an upper bound for the Hausdorff dimension of a set generally relies on finding a good δ-cover for the set and this is how we proceed here. In fact, Lemma 1.4.6 implies that $(T_i)_{i\in\Sigma_n}$ provides a suitable cover for large n and so there is little work needed to complete the proof.

Theorem 1.7.1 *Assume (W2), (R1) and (R2), then*

$$\dim_H(T) \leq \alpha, \qquad \mathbf{P}\text{-}a.s.$$

Proof: $\mathbf{P}$-a.s., by Lemma 1.4.6, we have that for large n, $(T_i)_{i\in\Sigma_n}$ is a δ-cover of T. Thus

$$\begin{aligned}
\mathbf{E}\left(\mathcal{H}^\theta(T)\right) &= \mathbf{E}\left(\lim_{\delta\to 0}\inf\left\{\sum_{i=1}^{\infty}(\mathrm{diam}_R A_i)^\theta :\ (A_i)_{i=1}^\infty \text{ is a } \delta\text{-cover of } T\right\}\right)\\
&\leq \mathbf{E}\left(\liminf_{n\to\infty}\sum_{i\in\Sigma_n}(\mathrm{diam}_R T_i)^\theta\right)\\
&\leq \liminf_{n\to\infty}\mathbf{E}\left(\sum_{i\in\Sigma_n}(\mathrm{diam}_R T_i)^\theta\right)\\
&= \liminf_{n\to\infty}\mathbf{E}\left(\sum_{i\in\Sigma_n}(l(i)W_i)^\theta\right),
\end{aligned}$$

where we have applied Fatou's lemma for the penultimate inequality. By Lemma 1.4.5 and Lemma 1.5.1, the expectation appearing in the bottom line is equal to

$\phi(\theta)^n\mathbf{E}((\mathrm{diam}_R T)^\theta)$, and the second of these factors is finite. Furthermore, for $\theta > \alpha$, $\phi(\theta) < \phi(\alpha) = 1$. Hence $\mathbf{E}(\mathcal{H}^\theta(T)) = 0$, and so $\mathcal{H}^\theta(T) = 0$, $\mathbf{P}$-a.s. The result follows from this using the characterisation of the Hausdorff dimension at (1.47). □

1.8 Stochastically self-similar measures

In Section 1.1, the concept of a self-similar measure was introduced. In our case, the randomness of (T, R) means that the corresponding natural measure will only be stochastically self-similar. To define such a measure on T, we initially define a measure on the address space Σ and then use the natural projection to map it on to T.

Generalising the definition at (1.20), let

$$Z_i^\theta(n) := \frac{\sum_{j\in\Sigma_n} l(ij)^\theta}{l(i)^\theta \phi(\theta)},$$

and note that $Z_i^\theta(n) \stackrel{d}{=} Z^\theta(n)$ for every $i \in \Sigma_*$. As noted in Section 1.3, $(Z^\theta(n))_{n\geq 0}$ is an $(\mathcal{F}_n)_{n\geq 0}$ martingale and so by the almost-sure martingale convergence theorem $Z^\theta(n) \to Z^\theta$, $\mathbf{P}$-a.s., for some random variable with $\mathbf{E}Z^\theta \in [0,1]$. Moreover, under the assumption (W2) and for $\theta \leq \alpha$, where α is defined by (1.48), we can show that $\mathbf{E}Z^\theta = 1$ using [44], Theorem 2.0. The same results will also apply to $Z_i^\theta := \lim_{n\to\infty} Z_i^\theta(n)$ for every $i \in \Sigma_*$, $\mathbf{P}$-a.s. From the definition of $Z_i^\theta(n)$, it may be deduced that we have the decomposition

$$Z_i^\theta = \frac{\sum_{j\in\Sigma_n} l(ij)^\theta Z_{ij}^\theta}{l(i)^\theta \phi(\theta)^n}, \tag{1.49}$$

and furthermore, $(Z_i^\theta)_{i\in\Sigma_n}$ is a collection of independent random variables for each n. We now show that the Z_i^θ are non-zero, $\mathbf{P}$-a.s. using a proof similar to that of Proposition 1.6.3.

Lemma 1.8.1 *Assume (W2) and let $\theta \leq \alpha$, then $\mathbf{P}(Z_i^\theta = 0) = 0$, $\forall i \in \Sigma_*$.*

Proof: Using the decomposition at (1.49) and the fact that $w(i) > 0$, $\mathbf{P}$-a.s., for each i, we obtain

$$\begin{aligned}
\mathbf{P}(Z_i^\theta = 0) &= \mathbf{P}\left(\frac{\sum_{j\in S} l(ij)^\theta Z_{ij}^\theta}{l(i)^\theta \phi(\theta)^n} = 0\right) \\
&= \mathbf{P}\left(Z_{ij}^\theta = 0, \text{ for } j \in S\right) \\
&= \prod_{j\in S} \mathbf{P}\left(Z_{ij}^\theta = 0\right) \\
&= \mathbf{P}(Z_i^\theta = 0)^N,
\end{aligned}$$

where we have also applied the independence of $(Z^\theta_{ij})_{j\in S}$ and the equality in distribution of $(Z^\theta_i)_{i\in\Sigma_*}$ for the third and fourth equalities respectively. Since $N\geq 2$, we must have that $\mathbf{P}(Z^\theta_i=0)\in\{0,1\}$. The assertion follows on noting that, as remarked in the comments preceding this lemma, under the assumption of (W2), $\mathbf{E}Z^\theta_i=1$. $\square$

Assume for the remainder of this section that (W2) holds. To define a measure on Σ it is sufficient to define it on the cylinder sets $i\Sigma:=\{ij:\, j\in\Sigma\}$. Define $\tilde{\mu}^\theta$ by,

$$\tilde{\mu}^\theta(i\Sigma)=\frac{Z^\theta_i l(i)^\theta}{Z^\theta\phi(\theta)^n},\qquad i\in\Sigma_n. \tag{1.50}$$

For $\theta\leq\alpha$, the decomposition identity at (1.49) and the fact that $Z^\theta\in(0,\infty)$, $\mathbf{P}$-a.s., imply that this defines a probability measure, $\mathbf{P}$-a.s. The reason for considering such a measure natural is that it can be considered as the limit measure of the measures $\tilde{\mu}^\theta_n$ on Σ_n which are defined by

$$\tilde{\mu}^\theta_n(i)=\frac{l(i)^\theta}{\sum_{j\in\Sigma_n}l(j)^\theta},\qquad i\in\Sigma_n.$$

On the finite sets Σ_n, these measures have the property of assigning weights proportional to a power of the edge lengths. Taking $\theta=\alpha$ removes the dependency on the word length, which suggests that the natural exponent of edge lengths to choose is α.

It is, in fact, the natural projection of $\tilde{\mu}^\alpha$ onto T that we shall utilise in the next section to prove lower bound results for the Hausdorff dimension of T. Assume now that (R1) and (R2) also hold, so that we may build (T,R). We shall denote the projection of $\tilde{\mu}^\alpha$ under the map $\pi:\Sigma\to T$, which was defined in the statement of Theorem 1.1.1, simply by $\mu^\alpha:=\tilde{\mu}^\alpha\circ\pi^{-1}$. It is possible to check that this is a non-atomic Borel probability measure on (T,R), which satisfies, for measurable $A\subseteq T$,

$$\mu^\alpha(A)=\lim_{n\to\infty}\sum_{i\in\Sigma_n:\,T_i\cap A\neq\emptyset}\tilde{\mu}^\alpha(i\Sigma).$$

In particular, it may be deduced that

$$\mu^\alpha(T_i)=\frac{Z^\alpha_i l(i)^\alpha}{Z^\alpha}, \tag{1.51}$$

which is the stochastic analogue of (1.12).

1.9 Hausdorff dimension lower bound

Proving a tight lower bound for the Hausdorff dimension of a set is often more of a challenge than proving the corresponding upper bound, and this is also the case here.

To prove that the α defined at (1.48) is a lower bound for the Hausdorff dimension of (T, R), we need to make more restrictive assumptions on the scaling factors, and we shall derive the result in two special cases only. We will apply the following standard density result, which is proved in [25], Proposition 4.9. For a metric space (X, d), we use the notation

$$B_d(x, r) := \{y \in X : \ d(x, y) < r\}$$

to represent the *open ball* of radius r about a point $x \in X$.

Lemma 1.9.1 *Let (X, d) be a metric space. Suppose that $A \subseteq X$ supports a measure μ with $\mu(A) \in (0, \infty)$ and there exists a constant $c_{1.19}$ such that*

$$\limsup_{\delta \to 0} \frac{\mu(B_d(x, \delta) \cap A)}{\delta^s} \leq c_{1.19}, \qquad \forall x \in A,$$

then $\mathcal{H}^s(A) \geq c_{1.19}^{-1} \mu(A)$.

So far, we have been able to use the fixed graphs $(\tilde{V}^n, \tilde{E}^n)$ to approximate T. As suggested by Remark 1.1, in general the lengths of edges within these graphs will vary widely as $n \to \infty$. In proving the lower bound for the Hausdorff dimension, it will be useful to to introduce graph approximations to T for which we have some more uniform control over the edge lengths. The approximation we use here is similar to that used in [32], Section 4, for proving results about a random recursive Sierpinski gasket.

We first introduce the notion of a cut-set. We say that $\Lambda \subseteq \Sigma_*$ is a *cut-set* if for every $i \in \Sigma$, there is a unique $j \in \Lambda$ with $i||j| = j$, and there exists an n such that $|j| \leq n$ for all $j \in \Lambda$. This final condition is included to ensure that there is only a countable number of cut-sets. Naturally associated with each cut-set is a graph with vertices in T. The vertex and edge set of the graph corresponding to the cut-set Λ are defined by

$$\tilde{V}^\Lambda := \{F_i(x) : \ x \in \tilde{V}^0, \ i \in \Lambda\}$$

and

$$\tilde{E}^\Lambda := \{\{x, y\} : \ x \in \tilde{V}^\Lambda, \ y \in \tilde{V}^\Lambda(x)\},$$

respectively. We now demonstrate that $(\tilde{V}^\Lambda, \tilde{E}^\Lambda)$ is a graph tree.

Lemma 1.9.2 *If Λ is a cut-set, then $(\tilde{V}^\Lambda, \tilde{E}^\Lambda)$ is a graph tree and for every edge $e \in \tilde{E}^\Lambda$, there exists a unique $e' \in \tilde{E}^0$ and $i \in \Sigma_\Lambda$ such that $e = F_i(e')$.*

Proof: That $(\tilde{V}^\Lambda, \tilde{E}^\Lambda)$ is a graph tree may be proved by repeating the argument of

Proposition 1.2.7, if we can demonstrate that $\tilde{V}^\Lambda$ is a fine subset of T. To do this, we note that it is possible to show that parts (a), (b) and (c) of Lemma 1.2.4 still hold when the set function F is replaced by F^Λ, where, for $A \subseteq T$,

$$F^\Lambda(A) := \bigcup_{i\in\Lambda} F_i(A).$$

The remaining claim is proved in the same way as Lemma 1.2.8. $\square$

For $\delta > 0$, we define a random cut-set, Σ_δ, by

$$\Sigma_\delta := \{i : \ l(i) \leq \delta < l(i|(|i|-1))\}.$$

Under the assumption (W2), Lemma 1.3.1(ii) guarantees that this is indeed a cut-set for all $\delta > 0$, $\mathbf{P}$-a.s. The graphs of interest to us will be those associated with these random cut-sets. For brevity, we will write $(\tilde{V}^\delta, \tilde{E}^\delta)$ to mean $(\tilde{V}^{\Sigma_\delta}, \tilde{E}^{\Sigma_\delta})$.

To be able to apply the density result of Lemma 1.9.1, we look for upper bounds on the measure $\mu^\alpha := \tilde{\mu}^\alpha \circ \pi^{-1}$, which was introduced in the previous section. Furthermore, we will use collections of the sets $(T_i)_{i\in\Sigma_\delta}$ to cover the balls $B_R(x,\delta)$. In a slight change of notation from (1.33) and (1.34), for $x \in T$, define

$$T_\delta(x) := \bigcup\{T_i\colon \ i \in \Sigma_\delta,\ x \in T_i\}$$

and a larger neighbourhood of x by

$$\tilde{T}_{\delta,\varepsilon}(x) := \bigcup\{T_i\colon \ i \in \Sigma_\delta,\ R(T_i \leftrightarrow T_\delta(x)) < \delta\varepsilon\}, \tag{1.52}$$

where for $A, B \subseteq T$, $R(A \leftrightarrow B) := \inf\{R(x,y) : \ x \in A,\ y \in B\}$. The number of sets making up this union is

$$N_{\delta,\varepsilon}(x) := \#\{i \in \Sigma_\delta\colon \ T_i \subseteq \tilde{T}_{\delta,\varepsilon}(x)\}.$$

It is clear that $B_R(x,\delta) \subseteq \tilde{T}_{\delta/\varepsilon,\varepsilon}(x)$. Noting that, for $i \in \Sigma_\delta$,

$$\mu^\alpha(T_i) = \frac{Z_i^\alpha l(i)^\alpha}{Z^\alpha} \leq \frac{\delta^\alpha}{Z^\alpha} \sup_{i\in\Sigma_\delta} Z_i^\alpha,$$

it follows that

$$\mu^\alpha(B_R(x,\delta)) \leq (Z^\alpha)^{-1}\varepsilon^{-\alpha}\delta^\alpha N_{\delta/\varepsilon,\varepsilon}(x) \sup_{i\in\Sigma_{\delta/\varepsilon}} Z_i^\alpha. \tag{1.53}$$

To complete the argument, we estimate the factors $\sup_{i\in\Sigma_\delta} Z_i^\alpha$ and $N_{\delta,\varepsilon}(x)$ separately. In bounding the first of these terms, we shall require some control over the growth of the mean of $|\Sigma_\delta|$. The next lemma provides this using a related age-dependent branching process.

Lemma 1.9.3 *Assume (W1). There exists a constant* $c_{1.20}$ *such that*

$$\mathbf{E}|\Sigma_\delta| \le c_{1.20}\delta^{-\alpha}, \qquad \forall \delta \in (0,1).$$

Proof: Consider the following branching process. Start at time 0 with one particle, labelled $\emptyset$. A particle i has N children at times $(\sigma_i - \ln w(ij))_{j\in S}$ where $\sigma_i := -\ln l(i)$ is the birth time of i. Label by ij the child born to i at $\sigma_i - \ln w(ij) \equiv -\ln l(ij)$, noting that children may not be labelled in birth order. It is not necessary to define the time of dying explicitly in this proof. The independence assumptions on the N-tuples $(w(ij))_{j\in S}$ mean that this setup describes a general branching process in the sense of [36], Chapter 6.

The relevance of this process is in the following observation. If Y_t is defined to be the random variable counting the births before time t then it is easy to check that

$$|\Sigma_\delta| \le NY_{-\ln\delta} \tag{1.54}$$

Noting that the Malthusian parameter for the branching process is precisely the α defined at (1.48), standard arguments then give that $\mathbf{E}Y_t \le c_{1.21}e^{\alpha t}$, for some constant $c_{1.21}$. Combining this bound with the inequality at (1.54) yields the result. $\square$

We now proceed with demonstrating that the rate of growth of $\sup_{i\in\Sigma_\delta} Z_i^\alpha$ is less than a power of $\ln\delta^{-1}$ as $\delta \to 0$. To allow us to apply Borel-Cantelli arguments to deduce $\mathbf{P}$-a.s. properties such as this, it is useful to choose a particular subsequence of δs to investigate. From here on, we consider $(\delta_n)_{n\ge 0}$, defined by $\delta_n := e^{-n}$.

Lemma 1.9.4 *Assume (W1). There exists a constant* $\theta_{1.4}$ *such that*

$$\limsup_{n\to\infty} n^{-\theta_{1.4}} \sup_{i\in\Sigma_{\delta_n}} Z_i^\alpha < \infty, \qquad \mathbf{P}\text{-}a.s.$$

In particular, if $\mathbf{P}(\sum_{i\in S} w(i)^\alpha = 1) = 1$*, then any* $\theta_{1.4} > 0$ *will suffice. Otherwise, we can take*

$$\theta_{1.4} := \inf\left\{\theta\in[0,1) : \sum_{i\in S} w(i)^{\frac{\alpha}{1-\theta}} \le 1,\ \mathbf{P}\text{-}.a.s.\right\},$$

where $\inf\emptyset := 1$.

Proof: If $\mathbf{P}(\sum_{i\in S} w(i)^\alpha = 1) = 1$ then $Z_i^\alpha \equiv 1$, $\mathbf{P}$-a.s. for all i and so the result is obvious. Assume now that $\mathbf{P}(\sum_{i\in S} w(i)^\alpha = 1) < 1$. Define a subset, $\tilde\Sigma_i$, of Σ_* by $\tilde\Sigma_i := \{ik\colon k\in\Sigma_*\}\backslash\{i\}$ and related σ-algebras by

$$\mathcal{F}_i := \sigma(w(j)\colon j\in\tilde\Sigma_i), \qquad \mathcal{G}_i := \sigma(w(j)\colon j\in\Sigma_*\backslash\tilde\Sigma_i).$$

By the independence assumptions on the $(w(j))_{j\in\Sigma_*}$, we have $\mathcal{F}_i \perp \mathcal{G}_i$. It is also straightforward to check that Z_i^α is $\mathcal{F}_i$ measurable and $\{\Sigma_\delta = \Lambda\} \in \mathcal{G}_i$ for any cut-set Λ containing i. Thus, for $i \in \Lambda$, with Λ a cut-set we have, for $\lambda \geq 0$,

$$\mathbf{P}(Z_i^\alpha > \lambda,\, \Sigma_\delta = \Lambda) = \mathbf{P}(\Sigma_\delta = \Lambda)\mathbf{P}(Z_i^\alpha > \lambda) = \mathbf{P}(\Sigma_\delta = \Lambda)\mathbf{P}(Z^\alpha > \lambda).$$

From which we may deduce, using the countability of cut-sets, for $\lambda \geq 0$,

$$\begin{aligned}
\mathbf{P}\left(\sup_{i\in\Sigma_\delta} Z_i^\alpha > \lambda\right) &= \sum_{\Lambda:\,\Lambda \text{ a cutset}} \mathbf{P}\left(\sup_{i\in\Sigma_\delta} Z_i^\alpha > \lambda,\, \Sigma_\delta = \Lambda\right) \\
&\leq \sum_{\Lambda:\,\Lambda \text{ a cutset}} \sum_{i\in\Lambda} \mathbf{P}\left(Z_i^\alpha > \lambda,\, \Sigma_\delta = \Lambda\right) \\
&= \sum_{\Lambda:\,\Lambda \text{ a cutset}} \sum_{i\in\Lambda} \mathbf{P}\left(Z^\alpha > \lambda\right)\mathbf{P}\left(\Sigma_\delta = \Lambda\right) \\
&= \mathbf{P}\left(Z^\alpha > \lambda\right) \sum_{\Lambda:\,\Lambda \text{ a cutset}} |\Lambda|\mathbf{P}\left(\Sigma_\delta = \Lambda\right) \\
&= \mathbf{P}\left(Z^\alpha > \lambda\right)\mathbf{E}|\Sigma_\delta|.
\end{aligned}$$

Since Z^α is the limit of a tree-martingale, we may check the conditions of [44], Theorem 2.1, to give us the following bound on the tail of its distribution. There exist constants $c_{1.22}, c_{1.23}$, such that

$$\mathbf{P}(Z^\alpha > \lambda) \leq c_{1.22} e^{-c_{1.23}\lambda^{1/\theta_{1.4}}}, \qquad \forall \lambda \geq 0.$$

Applying this estimate and Lemma 1.9.3, we obtain

$$\sum_{n=1}^\infty \mathbf{P}\left(n^{-\theta_{1.4}} \sup_{i\in\Sigma_{\delta_n}} Z_i^\alpha > \lambda\right) \leq \sum_{n=1}^\infty c_{1.24} e^{-c_{1.23} n \lambda^{1/\theta_{1.4}} + n\alpha},$$

which is finite for λ chosen suitably large. The result follows from this by an application of the Borel-Cantelli lemma. □

Estimating $N_{\delta,\varepsilon}(x)$ is more difficult and to do so we require a bound on the maximum number of sets from $(T_i)_{i\in\Sigma_\delta}$ that intersect each other. In the next lemma, we show that it is possible to bound this number uniformly over cut-sets. For a cut-set Λ, we will use the notation $\Delta_\Lambda := \sup_{x\in T} \#\{i :\, x \in T_i,\, i \in \Lambda\}$.

Lemma 1.9.5 *Let Λ be a cut-set, then*
(a) $\Delta_\Lambda \leq N|V^0|$.
(b) if $i \in \Lambda$, $\#\{j \in \Lambda :\, T_i \cap T_j \neq \emptyset\} \leq N|V^0|^2$.
Proof: Let $n = \max_{j\in\Lambda} |j|$ be the length of the longest word in Λ. By [9], Proposition

5.2.1, we have that $\Delta_{\Sigma_n} \leq N|V^0|$. However, each of the sets of the form T_i, $i \in \Sigma_n$, is contained in at most one of the sets T_i, $i \in \Lambda$, and so $\Delta_\Lambda \leq \Delta_{\Sigma_n}$, which completes the proof of (a). For part (b), note that

$$T_i \cap \{x : \ x \in T_j \text{ for some } j \in \Lambda, j \neq i\} \subseteq V_i^0.$$

There are $|V^0|$ elements of V_i^0. By part (a), each of the $x \in V_i^0$ can be contained in at most $N|V^0|$ of the $(T_j)_{j \in \Lambda}$, which proves the result. $\square$

We are now in a position to be able to prove the lower Hausdorff dimension bound in our first special case. The assumptions that we use here include the technical ones of (W1), (W2) and (M) which allow us to construct the resistance metric, apply Lemma 1.9.4 to control the variables $(Z_i^\alpha)_{i \in \Sigma_*}$, and also imply $R_i^e \equiv 1$, $\mathbf{P}$-a.s., which eliminates one random variable from our consideration. The assumption that is most specifically related to the problems which arise in the computation of a lower Hausdorff dimension bound is (W3a). Calculations of this kind become difficult if parts of the fractal become, in some sense, too small too quickly. By bounding the scaling factors uniformly below, we are able to prevent this from occurring here.

Theorem 1.9.6 *Assume (W1), (W2), (W3a) and (M), then*

$$\dim_H(T) \geq \alpha, \qquad \mathbf{P}\text{-}a.s.$$

Proof: It follows from (1.3) that intersection of distinct sets from $(T_i)_{i \in \Sigma_\delta}$ can only happen at points in $\cup_{i \in \Sigma_\delta} V_i^0$. Hence, using the tree structure of $(\tilde{V}^\delta, \tilde{E}^\delta)$ and the shortest path property of R, it is possible to obtain that, for $i, j \in \Sigma_\delta$,

$$R(T_i \leftrightarrow T_j) = \min_{x \in \tilde{V}_i^0, \, y \in \tilde{V}_j^0} \sum_{e \in \tilde{E}^\delta} R(e_+, e_-) \mathbf{1}_{\{G_e \subseteq G_{xy}\}}. \tag{1.55}$$

By Lemma 1.9.2, if $e \in \tilde{E}^\delta$, then $e = F_i(e')$ for some $e' \in \tilde{E}^0$, $i \in \Sigma_\delta$. Similarly to (1.31), this implies that $R(e_+, e_-) = l(i) R_i^{e'} / H_{e'}$. Since, by assumption (M), $R_i^{e'} \equiv 1$ and, by assumption (W3a), $l(i) \geq \delta\varepsilon$, this is bounded below by $\delta\varepsilon / H^*$, where $H^* := \max\{H_{e'} : \ e' \in \tilde{E}^0\}$. Consequently, if $R(T_i \leftrightarrow T_j) < \delta\varepsilon / H^*$, then $R(T_i \leftrightarrow T_j) = 0$.

Now fix $x \in T$. Suppose $T_i \subseteq \tilde{T}_{\delta, \varepsilon/H^*}(x)$ for some $i \in \Sigma_\delta$. By the previous paragraph, we must have $T_i \cap T_j \neq \emptyset$ for some $T_j \subseteq T_\delta(x)$, $j \in \Sigma_\delta$. By Lemma 1.9.5, there at most $N|V^0|$ sets satisfying $T_j \subseteq T_\delta(x)$, $j \in \Sigma_\delta$, and each of these can intersect with at most $N|V^0|^2$ of the sets $(T_i)_{i\in\Sigma_\delta}$. Hence $N_{\delta,\varepsilon/H^*}(x) \leq N^2|V^0|^3$. Substituting this into the bound of (1.53), for $\delta \in [\varepsilon e^{-(n+1)}/H^*, \varepsilon e^{-n}/H^*)$, we obtain

$$\begin{aligned}\mu^\alpha(B_R(x,\delta)) &\leq \frac{|N|^2|V^0|^3(H^*)^\alpha e^\alpha}{\varepsilon^\alpha Z^\alpha}\,\delta^\alpha \sup_{i\in\Sigma_{\delta_n}} Z_i^\alpha \\ &\leq c_{1.25}\delta^\alpha(\ln\delta^{-1})^{\theta_{1.4}},\end{aligned}$$

where we have applied Lemma 1.9.4 for the second inequality. Thus, if $s < \alpha$, then $\limsup_{\delta\to 0}\delta^{-s}\mu^\alpha(B_R(x,\delta)) = 0$ for all $x \in T$, $\mathbf{P}$-a.s., and so the density result of Lemma 1.9.1 implies the result. $\square$

Remark 1.7 *By Corollary 1.6.2, the assumption (M) implies (R1) and (R2). Thus, combining this result and the upper bound for the Hausdorff dimension of Theorem 1.7.1, we have under the assumptions of this theorem that* $\dim_H(T) = \alpha$, $\mathbf{P}$*-a.s.*

For the second special case in which we prove a Hausdorff dimension lower bound, we assume (W3b). Again, this is an assumption which stops the fractal getting too small too quickly. Rather than bounding them uniformly below, as is the case under the assumption (W3a), we assume independence of the scaling factors and restrict the amount of build up of mass close to zero in the distributions of the scaling factors. This independence allows us to use a percolation-type argument, which enables us to avoid having to impose a uniform lower bound, which is an assumption that is often used to prove results of this type. If $w(i)$ has distribution function Φ, then the inequality of assumption (W3b) is equivalent to

$$\Phi(\varepsilon x) \leq p\Phi(x), \qquad \forall x \in (0,1].$$

From this, it is easy to see that if Φ is approximately polynomial (i.e. there exist constants $c_{1.26}, c_{1.27}$ such that $c_{1.26}x^n \leq \Phi(x) \leq c_{1.27}x^n$), then assumption (W3b) holds. An example of when the build up of mass is too great for this to hold is the distribution function $\Phi(x) = (1-\ln x)^{-1}$.

We now use the alternative description of (W3b) provided by Lemma 1.6.6 to show that, under the assumption of finite negative moments of the R_i^e, the inequality of (W3b) holds if the $w(i)$ are multiplied by the resistance perturbations. We shall use the ε_0 that arises in this lemma to describe what constitutes a small edge of $(\tilde{V}^\delta, \tilde{E}^\delta)$.

Lemma 1.9.7 *Assume (W2), (W3b), (R1), (R2) and (R3). Given* $q \in (0,1)$*, there exists* $\varepsilon_0 \in (0,1)$ *such that*

$$\mathbf{P}\left(\frac{w(i)R_i^e}{H_e} \leq \varepsilon_0 x \text{ for some } e \in \tilde{E}^0 \,\middle|\, w(i) \leq x\right) \leq q, \qquad \forall x \in (0,1].$$

Proof: Assume (W3b) holds. By Lemma 1.6.6, we can find $\varepsilon_0, \beta > 0$ such that

$$\mathbf{E}\left(\left(1-\frac{x^\beta \varepsilon_0}{w(i)^\beta}\right)\mathbf{1}_{\{w(i)\leq x\}}\right) \geq 0, \qquad \forall x \in (0,1], i \in \Sigma_*\backslash\{\emptyset\}. \tag{1.56}$$

Note also that (R3) implies, after reducing β if necessary, $\mathbf{E}((R_i^e)^{-\beta}) < c_{1.28}$, for all $e \in \tilde{E}^0$, $i \in \Sigma_*$. Furthermore, the inequality (1.56) will still hold if we reduce ε_0. Hence, for $i \in \Sigma_*\backslash\{\emptyset\}$, $e \in \tilde{E}^0$, $x \in (0,1]$,

$$\begin{aligned}
\mathbf{P}\left(\frac{w(i)R_i^e}{H_e} \leq \varepsilon_0 x,\ w(i) \leq x\right) &\leq \mathbf{E}\left(\left(\frac{H_e \varepsilon_0 x}{w(i)R_i^e}\right)^\beta \mathbf{1}_{\{w(i)\leq x\}}\right) \\
&= (H_e\varepsilon_0)^\beta \mathbf{E}((R_\emptyset^e)^{-\beta})\mathbf{E}\left(\frac{x^\beta}{w(i)^\beta}\mathbf{1}_{\{w(i)\leq x\}}\right) \\
&\leq c_{1.29}\varepsilon_0^\beta \mathbf{P}\left(w(i) \leq x\right),
\end{aligned}$$

where for the final step we apply the inequality at (1.56). It follows that, for $i \in \Sigma_*\backslash\{\emptyset\}$,

$$\mathbf{P}\left(\frac{w(i)R_i^e}{H_e} \leq \varepsilon_0 x \text{ for some } e \in \tilde{E}^0 \,\middle|\, w(i) \leq x\right) \leq c_{1.29}|\tilde{E}^0|\varepsilon_0^\beta, \qquad \forall x \in (0,1].$$

Thus the result holds for ε_0 chosen suitably small. □

Henceforth, we shall consider q to be a deterministic constant and choose ε_0 so that the claim of the previous lemma holds. For reasons that will become clear in the proof of Lemma 1.9.9, we will assume that q is strictly less than $2^{-N|V^0|^2}(N|V^0|^2)^{-1}$. To bound $N_{\delta,\varepsilon_0}(x)$ we will show that the largest cluster of sets from $(T_i)_{i\in\Sigma_\delta}$ which contain a small edge is not too large. It is convenient to use the language of percolation theory to describe the setting for the next part of the discussion. We first define the events $(A_i(x))_{i\in\Sigma_*}$ by

$$A_i(x) := \left\{\frac{l(i)R_i^e}{H_e} \leq x \text{ for some } e \in \tilde{E}^0\right\}.$$

When it is clear that we are considering only $i \in \Sigma_\delta$, we will adopt the notation $A_i := A_i(\varepsilon_0\delta)$. For $i \in \Sigma_\delta$, we call the set T_i *open* if A_i occurs, and *closed* otherwise. Thus the open T_i are those sets which contain a small edge of $(\tilde{V}^\delta, \tilde{E}^\delta)$.

Consider the random variable

$$H_\delta := (\Sigma_\delta;\ (l(i|(|i|-1)))_{i\in\Sigma_\delta}).$$

We shall be conditioning on H_δ; the informal motivation for doing so is the following. In the proof of Lemma 1.9.3 we introduced a branching process where the individual i is born at time $-\ln l(i)$. Hence if we stop the branching process at time $-\ln\delta$ (and can not see into the future), then we will be able to ascertain the value of H_δ. However, we will not be able to observe the exact values of $l(i)$ for $i \in \Sigma_\delta$. So, in this sense, we can consider H_δ to be the information about the weighted graph $(\tilde{V}^\delta, \tilde{E}^\delta)$ available from the branching process at time $-\ln\delta$.

We now make precise the nature of the percolation-type behaviour that the independence of the $w(i)$s under the assumption (W3b) induces on the open/closed sets of $(T_i)_{i\in\Sigma_\delta}$. Note that the result provides an upper bound on the probability of a set from $(T_i)_{i\in\Sigma_\delta}$ being open which is independent of δ. This scale-invariance property will be of particular importance for the arguments that follow.

Lemma 1.9.8 *Assume (W2), (W3b), (R1), (R2), (R3). Let $\delta \in (0,1)$. Conditionally on H_δ, the sets $(T_i)_{i\in\Sigma_\delta}$ are open/closed independently and, for $i \in \Sigma_\delta$,*

$$\mathbf{P}\left(A_i \mid H_\delta\right) \leq q, \qquad \mathbf{P}\text{-}a.s.,$$

and, for $s \geq 1$,

$$\mathbf{E}\left(s^{\mathbf{1}_{\{A_i\}}} \mid H_\delta\right) \leq 1 - q + sq, \qquad \mathbf{P}\text{-}a.s. \tag{1.57}$$

Proof: Suppose that $i^1, \dots, i^n$ are distinct elements of Σ_δ. Applying the independence of the $(w(i))_{i\in\Sigma_*\setminus\{\emptyset\}}$, elementary arguments yield

$$\begin{aligned}&\mathbf{P}\left(A_{i^1}, \dots, A_{i^n} \mid H_\delta\right)\\ &= \prod_{m=1}^{n} \mathbf{P}\left(\frac{w(i^m)R^e_{i^m}}{H_e} \leq \frac{\varepsilon_0\delta}{x} \text{ for some } e \in \tilde{E}^0 \,\middle|\, w(i^m) \leq \frac{\delta}{x}\right)_{x=l(i^m|(|i^m|-1))}.\end{aligned}$$

This implies the independence claim. Consider the case $n=1$, and write $i = i^1$. Since $i \in \Sigma_\delta$, we must have $l(i|(|i|-1)) > \delta$. Hence we can apply the bound of Lemma 1.9.7 to the above expression to obtain that $\mathbf{P}(A_i \mid H_\delta) \leq q$, $\mathbf{P}$-a.s. The generating function bound of (1.57) is a simple consequence of this. □

We now introduce an algorithm to find the largest cluster of open sets of the form $(T_i)_{i\in\Sigma_\delta}$. We shall work on the graph $(\Sigma_\delta, \Gamma_\delta)$, where the edge set Γ_δ is defined by

$$\Gamma_\delta := \{\{i,j\} : \, i,j \in \Sigma_\delta, \; T_i \cap T_j \neq \emptyset, \; T_i, T_j \text{ open}\}.$$

We shall write $\mathcal{C}(i)$ for the component of $(\Sigma_\delta, \Gamma_\delta)$ which contains the vertex i. Clearly, if T_i is closed, then $\mathcal{C}(i) = \{i\}$. The following argument to find the size of the largest cluster is inspired by similar procedures used in [37] to find the size of the largest cluster of a random digraph, and in [6] to find the size of the largest cluster of a complete graph with edge percolation.

Let $i \in \Sigma_\delta$ and set $L_0 := \{i\}$, $D_0 := \emptyset$. For $n \geq 1$, we define L_n, D_n inductively. Assume we are given L_n, D_n. If $L_n \neq \emptyset$, then pick a vertex $j \in L_n$ (we can assume that there is a deterministic rule for doing this), and set

$$\begin{aligned} L_{n+1} &:= L_n \cup \{k \in \Sigma_\delta : \ k \notin L_n \cup D_n, \ \{j,k\} \in \Gamma_\delta\} \backslash \{j\} \\ D_{n+1} &:= D_n \cup \{j\}. \end{aligned}$$

If $L_n = \emptyset$, then set $L_{n+1} := \emptyset$, $D_{n+1} := D_n$.

It is a little unclear from this description as to exactly what the algorithm is doing and so we now try to provide a more intuitive description in terms of a branching process related to Σ_δ. Call i a *live* vertex. For the first step, connect to i all those vertices in Σ_δ that are joined to i by an edge in Γ_δ. Call these vertices live and i *dead*. At an arbitrary stage, pick a live vertex, j, and connect to it all those vertices which we have not yet considered and are connected to j by an edge in Γ_δ. Call the new vertices in our branching process live and j dead. Continue until we have no live vertices to pick from. At the point of termination, the collection of dead vertices contains exactly the vertices of $\mathcal{C}(i)$.

In our notation, L_n represents the live vertices and D_n the dead ones. Since we can pick each vertex in Σ_δ only once in the algorithm, we must have $D_{|\Sigma_\delta|+1} = \mathcal{C}(i)$. However, the algorithm may effectively terminate before this stage, giving that $|D_n| = n \wedge \tau$, where $\tau := \inf\{n : \ L_n = \emptyset\}$. Necessarily $L_{|\Sigma_\delta|+1} = \emptyset$, and so this infimum is well-defined and finite. In particular, we must have $|\mathcal{C}(i)| = \tau$.

Using this algorithm, we are able to obtain a tail estimate for the distribution of $|\mathcal{C}(i)|$, conditional on H_δ. Note that this result is scale-invariant; the tail bound on the size of a cluster does not depend on δ.

Lemma 1.9.9 *Assume (W2), (W3b), (R1), (R2) and (R3). Let $\delta \in (0,1)$. There exists a deterministic constant $c_{1.30}$, not depending on δ, such that, for $i \in \Sigma_\delta$,*

$$\mathbf{P}(|\mathcal{C}(i)| > n \mid H_\delta) \leq e^{-c_{1.30} n}, \qquad \mathbf{P}\text{-}a.s.$$

Proof: Choose $i \in \Sigma_\delta$ and use the algorithm described prior to this lemma to construct $(L_n, D_n)_{n \geq 0}$. Given L_n, D_n, the number of new live vertices in the $(n+1)$st step of the algorithm is

$$Z_n := \begin{cases} \#\{k \in \Sigma_\delta : \ k \notin L_n \cup D_n, \ \{j,k\} \in \Gamma_\delta\}, & \text{if } L_n \neq \emptyset, \\ 0, & \text{if } L_n = \emptyset, \end{cases}$$

where $j = j(L_n)$ is the vertex chosen from L_n in the algorithm. On $\{L_n = \emptyset\}$, for $s \geq 1$, $\mathbf{E}\left(s^{Z_n} \,\middle|\, H_\delta, L_n, D_n\right) = 1$, $\mathbf{P}$-a.s. On $\{L_n \neq \emptyset\}$ with $j = j(L_n)$, using the independence and generating function bound of Lemma 1.9.8, for $s \geq 1$, we have

$$\begin{aligned} \mathbf{E}\left(s^{Z_n} \,\middle|\, H_\delta, L_n, D_n\right) &\leq \prod_{k \in \Sigma_\delta:\, k \notin L_n \cup D_n,\, T_j \cap T_k \neq \emptyset} \mathbf{E}\left(s^{\mathbf{1}_{\{T_k \text{ open}\}}} \,\middle|\, H_\delta\right) \\ &\leq (1 - q + sq)^{N|V^0|^2}, \qquad \mathbf{P}\text{-a.s.,} \end{aligned} \tag{1.58}$$

where for the last line we have used the bound of Lemma 1.9.5 on the maximum number of sets of the form $(T_k)_{k \in \Sigma_\delta}$ that can intersect with T_j. Hence, because this upper bound is larger than 1, we have that $\mathbf{E}\left(s^{Z_n} \,\middle|\, H_\delta, L_n, D_n\right) \leq (1 - q + sq)^{N|V^0|^2}$, $\mathbf{P}$-a.s.

For $n \leq \tau$ we have $|L_n| = |L_{n-1}| + Z_{n-1} - 1$, and so, for $s \geq 1$,

$$\begin{aligned} \mathbf{E}\left(s^{|L_n|}\mathbf{1}_{\{|L_n|>0\}} \,\middle|\, H_\delta\right) &\leq \mathbf{E}\left(s^{|L_n|}\mathbf{1}_{\{|L_{n-1}|>0\}} \,\middle|\, H_\delta\right) \\ &= \mathbf{E}\left(s^{|L_{n-1}|} s^{|L_n| - |L_{n-1}|}\mathbf{1}_{\{|L_{n-1}|>0\}} \,\middle|\, H_\delta\right) \\ &= \mathbf{E}\left(s^{|L_{n-1}|}\mathbf{1}_{\{|L_{n-1}|>0\}} \mathbf{E}\left(s^{Z_{n-1}-1} \,\middle|\, H_\delta, L_{n-1}, D_{n-1}\right) \,\middle|\, H_\delta\right) \\ &\leq s^{-1}(1 - q + sq)^{N|V^0|^2} \mathbf{E}\left(s^{|L_{n-1}|}\mathbf{1}_{\{|L_{n-1}|>0\}} \,\middle|\, H_\delta\right), \end{aligned}$$

where we use the inequality at (1.58) for the final bound and we have also used the fact that $\{|L_n| > 0\} = \{\tau > n\}$. Applying this repeatedly yields

$$\mathbf{E}\left(s^{|L_n|}\mathbf{1}_{\{|L_n|>0\}} \,\middle|\, H_\delta\right) \leq s^{-n}(1 - q + sq)^{nN|V^0|^2}, \qquad \mathbf{P}\text{-a.s.}$$

Consequently, $\mathbf{P}$-a.s., for $s \geq 1$,

$$\begin{aligned} \mathbf{P}(|\mathcal{C}(i)| > n \mid H_\delta) &= \mathbf{P}(|L_n| > 0 \mid H_\delta) \\ &\leq \mathbf{E}\left(s^{|L_n|}\mathbf{1}_{\{|L_n|>0\}} \,\middle|\, H_\delta\right) \\ &\leq s^{-n}(1 - q + sq)^{nN|V^0|^2}. \end{aligned}$$

This is minimised by $s = (1-q)/q(N|V^0|^2 - 1)$, which is greater than 1, because of the upper bound we have assumed on q. Substituting for this value of s, we obtain that there is a strictly positive constant $c_{1.28}$, such that, $\mathbf{P}$-a.s.,

$$\mathbf{P}(|\mathcal{C}(i)| > n \mid H_\delta) \leq \left(q 2^{N|V^0|^2} N|V^0|^2\right)^n \leq e^{-c_{1.30} n}.$$

$\square$

This lemma is easily extended to give a tail estimate for the distribution of the size of the *largest component*, $\mathcal{C}_\delta := \sup_{i\in\Sigma_\delta} \mathcal{C}(i)$. We also prove an almost-sure convergence result.

Lemma 1.9.10 *Assume (W1), (W2), (W3b), (R1), (R2) and (R3). Let* $\delta \in (0,1)$. *There exist constants* $c_{1.31}, c_{1.32}$ *such that*

$$\mathbf{P}(\mathcal{C}_\delta > n) \leq c_{1.31} e^{-c_{1.32} n} \delta^{-\alpha}, \qquad \forall \delta \in (0,1].$$

Furthermore,

$$\limsup_{n\to\infty} n^{-1} \mathcal{C}_{\delta_n} < \infty, \qquad \mathbf{P}\text{-}a.s.$$

Proof: Applying the conditional tail distribution of Lemma 1.9.9, we have

$$\begin{aligned}
\mathbf{P}(\mathcal{C}_\delta > n) &= \mathbf{E}\left(\mathbf{P}(\mathcal{C}_\delta > n | H_\delta)\right) \\
&\leq \mathbf{E}\left(\sum_{i\in\Sigma_\delta} \mathbf{P}(|\mathcal{C}(i)| > n | H_\delta)\right) \\
&\leq \mathbf{E}(|\Sigma_\delta|) e^{-c_{1.30} n},
\end{aligned}$$

and so the first assertion follows from Lemma 1.9.3. A simple Borel-Cantelli argument yields the second part of the lemma. □

We are now able to prove the lower bound for the Hausdorff dimension of T in the second special case.

Theorem 1.9.11 *Assume (W1), (W2), (W3b), (R1), (R2), (R3), then*

$$\dim_H(T) \geq \alpha, \qquad \mathbf{P}\text{-}a.s.$$

Proof: As at (1.55), the distance between sets of the form $(T_i)_{i\in\Sigma_\delta}$ is the weighted graph distance between the corresponding vertices in $(\tilde{V}^\delta, \tilde{E}^\delta)$. Hence if it happens that $R(T_i \leftrightarrow T_j) < \delta\varepsilon_0$, then the shortest path between a vertex of $\tilde{V}_i^0$ and a vertex of $\tilde{V}_j^0$ contains only edges contained in open sets from $(T_k)_{k\in\Sigma_\delta}$. Thus, if $T_k \subseteq \tilde{T}_{\delta,\varepsilon_0}(x)$ for $x \in T$, then there exists $i \in \Sigma_\delta$, $j \in \mathcal{C}(i)$ such that $T_k \cap T_j \neq \emptyset$ and $T_i \cap T_\delta(x) \neq \emptyset$. It follows from the bounds on the number of set intersections proved in Lemma 1.9.5 that

$$N_{\delta,\varepsilon_0}(x) \leq N^3 |V^0|^5 \mathcal{C}_\delta,$$

and this bound is uniform in x. Consequently, for $\delta \in [\varepsilon_0 e^{-(n+1)}, \varepsilon_0 e^{-n})$, the bound at (1.53) implies

$$\mu^\alpha(B_R(x,\delta)) \leq \frac{e^\alpha N^3 |V^0|^5}{\varepsilon_0^{-\alpha} Z^\alpha} \delta^\alpha \mathcal{C}_{\delta_n} \sup_{i \in \Sigma_{\delta_n}} Z_i^\alpha.$$

Applying Lemmas 1.9.3 and 1.9.10, we have that $\mathbf{P}$-a.s., there exists a constant $c_{1.33}$ such that

$$\mu^\alpha(B_R(x,\delta)) \leq c_{1.33} \delta^\alpha (\ln \delta^{-1})^{1+\theta_{1.4}} \qquad \forall x \in T, \delta \in (0, \varepsilon_0). \tag{1.59}$$

Thus, for $s < \alpha$, $\limsup_{\delta \to 0} \delta^{-s} \mu^\alpha(B_R(x,\delta)) = 0$, $\forall x \in T$, $\mathbf{P}$-a.s. The result is obtained by applying Lemma 1.9.1. $\square$

Remark 1.8 *Combining this result and Theorem 1.7.1, under the assumptions of this theorem, we have* $\dim_H(T) = \alpha$, $\mathbf{P}$*-a.s.*

1.10 Measure bounds

The proofs of the Hausdorff dimension lower bound in the previous section involved establishing an upper bound on $\mu^\alpha(B_R(x,r))$, the measure of a ball of radius r, where μ^α is the stochastically self-similar measure introduced in Section 1.8. Under the second set of assumptions for which we were able to prove the Hausdorff dimension lower bound, we prove a corresponding lower bound. We shall see in Chapter 2 how the measure bounds we obtain here immediately imply transition density bounds for the diffusion naturally associated with the random Dirichlet form $(\mathcal{E}, \mathcal{F})$ constructed in Section 1.4.

When $|V^0| = 2$, and the scaling factors are uniformly bounded away from one, we can prove a tighter lower measure bound than in the general case. Crucially, under these conditions we are able to deduce an exponential bound for the tail of the distribution of the diameter of (T, R). It is important for the proof of the measure bounds that this implies an almost-sure upper bound for $\delta^{-1} \sup_{i \in \Sigma_\delta} \mathrm{diam}_R T_i$ (along the subsequence δ_n) which is polynomial in $\ln \delta^{-1}$.

Lemma 1.10.1 *Suppose* $|V^0| = 2$ *and the assumptions (W1) and (W4) hold.*
(a) There exist constants $c_{1.34}, c_{1.35}$ *and* $\theta_{1.5}$ *such that*

$$\mathbf{P}(\mathrm{diam}_R T \geq x) \leq c_{1.34} e^{-c_{1.35} x^{1/\theta_{1.5}}}, \qquad \forall x \geq 1.$$

(b) $\mathbf{P}$*-a.s., there exists a constant* $c_{1.36}$ *such that*

$$\sup_{i\in\Sigma_{\delta_n}} \mathrm{diam}_R T_i \leq c_{1.36}\delta_n n^{\theta_{1.5}},$$

where $\theta_{1.5}$ *is the constant of part (a).*

Proof: When $|V^0|=2$, R_i is the precisely the limit of a tree-martingale and we can apply [44] , Theorem 2.1, to obtain that, for $i\in\Sigma_*$,

$$\mathbf{P}(R_i\geq x)\leq c_{1.37}e^{-c_{1.38}x^{1/\theta_{1.5}}}, \qquad \forall x\geq 0, \tag{1.60}$$

for some constants $c_{1.37}, c_{1.38}$ and $\theta_{1.5}$. Observe that the upper bound for $\mathrm{diam}_{R'}\tilde{V}^*$ at (1.32) implies that $\mathrm{diam}_R T\leq 2N|\tilde{E}^0|\sum_{n=0}^\infty \sup_{i\in\Sigma_n} l(i)R_i$. Hence, under (W4),

$$\mathbf{P}(\mathrm{diam}_R T\geq x)\leq \mathbf{P}\left(2N|\tilde{E}^0|\sum_{n=0}^\infty \sup_{i\in\Sigma_n}\eta^n R_i\geq x\right),$$

where η is a constant strictly less than one. Now choose $\varepsilon>0$ small enough so that $\eta+\varepsilon<1$ and define $c_{1.39}:=(1-\eta-\varepsilon)/2N|\tilde{E}^0|$. For $x\geq 1$, set $x_n:=c_{1.39}(\eta+\varepsilon)^n x$. Clearly, if $\sup_{i\in\Sigma_n}\eta^n R_i<x_n$ for each n, then $2N|\tilde{E}^0|\sum_{n=0}^\infty \sup_{i\in\Sigma_n}\eta^n R_i<x$. Thus

$$\begin{aligned}
\mathbf{P}(\mathrm{diam}_R T\geq x) &\leq \sum_{n=0}^\infty \mathbf{P}\left(\sup_{i\in\Sigma_n}\eta^n R_i\geq x_n\right)\\
&\leq \sum_{n=0}^\infty N^n \mathbf{P}(R_i\geq x_n\eta^{-n})\\
&\leq \sum_{n=0}^\infty c_{1.37}e^{n\ln N-c_{1.40}\left(\frac{\eta+\varepsilon}{\eta}\right)^{n/\theta_{1.5}}x^{1/\theta_{1.5}}}\\
&\leq c_{1.41}e^{-c_{1.42}x^{1/\theta_{1.5}}},
\end{aligned}$$

where the third inequality is an application of the tail bound at (1.60), and the final inequality requires some elementary analysis. The proof of (b) requires a Borel-Cantelli argument similar to the proof of Lemma 1.9.4. □

Before reaching the main result of this section, Theorem 1.10.3, we present an almost-sure lower bound for $\inf_{i\in\Sigma_{\delta_n}} Z_i^\alpha$, where $(Z_i^\alpha)_{i\in\Sigma_*}$ are the random variables introduced in Section 1.8.

Lemma 1.10.2 *Assume (W1), (W2) and (W3b) hold. There exists a constant* $\theta_{1.6}$ *such that,* $\mathbf{P}$*-a.s., there exists a constant* $c_{1.43}$ *that satisfies*

$$\inf_{i\in\Sigma_{\delta_n}} Z_i^\alpha\geq c_{1.43}n^{-\theta_{1.6}}, \qquad \forall n\geq 1.$$

Proof: Since Z_i^α is the limit of a tree-martingale, we can repeat the argument of Lemma 1.6.5 to obtain that for some constants $c_{1.44}, c_{1.45}, \theta_{1.6}$, we have $\mathbf{P}(Z_i^\alpha \leq x) \leq c_{1.44}e^{-c_{1.45}x^{-1/\theta_{1.6}}}$, for all $x \geq 0$. Again, the result follows by a Borel-Cantelli argument similar to the proof of Lemma 1.9.4. □

We are now able to prove the measure bounds for (T, R, μ^α). In the statement of the result we use the notation $\ln_1 x := 1 \vee \ln x$, which represents the cut-off logarithm function.

Theorem 1.10.3 *Assume (W1), (W2), (W3b), (R1), (R2) and (R3).*
(a) Fix $\varepsilon > 0$. There exists a constant $\theta_{1.7}$ such that, $\mathbf{P}$-a.s., there exist constants $c_{1.46}, c_{1.47}$, that satisfy

$$c_{1.46}r^{\alpha+\varepsilon} \leq \mu^\alpha(B_R(x,r)) \leq c_{1.47}r^\alpha \left(\ln_1 r^{-1}\right)^{\theta_{1.7}},$$

for every $x \in T$, $r \in (0, \mathrm{diam}_R T]$.
(b) Suppose further that $|V^0| = 2$ and (W4) holds. There exist constants $\theta_{1.8}, \theta_{1.9}$ such that, $\mathbf{P}$-a.s., there exist constants $c_{1.48}, c_{1.49}$, satisfying

$$c_{1.48}r^\alpha \left(\ln_1 r^{-1}\right)^{-\theta_{1.8}} \leq \mu^\alpha(B_R(x,r)) \leq c_{1.49}r^\alpha \left(\ln_1 r^{-1}\right)^{\theta_{1.9}},$$

for every $x \in T$, $r \in (0, \mathrm{diam}_R T]$.

Proof: The common upper bound of (a) and (b) was proved at (1.59) for $r \in (0, \varepsilon_0)$. This is easily extended to hold for $r \in (0, \mathrm{diam}_R T]$, because μ^α is a probability measure on T.

We now prove the lower bound of (b). Under the assumptions of the lemma, we have that, $\mathbf{P}$-a.s.,

$$\mathcal{C}_{\delta_n} \leq c_{1.50}n, \qquad \sup_{i \in \Sigma_{\delta_n}} \mathrm{diam}_R T_i \leq c_{1.34}n^{\theta_{1.5}}\delta_n, \qquad \inf_{i \in \Sigma_{\delta_n}} Z_i^\alpha \geq c_{1.43}n^{-\theta_{1.6}}, \qquad \forall n \geq 1,$$

by Lemmas 1.9.10, 1.10.1 and 1.10.2 respectively. Recall the definition of $\tilde{T}_{\delta,\lambda}(x)$ from (1.52). From the above inequalities, we have that, for $n \geq 0$, $x \in T$,

$$\mathrm{diam}_R \tilde{T}_{\delta_n,\lambda}(x) \leq 2 \sup_{i \in \Sigma_{\delta_n}} \mathrm{diam}_R T_i + \delta_n \lambda \leq (2c_{1.34}n^{\theta_{1.5}} + \lambda)\delta_n,$$

which implies that $\tilde{T}_{\delta_n,\lambda}(x) \subseteq B(x, (2c_{1.34}n^{\theta_{1.5}} + \lambda)\delta_n)$. To complete the proof of the lower bound of (b), we shall establish a lower bound for the measure of a set of the form $\tilde{T}_{\delta_n,\lambda}(x)$. If every set $T_i \subseteq \tilde{T}_{\delta_n,\lambda}(x)$, $i \in \Sigma_{\delta_n}$ is open (in the sense of Section 1.9), we clearly have $N_{\delta_n,\lambda}(x) \leq \mathcal{C}_{\delta_n}$. Hence

$$\mathrm{diam}_R \tilde{T}_{\delta_n,\lambda}(x) \leq \mathcal{C}_{\delta_n} \sup_{i \in \Sigma_{\delta_n}} \mathrm{diam}_R T_i \leq c_{1.51} n^{1+\theta_{1.5}} \delta_n. \tag{1.61}$$

Choose $n_0 \geq 1$ such that $4c_{1.51} n_0^{1+\theta_{1.5}} \delta_{n_0} < \mathrm{diam}_R T$ and define $\lambda_n := 2c_{1.51} n^{1+\theta_{1.5}}$. Since $B_R(x, \delta_n \lambda_n) \subseteq \tilde{T}_{\delta_n,\lambda_n}(x)$, we must have, for $n \geq n_0$,

$$\mathrm{diam}_R \tilde{T}_{\delta_n,\lambda_n}(x) \geq \lambda_n \delta_n > c_{1.51} n^{1+\theta_{1.5}} \delta_n,$$

which contradicts (1.61). Hence there exists an $i \in \Sigma_{\delta_n}$ such that $T_i \subseteq \tilde{T}_{\delta_n,\lambda_n}(x)$ and T_i is closed. It follows that

$$\mu^\alpha(\tilde{T}_{\delta_n,\lambda_n}(x)) \geq \mu^\alpha(T_i) = \frac{Z_i^\alpha l(i)^\alpha}{Z^\alpha} \geq \frac{c_{1.52} \delta_n^\alpha n^{-\theta_{1.6}}}{\sup_{j \in \Sigma_{\delta_n}} R_j}, \tag{1.62}$$

because $l(i) \geq \varepsilon_0 \delta_n H^e / R_i$, by the definition of T_i being closed. From (1.60), we have an exponential tail bound for the distribution of R_i. Consequently, we can use a Borel-Cantelli argument to deduce that $\sup_{i \in \Sigma_{\delta_n}} R_i \leq c_{1.53} n^{\theta_{1.7}}$, for $n \geq 0$, $\mathbf{P}$-a.s., for some $\theta_{1.7}$. Combining this fact with (1.62) and the observation that $\tilde{T}_{\delta_n,\lambda_n}(x) \subseteq B(x, (2c_{1.34} n^{\theta_{1.5}} + \lambda_n)\delta_n)$, we obtain, $\mathbf{P}$-a.s.,

$$\mu^\alpha(B_R(x, (2c_{1.34} n^{\theta_{1.5}} + \lambda_n)\delta_n)) \geq c_{1.54} \delta_n^\alpha n^{-(\theta_{1.6}+\theta_{1.7})}, \qquad \forall n \geq n_0, \quad x \in T.$$

Some elementary manipulation allows it to be deduced from this that, $\mathbf{P}$-a.s., for some $r_0 > 0$, there exists a constant $c_{1.55}$ such that, for $r \in (0, r_0)$,

$$\mu^\alpha(B(x,r)) \geq c_{1.55} r^\alpha (\ln_1 r^{-1})^{-(\alpha(1+\theta_{1.5})+\theta_{1.6}+\theta_{1.7})}, \qquad \forall x \in T.$$

The bound is easily extended to $r \in (0, \mathrm{diam}_R T]$

The proof of the lower bound of (a) is similar and so we omit it here. We do note, however, that the small order polynomial term, r^ε, arises because we are no longer able to deduce exponential tail bounds for the distributions of $\mathrm{diam}_R T$ and R_i^e. Instead, because we know from Lemma 1.5.1 and by assumption (R2) that both random variables have finite positive moments of all orders, we can obtain tail bounds of the form

$$\mathbf{P}(\mathrm{diam}_R T \geq x) \leq c_{1.56} x^{-\theta}, \qquad \mathbf{P}(R_i^e \geq x) \leq c_{1.57} x^{-\theta},$$

for arbitrarily large θ. Following a Borel-Cantelli argument similar to Lemma 1.9.4, these translate to almost-sure results of the form, for $\varepsilon > 0$, $\mathbf{P}$-a.s., there exist constants $c_{1.58}, c_{1.59}$ such that, for $n \geq 0$,

$$\sup_{i \in \Sigma_{\delta_n}} \mathrm{diam}_R T_i \leq c_{1.58} \delta_n^{1-\varepsilon}, \qquad \sup_{e \in \tilde{E}^0} \sup_{i \in \Sigma_{\delta_n}} R_i^e \leq c_{1.59} \delta_n^{-\varepsilon}.$$

The rest of the argument is unchanged. $\square$

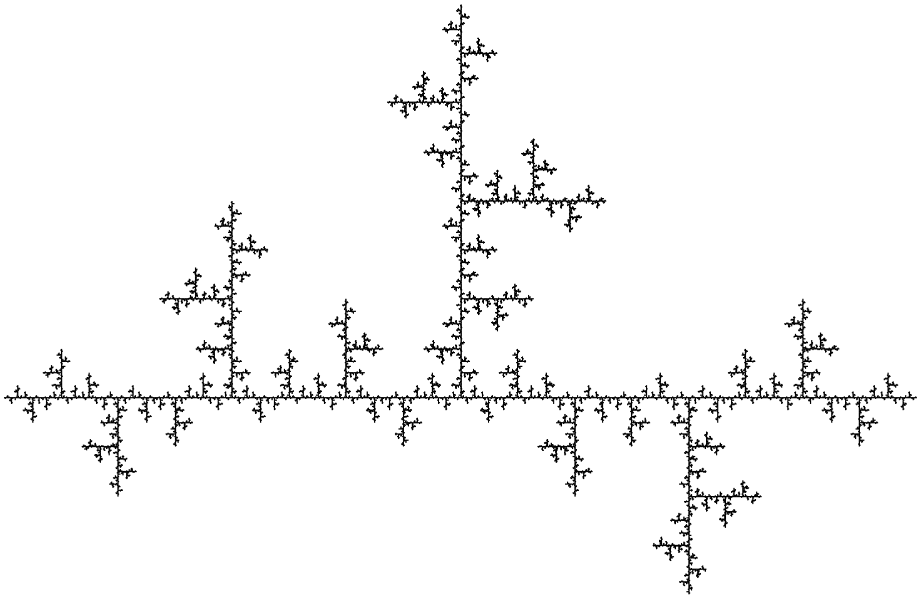

Figure 1.1: P.c.f.s.s. dendrite with $|V^0| = 2$.

1.11 Examples

We now present three examples to illustrate the kinds of sets and scaling factors to which our results apply and demonstrate that we really are generalising from the deterministic case.

Example 1.1 *Two point* V^0

Let $X = \mathbb{R}^2$, $S = \{1, 2, 3\}$ and define

$$F_1(x, y) = \frac{1}{2}(1 - x, y), \quad F_2(x, y) = \frac{1}{2}(1 + x, -y), \quad F_3(x, y) = \left(\frac{1}{2} + cy, cx\right), \quad (1.63)$$

where $c \in (0, 1/2)$ is a constant. The self-similar set T corresponding to these similitudes is shown in Figure 1.1 and has $V^0 = \{(0, 0), (1, 0)\} = \tilde{V}^0$. Since there is only one edge in the graph, the only resistance form on $\tilde{V}^0$ is (up to multiplicative constants):

$$D(f, f) = (f(x_1) - f(x_2))^2 ,$$

where $x_1 := (0, 0)$, $x_2 = (1, 0)$. All the regular harmonic structures for this set are obtained by choosing $\mathbf{r} \in (0, 1)^3$ with $r_1 + r_2 = 1$.

Since $|V^0| = 2$, it follows from the results of Sections 1.4 and 1.6 that, to construct the random Dirichlet form and resistance metric on T, we simply need the scaling factors to satisfy (W1) and (W2). In particular, we need

$$\mathbf{E}\,(w(1) + w(2)) = 1, \qquad \mathbf{E}w(3) < 1, \qquad (1.64)$$

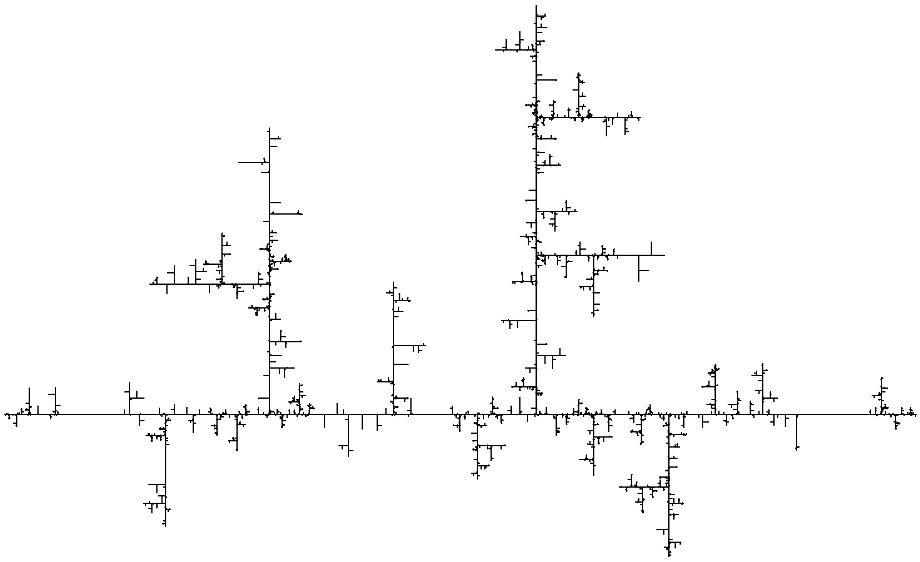

Figure 1.2: Random p.c.f.s.s. dendrite.

and

$$\sum_{i=1}^{3} \mathbf{P}(w(i) = 1) < 1. \tag{1.65}$$

These conditions also allow the application of Theorem 1.7.1 to deduce that α is an upper bound for the Hausdorff dimension of T in the resistance metric, $\mathbf{P}$-a.s.

We now discuss the two sets of conditions for which we have proved that α is also a lower bound for the Hausdorff dimension. First, because $|\tilde{E}^0| = 1$, the matrix M introduced at (1.39) reduces to a real-valued random variable. Specifically, $M = w(1) + w(2)$. Thus, for the assumption (M) to hold, we require $w(1) + w(2) = 1$, $\mathbf{P}$-a.s. Note that this immediately implies the left hand equality of (1.64) and the inequality at (1.65). Hence, by Theorem 1.9.6, we have that $\dim_H(T) = \alpha$, $\mathbf{P}$-a.s., whenever

$$w(1) + w(2) = 1, \quad \mathbf{P}\text{-a.s.}, \qquad \mathbf{E}w(3) < 1,$$

and the scaling factors are bounded away from zero uniformly. Figure 1.2 shows $(\tilde{V}^9, \tilde{E}^9)$ for $w(1) \sim U[\varepsilon, 1-\varepsilon]$, $w(2) = 1 - w(1)$, $w(3) = w(1) \wedge w(2)$, which are scaling factors that satisfy these conditions. The figure is drawn so that lengths of edges in the picture are equal to the resistance between end-points. As to be expected from Proposition 1.3.2, it is already noticeable at this stage of the construction that there is a significant difference between the shortest and longest edges of the graph.

The second set of conditions are perhaps more interesting because we are able to remove the uniform lower bound on the scaling factors. The assumptions we made in Theorem 1.9.11 were (W1), (W2), (W3b), (R1), (R2) and (R3). However, the results of Section 1.6 render the conditions on the resistance perturbations surplus to requirements, and we merely need to choose scaling factors which satisfy (W1), (W2) and (W3b) to construct the Dirichlet form and resistance metric and show that $\dim_H(T) = \alpha$, $\mathbf{P}$-a.s. For example, we can simply take $(w(i))_{i=1}^3$ to be independent $U(0,1)$ random variables. In this case the Hausdorff dimension of T is $\mathbf{P}$-a.s. equal to the solution of

$$3\int_0^1 x^\alpha dx = 1,$$

which we can solve explicitly to obtain $\alpha = 2$. Observe that the associated deterministic construction of (T, R), using $r_i = \mathbf{E}w(i) = 1/2$, $i = 1, 2, 3$, has Hausdorff dimension $\ln 3/\ln 2$. This demonstrates a significant difference between the random and deterministic metric spaces, despite their topological equivalence.

Possibly the most important example choice of scaling factors for which we may construct the Dirichlet form and resistance metric is when $(w(i))_{i=1}^3$ are the square roots of a triple of Dirichlet $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ random variables. Precisely, this means that $(w(i))_{i=1}^3 \stackrel{d}{=} (\Delta_i^{1/2})_{i=1}^3$, where $(\Delta_i)_{i=1}^3$ is a random variable that takes values in the simplex $\{(x_1, x_2, x_3) \in [0,1]^3 : \, x_1 + x_2 + x_3 = 1\}$ and has density

$$\frac{1}{2\pi\sqrt{x_1 x_2 x_3}}.$$

In this case, the random set we construct is a realisation of the continuum random tree of Aldous, (see Chapter 3 for a definition and, for a proof of this correspondence, see Appendix A). Furthermore, we note that, although the Hausdorff dimension results of this chapter do not apply to these scaling factors, we have

$$\mathbf{E}\sum_{i=1}^3 w(i)^2 = 1,$$

and so $\alpha = 2$, which is known to be the Hausdorff dimension of the continuum random tree. Finally, note that in Chapter 3 we show that the measures of balls in the continuum random tree exhibit fluctuations of logarithmic order, suggesting that it will not be possible to significantly tighten the measure bounds of Theorem 1.10.3(b) in general.

Example 1.2 *Vicsek Set*

Let $X = \mathbb{R}^2$, $S = \{1, 2, 3, 4, 5\}$ and v_1, v_2, v_3, v_4, $v_5 = (0,0)$, $(1,0)$, $(1,1)$, $(0,1)$, $(1/2, 1/2)$ respectively. Set $F_i(x) = (x + 2v_i)/3$, for $i = 1, \dots, 5$. The self-similar set T with respect to $\{F_1, \dots, F_5\}$ is called the Vicsek set. The vertex set $V^0 = \{v_1, v_2, v_3, v_4\}$ is not a fine subset of T and so we do need to add the branch point

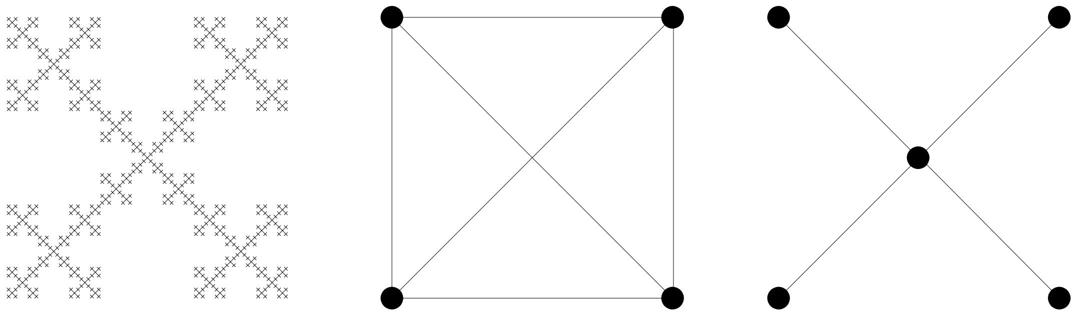

Figure 1.3: Vicsek set, (V^0, E^0) and $(\tilde{V}^0, \tilde{E}^0)$.

v_5 to make up $\tilde{V}^0$, see Figure 1.3. We will denote the edges in $\tilde{E}^0$ by $e_i := \{v_i, v_5\}$, $i = 1, 2, 3, 4$. For this set, we obtain harmonic structures by taking $H_e = 1$, $\forall e \in \tilde{E}^0$, and choosing $\mathbf{r}$ to be an element of the three-dimensional set

$$\{\mathbf{r} : \ 0 < r_i < 1, \ r_1 + r_3 = 1 - r_5 = r_2 + r_4\}. \tag{1.66}$$

Of course, there are other choices of $(H_e)_{e \in \tilde{E}^0}$ which lead to different relationships between the scaling factors.

To construct the Dirichlet form of Section 1.4, we require a choice of scaling factors for which the assumptions (R1) and (R2) hold. Consider the branching process introduced at the end of Section 1.6, started from an initial ancestor $(\emptyset, e_1)$. The first three generations of the process are shown partially in Figure 1.4. We note that the positions of $(15, e_3)$ relative to $(1, e_3)$ and the position of $(15, e_1)$ relative to $(1, e_1)$ both depend on $w(15)$. Hence the offspring of $(1, e_1)$ are not independent of the offspring of $(1, e_3)$, and the branching process does not fit the multi-type branching random walk framework. Consequently, we cannot apply results from that area of study to deduce when (R1) and (R2) hold. Note that this is consistent with the discussion of the multi-type branching random walk in Section 1.6, because the condition (1.46) is not satisfied for the Vicsek set.

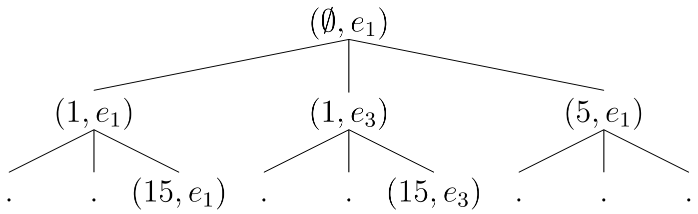

Figure 1.4: Vicsek set family tree.

It is possible, however, to check that whenever $(w(i))_{i=1}^5$ takes values in the set at (1.66), $\mathbf{P}$-a.s., then we may construct the random Dirichlet form and show that $\dim_H(T) \leq \alpha$, $\mathbf{P}$-a.s. in the associated resistance metric. In fact, the only extra condition we need to impose to deduce that $\dim_H(T) \geq \alpha$, $\mathbf{P}$-a.s., is that the scaling

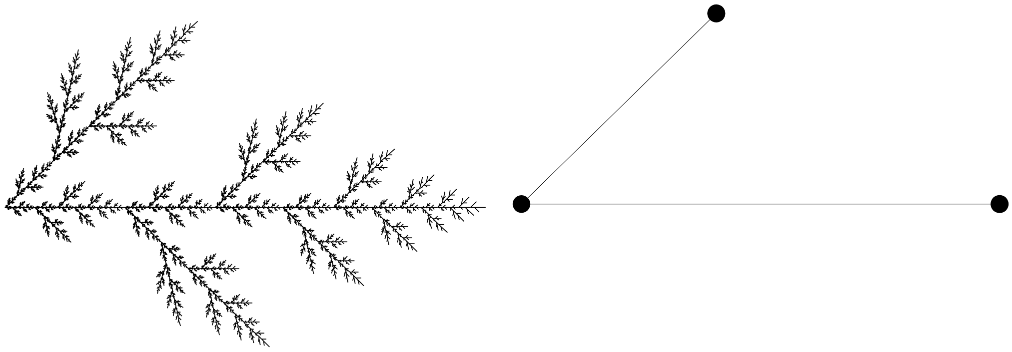

Figure 1.5: Hata's tree-like set and $(\tilde{V}^0, \tilde{E}^0)$.

factors are bounded away from zero uniformly. For example, we can take $w(5) \sim U[\varepsilon, 1-\varepsilon]$ and set $w(i) = (1 - w(5))/2$, $i = 1, 2, 3, 4$.

Example 1.3 *Hata's tree-like set*

Let $X = \mathbb{C}$. Set $F_1(z) = c\overline{z}$, $F_2(z) = (1 - |c|^2)\overline{z} + |c|^2$, where $|c|, |1-c| \in (0,1)$. The self-similar set T with respect to $\{F_1, F_2\}$ is called Hata's tree-like set. One may check that $V^0 = \{c, 0, 1\}$, and that the graph $(\tilde{V}^0, \tilde{E}^0)$ is as shown in Figure 1.5. Note that this graph differs from (V^0, E^0) by having an edge removed. If we define D by $H_{0c} = h$ and $H_{01} = 1$ and also define $\mathbf{r} = (r, 1 - r^2)$, then $(D, \mathbf{r})$ is a regular harmonic structure if and only if $hr = 1$ and $r \in (0,1)$, see [39], Examples 1.3.16 and 3.1.6. Since this is the only regular harmonic structure, the set at (1.40) contains only one point and the choice of scaling factors for which assumption (M) holds is the deterministic case, $w(1) \equiv r$, $w(2) \equiv 1 - r^2$, $\mathbf{P}$-a.s.

However, this set does satisfy the the condition at (1.46) and so we can use multi-type branching random walk arguments to find conditions upon the scaling factors that allow us to deduce that the resistance perturbation assumptions (R1), (R2) and (R3) hold. Figure 1.6 shows the first three generations of the family tree, started from $(\emptyset, e_1)$, where $e_1 := \{0, 1\}$, $e_2 := \{0, c\}$.

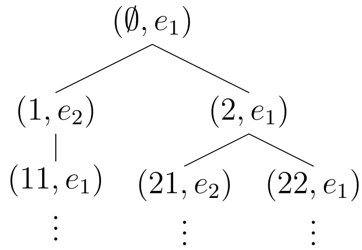

Figure 1.6: Hata family tree.

The relevant matrix of Laplace transforms may be calculated to be equal to

$$\overline{M}(\theta) = \mathbf{E} \begin{pmatrix} w(2)^\theta & h^{-\theta} w(1)^\theta \\ h^\theta w(1)^\theta & 0 \end{pmatrix}.$$

This has Perron-Frobenius (maximum positive) eigenvalue

$$\rho(\theta) = \frac{1}{2}\left\{\mathbf{E}w(2)^\theta + \sqrt{\left(\mathbf{E}w(2)^\theta\right)^2 + 4\left(\mathbf{E}w(1)^\theta\right)^2}\right\}.$$

Under the assumption (W1), $\overline{M}(1)$ is regular stochastic and furthermore,

$$\rho'(1) = \frac{2r}{1+r^2}\mathbf{E}w(1)\ln w(1) + \frac{1}{1+r^2}\mathbf{E}w(2)\ln w(2) < 0.$$

These conditions allow us to apply Theorem 1 of [42] to show that $R_i^e(n)$, as defined at (1.28), converges almost-surely and in mean to R_i^e, for $e \in \tilde{E}^0$, $i \in \Sigma_*$. Thus the limit defining R_i^e exists and is finite $\mathbf{P}$-a.s. In fact, we remark that by adapting the argument of [54], Proposition 2.2, it is possible to show that $\rho'(1) < 0$ in general under (W1) and the regularity of $\overline{M}(1)$, see Appendix B. To demonstrate that R_i^e is non-zero, we can use the irreducibility of $\overline{M}(1)$ in an argument which is analogous to the single-type case, see Lemma 1.6.3. Consequently, (W1) implies (R1).

In general, to show that R_i^e has finite positive moments, it is possible to apply an argument similar to the proof of [54], Theorem 2.1(ii). However, for this set, we can remove the necessity for this by reducing the problem to the single-type situation. Observe that we can write,

$$R_\emptyset^{e_1} = w(2)R_2^{e_1} + w(1)w(11)R_{11}^{e_1},$$

and so we can use the single-type results to obtain that under (W1), $\mathbf{E}((R_\emptyset^{e_1})^d)$ is finite for $d > 0$. Since $R_\emptyset^{e_2} = w(1)R_1^{e_1}$, this means that (W1) also implies (R2). Assume further that (W3b) holds. It follows from Lemma 1.6.6, the random variables $(w(i))_{i\in\Sigma_*}$ are independent have finite negative moments of some order. Thus the same is true for $w(2)$ and $w(1)w(11)$, and so we can repeat the proof of Lemma 1.6.5 to show that the distribution of $R_\emptyset^{e_1}$ has exponential tails at zero. Hence $\mathbf{E}((R_\emptyset^{e_1})^{-d})$ is finite for all $d > 0$. Again, because $R_\emptyset^{e_2} = w(1)R_1^{e_1}$, and each of these factors has finite negative moments for some power, then so does $R_\emptyset^{e_2}$. Hence (R3) holds.

In summary, whenever $(w(i))_{i=1}^2$ are independent $(0,1]$ random variables which satisfy

$$(\mathbf{E}w(1))^2 + \mathbf{E}w(2) = 1, \qquad \sum_{i=1}^2 \mathbf{P}(w(i) = 1) < 1,$$

and the tail inequality of (W3b), we can construct the random Dirichlet form and resistance metric, and deduce that $\dim_H(T) = \alpha$, $\mathbf{P}$-a.s.

Chapter 2

Heat kernel estimates for a resistance form with non-uniform volume growth

In this chapter, we consider the general problem of estimating the heat kernel on measure-metric spaces equipped with a resistance form. As outlined in the first chapter, such spaces admit a corresponding resistance metric that reflects the conductivity properties of the set. In this situation, it has been proved that when there is uniform polynomial volume growth with respect to the resistance metric, the behaviour of the on-diagonal part of the heat kernel is completely determined by this rate of volume growth. However, recent results have shown that for certain random fractal sets, there are global and local (point-wise) fluctuations in the volume of balls of radius r as $r \to 0$ and so these uniform results do not apply. Motivated by these examples, we present global and local on-diagonal heat kernel estimates when the volume growth is not uniform, and demonstrate that when the volume fluctuations are non-trivial, there will be non-trivial fluctuations of the same order (up to exponents) in the short-time heat kernel asymptotics. We also provide bounds for the off-diagonal part of the heat kernel. These results apply to deterministic and random self-similar fractals, and metric space dendrites.

2.1 Background and notation

We start by introducing the general framework and notation that we will use throughout the chapter. Let (X, d) be a locally compact, separable, path-connected metric space, and μ be a non-negative Borel measure on X, finite on compact sets and strictly positive on non-empty open sets. Assume that $(\mathcal{E}, \mathcal{F})$ is a local, regular Dirichlet form on $L^2(X,\mu)$ and that its extended Dirichlet space $(\mathcal{E},\mathcal{F}_e)$ is a resistance form on X, (see Appendix C for a definition of $\mathcal{F}_e$). As discussed in Chapter 1, resistance forms arise naturally from self-similar fractals and metric space dendrites, and a precise definition is given in Section 1.1. A further discussion of the connection between resistance and Dirichlet forms appears in Appendix C.

Generalising the definition at (1.8), define the resistance function R by

$$R(A,B)^{-1} := \inf\{\mathcal{E}(f,f) : f \in \mathcal{F}, f|_A = 1, f|_B = 0\}, \tag{2.1}$$

for disjoint subsets A, B of X. If we set $R(x,y) = R(\{x\},\{y\})$, for $x \neq y$, and $R(x,x) = 0$, then using the fact that $(\mathcal{E},\mathcal{F}_e)$ is a resistance form, it may be shown that the function $R : X \times X \to [0,\infty)$ is a metric on X. This metric is called the *resistance metric*, and we shall assume that the topology induced by R is compatible with the topology induced by d. Note that, in the electrical network interpretation of a quadratic form, the right hand side of (2.1) is precisely the effective conductivity between the sets A and B. Thus the resistance function represents the effective resistance between sets. In this chapter, we shall denote by $B(x,r)$ the path-connected component of the resistance ball of radius r around x containing x.

Given a Dirichlet form, there is a natural way to associate it with a non-negative self-adjoint operator, $-\mathcal{L}$, which has a domain dense in $L^2(X,\mu)$ and satisfies

$$\mathcal{E}(f,g) = -\int_X f\mathcal{L}g d\mu, \qquad \forall f \in \mathcal{F}, g \in \mathcal{D}(\mathcal{L}).$$

Through this association, we may define a related reversible strong Markov process, $((X_t)_{t\geq 0}, \mathbf{P}_x, x \in X)$, with semi-group given by $P_t := e^{t\mathcal{L}}$, ([27], Theorem 6.2.1). By repeating the argument of Proposition 1.5.3, it is possible to show that every point in X has strictly positive capacity (to extend the proof to the non-compact case, integrate (1.38) over $B(x,1)$ rather than the whole space). Consequently, it may be deduced that the process associated with $(\mathcal{E},\mathcal{F})$ is unique ([27], Theorem 4.3.6), and because our Dirichlet form is local, our process is a diffusion ([27], Theorem 6.2.2). Note that the well-known inequality for resistance forms that is stated below at (2.17) and the assumption that the topologies of d and R are compatible imply that $P_t(C_b(X)) \subseteq C_b(X)$, where $C_b(X)$ is the space of continuous bounded functions on X. In particular, when (X,d) is compact, it follows from this that the semi-group $(P_t)_{t\geq 0}$ is Feller. Finally, we prove in Section 2.5 that there exists a version of the transition density p_t, for each $t > 0$, and it is this that will be the object of interest in this chapter. Apart from in Section 2.5, we shall refer to it as the heat kernel or transition density interchangeably.

In the resistance form setting, it has been established that knowledge of the volume growth with respect to the resistance metric can be extremely useful in determining the behaviour of the heat kernel. One widely applicable way of describing volume growth is the idea of volume doubling. To introduce this, suppose that we have a strictly increasing function V, with $V(0)=0$, that satisfies the *doubling* condition:

$$V(2r) \leq C_u V(r). \tag{2.2}$$

We say our measure-metric space, X, has *uniform volume doubling* if we can find a function V satisfying the above properties and also $c_{2.1}V(r) \leq V(x,r) \leq c_{2.2}V(r)$, for every $x \in X$ and $r \in [0, R_X+1)$, where $V(x,r) := \mu(B(x,r))$, and R_X is the diameter of X with respect to the resistance metric, which may be infinite. This uniform volume growth condition includes any space with uniform polynomial volume growth, but excludes exponential growth.

In [41], for a measure-metric space satisfying the conditions of this chapter, Kumagai proves that uniform volume doubling implies that there exists a constant $T_X > 0$ depending only on (X,R) such that the following upper bound on the heat kernel holds: for $x, y \in X$, $t \in (0, T_X]$,

$$p_t(x,y) \leq \frac{c_{2.3}h^{-1}(t)}{t} e^{-\frac{R(x,y)}{c_{2.3}V^{-1}(t/R(x,y))}}, \tag{2.3}$$

where $h(r) := rV(r)$ occurs as a time scale function. It is also demonstrated that a near diagonal lower bound of the form

$$p_t(x,y) \geq \frac{c_{2.4}h^{-1}(t)}{t}, \qquad \text{for } h(c_{2.4}R(x,y)) \leq t, \tag{2.4}$$

holds for $t \in (0, T_X]$. In particular, uniform volume doubling determines that the on-diagonal part of the heat kernel is given up to constant multiples by $h^{-1}(t)/t$.

A major motivation for investigating the properties of the heat kernel on measure-metric spaces equipped with a resistance form is provided by fractal spaces of the type discussed in Section 1.1. For these sets, the high degree of symmetry allows it to be deduced that uniform volume doubling holds, and thus the results of [41] immediately apply. However, this uniformity of volume growth does not necessarily occur in the random fractal setting. In [33], Hambly and Jones prove that for a class of random recursive fractals we can do no better than to bound the measures of balls by

$$c_{2.5}V(r)(\ln r^{-1})^{-a_1} \leq V(x,r) \leq c_{2.6}V(r)(\ln r^{-1})^{a_2},$$

where $V(r) = r^\alpha$, and a_1, a_2 are strictly positive constants. Note that, although in [33] the volume growth is presented in terms of the original metric, it is straightforward to show that the same kinds of fluctuations occur when we consider the resistance metric balls. This unevenness, caused by the random construction mechanism, means that the uniform results do not apply. In fact, there is not even local (point-wise) volume doubling in this example. In [34], Hambly and Kumagai show that the best possible upper and lower bounds for the on-diagonal part of the heat kernel on a random Sierpinski gasket are not asymptotically multiples of each other and also exhibit logarithmic fluctuations.

The main purpose of this chapter is to approach the problem of having non-uniform volume growth more generally. We make no assumptions on the specific structure of our measure-metric space and place only weak conditions on the fluctuations we use in the volume growth condition, see Section 2.2. The argument we use follows closely that of Kumagai, [41], for the case of uniform volume doubling, although more work is required to deal with the fluctuations. As one would expect, by considering the problem in such generality, the results we get are not as sharp as those obtained in specific cases. However, we demonstrate that the loss of accuracy can only be in the exponents of the correction terms. We shall discuss this further in Section 2.9 for some particular examples. The advantage of taking this approach is that we are able to deduce widely applicable bounds, and a particularly nice feature of the results we obtain is that the correction terms of the heat kernel bounds depend on the correction terms of the measure bounds in simple, explicit ways. For example, if we have logarithmic corrections to the measure, our results imply that there are no worse than logarithmic corrections to the heat kernel.

The estimation of heat kernels has, of course, been of interest in various other settings. Aronson, [7], derived upper and lower bounds on the heat kernel for an elliptic operator in $\mathbb{R}^n$ and since then, the behaviour of the heat kernel for elliptic operators on Riemannian manifolds has been studied extensively, see [31] for an introduction to this area. Closely related to this, through discretisation techniques, is the estimation of heat kernels on graphs, where for these spaces, heat kernels are most easily thought of as the transition densities of the associated simple random walks. By considering a graph to be an electrical network, where each edge has resistance one, then we can define the resistance metric by taking $R(x, y)$ to be the effective resistance between vertices x and y. In this case, if we have uniform volume doubling in the resistance metric, then suitable modifications of the results obtained by Kumagai for resistance forms allow it to be deduced that the on-diagonal part of the (discrete time) heat kernel behaves like $h^{-1}(n)/n$, for large n.

For a graph, it is not always straightforward to calculate the resistance between points, and the more natural distance to use is the shortest path length metric, d. Furthermore, the volume growth with respect to d can sometimes be very different from that with respect to R. For example, on the integer lattice $\mathbb{Z}^2$, there is uniform volume doubling in the metric d, as the volume grows like r^2, whereas in the resistance metric, the volume grows exponentially in r. Since the distance d is easier to calculate, there has been a great deal of effort put into establishing heat kernel estimates using knowledge of the volume growth with respect to d. As shown in [12], the information contained by the volume growth in d is insufficient to characterise the heat kernel behaviour, and a range of outcomes is possible. However, for fractal-type graphs the resistance and shortest path metrics are often more closely linked, with some kind of power law between the two holding. In fact, when the volume growth is polynomial (in d), in [13] it is shown that double-sided (sub-Gaussian) heat kernel estimates hold if and only if such a connection holds. The relationship between d and R is most obvious in the case of graph trees, where the two are, in fact, identical. Consequently, it is to fractal-type graphs and graph trees that the resistance form results are most easily adapted.

By analogy with the random recursive fractals of [33] and [34], one might expect that the kind of uniform volume growth that holds for many fractal-type graphs does not hold when random variants are considered. In fact, this has already been proved in the case of the incipient infinite cluster of critical percolation on the binary tree, where local fluctuations of order $\ln \ln r$ about a leading order r^2 term occur in $V(x, r)$, see [14]. Note that, since this structure is a graph tree, this is the volume growth with respect to the resistance metric. In the same article, it was shown that these measure fluctuations lead to fluctuations of log-logarithmic order in the heat kernel, which mirrors the results of this chapter. As in the uniform volume doubling case, it should be a matter of making simple modifications to the techniques used here for resistance forms to exhibit fluctuation results for graphs more generally.

Of greater relevance to our situation are dendrites. For these sets, it was shown by Kigami, [38], that any shortest path metric, d, is in fact a resistance metric for some resistance form. Thus, for these sets, the volume growth in the original metric, d, and in the resistance metric, R, coincides. Moreover, using the simple structure of these spaces, under uniform volume doubling, it is also possible to obtain a lower bound for the heat kernel of the same form as (2.3), see [41]. Although the assumptions that make a space a dendrite are restrictive, there are many interesting examples, including the random self-similar dendrites constructed in Chapter 1. For discussion of how the results of this chapter apply to these sets, see Section 2.9. An example of particular importance is the continuum random tree, which is a random dendrite that arises naturally as the scaling limit of various families of random graph trees. In Chapter 3, detailed measure asymptotics are proved for this set, allowing us to apply the results of this chapter to deduce corresponding heat kernel estimates.

2.2 Volume fluctuations

In this section, we make precise the volume growth condition that we shall presuppose for the remainder of the chapter. First, as in the previous section, let V be a strictly increasing function, with $V(0) = 0$, that satisfies the doubling condition of (2.2). We will define $\beta_u := \ln C_u / \ln 2$ to be the *upper volume growth exponent*, and continue to use the notation $h(r) := rV(r)$. Secondly, we assume that there exist *volume fluctuation functions* $f_l, f_u : [0, R_X + 1) \to [0, \infty]$ such that

$$f_l(r)V(r) \leq V(x,r) \leq f_u(r)V(r), \qquad \forall x \in X, r \in [0, R_X + 1), \tag{2.5}$$

where $V(x,r) := \mu(B(x,r))$, and $B(x,r)$ is the path-connected component of the resistance ball containing x, as in Section 2.1. Typically, we are considering the case when the volume growth is primarily determined by V and the functions f_l and f_u are lower order fluctuations. This is formalised in the conditions given below on f_l and f_u, although it is possibly more enlightening to refer to the examples in Section 2.9. We will use the notation $V_l(r)$, $V_u(r)$ to represent $f_l(r)V(r)$, $f_u(r)V(r)$ respectively. Similarly, we define $h_l(r) = rV_l(r)$ and $h_u(r) := rV_u(r)$. The restrictions we make on f_l and f_u are the following:

(i) $f_l(r)^{-1}, f_u(r) = O(r^{-\varepsilon})$, as $r \to 0$, for some $\varepsilon > 0$.

(ii) $f_l(r)$ is increasing, $f_u(r)$ is decreasing.

(iii) $f_l(r)^{1/b}, f_u(r)^{-1/b}$ are concave on $[0, r_0]$, for some $b, r_0 > 0$.

Here, b and ε are constants upon which we will place upper bounds in Sections 2.3 and 2.4. Without loss of generality, by rescaling if necessary, we can assume further that $f_l \leq 1$ and $f_u \geq 1$. It turns out that the ratio of f_l to f_u is particularly useful in stating our main results, and we shall notate it as follows

$$g(r) := \frac{f_l(r)}{f_u(r)}.$$

By the assumptions on f_l and f_u, we have that g is increasing, ≤ 1 and $g(r)^{-1} = O(r^{-2\varepsilon})$ as $r \to 0$.

2.3 Statement of on-diagonal results

We are now ready to present the first results of this chapter, which explain the behaviour of the on-diagonal part of the heat kernel when the volume growth of the previous section is assumed. The upper bound on the constants b and ε, which appear in the conditions of the volume fluctuation functions f_l and f_u, that we require is the following:

$$b,\ \varepsilon < \frac{1}{4(2+\beta_u)}. \tag{2.6}$$

We also define θ_1 to be a constant that satisfies

$$\theta_1 > \frac{(3+2b+2\beta_u)(2+\beta_u)}{1-2b(3+2b+2\beta_u)}. \tag{2.7}$$

This is an exponent that arises in the course of establishing the following on-diagonal heat kernel bounds, which are proved in Section 2.6 as Propositions 2.6.1 and 2.6.8.

Theorem 2.3.1 *There exist constants $t_0 > 0$ and $c_{2.6}, c_{2.7}, c_{2.8}$ such that*

$$c_{2.6}\frac{h^{-1}(t)}{t}g(h^{-1}(t))^{\theta_1} \leq p_t(x,x) \leq c_{2.7}\frac{h_l^{-1}(t)}{t} \leq c_{2.8}\frac{h^{-1}(t)}{t}f_l(h^{-1}(t))^{-1},$$

for all $x \in X, t \in (0,t_0)$. If $R_X = \infty$ then we may take $t_0 = \infty$, otherwise t_0 is finite.

Remark 2.1 *The bound on the right hand side of this theorem is in general strictly worse than the bound involving $h_l^{-1}(t)$. However, we include it here because it demonstrates clearly that the type of fluctuations in the heat kernel are no worse than those in the measure.*

The next result shows that, if there actually are asymptotic fluctuations in the measure of the order of f_l and f_u, then there will be spatial fluctuations in the heat kernel asymptotics.

Theorem 2.3.2 *If*

$$0 < \liminf_{r\to 0}\inf_{x\in X}\frac{V(x,r)}{V_l(r)} \leq \limsup_{r\to 0}\inf_{x\in X}\frac{V(x,r)}{V_l(r)} < \infty, \tag{2.8}$$

and

$$0 < \liminf_{r\to 0}\sup_{x\in X}\frac{V(x,r)}{V_u(r)} \leq \limsup_{r\to 0}\sup_{x\in X}\frac{V(x,r)}{V_u(r)} < \infty; \tag{2.9}$$

then

$$0 < \liminf_{t\to 0}\inf_{x\in X}\frac{tp_t(x,x)}{h^{-1}(t)g(h^{-1}(t))^{\theta_1}}, \quad \limsup_{t\to 0}\inf_{x\in X}\frac{tp_t(x,x)}{h_u^{-1}(t)} < \infty, \tag{2.10}$$

and

$$0 < \liminf_{t\to 0} \sup_{x\in X} \frac{tp_t(x,x)}{h_l^{-1}(t)} \leq \limsup_{t\to 0} \sup_{x\in X} \frac{tp_t(x,x)}{h_l^{-1}(t)} < \infty. \tag{2.11}$$

Remark 2.2 *Note that we have non-trivial fluctuations in the measure if and only if $V_u(r)/V_l(r) \to \infty$ as $r \to 0$. This is equivalent to $h_l^{-1}(t)/h_u^{-1}(t) \to \infty$ as $t \to 0$, which implies that there are non-trivial fluctuations in the heat kernel over space.*

2.4 Statement of off-diagonal results

To obtain the off-diagonal heat kernel bounds we shall assume again that we have volume growth bounded as at (2.5). We also need two extra conditions and we introduce those now. We shall be slightly stricter about how the function $V(r)$ behaves for small r. We shall assume that there exist constants $R'_X > 0$, $C_l > 1$ such that

$$C_l V(r) \leq V(2r), \qquad \forall r \leq R'_X, \tag{2.12}$$

and define $\beta_l := \ln C_l / \ln 2$, the *lower volume growth exponent*. Comparing this to equation (2.2) means that we must have $\beta_l \leq \beta_u$. This condition ensures that V increases suitably quickly near 0, and is sometimes referred to in the literature as the *anti-doubling* property. We shall also tighten the conditions on b and ε to

$$b,\ \varepsilon < \frac{\beta_l}{8(2+\beta_u)^2}, \tag{2.13}$$

and define θ_1, θ_2 and θ_3 to be exponents satisfying

$$\frac{\beta_l}{2b} \wedge \frac{\beta_l}{2\varepsilon} > \theta_1 > \frac{(3+2b+2\beta_u)(2+\beta_u)}{1-2b(3+2b+2\beta_u)}, \tag{2.14}$$

$$\theta_2 > \frac{\theta_1(1+\beta_l)}{\beta_l - 2b\theta_1}, \tag{2.15}$$

$$\theta_3 = (3+2b+2\beta_u)(1+2\beta_l^{-1}).$$

Note that our assumptions on b and ε at (2.13) mean that it is indeed possible to choose θ_1 satisfying (2.14). Furthermore, we can choose θ_1 that is consistent with (2.7) and (2.14), we have merely added an upper bound.

Under these assumptions, we are able to deduce the following result for the off-diagonal parts of the heat kernel. It is proved in Section 2.7 as Propositions 2.7.3 and 2.7.6. In the statement of the result we use the *chaining condition* (CC), which is defined as follows: there exists a constant $c_{2.9}$ such that, for all $x, y \in X$, and all $n \in \mathbb{N}$, there exists $\{x_0, x_1, \dots, x_n\} \subseteq X$ with $x_0 = x$, $x_n = y$ such that

$$R(x_{i-1}, x_i) \leq c_{2.9} \frac{R(x,y)}{n}, \qquad \forall 1 \leq i \leq n.$$

When this assumption holds, the following bounds show that the exponential decay away from the diagonal differs from the uniform case by a factor that is of an order no greater than the measure fluctuations (up to exponents).

Theorem 2.4.1 *There exist constants $t_1 > 0$ and $c_{2.10}, c_{2.11}$ such that*

$$p_t(x,y) \leq c_{2.10} \frac{h^{-1}(t)}{t} f_l(h^{-1}(t))^{-1} e^{-c_{2.11} \frac{R}{V^{-1}(t/R)} g(V^{-1}(t/R))^{\theta_3}},$$

for all $x, y \in X$, $t \in (0, t_1)$, where $R := R(x,y)$.

Furthermore, if (CC) holds, then there exist constants $t_2 > 0$ and $c_{2.12}, c_{2.13}$ such that

$$p_t(x,y) \geq c_{2.12} \frac{h^{-1}(t)}{t} g(h^{-1}(t))^{\theta_1} e^{-c_{2.13} \frac{R}{V^{-1}(t/R)} g(V^{-1}(t/R))^{-\theta_2}},$$

for all $x, y \in X$, $t \in (0, t_2)$, where $R := R(x,y)$.

Note that, if $R'_X = \infty$, then we may take $t_1 = t_2 = \infty$, otherwise t_1 and t_2 are finite.

Remark 2.3 *We note that the results of Sections 2.3 and 2.4 reduce to those obtained by Kumagai in [41] when f_l is bounded away from 0 and f_u is bounded above by a finite constant. The extension of the near-diagonal lower bound of (2.4) is proved in Lemma 2.7.4.*

Remark 2.4 *Choosing θ_1 and θ_2 closer to the lower bound will give tighter bounds asymptotically.*

Remark 2.5 *The chaining condition is not necessary to obtain the off-diagonal upper bound. However, as is remarked in [41], Section 5, by Kumagai, even for the case of uniform volume doubling, the bound is not optimal in general when (CC) does not hold, which is often. We note that the chaining condition holds most obviously when X is a dendrite.*

2.5 Existence of the transition density

In this section, we prove the existence of a transition density for $(P_t)_{t>0}$, using a result appearing in [30], by Grigor′yan. The key step is establishing the ultracontractivity of the semi-group in our setting. We shall start by defining this property and the other standard terms that will be used in this section.

A self-adjoint semi-group $(P_t)_{t>0}$ is said to be *ultracontractive* if there exists a positive, decreasing function $\gamma(t)$ on $(0, \infty)$ such that

$$\|P_t f\|_2 \leq \gamma(t)\|f\|_1, \qquad \forall f \in L^1(X, \mu) \cap L^2(X, \mu). \tag{2.16}$$

This property is particularly appealing for a semi-group, and as we explain below, it immediately guarantees the existence of a transition density for $(\mathcal{E}, \mathcal{F})$. In the resistance form setting, we show in Proposition 2.5.2 that the only condition needed to deduce ultracontractivity is a suitable uniform lower bound on the volume of resistance balls.

A family $(p_t)_{t>0}$ of $\mu \times \mu$-measurable functions on $X \times X$ is called a (symmetric) *transition density* of the semi-group $(P_t)_{t>0}$ (alternatively, of the form $(\mathcal{E}, \mathcal{F})$) if there exists $X' \subseteq X$ with $\mu(X \backslash X') = 0$ such that, for any bounded measurable function f,

$$P_t f(x) = \int_X p_t(x, y) f(y) \mu(dy), \qquad \forall x \in X', t > 0,$$

$$p_t(x, y) = p_t(y, x) \qquad \forall x, y \in X, t > 0,$$

and

$$p_{s+t}(x, y) = \int_X p_s(x, z) p_t(z, y) \mu(dz), \qquad \forall x, y \in X, s, t > 0.$$

Similarly, a family $(\tilde{p}_t)_{t>0}$ of $\mu \times \mu$-measurable functions on $X \times X$ is called a *heat kernel* of $(P_t)_{t>0}$ if $\tilde{p}_t$ is an integral kernel of P_t for each $t > 0$. Clearly, this only defines a heat kernel up to a μ-null set. The extra conditions on the transition density mean that it is defined everywhere in X and is also a heat kernel. Consequently, for an arbitrary heat kernel our results only apply μ-almost-everywhere.

Before we prove the existence of a transition density for $(\mathcal{E}, \mathcal{F})$, we state the crucial lemma that we will apply, the proof of which relies on the Riesz representation theorem. It should be noted that the argument we use for our main result, Proposition 2.5.2, is standard, and is similar to the proof of the heat kernel upper bound proved in [41], Proposition 4.1. In the proof, we will utilise the following observation, which we recall from (1.36). In particular, we have that

$$|f(x) - f(y)|^2 \leq R(x, y)\mathcal{E}(f, f), \qquad \forall x, y \in X, f \in \mathcal{F}_e. \tag{2.17}$$

This inequality, together with the assumption that the topologies induced by R and d are compatible, means that $\mathcal{F}_e \subseteq C(X)$, where $C(X)$ is the space of continuous functions on (X, d).

Lemma 2.5.1 *([30], Lemma 8.1) If the semi-group $(P_t)_{t>0}$ is ultracontractive, then it admits a transition density.*

Proposition 2.5.2 *There exists a transition density $(p_t)_{t>0}$ for $(P_t)_{t>0}$, and moreover, for each $t > 0$, $p_t(x, y)$ is jointly continuous in x and y.*

Proof: By rescaling, to demonstrate that $(P_t)_{t>0}$ is ultracontractive, it is sufficient to check that (2.16) holds for every $f \in L^1(X, \mu) \cap L^2(X, \mu)$ with $||f||_1 = 1$. Consequently, we take f to be a function satisfying these conditions, and we denote $f_t := P_t f$. By standard semi-group theory, we note that $f_t \in \mathcal{D}(\mathcal{L}) \subseteq \mathcal{F}$ for every $t > 0$, where $\mathcal{D}(\mathcal{L})$ is the domain of the generator of $(P_t)_{t>0}$. Now observe that we must have, for every $x \in X$, $r, t > 0$,

$$\int_{B(x,r)} |f_t(y)|\mu(dy) \leq ||f_t||_1 \leq ||f||_1 = 1.$$

Hence there must exist a $y \in B(x, r)$ such that $|f_t(y)| \leq V(x, r)^{-1} \leq V_l(r)^{-1}$, where we apply the volume bound of (2.5) for the second inequality. Combining this result with the inequality that was stated at (2.17), it is possible to deduce that

$$\begin{aligned} \frac{1}{2}|f_t(x)|^2 &\leq |f_t(y)|^2 + |f_t(x) - f_t(y)|^2 \\ &\leq V_l(r)^{-2} + r\mathcal{E}(f_t, f_t). \end{aligned}$$

We now define $\psi(t) := ||f_t||_2^2$, which is a positive decreasing function. The above inequality allows us to write

$$\begin{aligned} \psi(t/2) &= \int_X f_{t/2}(x) f_{t/2}(x)\mu(dx) \\ &\leq \int_X |f(x) f_t(x)|\mu(dx) \\ &\leq 2^{1/2}(V_l(r)^{-2} + r\mathcal{E}(f_t, f_t))^{1/2}, \end{aligned}$$

where for the final inequality we use the fact that $||f||_1 = 1$. Applying established results for semi-groups, we have that $\psi'(t) = -2\mathcal{E}(f_t, f_t)$. Thus the above inequality may be rearranged to give

$$\psi'(t) \leq \frac{2V_l(r)^{-2} - \psi(t)^2}{r},$$

where we also apply the fact that $\psi(t) \leq \psi(t/2)$. By following the proof of [41], Proposition 4.1, we are able to deduce from this differential inequality the existence of constants $c_{2.14}, t_3 > 0$ such that

$$\psi(t) \leq c_{2.14} \frac{h_l^{-1}(t)}{t}, \qquad \forall t \in (0, t_3),$$

which implies that in (2.16) we may take $\gamma(t) = (c_{2.14} h_l^{-1}(t)/t)^{1/2}$ for $t \in (0, t_3)$. Hence $(P_t)_{t>0}$ is ultracontractive, and so, by Lemma 2.5.1, it admits a transition density $(p_t)_{t>0}$.

To prove the continuity of p_t for each $t > 0$, we first observe that $p_t(x, \cdot) = P_{t/2} p_{t/2}(x, \cdot)$. This implies that $p_t(x, \cdot) \in \mathcal{D}(\mathcal{L}) \subseteq \mathcal{F}_e$, and in particular we must have $\mathcal{E}(p_t(x, \cdot), p_t(x, \cdot)) < \infty$. Consequently, we can apply the inequality at (2.17) and the symmetry of the transition density to deduce the desired continuity result. □

2.6 Proof of on-diagonal heat kernel bounds

In this section, we determine bounds for the on-diagonal part of the heat kernel. We start with the proof of the upper bound. As is often the case, this is relatively straightforward to obtain. It is the lower bound which requires more work and the remainder of the section is dedicated to this. A result of interest in its own right is Proposition 2.6.6, where we present bounds for the expected time to exit a ball.

Proposition 2.6.1 *There exist constants $t_4 > 0$ and $c_{2.15}$ such that*

$$p_t(x, x) \leq c_{2.15} \frac{h_l^{-1}(t)}{t} \leq c_{2.15} \frac{h^{-1}(t)}{t} f_l(h^{-1}(t))^{-1}, \qquad \forall x \in X, t \in (0, t_4).$$

If $R_X = \infty$, then we may take $t_4 = \infty$, otherwise t_4 is finite.

Proof: The proof of the analogous upper bound in [41], Proposition 4.1, uses only that a multiple of $V(r)$ is a lower bound for $V(x, r)$. In our case, the lower bound that is appropriate is $V_l(r)$, and we may repeat the argument to obtain

$$p_{2h_l(r)}(x, x) \leq \frac{2r}{h_l(r)}, \qquad \forall r \in [0, R_X). \tag{2.18}$$

Define $t_0 := h_l(R_X)$. Then, for $t < t_0$ we can find $r < R_X$ such that $t = 2h_l(r)$, and under this parametrisation we find $p_t(x, x) \leq c_{2.15} h_l^{-1}(t)/t$, which is the first inequality.

We now claim that

$$h(f_l(r)r) \leq h_l(r) \leq h(r). \tag{2.19}$$

Noting that V is increasing and $f_l(r) \leq 1$ we must have

$$h(f_l(r)r) = f_l(r)rV(f_l(r)r) \leq f_l(r)rV(r) = h_l(r),$$

which is the left hand inequality. To prove the right hand inequality we simply note that $h_l(r) = rf_l(r)V(r) \leq rV(r) = h(r)$. Thus the claim does indeed hold. If we now define r by $t = h_l(r)$, we have from (2.19) that $h(f_l(r)r) \leq t \leq h(r)$, and applying h^{-1} to this yields

$$f_l(r)h_l^{-1}(t) = f_l(r)r \leq h^{-1}(t) \leq r. \tag{2.20}$$

With this choice of r, the upper bound on the transition density given at (2.18) implies that

$$p_t(x,x) \leq c_{2.15}\frac{h_l^{-1}(t)}{t} \leq c_{2.15}\frac{h^{-1}(t)}{t}f_l(r)^{-1},$$

where we have applied the left hand inequality of (2.20). To complete the proof we use the right hand inequality of (2.20) to deduce that $f_l(h^{-1}(t)) \leq f_l(r)$. □

The aim of the subsequent four lemmas is to deduce bounds on the effective resistance from the centre of a ball to its surface. We start by proving two lemmas which explain how to move factors in and out of the functions V, f_l and f_u, and which will be used repeatedly later in the chapter. Following this, Lemma 2.6.4 is a version of the result proved in [11], Lemma 2.7. We show how we can bound the size of a cover of a ball with suitably scaled smaller balls. The result of interest is easily deduced from this estimate, and appears as Lemma 2.6.5.

Lemma 2.6.2 *Let* $\Lambda \geq 1$, *then* $V(\Lambda r) \leq C_u\Lambda^{\beta_u}V(r)$.
Proof: Let $n = \lceil \ln\Lambda/\ln 2\rceil$ and then, using the doubling property of V, (2.2), we have $V(\Lambda r) \leq C_u^n V(2^{-n}\Lambda r) \leq C_u^{1+\ln\Lambda/\ln 2}V(r) = C_u\Lambda^{\beta_u}V(r)$. □

Lemma 2.6.3 *There exist constants* $c_{2.16}, c_{2.17}$ *such that*

$$f_l(\lambda r) \geq c_{2.16}\lambda^b f_l(r), \qquad \forall\lambda \in [0,1], r \in [0, R_X + 1),$$

$$f_u(\lambda r) \leq c_{2.17}\lambda^{-b} f_u(r), \qquad \forall\lambda \in [0,1], r \in [0, R_X + 1).$$

Proof: We shall only prove the result for the f_l. The result for f_u is proved by applying the same argument to $1/f_u$. By assumption, $f_l^{1/b}$ is concave and positive on $[0, r_0]$ and so, for $\lambda \in [0,1]$, $r \in [0, r_0]$,

$$f_l^{1/b}(\lambda r) \geq \lambda f_l^{1/b}(r) + (1-\lambda)f_l^{1/b}(0) \geq \lambda f_l^{1/b}(r).$$

Thus we have the result for $r \in [0, r_0]$. Now, define $f_l(\tilde{R}_X) := \lim_{r \uparrow \tilde{R}_X} f_l(r)$, where $\tilde{R}_X := R_X + 1$, which exists in $(0, 1]$ by the boundedness and monotonicity of f_l. We also have that $f_l(r) \leq f_l(\tilde{R}_X)$, for every $r \in [0, \tilde{R}_X)$. Hence, using the result already established for small r, we can deduce, $\forall \lambda \in [0, 1], r \in [r_0, \tilde{R}_X)$, that

$$f_l(\lambda r) \geq f_l(\lambda r_0) \geq \frac{f_l(r_0)}{f_l(\tilde{R}_X)} \lambda^b f_l(r),$$

which completes the proof. □

Lemma 2.6.4 *Fix $\varepsilon \in (0, 1/2]$. For any $r > 0, x \in X$, we can find a cover of $B(x, r)$ consisting of fewer than M balls of radius εr, where*

$$M := c_{2.18} g(r)^{-1},$$

with $c_{2.18}$ a constant (depending on ε).

Proof: Let $x_1 \in B(x, r)$ and choose $x_2, x_3, \dots$ by letting x_{i+1} be any point in $B(x, r) \setminus \cup_{j=1}^{i} B(x_j, \varepsilon r)$. We do this until we can no longer proceed. Note that we must have the $B(x_i, \varepsilon r/2)$ disjoint and also $\cup_{i=0}^{m} B(x_i, \varepsilon r/2) \subseteq B(x, r(1+\varepsilon/2))$, where m is the number of balls selected for the cover. It follows that

$$\begin{aligned} m V_l(\varepsilon r/2) &\leq \mu\left(\bigcup_{i=0}^{m} B(x_i, \varepsilon r/2)\right) \\ &\leq \mu\left(B(x, r(1+\varepsilon/2))\right) \\ &\leq V_u(r(1+\varepsilon/2)). \end{aligned}$$

Now, by applying Lemma 2.6.2 and Lemma 2.6.3, we have $V_l(\varepsilon r/2) \geq c_{2.19} V_l(r)$, and also $V_u(r(1+\varepsilon/2)) \leq c_{2.20} V_u(r)$. Hence we must have $m \leq c_{2.18} f_u(r) f_l(r)^{-1} = c_{2.18} g(r)^{-1}$, and so the assertion is proved. □

Lemma 2.6.5 *There is a constant $c_{2.21}$ such that, for all $r \in [0, R_X/2), x \in X$,*

$$c_{2.21} r g(r)^2 \leq R(x, B(x, r)^c) \leq r.$$

Proof: This result may be proved by repeating exactly the same argument as was used in [41], Lemma 4.1, with the cover size being determined by Lemma 2.6.4. □

We shall now prove bounds on the expected exit time of a resistance ball. For $A \subseteq X$, we shall define

$$T_A := \inf\{t \geq 0 : \ X_t \notin A\}$$

to be the *first exit time* from A. Furthermore, in subsequent results, we use the notation $\mathbf{E}_x$ to represent the expectation under the measure $\mathbf{P}_x$.

Proposition 2.6.6 *There exists a constant $c_{2.22}$ such that*

$$\mathbf{E}_{x_0} T_{B(x_0,r)} \geq c_{2.22} h_l(rg(r)^2), \qquad \forall x_0 \in X, r \in [0, R_X/2)$$

$$\mathbf{E}_x T_{B(x_0,r)} \leq h_u(r), \qquad \forall x, x_0 \in X, r \in [0, R_X/2).$$

Proof: Fix $x_0 \in X, r \in [0, R_X/2)$ and let $B := B(x_0, r)$. Then, as in [41], Proposition 4.2, it may be deduced that there exists a Green kernel $g_B(\cdot,\cdot)$ for the process killed on exiting B that satisfies

$$\mathcal{E}(g_B(x,\cdot), g_B(x,\cdot)) = g_B(x,x), \tag{2.21}$$

$$g_B(x,x) = R(x, B^c), \tag{2.22}$$

$$g_B(x,y) \leq g_B(x,x), \tag{2.23}$$

$$\mathbf{E}_x T_B = \int_B g_B(x,y)\mu(dy), \tag{2.24}$$

for all $x, y \in X$.

By the inequality at (2.17) for the function $g_B(x_0,\cdot)$, one has that

$$|g_B(x_0,y) - g_B(x_0,x_0)|^2 \leq R(x_0,y)\mathcal{E}(g_B(x_0,\cdot), g_B(x_0,\cdot)).$$

By using properties (2.21) and (2.22) it follows that

$$\left(1 - \frac{g_B(x_0,y)}{g_B(x_0,x_0)}\right)^2 \leq \frac{R(x_0,y)}{R(x_0,B^c)}.$$

Using (2.23) and the lower bound on $R(x, B(x_0,r)^c)$ obtained in Lemma 2.6.5, it may be deduced from the above inequality that for some constant $c_{2.23}$, if $y \in B(x_0, c_{2.23} rg(r)^2)$, then $g_B(x_0,y) \geq \frac{1}{2} g_B(x_0,x_0)$. So, by the representation of $\mathbf{E}_{x_0} T_B$ given at (2.24), we have

$$\begin{aligned}
\mathbf{E}_{x_0} T_{B(x_0,r)} &\geq \frac{1}{2} R(x_0, B^c) V(x_0, c_{2.23} rg(r)^2) \\
&\geq \frac{1}{2} c_{2.23} rg(r)^2 V_l(c_{2.23} rg(r)^2) \\
&\geq c_{2.22} h_l(rg(r)^2),
\end{aligned}$$

which proves the lower bound. For the upper bound, we proceed as in [41], Proposition 4.2, to obtain for $x \in X$, $\mathbf{E}_x T_{B(x_0,r)} \leq rV(x_0,r)$, which immediately implies the result by the volume bounds at (2.5). $\square$

We now present a bound on the tail of the exit time distribution which will be sufficient for obtaining the on-diagonal lower bound for the heat kernel. The extra anti-doubling assumption we make on the volume growth for the off-diagonal bounds will also allow us to write this bound in a way that avoids using the rather awkward function q. This bound is presented in Proposition 2.7.2.

Lemma 2.6.7 *There exist constants $c_{2.24}, c_{2.25}, c_q$ such that*

$$\mathbf{P}_x(T_{B(x,r)} \leq t) \leq c_{2.24} e^{-c_{2.25} \frac{r}{q^{-1}(t/r)} g(q^{-1}(t/r))^{\gamma_1}}, \qquad \forall x \in X, r \in (0, R_X/2), t > 0,$$

where $q(r) := c_q g(r)^{2\gamma_1} V_u(r)$ and $\gamma_1 := 3 + 2b + 2\beta_u$.

Proof: The proof follows a standard pattern and involves the application of [10], Lemma 1.1, to strengthen a simple linear bound to an exponential one. We start by deducing the relevant linear bound. By Proposition 2.6.6, we have $\mathbf{E}_x T_{B(x,r)} \geq c_{2.22} h_l(g(r)^2 r)$, $\forall x \in X$, $r \in (0, R_X/2)$, from which we may deduce that

$$\mathbf{E}_x T_{B(x,r)} \geq c_{2.26} r g(r)^{2(1+b+\beta_u)} f_l(r) V(r), \tag{2.25}$$

by using Lemmas 2.6.2 and 2.6.3, and moreover, we can assume that $c_{2.26} \in (0, \frac{1}{2})$. Furthermore, we may use the Markov property of $(X_t)_{t \geq 0}$ to deduce that

$$\mathbf{E}_x T_{B(x,r)} \leq t + \mathbf{E}_x \mathbf{1}_{\{T_{B(x,r)} > t\}} \mathbf{E}_{X_t} T_{B(x,r)}. \tag{2.26}$$

Since $\mathbf{E}_{x_0} T_{B(x,r)} \leq h_u(r)$, comparing (2.25) and (2.26) yields

$$c_{2.26} r g(r)^{2(1+b+\beta_u)} f_l(r) V(r) \leq t + \mathbf{P}_x(T_{B(x,r)} > t) h_u(r),$$

which we may rearrange to obtain

$$\mathbf{P}_x(T_{B(x,r)} \leq t) \leq 1 - c_{2.26} g(r)^{3+2b+2\beta_u} + \frac{t}{h_u(r)},$$

our linear bound.

To get the exponential bound requires a kind of chaining argument which we describe now. Let $n \geq 1$ and define stopping times σ_i, $i \geq 0$, by

$$\sigma_0 = 0, \quad \sigma_{i+1} = \inf\{s \geq \sigma_i \ : \ R(X_s, X_{\sigma_i}) > r/n\}.$$

Let $\tau_i = \sigma_i - \sigma_{i-1}$, $i \geq 1$. Let $\mathcal{F}_t$ be the filtration generated by $\{X_s : s \leq t\}$ and $\mathcal{G}_m = \mathcal{F}_{\sigma_m}$. Our linear bound gives

$$\begin{aligned}
\mathbf{P}_x(\tau_{i+1} \leq t | \mathcal{G}_i) &\leq \mathbf{P}_{X_{\sigma_i}}(T_{B(X_{\sigma_i}, r/n)} \leq t) \\
&\leq 1 - c_{2.26} g(r/n)^{3+2b+2\beta_u} + \frac{t}{h_u(r/n)} \\
&= p(r/n) + \frac{t}{h_u(r/n)},
\end{aligned}$$

where $p(r) := 1 - c_{2.26} g(r)^{\gamma_1} \in (\frac{1}{2}, 1)$ for $r > 0$. By continuity, we have $R(X_{\sigma_i}, X_{\sigma_{i+1}}) = r/n$ and so $R(X_0, X_t) \leq r$, for every $t \in [0, \sigma_n]$, which means that $\sigma_n = \sum_{i=1}^n \tau_i \leq T_{B(X_0, r)}$. Thus, by [10], Lemma 1.1,

$$\begin{aligned}
\ln \mathbf{P}_x(T_{B(x,r)} \leq t) &\leq 2\sqrt{\frac{nt}{p(r/n) h_u(r/n)}} - n \ln \frac{1}{p(r/n)} \\
&\leq 4\sqrt{\frac{nt}{h_u(r/n)}} - c_{2.26} n g(r/n)^{\gamma_1},
\end{aligned}$$

where we have used the inequality $\ln(1-x) \leq -x$ for $x \in [0, 1)$.

Let $c_q = \frac{c_{2.26}^2}{64}$ so that q is fixed. Now q may be rewritten as

$$q(r) = c_q f_l(r)^{2\gamma_1} f_u(r)^{1-2\gamma_1} V(r).$$

Since $2\gamma_1 > 0 > 1 - 2\gamma_1$, each of the terms in the product is increasing and strictly positive, with V strictly increasing. Thus q is strictly increasing and q^{-1} may be defined sensibly on the appropriate domain.

We consider first the case $r \geq q^{-1}(t/r)$. Define

$$\begin{aligned}
n_0 &:= \sup\{n : 8\sqrt{\frac{nt}{h_u(r/n)}} \leq c_{2.26} n g(r/n)^{\gamma_1}\} \\
&= \sup\{n : n q^{-1}(t/r) \leq r\}.
\end{aligned}$$

By assumption, we have $n_0 \geq 1$ and because $q^{-1}(t/r) > 0$ we must also have $n_0 < \infty$. Thus

$$n_0 \leq \frac{r}{q^{-1}(t/r)} < n_0 + 1,$$

from which it follows that

$$\begin{aligned}
\ln \mathbf{P}_x(T_{B(x,r)} \leq t) &\leq -c_{2.27} \left(\frac{r}{q^{-1}(t/r)} - 1 \right) g(q^{-1}(t/r))^{\gamma_1} \\
&\leq -c_{2.27} \left(\frac{r}{q^{-1}(t/r)} \right) g(q^{-1}(t/r))^{\gamma_1} + c_{2.27},
\end{aligned}$$

which yields the result in this case. If $r < q^{-1}(t/r)$ then

$$\frac{r}{q^{-1}(t/r)} g(q^{-1}(t/r))^{\gamma_1} \leq 1,$$

and so we have the result by choosing $c_{2.24}$ sufficiently large. $\square$

We are now ready to prove the on-diagonal lower bound. In the proof, we will use the following observation, which is an immediate consequence of Lemma 2.6.3: there is a constant $c_{2.28}$ such that

$$g(\lambda r) \geq c_{2.28} \lambda^{2b} g(r), \qquad \forall \lambda \in [0,1], r \in [0, R_X). \tag{2.27}$$

Proposition 2.6.8 *There exists a constant $c_{2.29}$ such that*

$$p_t(x,x) \geq c_{2.29} \frac{h^{-1}(t)}{t} g(h^{-1}(t))^{\theta_1}, \qquad \forall x \in X, t > 0,$$

where θ_1 is chosen to satisfy (2.7).

Proof: Using Cauchy-Schwarz,

$$\begin{aligned} \mathbf{P}_x(T_{B(x,r)} > t)^2 &\leq \mathbf{P}_x(X_t \in B(x,r))^2 \\ &= \left(\int_{B(x,r)} p_t(x,z) \mu(dz) \right)^2 \\ &\leq V(x,r) p_{2t}(x,x) \\ &\leq V_u(r) p_{2t}(x,x). \end{aligned} \tag{2.28}$$

We prove the result by choosing a suitable r in this inequality. We shall consider the cases for small and large t separately. Define

$$\gamma_2 := \frac{\theta_1 - 2\gamma_1}{\beta_u + 4b\gamma_1}.$$

We then have $\gamma_1 - \gamma_2(1 - 2b\gamma_1) < 0$. Noting that, if $g(r) \not\to 0$, $f_l(r)$ is bounded below by a strictly positive constant and $f_u(r)$ is bounded above by a finite constant. This means that we have uniform volume doubling and the result is given in [41], Proposition 4.3. Thus we may assume $g(r) \to 0$ as $r \to 0$, and so we can choose $r' < R_X/2$ such that

$$c_{2.24} e^{-c_{2.30} g(r)^{\gamma_1 - \gamma_2(1-2b\gamma_1)}} \leq \frac{1}{2}, \qquad \forall r \leq r',$$

where $c_{2.30} := c_{2.25} c_{2.28}$. Now, define $t' := r' q(r' g(r')^{\gamma_2})$. For $t \leq t'$ we can find $r \leq r'$ such that $t = r q(r g(r)^{\gamma_2})$, and use Lemma 2.6.7 to deduce that

$$\mathbf{P}_x(T_{B(x,r)} \leq t) \leq c_{2.24} e^{-c_{2.25} g(r)^{-\gamma_2} g(r g(r)^{\gamma_2})^{\gamma_1}} \leq c_{2.24} e^{-c_{2.30} g(r)^{\gamma_1 - \gamma_2(1-2b\gamma_1)}} \leq \frac{1}{2},$$

where we have applied the inequality at (2.27) for the second inequality. Thus (2.28) gives that $p_{2t}(x,x) \geq 1/4V_u(r)$. After substituting the definition of q and manipulating we find that

$$\begin{aligned} t &= c_q r V_u(rg(r)^{\gamma_2}) g(rg(r)^{\gamma_2})^{2\gamma_1} \\ &\geq c_{2.31} r g(r)^{\theta_1} V_u(r), \end{aligned}$$

and hence $p_{2t}(x,x) \geq c_{2.31} r g(r)^{\theta_1}/4t$. We also have

$$t \leq c_q r V_u(r) g(r)^{2\gamma_1} \leq c_q r V(r) g(r)^{2\gamma_1 - 1} \leq c_q h(r) \leq h(c_{2.32} r),$$

noting that $2\gamma_1 > 1$ and taking $c_{2.32} = \max\{1, c_q\}$. Consequently, $h^{-1}(t) \leq c_{2.32} r$ and so

$$p_t(x,x) \geq p_{2t}(x,x) \geq c_{2.33} \frac{h^{-1}(t)}{t} g(h^{-1}(t))^{\theta_1},$$

using that $p_t(x,x)$ is decreasing in t. Hence we have the bound for $t \leq t'$.

Before proceeding we note that $rg(r)^{-\gamma_1} = O(r^{1-2\varepsilon\gamma_1}) \to 0$, as $r \to 0$, because, by the bound on ε and b at (2.6), $2\varepsilon\gamma_1 < 1$. Therefore, we can choose $\tilde{r}$ less than 1 such that

$$c_{2.24} e^{-c_{2.25} \frac{1}{\tilde{r}} g(\tilde{r})^{\gamma_1}} \leq \frac{1}{2}.$$

Choose $t'' := q(\tilde{r})$. Now let $t \geq t''$ and define r by $t = rq(r\tilde{r})$. The right hand side of this equation is increasing and so, because t is bounded below (by t''), we can assume that r is bounded below by 1. Hence applying Lemma 2.6.7 gives

$$\begin{aligned} \mathbf{P}_x(T_{B(x,r)} \leq t) &\leq c_{2.24} e^{-c_{2.25} \frac{r}{q^{-1}(t/r)} g(q^{-1}(t/r))^{\gamma_1}} \\ &\leq c_{2.24} e^{-c_{2.25} \frac{r}{q^{-1}(t/r)} g(q^{-1}(t/r)/r)^{\gamma_1}} \\ &= c_{2.24} e^{-c_{2.25} \frac{1}{\tilde{r}} g(\tilde{r})^{\gamma_1}} \\ &\leq \frac{1}{2}. \end{aligned}$$

Hence we also have $p_{2t}(x,x) \geq 1/4V_u(r)$ in this case, by (2.28). By bounding t in a similar way to the case $t \leq t'$ it may be deduced from this that

$$p_t(x,x) \geq c_{2.34} \frac{h^{-1}(t)}{t} g(h^{-1}(t))^{2\gamma_1},$$

and so we have the bound in this case, because $2\gamma_1 \leq \theta_1$. Finally, for $t \in (t', t'')$ we may obtain the result by choosing $c_{2.29}$ small enough. □

We conclude this section by proving the fluctuation results of Theorem 2.3.2.

Proof of Theorem 2.3.2: The left hand inequality of (2.10) and the right hand inequality of (2.11) are immediate corollaries of Propositions 2.6.1 and 2.6.8. We now prove the right hand inequality of (2.10). As at (2.18), we repeat the argument of [41] to obtain

$$p_{2rV(x,r)}(x,x) \leq \frac{2}{V(x,r)}, \qquad \forall x \in X, r \in [0, R_X). \tag{2.29}$$

Hence, because $p_t(x,x)$ is decreasing in t, this means that

$$\inf_{x\in X} p_{2h_u(r)}(x,x) \leq \inf_{x\in X} p_{2rV(x,r)}(x,x) \leq \frac{2}{\sup_{x\in X} V(x,r)} \leq \frac{c_{2.35}}{V_u(r)},$$

for all $r \in [0, R_X)$, where we use the assumption at (2.9) for the final inequality. Setting $r = h_u^{-1}(t/2)$, we obtain $\inf_{x\in X} p_t(x,x) \leq c_{2.36} h_u^{-1}(t)/t$, which gives the result.

It remains to prove the left hand inequality of (2.11). The majority of the proof of this consists of repeating arguments that are almost identical to those we have seen already, and so we omit many of the details here. By the assumption at (2.8), we can find a sequence $(x_n, r_n)_{n\in\mathbb{N}}$ such that $x_n \in X$, $r_n \to 0$ and $V(x_n, r_n) \leq c_{2.37} V_l(r_n)$. By proceeding similarly to the proofs of Lemmas 2.6.4 and 2.6.5, it may be deduced that

$$c_{2.38} r_n \leq R(x_n, B(x_n, r_n)^c) \leq r_n, \qquad \forall n \in \mathbb{N}.$$

Using this result, by following the argument of Proposition 2.6.6, we find that

$$\mathbf{E}_{x_n} T_{B(x_n,r_n)} \geq c_{2.39} h_l(r_n) \qquad \forall n \in \mathbb{N},$$

and

$$\mathbf{E}_x T_{B(x_n,r_n)} \leq c_{2.37} h_l(r_n), \qquad \forall x \in X, n \in \mathbb{N}.$$

Thus, by utilising the Markov property of $(X_t)_{t\geq 0}$ as at (2.26), it follows that

$$\mathbf{P}_{x_n}(T_{B(x_n,r_n)} \leq t) \leq 1 - \frac{c_{2.39}}{c_{2.37}} + \frac{t}{c_{2.37} h_l(r_n)},$$

and, in particular,

$$\mathbf{P}_{x_n}\left(T_{B(x_n,r_n)} \leq \frac{c_{2.39}}{2} h_l(r_n)\right) \leq 1 - \frac{c_{2.39}}{2c_{2.37}} < 1, \qquad \forall n \in \mathbb{N}.$$

The Cauchy-Schwarz equation at (2.28) applied to x_n, r_n and $t_n = c_{2.39} h_l(r_n)/2$ will then imply that

$$\sup_{x\in X} p_{t_n}(x,x) \geq p_{t_n}(x_n,x_n) \geq \frac{c_{2.40}}{V(x_n,r_n)} \geq \frac{c_{2.40}}{c_{2.37} V_l(r_n)} \geq \frac{c_{2.41} h_l^{-1}(t_n)}{t_n}.$$

Noting that $t_n \to 0$, this completes the proof. $\square$

2.7 Proof of off-diagonal heat kernel bounds

Throughout this section, we shall be assuming the extra anti-doubling condition on the volume growth, (2.12), and the tighter upper bound on b and ε, (2.13), that was stated in Section 2.4. These restrictions allow us to obtain the off-diagonal estimates stated there. We start by presenting a counterpart to Lemma 2.6.2 for small λ, which the extra volume growth condition implies.

Lemma 2.7.1 *Let $\lambda \leq 1$, then $V(\lambda r) \leq C_l \lambda^{\beta_l} V(r)$, for every $r \leq R'_X$.*
Proof: This follows a similar argument to the proof of Lemma 2.6.2. □

As is usually the case in situations similar to this, the off-diagonal upper bound is relatively straightforward to obtain from the upper bounds for the on-diagonal part of the heat kernel and the tail of the exit time distribution of resistance balls. However, before proceeding with the proof of the off-diagonal upper bound, it will be useful to write the result of Lemma 2.6.7 in a slightly clearer form.

Proposition 2.7.2 *If $R'_X = \infty$, let $t_5 = \infty$, otherwise fix $t_5 \in (0, \infty)$. Then there exist constants $c_{2.42}, c_{2.43}$ such that*

$$\mathbf{P}_x(T_{B(x,r)} \leq t) \leq c_{2.42} e^{-c_{2.43} \frac{r}{V^{-1}(t/r)} g(V^{-1}(t/r))^{\theta_3}}, \quad \forall x \in X, r \in (0, R_X), t \in (0, t_5).$$

Proof: In Lemma 2.6.7 we obtained a bound for the relevant probability in terms of the function q^{-1} when $r \in (0, R_X/2)$. This result is easily extended to $r \in (0, R_X)$ by adjusting the constants as necessary, and we shall take this as our starting point. To establish the proposition, we use Lemma 2.7.1 to compare q^{-1} to functions of V^{-1} and g only. Recall $q(r) = c_q g(r)^{2\gamma_1} V_u(r)$, and so for $r \leq R'_X$, we have $q(r) \geq V(c_{2.44} r g(r)^{2\gamma_1/\beta_l})$, for some constant $c_{2.44}$. Thus

$$V^{-1}(t/r) \geq c_{2.44} q^{-1}(t/r) g(q^{-1}(t/r))^{2\gamma_1/\beta_l}, \tag{2.30}$$

for $t/r \leq q(R'_X)$. We also have the following upper bound on q

$$q(r) \leq c_q V(r) g(r)^{2\gamma_1 - 1} \leq c_q V(r) \leq V(c_{2.45} r),$$

where $c_{2.45} = \max\{(c_q C_l)^{1/\beta_l}, 1\}$, which holds whenever $c_{2.45} r \leq R'_X$. Thus

$$V^{-1}(t/r) \leq c_{2.45} q^{-1}(t/r), \tag{2.31}$$

for $t/r \leq q(c_{2.45}^{-1}R'_X)$. Combining the bounds at (2.30) and (2.31) we find that

$$\frac{r}{q^{-1}(t/r)}g(q^{-1}(t/r))^{\gamma_1} \geq c_{2.46}\frac{r}{V^{-1}(t/r)}g(V^{-1}(t/r))^{\gamma_1(1+2\beta_l^{-1})}$$

for all $t/r \leq q(c_{2.47}R'_X)$, where $c_{2.47} := \min\{c_{2.45}^{-1}, 1\}$. Thus we have the result when $R'_X = \infty$. Assume now $R'_X < \infty$ and fix $t_5 < \infty$. The previous equation gives us the result when $t/r \leq q(c_{2.47}R'_X)$ and so we can assume that this does not hold. Hence

$$\frac{r}{V^{-1}(t/r)}g(V^{-1}(t/r))^{\theta_3} \leq \frac{t_5}{q(c_{2.47}R'_X)V^{-1}(q(c_{2.47}R'_X))}, \qquad \forall t < t_5,$$

and so the result will hold on choosing $c_{2.42}$ suitably large. $\square$

Proposition 2.7.3 *We can find a $t_6 > 0$ such that the following holds: there exist constants $c_{2.48}, c_{2.49}$ such that, if $x, y \in X$, $t \in (0, t_6)$,*

$$p_t(x,y) \leq c_{2.48}\frac{h^{-1}(t)}{t}f_l(h^{-1}(t))^{-1}e^{-c_{2.49}\frac{R}{V^{-1}(t/R)}g(V^{-1}(t/R))^{\theta_3}},$$

where $R = R(x,y)$. If $R'_X = \infty$, then we can take $t_6 = \infty$, otherwise $t_6 \in (0,\infty)$.

Proof: Once we have the on-diagonal bound, Lemma 2.6.1, and the exponential bound for the exit time distribution, Lemma 2.7.2, the proof is standard, see [9], Theorem 3.11. $\square$

We now start to work towards the full lower bound. The first step is deducing a near diagonal result using a modulus of continuity argument. This is the extension of the result obtained by Kumagai in the uniform volume doubling case, as stated at (2.4).

Lemma 2.7.4 *There exist constants $c_{2.50}, c_{2.51}$ such that, whenever $x, y \in X$ satisfy*

$$R(x,y) \leq c_{2.50}h^{-1}(t)g(h^{-1}(t))^{\theta_1},$$

we have

$$p_t(x,y) \geq c_{2.51}\frac{h^{-1}(t)}{t}g(h^{-1}(t))^{\theta_1}, \qquad \forall t > 0.$$

Proof: The proof is again standard. For any $x \in X$, $t > 0$, it is known that the transition density satisfies $\mathcal{E}(p_t(x,\cdot), p_t(x,\cdot)) \leq p_t(x,x)/t$. For a proof, see [9], Proposition 4.16. In conjunction with the inequality at (2.17), we obtain from this that

$$|p_t(x,x) - p_t(x,y)|^2 \leq R(x,y)\mathcal{E}(p_t(x,\cdot), p_t(x,\cdot)) \leq R(x,y)\frac{p_t(x,x)}{t}.$$

Thus

$$\begin{aligned} p_t(x,y) &\geq p_t(x,x) - |p_t(x,x) - p_t(x,y)| \\ &\geq p_t(x,x)\left(1 - \sqrt{\frac{R(x,y)}{tp_t(x,x)}}\right) \\ &\geq \frac{1}{2}p_t(x,x), \end{aligned}$$

whenever $4R(x,y) \leq tp_t(x,x)$. Consequently, the result may be obtained by applying the on-diagonal lower bound obtained in Proposition 2.6.8. $\square$

To prove the full lower bound we shall assume the chaining condition as defined in Section 2.4. We shall use the standard chaining argument to extend the near diagonal lower bound to the full bound. The main complication caused by the perturbations is in choosing a suitable number of pieces into which to break the path. The aim of the following lemma is to check that the number that we do choose is sensibly defined.

Lemma 2.7.5 *Fix $c_{2.52}$. Let $x, y \in X$ and $t > 0$. If we define $N = N(x,y,t)$ by*

$$N := \inf\{n \in \mathbb{N} : \frac{R(x,y)}{n} \leq c_{2.52} h^{-1}(t/n) g(h^{-1}(t/n))^{\theta_1}\},$$

then N is well-defined and finite for each pair $x, y \in X$.

Proof: Note first that $h^{-1}(t)/t = 1/V(h^{-1}(t))$, so we can rewrite N as

$$N = \inf\{n \in \mathbb{N} : \frac{R(x,y)}{t} \leq \frac{c_{2.52}}{V(h^{-1}(t/n))} g(h^{-1}(t/n))^{\theta_1}\}.$$

It is clear that $h^{-1}(t/n) \to 0$ as $n \to \infty$ and so, to prove the lemma, it suffices to show that $V(r)g(r)^{-\theta_1} \to 0$ as $r \to 0$. By Lemma 2.7.1 we have

$$V(r)g(r)^{-\theta_1} \leq C_l V(rg(r)^{-\theta_1/\beta_l}),$$

for $rg(r)^{-\theta_1/\beta_l} \leq R'_X$. We note that, using the assumptions of Sections 2.2 and 2.4, we have $rg(r)^{-\theta_1/\beta_l} = O(r^{1-2\varepsilon\theta_1/\beta_l}) \to 0$ as $r \to 0$, and so the result does indeed hold. $\square$

We are now ready to state and prove the full lower bound. We now assume that the chaining condition, (CC), holds.

Proposition 2.7.6 *There exist constants $t_7 > 0$ and $c_{2.53}, c_{2.54}$ such that, if $x, y \in X$, $t \in (0, t_7)$,*

$$p_t(x,y) \geq c_{2.53} \frac{h^{-1}(t)}{t} g(h^{-1}(t))^{\theta_1} e^{-c_{2.54} \frac{R}{V^{-1}(t/R)} g(V^{-1}(t/R))^{-\theta_2}},$$

where $R = R(x,y)$. If $R'_X = \infty$ then we may take $t_7 = \infty$, otherwise t_7 will be finite.

Proof: Let $x, y \in X$ and $R = R(x,y)$. If $R \leq c_{2.50} h^{-1}(t) g(h^{-1}(t))^{\theta_1}$ then we have the result by Lemma 2.7.4 immediately. Thus we need only consider the case $R > c_{2.50} h^{-1}(t) g(h^{-1}(t))^{\theta_1}$. We shall apply a standard chaining argument, using the previous lemma to select the length of the path. Define

$$N = \inf\{n \in \mathbb{N} : \frac{R}{n} \leq \frac{c_{2.50}}{3c_{2.9}} h^{-1}(t/n) g(h^{-1}(t/n))^{\theta_1}\}.$$

Lemma 2.7.5 and the assumption on R imply that $N \in (1, \infty)$. By the chaining condition we can find a path $x = x_0, x_1, \ldots, x_N = y$ such that

$$R(x_{i-1}, x_i) \leq \frac{c_{2.9} R}{N}, \qquad i = 1, \ldots, N.$$

If we set $\delta = c_{2.50} h^{-1}(t/N) g(h^{-1}(t/N))^{\theta_1}$, then by the definition of N, this inequality implies that $R(x_{i-1}, x_i) \leq \delta/3$, for $i = 1, \ldots, N$. Thus, if $z_i \in B(x_i, \delta/3)$, we have

$$R(z_{i-1}, z_i) \leq \delta, \qquad i = 1, \ldots, N,$$

and so we may apply the near diagonal estimate to obtain

$$p_{t/N}(z_{i-1}, z_i) \geq c_{2.51} \frac{N h^{-1}(t/N)}{t} g(h^{-1}(t/N))^{\theta_1}. \tag{2.32}$$

This is the first ingredient that we shall require to apply the chaining argument. The other is a lower bound on the measures of the balls $B(x_i, \delta/3)$. Using the assumption (2.5),

$$\begin{aligned} V(x_i, \delta/3) &\geq V_l(\delta/3) \\ &= V_l(c_{2.50} h^{-1}(t/N) g(h^{-1}(t/N))^{\theta_1}/3) \\ &\geq c_{2.55} V(h^{-1}(t/N)) g(h^{-1}(t/N))^{1+\theta_1(b+\beta_u)}, \end{aligned} \tag{2.33}$$

where we have applied Lemmas 2.6.2 and 2.6.3 to obtain the second inequality.

By using the Chapman-Kolmogorov equation for the transition densities of the process X we obtain the following chaining inequality

$$p_t(x,y) \geq \int_{B(x_1, \delta/3)} \mu(dz_1) \ldots \int_{B(x_{N-1}, \delta/3)} \mu(dz_{N-1}) \prod_{i=1}^{N} p_{t/N}(z_{i-1}, z_i).$$

If we then combine this with the bounds at (2.32) and (2.33) we obtain

$$\begin{aligned} p_t(x,y) &\geq c_{2.51} \frac{N h^{-1}(t/N)}{t} g(h^{-1}(t/N))^{\theta_1} \\ &\qquad \times \left(c_{2.51} c_{2.55} g(h^{-1}(t/N))^{1+\theta_1(b+\beta_u+1)} \right)^{N-1}, \end{aligned}$$

where we have used the identity $h^{-1}(t)/t = 1/V(h^{-1}(t))$. The definition of N and the assumption that $R > c_{2.50}h^{-1}(t)g(h^{-1}(t))^{\theta_1}$ may be combined to give

$$\frac{Nh^{-1}(t/N)}{t}g(h^{-1}(t/N))^{\theta_1} \geq 3c_{2.9}\frac{h^{-1}(t)}{t}g(h^{-1}(t))^{\theta_1},$$

yielding

$$p_t(x,y) \geq c_{2.56}\frac{h^{-1}(t)}{t}g(h^{-1}(t))^{\theta_1}e^{-c_{2.57}N(1-c_{2.58}\ln g(h^{-1}(t/N)))}. \tag{2.34}$$

To complete the argument we look for bounds on the terms involving N. Since we know that $N > 1$ we can deduce, because $h^{-1}(t)$ is increasing,

$$\frac{R}{N} = \frac{R}{N-1}\frac{N-1}{N} \geq \frac{c_{2.50}}{6c_{2.9}}h^{-1}(t/N)g(h^{-1}(t/N))^{\theta_1},$$

which we can rewrite as

$$\frac{t}{R} \leq c_{2.59}V(h^{-1}(t/N))g(h^{-1}(t/N))^{-\theta_1}. \tag{2.35}$$

Since $g(r)^{-1} = O(r^{-2\varepsilon})$ and $2\varepsilon\theta_1/\beta_l < 1$, we can find a $t_7 > 0$ such that

$$c_{2.60}h^{-1}(t)g(h^{-1}(t))^{-\theta_1/\beta_l} \leq R'_X, \qquad \forall t < t_7, \tag{2.36}$$

where $c_{2.60} := \max\{C_l^{1/\beta_l}, (C_l c_{2.59})^{1/\beta_l}\}$. Note that if $R'_X = \infty$ we may take $t_7 = \infty$. Clearly this also implies that $c_{2.60}h^{-1}(t/N)g(h^{-1}(t/N))^{-\theta_1/\beta_l} \leq R'_X$, for $t < t_7$. Thus applying Lemma 2.7.1 to (2.35) gives

$$\frac{t}{R} \leq V(c_{2.60}h^{-1}(t/N)g(h^{-1}(t/N))^{-\theta_1/\beta_l}), \tag{2.37}$$

and so

$$g(\tilde{V}) \leq c_{2.61}g(h^{-1}(t/N))^{1-2b\theta_1/\beta_l}, \tag{2.38}$$

where $\tilde{V} := V^{-1}(t/R)$. By using this in (2.35) we find

$$V(h^{-1}(t/N)) \geq c_{2.62}\frac{t}{R}g(\tilde{V})^{\theta_1\beta_l/(\beta_l-2b\theta_1)},$$

and moreover, equations (2.36) and (2.37) imply that $\tilde{V} \leq R'_X$. These facts allow us to deduce, after some manipulation and the use of Lemma 2.7.1, that

$$h^{-1}(t/N) \geq V^{-1}(c_{2.62}\frac{t}{R}g(\tilde{V})^{\theta_1\beta_l/(\beta_l-2b\theta_1)}) \geq c_{2.63}\tilde{V}g(\tilde{V})^{\theta_1/(\beta_l-2b\theta_1)}.$$

Consequently,

$$\frac{t}{N} \geq c_{2.62}c_{2.63}\frac{t}{R}\tilde{V}g(\tilde{V})^{\theta_1(\beta_l+1)/(\beta_l-2b\theta_1)},$$

which is equivalent to

$$N \leq c_{2.64}\frac{R}{\tilde{V}}g(\tilde{V})^{-\theta_1(\beta_l+1)/(\beta_l-2b\theta_1)}. \tag{2.39}$$

Substituting the bounds of (2.38) and (2.39) into the lower bound we established at (2.34) yields

$$\begin{aligned} p_t(x,y) &\geq c_{2.56}\frac{h^{-1}(t)}{t}g(h^{-1}(t))^{\theta_1}e^{-c_{2.65}\frac{R}{\tilde{V}}g(\tilde{V})^{-\theta_1(\beta_l+1)/(\beta_l-2b\theta_1)}(1-c_{2.66}\ln g(\tilde{V}))}, \\ &\geq c_{2.56}\frac{h^{-1}(t)}{t}g(h^{-1}(t))^{\theta_1}e^{-c_{2.67}\frac{R}{\tilde{V}}g(\tilde{V})^{-\theta_2}}, \qquad \forall t < t_7, \end{aligned}$$

which is the desired result. □

2.8 Local fluctuations

In [34], Hambly and Kumagai demonstrated that for certain random recursive Sierpinski gaskets, as well as spatial fluctuations, the heat kernel will undergo fluctuations in time μ-almost-everywhere in X. In this section, we look to generalise this result by showing that these local fluctuations in the heat kernel result from local fluctuations in the measure.

Again, we shall be working with the measure-metric space (X, d, μ) and the volume function V. We shall denote the local fluctuations by $\tilde{f}_l$ and $\tilde{f}_u$ and assume that these satisfy the same properties as did f_l and f_u respectively. In fact, the results proved here may be obtained using slightly weaker assumptions, but we omit these for brevity. We shall use $\tilde{V}_l(r)$, $\tilde{V}_u(r)$, $\tilde{h}_l(r)$ and $\tilde{h}_u(r)$ to denote $\tilde{f}_l(r)V(r)$, $\tilde{f}_u(r)V(r)$, $r\tilde{V}_l(r)$ and $r\tilde{V}_u(r)$ respectively.

For the following theorem, we impose conditions on the point-wise behaviour of the volume growth. Since these assumptions do not imply uniform volume bounds, we cannot establish a lower bound on $R(x, B(x,r)^c)$ in the way we did in Lemma 2.6.5. As we need some kind of global control on this, we simply suppose that it is bounded below by a multiple of r. Note that this is a stricter condition than the one established at Lemma 2.6.5 when we had global bounds on the measure.

Theorem 2.8.1 *If*

$$0 < \liminf_{r\to 0}\frac{V(x,r)}{\tilde{V}_l(r)} < \infty, \qquad 0 < \limsup_{r\to 0}\frac{V(x,r)}{\tilde{V}_u(r)},$$

and
$$0 < \liminf_{r\to 0} \frac{R(x, B(x,r)^c)}{r} \tag{2.40}$$
for μ-almost-every $x \in X$; then
$$\liminf_{t\to 0} \frac{tp_t(x,x)}{\tilde{h}_u^{-1}(t)} < \infty, \tag{2.41}$$
and
$$0 < \limsup_{t\to 0} \frac{tp_t(x,x)}{\tilde{h}_l^{-1}(t)} < \infty, \tag{2.42}$$
for μ-almost-every $x \in X$.

Proof: The bound at (2.41) is proved by applying the inequality at (2.29) in exactly the same way as in the proof of the corresponding global bound. A similar argument is also used to prove the upper bound of (2.42).

The assumption on R at (2.40) allows us to deduce that for μ-almost-every $x \in X$, there exists a sequence $r_n \to 0$ such that
$$\mathbf{E}_x T_{B(x,r_n)} \geq c_{2.68} \tilde{h}_l(r_n), \qquad \forall n \in \mathbb{N},$$
and
$$\mathbf{E}_y T_{B(x,r_n)} \leq c_{2.69} \tilde{h}_l(r_n), \qquad \forall y \in X, n \in \mathbb{N},$$
by following the argument of Proposition 2.6.6. The result at (2.42) follows from this by applying the Markov property of our process and the Cauchy-Schwarz inequality as we did for the analogous global bound. □

Remark 2.6 *Using the techniques of this chapter, it is not enough to assume that $\limsup_{r\to 0}(V(x,r)/\tilde{V}_u(r)) < \infty$ to establish a lower bound on $p_t(x,x)$ that holds for all small t. The problem arises because we are unable to emulate the chaining argument that was used in Proposition 2.6.7 to establish an exponential tail for the distribution of the exit time from a ball.*

Remark 2.7 *Similar to the remark made after Theorem 2.3.2, we note there are non-trivial local fluctuations in the measure if and only if $\tilde{V}_u(r)/\tilde{V}_l(r) \to \infty$ as $r \to 0$. This is equivalent to $\tilde{h}_l^{-1}(t)/\tilde{h}_u^{-1}(t) \to \infty$ as $t \to 0$, which implies that there are non-trivial local fluctuations in the heat kernel.*

2.9 Examples

In this section, to illustrate the results, we look at two specific examples of correction terms and present the conclusions for two different types of random sets. In Examples 2.1 and 2.2 we shall take $V(r) = r^\alpha$ for some $\alpha > 0$, so that $\beta_l = \beta_u = \alpha$. For simplicity, we assume that $R_X = \infty$ and the chaining condition holds. In this case, we have $V^{-1}(t) = t^{1/\alpha}$ and $h^{-1}(t) = t^{1/(1+\alpha)}$. Furthermore, in the case of uniform volume growth with volume doubling, we can use the results of Kumagai to show that

$$c_{2.70} t^{-\frac{\alpha}{\alpha+1}} e^{-c_{2.71}\left(\frac{R^{\alpha+1}}{t}\right)^{1/\alpha}} \leq p_t(x,y) \leq c_{2.72} t^{-\frac{\alpha}{\alpha+1}} e^{-c_{2.73}\left(\frac{R^{\alpha+1}}{t}\right)^{1/\alpha}}$$

for this choice of volume growth function.

Example 2.1 *Polynomial corrections*

We first discuss the case of arbitrary polynomial corrections. We shall assume that given $\delta > 0$, there exist constants $c_{2.74}, c_{2.75}$ such that

$$c_{2.74} r^\alpha (r^\delta \wedge 1) \leq V(x,r) \leq c_{2.75} r^\alpha (r^{-\delta} \vee 1), \qquad \forall x \in X, r \geq 0,$$

so that $f_l(r) = c_{2.74}(r^\delta \wedge 1)$ and $f_u(r) = c_{2.75}(r^{-\delta} \vee 1)$. If we set $\varepsilon = b = \delta$, then f_l and f_u satisfy the conditions for the full bounds when $\delta < \alpha/8(3+\alpha)^2$. We can then also choose

$$\theta_1 = 4(2+\alpha)^2, \quad \theta_2 = \frac{4(2+\alpha)^3}{\alpha - 8\delta(2+\alpha)^2}, \quad \theta_3 = (3 + 2\delta + 2\alpha)(1 + 2\alpha^{-1}),$$

and apply Theorem 2.4.1 to obtain that

$$c_{2.76} t^{-\frac{\alpha - 2\delta\theta_1}{\alpha+1}} e^{-c_{2.77}\left(\frac{R^{1+\alpha-2\delta\theta_2}}{t^{1-2\delta\theta_2}}\right)^{1/\alpha}} \leq p_t(x,y) \leq c_{2.78} t^{-\frac{\alpha+\delta}{\alpha+1}} e^{-c_{2.79}\left(\frac{R^{1+\alpha+2\delta\theta_3}}{t^{1+2\delta\theta_3}}\right)^{1/\alpha}},$$

for appropriate t, x, y. We note that $\delta, 2\delta\theta_1, 2\delta\theta_2, 2\delta\theta_3 \to 0$ as $\delta \to 0$, and so, by taking δ small enough, we can write down bounds with arbitrarily small polynomial correction terms.

Example 2.2 *Logarithmic fluctuations*

Assume now that

$$0 < \liminf_{r\to 0} \inf_{x \in X} \frac{V(x,r)}{V(r)(\ln r^{-1})^{-a_1}} \leq \limsup_{r \to 0} \inf_{x\in X} \frac{V(x,r)}{V(r)(\ln r^{-1})^{-a_1}} < \infty, \qquad (2.43)$$

and

$$0 < \liminf_{r\to 0} \sup_{x \in X} \frac{V(x,r)}{V(r)(\ln r^{-1})^{a_2}} \leq \limsup_{r \to 0} \sup_{x\in X} \frac{V(x,r)}{V(r)(\ln r^{-1})^{a_2}} < \infty; \qquad (2.44)$$

for some $a_1, a_2 \in (0,\infty)$. As we noted in Section 2.1, this is an example that arises naturally in the random recursive fractal setting. We have $f_l(r) = c_{2.80}(\ln r^{-1})^{-a_1}$ and $f_u(r) = c_{2.81}(\ln r^{-1})^{a_2}$, which satisfy the conditions for any $\varepsilon, b > 0$. Thus, by applying Theorem 2.4.1, we can deduce full heat kernel bounds with $\theta_1, \theta_2, \theta_3$ arbitrarily close to the lower bounds of

$$\theta_1 > (3+2\alpha)(2+\alpha),\ \theta_2 > \frac{(3+2\alpha)(2+\alpha)(1+\alpha)}{\alpha},\ \theta_3 > (3+2\alpha)(1+2\alpha^{-1}),$$

as long as θ_1, θ_2 satisfy (2.15). Thus our results show that the correction terms in the heat kernel will be of logarithmic order. In fact, because we know the functions explicitly, by repeating the same arguments as in previous sections more carefully, we can improve these exponents. Theorem 2.3.2 allows us to deduce that the on-diagonal part of the heat kernel satisfies

$$0 < \liminf_{t\to 0} \inf_{x\in X} \frac{p_t(x,x)}{t^{-\frac{\alpha}{\alpha+1}}(\ln t^{-1})^{-\frac{\alpha(2\alpha+3)(\alpha+2)a_0+a_2}{\alpha+1}}}, \tag{2.45}$$

$$\limsup_{t\to 0} \inf_{x\in X} \frac{p_t(x,x)}{t^{-\frac{\alpha}{\alpha+1}}(\ln t^{-1})^{-\frac{a_2}{\alpha+1}}} < \infty$$

and

$$0 < \liminf_{t\to 0} \sup_{x\in X} \frac{p_t(x,x)}{t^{-\frac{\alpha}{\alpha+1}}(\ln t^{-1})^{-\frac{a_1}{\alpha+1}}} \le \limsup_{t\to 0} \sup_{x\in X} \frac{p_t(x,x)}{t^{-\frac{\alpha}{\alpha+1}}(\ln t^{-1})^{-\frac{a_1}{\alpha+1}}} < \infty,$$

where $a_0 := a_1 + a_2$, and we have sharpened the exponent θ_1.

Example 2.3 *Random recursive Sierpinski gaskets*

We now compare the above results for logarithmic corrections to those that are known to hold for the random recursive Sierpinski gasket described in [33]. The gasket does not satisfy the chaining condition, but since we do not need this for the on-diagonal results, our results still apply. As noted in Section 2.1, for this gasket, the results of [33] may be adapted to show there are fluctuations in the measure of resistance balls of the type described at (2.43) and (2.44) for some $a_1, a_2 > 0$.

Our results for the asymptotics of $\sup_{x\in X} p_t(x,x)$ are tight and agree with those found in [34] by Hambly and Kumagai for these random sets. We also have that the upper bound on $\inf_{x\in X} p_t(x,x)$ agrees with the result proved there. We observe that the heat kernel bounds obtained for this gasket in [32] imply that

$$0 < \liminf_{t\to 0} \inf_{x\in X} \frac{p_t(x,x)}{t^{-\frac{\alpha}{\alpha+1}}(\ln t^{-1})^{-\frac{\alpha a_0+a_2}{\alpha+1}}},$$

and so the lower bound at (2.45) has a strictly worse exponent than is optimal. The main reason for this is that, because we have not taken into account the structure of the space, our lower bound on $R(x, B(x,r)^c)$ is not tight. Using results of [32], we deduce that $c_{2.82} r \leq R(x, B(x,r)^c)$ for this gasket, whereas Lemma 2.6.5 only allows us to obtain $c_{2.83} r (\ln r^{-1})^{-2a_0} \leq R(x, B(x,r)^c)$.

We note that, because $c_{2.82} r \leq R(x, B(x,r)^c)$, the local measure results proved in [33] may also be adapted to enable us to apply Theorem 2.8.1 to demonstrate there are fluctuations in time for the heat kernel on this gasket with $\tilde{f}_l(r) = c_{2.84}(\ln\ln r^{-1})^{-a_1}$ and $\tilde{f}_u(r) = c_{2.85}(\ln\ln r^{-1})^{a_2}$. That fluctuations of this kind exist was first proved in [34], and it may be readily observed that the bounds of Theorem 2.8.1 agree with the corresponding results of that paper. Finally, as was noted in the remark following Theorem 2.8.1, we are unable to establish a local lower bound for $p_t(x,x)$ for small t in the general case, whereas, by taking into account the specific structure of the sets involved, Hambly and Kumagai are able to do so in this particular example.

Example 2.4 *Random self-similar dendrites*

Supporting a resistance form, the dendrites of Chapter 1 fit neatly into the framework of this chapter, and the fact that the metric R that we constructed on these random sets is a shortest path metric means that the chaining condition is immediately satisfied. Note that, by [38], Lemma 5.7, because (T, R) is compact, it is separable, $\mathbf{P}$-a.s., and so all of the conditions on the metric space that we have assumed in this chapter are indeed satisfied. Moreover, if we assume that the measure of interest is the self-similar one, μ^α, as introduced in Section 1.8, then this also satisfies the conditions necessary to apply the results proved here, $\mathbf{P}$-a.s. In particular, under the assumptions of Theorem 1.10.3(a), we are able to obtain the full off-diagonal bounds for the heat kernel on the dendrite with arbitrarily small polynomial corrections, as in Example 2.1. Similarly, for the assumptions of Theorem 1.10.3(b), we are able to obtain the full off-diagonal bounds with only logarithmic corrections to the on-diagonal part and the exponential rate of decay of the heat kernel. In Chapter 3, we show that for the continuum random tree, which is a particular self-similar dendrite (see Appendix A), global logarithmic fluctuations in the heat kernel actually do occur, and also deduce local results similar to those of Section 2.8.

Chapter 3

Volume growth and heat kernel estimates for the continuum random tree

In this chapter, we prove global and local (point-wise) volume and heat kernel bounds for the continuum random tree. We demonstrate that there are almost-surely logarithmic global fluctuations and log-logarithmic local fluctuations in the volume of balls of radius r about the leading order polynomial term as $r \to 0$. We also show that the on-diagonal part of the heat kernel exhibits corresponding global logarithmic fluctuations as $t \to 0$ almost-surely, and provide a description of the local behaviour. Furthermore, we demonstrate that this quenched (almost-sure) behaviour contrasts with the local annealed (averaged over all realisations of the tree) volume and heat kernel behaviour, which is smooth. Finally, we explain how already established results about dendrites and measure-metric spaces may be applied to the continuum random tree to construct a process, which is the Brownian motion on the continuum random tree.

3.1 Background and statement of main results

The continuum random tree has, since its introduction by Aldous in [1], become an important object in modern probability theory. As well as being the scaling limit of a variety of discrete tree-like objects, see [1], [3], by a suitable random embedding into $\mathbb{R}^d$, it is possible to describe the support of the integrated super-Brownian excursion (ISE) using the continuum random tree ([4]). With growing evidence ([35]) to support the fact that the incipient infinite cluster of percolation in high dimensions at criticality scales to the ISE, we hope that the results proved here will eventually contribute to the understanding of the asymptotic behaviour of random walks on these lattice objects.

We shall denote by $\mathcal{T}$ the continuum random tree, which is a random set defined on an underlying probability space with probability measure $\mathbf{P}$ (we shall write $\mathbf{E}$ for the expectation under $\mathbf{P}$). It has a natural metric, $d_{\mathcal{T}}$, and a natural volume measure, μ. The existence of Brownian motion on $\mathcal{T}$, as defined in Section 5.2 of [2], has already been proved by Krebs, who constructed a process via a Dirichlet form on $\mathcal{T}$, which was defined as the limit of differential operators, and then Brownian motion was a time change of this process, see [40]. We provide an alternative construction, using the resistance form techniques developed by Kigami in [38] to define a local, regular Dirichlet form on the measure-metric space $(\mathcal{T}, d_{\mathcal{T}}, \mu)$. Given this Dirichlet form, standard results allow the construction of an associated Markov process, $X = (X_t)_{t\geq 0}$ with invariant measure μ, which we show is actually Brownian motion on $\mathcal{T}$. The construction used here seems more natural, allowing us to define the Dirichlet form for Brownian motion directly. Furthermore, the arguments we use to deduce our process satisfies the properties of Brownian motion are more concise, using more recently developed techniques for resistance forms, rather than limiting arguments.

Once a Markov process is established on $\mathcal{T}$, it is natural to ask whether it has a transition density, and if it does, what form does the transition density take? As discussed in Chapter 2, the current literature on measure-metric spaces equipped with a resistance form indicates that an important part of the answer to this question is the volume growth of the space with respect to the resistance metric. Consequently, an understanding of this volume growth for the continuum random tree is required for us to proceed.

It was also noted in the previous chapter that, for certain random subsets, the kind of uniform volume growth that is often assumed does not apply. In particular, for a class random recursive fractals, the volume of balls of radius r have fluctuations of order of powers of $\ln r^{-1}$ about a leading order polynomial term, r^{α}, [33]. With the random self-similarity of the continuum random tree ([5], and also Appendix A), it is reasonable to expect similar behaviour for the continuum random tree, and we shall prove this is the case in the course of this chapter. In fact, it has already been shown that the continuum random tree and a class of recursive fractals do exhibit the same form of Hausdorff measure function, $r^{\alpha}(\ln\ln r^{-1})^{\theta}$, with $\alpha = 2$, $\theta = 1$ in the case of the continuum random tree (see [20], Corollary 1.2 and [29], Theorem 5.2). Note that this tells us that the Hausdorff dimension of the continuum random tree is 2 and we will expect to see a leading order term of r^2 in the volume estimates.

Henceforth, define the *open ball* of radius r around the point $\sigma \in \mathcal{T}$ to be

$$B(\sigma, r) := \{\sigma' : \, d_{\mathcal{T}}(\sigma, \sigma') < r\}.$$

In the annealed case (Theorem 3.1.1), we calculate the volume of a ball of radius r around the root exactly. The expression we obtain is easily seen to be asymptotically equal to $2r^2$ as $r \to 0$. In the quenched case (Theorem 3.1.2), the behaviour is not as smooth and we see fluctuations in the volume growth of logarithmic order, which confirm the expectations of the previous paragraph. Although it is tight enough to demonstrate the order of the fluctuations, we remark that the upper bound for $\inf_{\sigma \in \mathcal{T}} \mu(B(\sigma, r))$ is almost certainly not optimal (as a consequence, neither are the corresponding lower heat kernel bounds). We conjecture that, up to constants, the lower bound for this quantity is sharp.

Theorem 3.1.1 *Let ρ be the root of $\mathcal{T}$, then*

$$\mathbf{E}(\mu(B(\rho, r))) = 1 - e^{-2r^2}, \qquad \forall r \geq 0.$$

Theorem 3.1.2 $\mathbf{P}$*-a.s., there exist constants $c_{3.1}, c_{3.2}, c_{3.3}, c_{3.4}$ such that*

$$c_{3.1} r^2 \ln_1 r^{-1} \leq \sup_{\sigma \in \mathcal{T}} \mu(B(\sigma, r)) \leq c_{3.2} r^2 \ln_1 r^{-1},$$

and

$$c_{3.3} r^2 \left(\ln_1 r^{-1}\right)^{-1} \leq \inf_{\sigma \in \mathcal{T}} \mu(B(\sigma, r)) \leq c_{3.4} r^2 \ln_1 \ln_1 r^{-1},$$

for $r \in (0, \mathrm{diam}\mathcal{T})$, where $\mathrm{diam}\mathcal{T}$ is the diameter of $(\mathcal{T}, d_{\mathcal{T}})$ and $\ln_1 x := \ln x \vee 1$.

Locally, we prove the following volume bounds, which show that the volume growth of a ball around a particular point demonstrates fluctuations about r^2 of the order of $\ln \ln r^{-1}$ asymptotically. This exactly mirrors the $\ln \ln r^{-1}$ local fluctuations exhibited by the random recursive fractals of [33]. We remark that the lim sup result has also been proved in the course of deriving the Hausdorff measure function of $\mathcal{T}$ in [20]. However, we include an alternative proof which applies properties of a Brownian excursion directly. Similarly to the global case, we conjecture that the best local lower bound for $\mu(B(\sigma, r))$ is actually a multiple of $r^2(\ln \ln r^1)^{-1}$ and the asymptotic result appearing on the left hand side of (3.1) is optimal.

Theorem 3.1.3 **P**-*a.s., we have*

$$0 < \limsup_{r \to 0} \frac{\mu(B(\sigma,r))}{r^2 \ln\ln r^{-1}} < \infty,$$

and also

$$0 < \liminf_{r \to 0} \frac{\mu(B(\sigma,r))}{r^2 \left(\ln\ln r^{-1}\right)^{-1}}, \qquad \liminf_{r \to 0} \frac{\mu(B(\sigma,r))}{r^2} < \infty, \tag{3.1}$$

for μ-a.e. $\sigma \in \mathcal{T}$.

The global volume bounds of Theorem 3.1.2 mean that the continuum random tree satisfies the non-uniform volume doubling of Chapter 2, **P**-a.s. These results immediately allow us to deduce the existence of a transition density for the Brownian motion on $\mathcal{T}$ and the following bounds upon it.

Theorem 3.1.4 **P**-*a.s., the Brownian motion $X = (X_t)_{t \geq 0}$ on $\mathcal{T}$ exists, and furthermore, it has a transition density $(p_t(\sigma,\sigma'))_{\sigma,\sigma' \in \mathcal{T}, t>0}$, that satisfies, for some constants $c_{3.5}, c_{3.6}, c_{3.7}, c_{3.8}$, $t_0 > 0$ and deterministic $\theta_1, \theta_2, \theta_3 \in (0,\infty)$,*

$$p_t(\sigma,\sigma') \geq c_{3.5} t^{-\frac{2}{3}} (\ln_1 t^{-1})^{-\theta_1} \exp\left\{ -c_{3.6} \left(\frac{d^3}{t}\right)^{1/2} \ln_1 \left(\frac{d}{t}\right)^{\theta_2} \right\}, \tag{3.2}$$

and

$$p_t(\sigma,\sigma') \leq c_{3.7} t^{-\frac{2}{3}} (\ln_1 t^{-1})^{1/3} \exp\left\{ -c_{3.8} \left(\frac{d^3}{t}\right)^{1/2} \ln_1 \left(\frac{d}{t}\right)^{-\theta_3} \right\}, \tag{3.3}$$

for all $\sigma, \sigma' \in \mathcal{T}$, $t \in (0, t_0)$, where $d := d_{\mathcal{T}}(\sigma,\sigma')$ and $\ln_1 x := \ln x \vee 1$.

This result demonstrates that the heat kernel decays exponentially away from the diagonal and there can be spatial fluctuations of no more than logarithmic order. The following theorem that we prove for the on-diagonal part of the heat kernel shows that global fluctuations of this order do actually occur.

Theorem 3.1.5 **P**-*a.s., there exist constants $c_{3.9}, c_{3.10}, c_{3.11}, c_{3.12}, t_1 > 0$ and deterministic $\theta_4 \in (0,\infty)$ such that for all $t \in (0, t_1)$,*

$$c_{3.9} t^{-2/3} (\ln\ln t^{-1})^{-14} \leq \sup_{\sigma \in \mathcal{T}} p_t(\sigma,\sigma) \leq c_{3.10} t^{-2/3} (\ln t^{-1})^{1/3}, \tag{3.4}$$

$$c_{3.11} t^{-2/3} (\ln t^{-1})^{-\theta_4} \leq \inf_{\sigma \in \mathcal{T}} p_t(\sigma,\sigma) \leq c_{3.12} t^{-2/3} (\ln t^{-1})^{-1/3}. \tag{3.5}$$

Locally, the results we obtain are not precise enough to demonstrate the $\mathbf{P}$-a.s. existence of fluctuations. However, they do show that there can only be fluctuations of log-logarithmic order, and combined with the annealed result of Proposition 3.1.7, they prove that log-logarithmic fluctuations occur with positive probability.

Theorem 3.1.6 $\mathbf{P}$*-a.s., for* μ*-a.e.* $\sigma \in \mathcal{T}$*, there exist constants* $c_{3.13}, c_{3.14}$*,* $t_2 > 0$ *such that for all* $t \in (0, t_2)$*,*

$$c_{3.13} t^{-2/3} (\ln \ln t^{-1})^{-14} \leq p_t(\sigma, \sigma) \leq c_{3.14} t^{-2/3} (\ln \ln t^{-1})^{1/3},$$

and also

$$\liminf_{t \to 0} \frac{p_t(\sigma, \sigma)}{t^{-2/3} (\ln \ln t^{-1})^{-1/3}} < \infty.$$

The final estimates we prove are annealed heat kernel bounds at the root of $\mathcal{T}$, which show that the expected value of $p_t(\rho, \rho)$ is controlled by $t^{-2/3}$ with at most $O(1)$ fluctuations as $t \to 0$.

Proposition 3.1.7 *Let* ρ *be the root of* $\mathcal{T}$*, then there exist constants* $c_{3.15}, c_{3.16}$ *such that*

$$c_{3.15} t^{-2/3} \leq \mathbf{E}\left(p_t(\rho, \rho)\right) \leq c_{3.16} t^{-2/3}, \qquad \forall t \in (0, 1).$$

At this point, a comparison with the results obtained by Barlow and Kumagai for the random walk on the incipient cluster for critical percolation on a regular tree, [14], is pertinent. First, observe that the incipient infinite cluster can be constructed as a particular branching process conditioned to never become extinct and the self-similar continuum random tree (see [2]) can be constructed as the scaling limit of a similar branching process. Note also that the objects studied here and by Barlow and Kumagai are both measure-metric space trees and so similar probabilistic and analytic techniques for estimating the heat kernel may be applied to them. Consequently, it is not surprising that the quenched local heat kernel bounds of [14] exhibit log-logarithmic differences similar to those obtained in this chapter and furthermore, the annealed heat kernel behaviour at the root is also shown to be the same in both settings. It should be noted, though, that the volume bounds which are crucial for obtaining these heat kernel bounds are proved in very different ways. Here we use Brownian excursion properties, whereas in [14], branching process arguments are applied. Unlike in [14], we do not prove annealed off-diagonal heat kernel bounds. This is primarily because there is no canonical way of labelling vertices (apart from the root) in the continuum random tree.

3.2 Preliminaries

3.2.1 Normalised Brownian excursion

An important part of the definition of the continuum random tree is the Brownian excursion, normalised to have length 1. In this section, we provide two characterisations of the law of the normalised Brownian excursion, which may be deduced from a standard Brownian motion on $\mathbb{R}$. We shall denote by $B = (B_t)_{t\geq 0}$ a standard, 1-dimensional Brownian motion starting from 0 under $\mathbf{P}$.

We begin by defining the space of excursions, U. First, let U' be the space of functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ for which there exists a $\tau(f) \in (0, \infty)$ such that

$$f(t) > 0 \qquad \Leftrightarrow \qquad t \in (0, \tau(f)).$$

We shall take $U := U' \cap C(\mathbb{R}_+, \mathbb{R}_+)$, the restriction to the continuous functions contained in U'. The space of excursions of length 1 is then defined to be the set $U^{(1)} := \{f \in U : \ \tau(f) = 1\}$.

Our first description of the law of W involves conditioning the Itô excursion law, which arises from the Poisson process of excursions of B. Since this law has been widely studied, we shall omit most of the technicalities here. For more details of excursion laws for Markov processes, the reader is referred to [52], Chapter VI.

Let L_t be the local time of B at 0, and $L_t^{-1} := \inf\{s > 0 : \ L_s > t\}$ be its right continuous inverse. Wherever $L_{t^-}^{-1} \neq L_t^{-1}$, we define $e_t \in U$ to be the (positive) excursion at local time t. In particular,

$$e_t(s) := \begin{cases} |B_{L_{t^-}^{-1}+s}|, & 0 \leq s \leq L_t^{-1} - L_{t^-}^{-1}, \\ 0, & s > L_t^{-1} - L_{t^-}^{-1}. \end{cases}$$

The set of excursions of B is denoted by $\Pi := \{(t, e_t) : \ L_{t^-}^{-1} \neq L_t^{-1}\}$. The key idea is that Π is a Poisson process on $(0, \infty) \times U$. More specifically, there exists a σ-finite measure, n, on U such that, under $\mathbf{P}$,

$$\#\left(\Pi \cap \cdot\right) \stackrel{d}{=} N(\cdot),$$

where N is a Poisson random measure on $(0, \infty) \times U$ with intensity $dt\, n(df)$. Bearing this result in mind, even though it has infinite mass, the measure n can be considered to be the "law" of the (unconditional) *Brownian excursion*.

We now describe the procedure for conditioning this measure. For $c > 0$, the *re-normalisation operator* $\Lambda_c : U \to U$ is defined by

$$\Lambda_c(f)(t) = \frac{1}{\sqrt{c}} f(ct), \qquad \forall t \geq 0, \ f \in U.$$

Clearly, if $f \in U$, then $\Lambda_{\tau(f)}(f) \in U^{(1)}$. For a measurable $A \subseteq U^{(1)}$, define the probability measure $n^{(1)}$ by

$$n^{(1)}(A) := \frac{n\left(\Lambda_{\tau(f)}(f) \in A, \tau(f) \geq 1\right)}{n\left(\tau(f) \geq 1\right)} \equiv n\left(\Lambda_{\tau(f)}(f) \in A \middle| \tau(f) \geq 1\right).$$

A $U^{(1)}$ valued process which has law $n^{(1)}$ is said to be a *normalised Brownian excursion.*

Our second description of the law of the normalised Brownian excursion is as the law of the normalised excursion straddling a fixed time. In fact, this description also allows an explicit construction of the normalised Brownian excursion, which we shall denote $W = (W_t)_{0 \leq t \leq 1}$. First, fix $T > 0$ and set $G_T := \sup\{t < T : \; B_t = 0\}$, $D_T := \inf\{t > T : \; B_t = 0\}$, which are not equal, $\mathbf{P}$-a.s. The excursion straddling T is then

$$Z_t := \begin{cases} |B_{G_T + t}|, & 0 \leq t \leq D_T - G_T, \\ 0, & t > D_T - G_T, \end{cases}$$

which takes values in U, $\mathbf{P}$-a.s. We can normalise Z to have length 1 by setting $W = \Lambda_{D_T - G_T}(Z)$, which takes values in $U^{(1)}$, $\mathbf{P}$-a.s. By comparing the density formula for $(G_T, D_T, (Z_t)_{t \geq 0})$ of [18], Theorem 6, with the finite dimensional density of $n^{(1)}$ (see [51], Chapter XII), it is elementary to show that the process W has the law $n^{(1)}$, and so the two descriptions of the law of the normalised Brownian excursion are equivalent.

3.2.2 Continuum random tree

The connection between trees and excursions is an area that has been of much recent interest. In this section, we look to provide a brief introduction to this link and also a definition of the continuum random tree, which is the object of interest of this chapter.

Given a function $f \in U$, we define a distance on $[0, \tau(f)]$ by setting

$$d_f(s,t) := f(s) + f(t) - 2m_f(s,t), \tag{3.6}$$

where $m_f(s,t) := \inf\{f(r) : \; r \in [s \wedge t, s \vee t]\}$. Then, we use the equivalence

$$s \sim t \qquad \Leftrightarrow \qquad d_f(s,t) = 0, \tag{3.7}$$

to define $\mathcal{T}_f := [0, \tau(f)]/\sim$. We can write this as $\mathcal{T}_f = \{\sigma_s : \; s \in [0, \tau(f)]\}$, where $\sigma_s := [s]$, the equivalence class containing s. It is then straightforward to check that

$$d_{\mathcal{T}_f}(\sigma_s, \sigma_t) := d_f(s,t),$$

defines a metric on $\mathcal{T}_f$, and also that $\mathcal{T}_f$ is a dendrite, as defined in the introduction of this thesis. Furthermore, the metric $d_{\mathcal{T}_f}$ is a shortest path metric on $\mathcal{T}_f$, which means that it is additive along the paths of $\mathcal{T}_f$. The root of the tree $\mathcal{T}_f$ is defined to be the equivalence class σ_0 and is denoted by ρ_f.

A natural volume measure to put on $\mathcal{T}_f$ is the projection of Lebesgue measure on $[0, \tau(f)]$. For open $A \subseteq \mathcal{T}_f$, let

$$\mu_f(A) := \lambda\left(\{t \in [0, \tau(f)] : \ \sigma_t \in A\}\right),$$

where, throughout this chapter, λ is the usual 1-dimensional Lebesgue measure. This defines a Borel measure on $(\mathcal{T}_f, d_{\mathcal{T}_f})$, with total mass equal to $\tau(f)$.

The *continuum random tree* is then simply the random dendrite that we get when the function f is chosen according to the law of the normalised Brownian excursion. This differs from the Aldous continuum random tree, which is based on the random function $2W$. Since this extra factor only has the effect of increasing distances by a factor of 2, our results will still apply to Aldous' tree. In keeping with the notation used so far in this section, the measure-metric space should be written $(\mathcal{T}_W, d_{\mathcal{T}_W}, \mu_W)$, the distance on $[0, \tau(W)]$, defined at (3.6), d_W, and the root, ρ_W. However, we shall omit the subscripts W with the understanding that we are discussing the continuum random tree in this case. We note that $\tau(W) = 1$, $\mathbf{P}$-a.s., and so $[0, \tau(W)] = [0, 1]$ and μ is a probability measure on $\mathcal{T}$, $\mathbf{P}$-a.s. Finally, it follows from the continuity of W that the diameter of $\mathcal{T}$, $\mathrm{diam}\mathcal{T}$, is finite $\mathbf{P}$-a.s.

3.2.3 Other notation

The *δ-level oscillations* of a function y on the interval $[s, t]$ will be written as

$$\mathrm{osc}(y, [s, t], \delta) := \sup_{\substack{r, r' \in [s,t]: \\ |r'-r| \leq \delta}} |y(r) - y(r')|.$$

We shall also continue to use the notation introduced in Theorems 3.1.2 and 3.1.4, $\ln_1 x := \ln x \vee 1$.

3.3 Annealed volume result at the root

The annealed volume result that we prove in this section follows easily from the expected occupation time of $[0, r)$ for normalised Brownian excursion.

Proof of Theorem 3.1.1: By definition, we have that

$$\begin{aligned}\mu(B(\rho,r)) &= \lambda\left(\{s:\, d_{\mathcal{T}}(\sigma_s,\sigma_0)<r\}\right)\\ &= \lambda\left(\{s:\, W_s<r\}\right)\\ &= \int_0^1 \mathbf{1}_{\{W_s<r\}}ds. \end{aligned}\tag{3.8}$$

An expression for the expectation of this random variable is obtained in [22], Section 3, giving

$$\mathbf{E}\left(\mu(B(\rho,r))\right)=\int_0^r 4ae^{-2a^2}da.$$

This integral is easily evaluated to give the desired result. □

Remark 3.1 *The characterisation of $\mu(B(\rho,r))$ at (3.8) has as a consequence that the asymptotic results of Theorem 3.1.3 also apply to the time spent in $[0,r)$ by the normalised Brownian excursion as $r\to 0$.*

3.4 Brownian excursion properties

In this section, we use sample path properties of a standard Brownian motion to deduce various sample path properties for the normalised Brownian excursion. The definitions of the random variables B, L_t^{-1}, Π, Z and W should be recalled from Section 3.2.1.

Lemma 3.4.1 $\mathbf{P}$-*a.s.,*

$$\limsup_{\delta\to 0}\frac{\mathrm{osc}(W,[0,1],\delta)}{\sqrt{\delta\ln\delta^{-1}}}<\infty.$$

Proof: From Levy's 1937 result ([43], Theorem 52.2) on the modulus of continuity of a standard Brownian motion, B, we may easily obtain

$$\limsup_{\delta\to 0}\frac{\mathrm{osc}(B,[s,t],\delta)}{\sqrt{\delta\ln\delta^{-1}}}<\infty,\qquad \forall 0\leq s<t<\infty,\qquad \mathbf{P}\text{-a.s.}\tag{3.9}$$

For $(t,e_t)\in\Pi$, we have that $\mathrm{osc}(e_t,\mathbb{R}_+,\delta)=\mathrm{osc}(B,[L_{t^-}^{-1},L_t^{-1}],\delta)$. Consequently, (3.9) implies that $e_t\in A$, for all $(t,e_t)\in\Pi$, $\mathbf{P}$-a.s., where

$$A:=\left\{f\in U:\,\limsup_{\delta\to 0}\frac{\mathrm{osc}(f,\mathbb{R}_+,\delta)}{\sqrt{\delta\ln\delta^{-1}}}<\infty\right\},$$

Hence $\mathbf{P}(N((0,\infty)\times A^c)>0)=0$, where N is the Poisson process with intensity $dt\, n(df)$, as described in Section 3.2.1. This means that $0=\int_0^\infty n(A^c)dt$, whence $n(A^c)=0$. Since A is invariant under the re-normalisation of excursions, it follows that $n^{(1)}(A^c)=0$. Since $n^{(1)}$ is the law of W, this completes the proof. □

Lemma 3.4.2 **P**-*a.s.,*

$$\limsup_{\delta\to 0}\frac{\inf_{t\in[0,1-\delta]}\mathrm{osc}(W,[t,t+\delta],\delta)}{\sqrt{\delta(\ln\delta^{-1})^{-1}}}<\infty. \tag{3.10}$$

Proof: We start by proving the corresponding result for a standard Brownian motion. Fix a constant $c_{3.17}$ and then, for $n \geq 0$,

$$\begin{aligned}
&\mathbf{P}\left(\inf_{j=0,\dots,2^n-1}\mathrm{osc}(B,[j2^{-n},(j+1)2^{-n}],2^{-n})\geq c_{3.17}2^{-\frac{n}{2}}n^{-\frac{1}{2}}\right)\\
&= \mathbf{P}\left(\mathrm{osc}(B,[j2^{-n},(j+1)2^{-n}],2^{-n})\geq c_{3.17}2^{-\frac{n}{2}}n^{-\frac{1}{2}},\ j=0,\dots,2^n-1\right)\\
&= \mathbf{P}\left(\mathrm{osc}(B,[0,2^{-n}],2^{-n})\geq c_{3.17}2^{-\frac{n}{2}}n^{-\frac{1}{2}}\right)^{2^n},
\end{aligned}$$

using the independent increments of a Brownian motion for the second equality. The probability in this expression is bounded above by

$$\mathbf{P}\left(\sup_{t\in[0,2^{-n}]}|B_t|\geq\frac{c_{3.17}}{2^{1+\frac{n}{2}}n^{\frac{1}{2}}}\right)=\mathbf{P}\left(T_{B_E(0,1)}\leq\frac{4n}{c_{3.17}^2}\right)\leq 1-c_{3.18}e^{-\frac{c_{3.19}n}{c_{3.17}^2}},$$

for some constants $c_{3.18}, c_{3.19}$, where $T_{B_E(0,1)}$ represents the exit time of a standard Brownian motion from a Euclidean ball of radius 1 about the origin. The distribution of this random variable is known explicitly, see [19], and the above tail estimate is readily deduced from the expression given there. Using the fact that $1-x\leq e^{-x}$ for $x\geq 0$ and summing over n, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathbf{P}\left(\inf_{j=0,\dots,2^n-1}\mathrm{osc}(B,[j2^{-n},(j+1)2^{-n}],2^{-n})\geq c_{3.17}2^{-\frac{n}{2}}n^{-\frac{1}{2}}\right)\\
&\qquad\leq\sum_{n=0}^{\infty}e^{-2^n c_{3.18}e^{-\frac{c_{3.19}n}{c_{3.17}^2}}},
\end{aligned}$$

which is finite for $c_{3.17}$ chosen suitably large. Hence Borel-Cantelli implies that, **P**-a.s., there exists a constant $c_{3.20}$ such that

$$\inf_{j=0,\dots,2^n-1}\mathrm{osc}(B,[j2^{-n},(j+1)2^{-n}],2^{-n})\leq c_{3.20}2^{-\frac{n}{2}}n^{-\frac{1}{2}},\qquad\forall n\geq 0.$$

Let $\delta\in(0,1]$, then $\delta\in[2^{-(n+1)},2^{-n}]$ for some $n\geq 0$. Hence

$$\begin{aligned}
\inf_{t\in[0,1-\delta]}\mathrm{osc}(B,[t,t+\delta],\delta) &\leq \inf_{t\in[0,1-2^{-n}]}\mathrm{osc}(B,[t,t+2^{-n}],2^{-n})\\
&\leq \inf_{j=0,\dots,2^n-1}\mathrm{osc}(B,[j2^{-n},(j+1)2^{-n}],2^{-n})\\
&\leq c_{3.20}2^{-\frac{n}{2}}n^{-\frac{1}{2}}\\
&\leq c_{3.21}\sqrt{\delta(\ln\delta^{-1})^{-1}},
\end{aligned}$$

which proves (3.10) holds when W is replaced by B. By rescaling, an analogous result holds for any interval with rational endpoints. By countability and a monotonicity argument, this is easily extended to **P**-a.s.,

$$\limsup_{\delta\to 0}\frac{\inf_{t\in[r,s-\delta]}\operatorname{osc}(B,[t,t+\delta],\delta)}{\sqrt{\delta(\ln\delta^{-1})^{-1}}}<\infty,\qquad \forall 0\leq r<s<\infty.$$

Applying a similar to argument to that used in the proof of Lemma 3.4.1, the result for excursions may be deduced from this by using the Poisson process of excursions and rescaling. □

Lemma 3.4.3 **P**-*a.s.,*

$$\lambda\left\{t\in[0,1]:\ \limsup_{\delta\to 0}\frac{|W_{t+\delta}-W_t|}{\sqrt{\delta\ln\ln\delta^{-1}}}<\infty\right\}=1.$$

Proof: An application of Fubini's theorem allows it to be deduced from the law of the iterated logarithm for the standard Brownian motion, B, that

$$\lambda\left\{t\geq 0:\ \limsup_{\delta\to 0}\frac{|B_{t+\delta}-B_t|}{\sqrt{\delta\ln\ln\delta^{-1}}}>1\right\}=0,\qquad \mathbf{P}\text{-a.s.}$$

For Z, the excursion straddling T, this implies

$$\begin{aligned}&\lambda\left\{t\in[0,D_T-G_T]:\ \limsup_{\delta\to 0}\frac{|Z_{t+\delta}-Z_t|}{\sqrt{\delta\ln\ln\delta^{-1}}}>1\right\}\\ &=\ \lambda\left\{t\in[G_T,D_T]:\ \limsup_{\delta\to 0}\frac{|B_{t+\delta}-B_t|}{\sqrt{\delta\ln\ln\delta^{-1}}}>1\right\}=0,\qquad \mathbf{P}\text{-a.s.}\end{aligned}$$

The result follows easily from this using the normalisation $W=\Lambda_{D_T-G_T}(Z)$, which we have by construction. □

Lemma 3.4.4 **P**-*a.s.,*

$$\limsup_{\delta\to 0}\frac{\int_0^1\mathbf{1}_{\{W_t<\delta\}}dt}{\delta^2\ln\ln\delta^{-1}}<\infty.$$

Proof: Letting Z be the excursion straddling T and using the notation $L:=D_T-G_T$, we can rewrite the moments of the time spent below level δ by the normalised Brownian excursion as follows. For $k\geq 0$,

$$\mathbf{E}\left(\left(\int_0^1\mathbf{1}_{\{W_t<\delta\}}dt\right)^k\right)=\mathbf{E}\left(L^{-k}\mathbf{E}\left(\left.\left(\int_0^L\mathbf{1}_{\{Z_t<\delta\sqrt{L}\}}dt\right)^k\right|L\right)\right).$$

The conditional expectation in this expression was shown in [18], Theorem 9, to be bounded above by $(k+1)!L^k\delta^{2k}$. Hence we have an upper bound of $(k+1)!\delta^{2k}$ for the non-conditional expectation. Thus, summing over k, this yields, for $\theta \in [0,1)$, $\delta > 0$,

$$\mathbf{E}\left(e^{\theta\delta^{-2}\int_0^1 \mathbf{1}_{\{W_t<\delta\}}dt}\right) \leq \sum_{k=0}^{\infty} \frac{(k+1)!\theta^k}{k!} = \frac{1}{(1-\theta)^2}. \tag{3.11}$$

Hence, for $\theta \in (0,1)$, $\delta_k := e^{-k}$,

$$\begin{aligned}\sum_{k=0}^{\infty} \mathbf{P}\left(\frac{\int_0^1 \mathbf{1}_{\{W_t<\delta_k\}}dt}{\delta_k^2 \ln\ln \delta_k^{-1}} \geq \frac{2}{\theta}\right) &\leq \sum_{k=0}^{\infty} \mathbf{E}\left(e^{\theta\delta_k^{-2}\int_0^1 \mathbf{1}_{\{W_t<\delta_k\}}dt - 2\ln\ln\delta_k^{-1}}\right) \\ &\leq \frac{1}{(1-\theta)^2}\sum_{k=0}^{\infty} k^{-2}.\end{aligned}$$

Since this is finite, the Borel-Cantelli lemma implies that

$$\limsup_{k\to\infty} \frac{\int_0^1 \mathbf{1}_{\{W_t<\delta_k\}}dt}{\delta_k^2 \ln\ln \delta_k^{-1}} < \infty, \qquad \mathbf{P}\text{-a.s.}$$

The lemma follows from this by a monotonicity argument. □

Lemma 3.4.5 **P**-*a.s.,*

$$\lambda\left\{t \in [0,1] : \liminf_{\delta\to 0} \frac{\sup_{s\in[0,\delta]} |W_{t+s} - W_t|}{\sqrt{\delta(\ln\ln\delta^{-1})^{-1}}} < \infty\right\} = 1.$$

Proof: In, [50], Section 6, Orey and Taylor prove that

$$\liminf_{\delta\to 0} \frac{\sup_{s\in[0,\delta]} |B_s|}{\sqrt{\delta(\ln\ln\delta^{-1})^{-1}}} = \frac{\pi}{2\sqrt{2}}, \qquad \mathbf{P}\text{-a.s..}$$

By applying Fubini's theorem and rescaling, as in the proof of Lemma 3.4.3, the lemma follows. □

3.5 Global upper volume bound

In this section, we prove a global upper volume bound for $\mathcal{T}$. The three main ingredients in the proof are the modulus of continuity result proved in Lemma 3.4.1, a bound on the tail of the distribution of the volume of a ball about the root, and the invariance under random re-rooting of the continuum random tree. We shall apply this final property repeatedly in later sections. For example, in the local upper volume bounds of Propositions 3.7.1 and 3.7.2, we proceed by investigating the volume of a ball around the root, ρ. Random re-rooting then allows the asymptotics we obtain at the root to be extended to μ-a.e. σ in $\mathcal{T}$. Before proving the main result of this section, we define precisely what we mean by re-rooting and state the invariance result that we will use.

Given W and $s \in [0,1]$, we define the shifted process $W^{(s)} = (W^{(s)}_t)_{0\leq t\leq 1}$ by

$$W^{(s)}_t := \begin{cases} W_s + W_{s+t} - 2m(s, s+t), & 0 \leq t \leq 1-s \\ W_s + W_{s+t-1} - 2m(s+t-1, s), & 1-s \leq t \leq 1. \end{cases}$$

The following lemma tells us that, if we select s according to the uniform distribution on $[0,1]$, independently of W, then the shifted process has the same distribution as the original, see [2], Section 2.7 for further discussion of the result.

Lemma 3.5.1 *If $W = (W_t)_{0\leq t\leq 1}$ is a normalised Brownian excursion and U is an independent $U[0,1]$ random variable, then $W^{(U)}$ has the same distribution as W.*

Written down in terms of excursions, it is not immediately clear what this result is telling us about the continuum random tree. Heuristically, it says that the distribution of the continuum random tree is invariant under random re-rooting, when the root is chosen from $\mathcal{T}$ so that it has law μ. Note that, similarly to (3.8), from the definition of the shifted process $W^{(s)}$ it may be deduced that, for $s \in [0,1]$,

$$\mu(B(\sigma_s, r)) = \int_0^1 \mathbf{1}_{\{W^{(s)}_t < r\}} dt.$$

From this expression, it is easy to deduce from the previous lemma that, if U is a $U[0,1]$ random variable independent of W, then $(\mu(B(\sigma_U, r)))_{r\geq 0}$ is equal in distribution to $(\mu(B(\rho, r)))_{r\geq 0}$.

Before proceeding with the main result of this section, we prove an exponential tail bound for the distribution of the volume of a ball of radius r about the root.

Lemma 3.5.2 *There exist constants $c_{3.22}, c_{3.23}$ such that, for all $r > 0, \lambda \geq 1$,*

$$\mathbf{P}\left(\mu(B(\rho, r)) \geq r^2\lambda\right) \leq c_{3.22} e^{-c_{3.23}\lambda}.$$

Proof: This estimate is a straightforward application of the inequality at (3.11), for it follows that, for all $\theta \in (0,1)$,

$$\mathbf{P}\left(\mu(B(\rho, r)) \geq r^2\lambda\right) \leq \mathbf{E}\left(e^{\theta r^{-2}\mu(B(\rho,r)) - \theta\lambda}\right) \leq \frac{e^{-\lambda\theta}}{(1-\theta)^2}.$$

□

Proposition 3.5.3 **P**-*a.s., there exists a constant* $c_{3.24}$ *such that*

$$\sup_{\sigma\in\mathcal{T}}\mu(B(\sigma,r))\leq c_{3.24}r^2\ln_1 r^{-1},\qquad \forall r\in(0,\mathrm{diam}\mathcal{T}).$$

Proof: In the proof, we shall denote $r_n:=e^{-n}$, $\delta_n:=r_n^2(\ln_1 r_n^{-1})^{-1}$, and also write $g(r):=r^2\ln_1 r^{-1}$. Furthermore, we introduce the notation, for $\lambda\in(0,1]$, $n_0\in\mathbb{N}$,

$$A_\lambda(n_0):=\left\{\mathrm{osc}(W,[0,1],\lambda\delta_n)\leq r_n,\ \forall n\geq n_0\right\},$$

which represents a collection of sample paths of W which are suitably regular for our purposes. We will start by showing that the claim holds on each set of the form $A_\lambda(n_0)$. Until otherwise stated, we shall assume that λ and n_0 are fixed. Now, consider the sets

$$B_n:=\left\{\sup_{\sigma\in\mathcal{T}}\mu(B(\sigma,r_n))>c_{3.25}g(r_n)\right\}\cap A_\lambda(n_0),$$

where $n\geq n_0$, and $c_{3.25}$ is a constant we will specify below. Clearly, on the event B_n, the random subset of $[0,1]$ defined by

$$\mathcal{I}_n:=\{t\in[0,1]:\ |t-s|<\lambda\delta_n \text{ for some } s\in[0,1] \text{ with } \mu(B(\sigma_s,r_n))\geq c_{3.25}g(r_n)\}$$

has Lebesgue measure no less than $\lambda\delta_n$. Thus, if U is a $U[0,1]$ random variable, independent of W, then

$$\mathbf{P}(U\in\mathcal{I}_n,\ B_n)=\mathbf{E}(\mathbf{P}(U\in\mathcal{I}_n|W)\mathbf{1}_{B_n})\geq\lambda\delta_n\mathbf{P}(B_n).$$

Moreover, on the event $\{U\in\mathcal{I}_n\}\cap B_n$, there exists an $s\in[0,1]$ for which both $|U-s|<\lambda\delta_n$ and $\mu(B(\sigma_s,r_n))\geq c_{3.25}g(r_n)$ are satisfied. Applying the modulus of continuity property that holds on $A_\lambda(n_0)$, for this s we have that $d_{\mathcal{T}}(\sigma_s,\sigma_U)\leq 3r_n$, and so $\mu(B(\sigma_U,4r_n))\geq c_{3.25}g(r_n)$. Hence the above inequality implies that

$$\begin{aligned}\lambda\delta_n\mathbf{P}(B_n)&\leq\mathbf{P}\left(\mu(B(\sigma_U,4r_n))\geq c_{3.25}g(r_n)\right)\\&=\mathbf{P}\left(\mu(B(\rho,4r_n))\geq c_{3.25}g(r_n)\right)\\&\leq c_{3.22}e^{-c_{3.23}c_{3.25}n/16},\end{aligned}$$

where we have applied the random re-rooting of Lemma 3.5.1 to deduce the equality, and the distributional tail bound of Lemma 3.5.2 to obtain the final line. As a consequence, we have that

$$\sum_{n\geq n_0}\mathbf{P}(B_n)\leq c_{3.22}\lambda^{-1}\sum_{n\geq n_0}ne^{2n}e^{-c_{3.23}c_{3.25}n/16},$$

which is finite for $c_{3.25}$ chosen suitably large. Appealing to Borel-Cantelli and applying a simple monotonicity argument, this implies that $\mathbf{P}$-a.s. on $A_\lambda(n_0)$

$$\limsup_{r\to 0} \frac{\sup_{\sigma\in\mathcal{T}} \mu(B(\sigma,r))}{r^2 \ln_1 r^{-1}} < \infty.$$

By countability, the same must be true on the set $A_\lambda := \cup_{n\geq 1} A_\lambda(n)$. Thus, to deduce the proposition, it remains to be shown that we can choose this set to be arbitrarily large. However, this is a simple consequence of the sample path property of the Brownian excursion that we proved in Lemma 3.4.1. In particular,

$$\begin{aligned}\mathbf{P}(A_\lambda^c) &= \mathbf{P}(\mathrm{osc}(W,[0,1],\lambda\delta_n) > r_n \text{ i.o.})\\ &\leq \mathbf{P}\left(\limsup_{\delta\to 0} \frac{\mathrm{osc}(W,[0,1],\delta)}{\sqrt{\delta \ln \delta^{-1}}} \geq \frac{c_{3.26}}{\sqrt{\lambda}}\right),\end{aligned}$$

for some constant $c_{3.26}$ that does not depend on λ. Letting $\lambda \to 0$, this probability converges to

$$\mathbf{P}\left(\limsup_{\delta\to 0} \frac{\mathrm{osc}(W,[0,1],\delta)}{\sqrt{\delta \ln \delta^{-1}}} = \infty\right),$$

which is equal to 0 by Lemma 3.4.1. This completes the proof. $\square$

3.6 Global lower volume bounds

In this section, we prove global lower bounds for the volume of balls of the continuum random tree. The estimates are simple corollaries of the excursion modulus of continuity results proved in Lemmas 3.4.1 and 3.4.2.

Proposition 3.6.1 $\mathbf{P}$-*a.s., there exists a constant* $c_{3.27}$ *such that*

$$\inf_{\sigma\in\mathcal{T}} \mu(B(\sigma,r)) \geq c_{3.27} r^2 \left(\ln_1 r^{-1}\right)^{-1}, \qquad \forall r \in (0, \mathrm{diam}\mathcal{T}).$$

Proof: For $s \in [0,1], r > 0$, define

$$\alpha_u(s,r) := \inf\{t \geq 0 : \, |W_{s+t} - W_s| > r\},$$

$$\alpha_l(s,r) := \inf\{t \geq 0 : \, |W_s - W_{s-t}| > r\},$$

where these expressions are defined to be $1-s$, s if the infimum is taken over an empty set, respectively. From the definition of d at (3.6), it is readily deduced that $d(s,t) \leq r$

for all $t \in (s - \alpha_l(s, r/3), s + \alpha_u(s, r/3))$, from which it follows that $d_{\mathcal{T}}(\sigma_s, \sigma_t) \leq r$ for all t in this range. Hence

$$\mu(B(\sigma_s, r)) \geq \alpha_l(s, r/3) + \alpha_u(s, r/3). \tag{3.12}$$

We now show how the right hand side of this inequality can be bounded below, uniformly in s, using the uniform modulus of continuity of the Brownian excursion. By Lemma 3.4.1, **P**-a.s. there exist constants $c_{3.28}, \eta \in (0, 1)$ such that

$$\mathrm{osc}(W, [0, 1], \delta) \leq c_{3.28}\sqrt{\delta \ln \delta^{-1}}, \qquad \forall \delta \in (0, \eta). \tag{3.13}$$

Set $r_1 = 3c_{3.28}\sqrt{\eta \ln \eta^{-1}}$ and then, for $r \in (0, r_1)$, it is possible to choose $\delta = \delta(r)$ to satisfy $r = 3c_{3.28}\sqrt{\delta \ln \delta^{-1}}$. It follows from the inequality at (3.13) that if $r \in (0, r_1)$ and $|W_s - W_t| > r/3$, then $|s - t| > \delta \geq c_{3.29} r^2 \left(\ln r^{-1}\right)^{-1}$, where $c_{3.29}$ is a constant depending only on $c_{3.28}$ and η. By definition, this implies that

$$\alpha_l(s, r/3) \geq c_{3.29} r^2 \left(\ln r^{-1}\right)^{-1} \wedge s, \qquad \alpha_u(s, r/3) \geq c_{3.29} r^2 \left(\ln r^{-1}\right)^{-1} \wedge (1 - s),$$

for all $s \in [0, 1]$, $r \in (0, r_1)$. Adding these two expressions and taking a suitably small constant, we obtain that **P**-a.s., there exists a constant $c_{3.30}$ such that

$$\inf_{s \in [0,1]} \left(\alpha_l(s, r/3) + \alpha_u(s, r/3)\right) \geq c_{3.30} r^2 \left(\ln_1 r^{-1}\right)^{-1}, \qquad \forall r \in (0, \mathrm{diam}\mathcal{T}),$$

where we use the fact that $\mathrm{diam}\mathcal{T}$ is **P**-a.s. finite. Taking infima in (3.12) and comparing with the above inequality completes the proof. □

Proposition 3.6.2 **P**-*a.s., there exists a constant* $c_{3.31}$ *such that*

$$\sup_{\sigma \in \mathcal{T}} \mu(B(\sigma, r)) \geq c_{3.31} r^2 \ln_1 r^{-1}, \qquad \forall r \in (0, \mathrm{diam}\mathcal{T}).$$

Proof: By following an argument similar to that used in the proof of the previous proposition to transfer the excursion result to a result about the volume of balls in the CRT, the proposition may be deduced from Lemma 3.4.2. □

3.7 Local volume bounds

In this section, we prove the local volume growth asymptotics of Theorem 3.1.3 using the properties of the normalised Brownian excursion that were derived in Section 3.4. We also complete the proof of the global volume bounds of Theorem 3.1.2.

Proposition 3.7.1 **P**-*a.s., we have*

$$0 < \limsup_{r \to 0} \frac{\mu(B(\sigma, r))}{r^2 \ln\ln r^{-1}} < \infty, \tag{3.14}$$

for μ-*a.e.* $\sigma \in \mathcal{T}$.

Proof: We shall start by proving the lower bound. Since the argument is close to that of Proposition 3.6.1, we omit some of the details. Now, if $t \in [0, 1)$ is a point which satisfies

$$\liminf_{\delta \to 0} \frac{\sup_{s\in[0,\delta]} |W_{t+s} - W_t|}{\sqrt{\delta(\ln\ln \delta^{-1})^{-1}}} < \infty,$$

then it may be deduced, using a similar argument to Proposition 3.6.1, that

$$\limsup_{r \to 0} \frac{\mu(B(\sigma_t, r))}{r^2 \ln\ln r^{-1}} > 0.$$

Thus

$$\begin{aligned} &\mu\left\{\sigma \in \mathcal{T} :\ \limsup_{r \to 0} \frac{\mu(B(\sigma, r))}{r^2 \ln\ln r^{-1}} > 0\right\} \\ &= \lambda\left\{t \in [0, 1] :\ \limsup_{r \to 0} \frac{\mu(B(\sigma_t, r))}{r^2 \ln\ln r^{-1}} > 0\right\} \\ &\geq \lambda\left\{t \in [0, 1) :\ \liminf_{\delta \to 0} \frac{\sup_{s\in[0,\delta]} |W_{t+s} - W_t|}{\sqrt{\delta(\ln\ln \delta^{-1})^{-1}}} < \infty\right\}. \end{aligned}$$

Since, by Lemma 3.4.5, the last line is equal to 1, **P**-a.s., the proof of the lower estimate is complete.

We now prove the upper bound. Recall from (3.8) that $\mu(B(\rho, r)) = \int_0^1 \mathbf{1}_{\{W_t < r\}} dt$. Hence, **P**-a.s.,

$$\limsup_{r \to 0} \frac{\mu(B(\sigma_0, r))}{r^2 \ln\ln r^{-1}} = \limsup_{r \to 0} \frac{\int_0^1 \mathbf{1}_{\{W_t < r\}} dt}{r^2 \ln\ln r^{-1}} < \infty, \tag{3.15}$$

by Lemma 3.4.4. This gives us the desired result at the root.

Setting $g(r) = r^2 \ln\ln r^{-1}$, and letting U be a $U[0,1]$ random variable, independent of W,

$$\begin{aligned}
&\mathbf{E}\left(\mu\left\{\sigma \in \mathcal{T} : \limsup_{r\to 0} \frac{\mu(B(\sigma,r))}{g(r)} < \infty\right\}\right) \\
&= \mathbf{E}\left(\lambda\left\{s \in [0,1] : \limsup_{r\to 0} g(r)^{-1} \int_0^1 \mathbf{1}_{\{W_t^{(s)} < r\}} dt < \infty\right\}\right) \\
&= \mathbf{E}\left(\int_0^1 \mathbf{1}_{\{\limsup_{r\to 0} g(r)^{-1} \int_0^1 \mathbf{1}_{\{W_t^{(s)}<r\}} dt < \infty\}} ds\right) \\
&= \int_0^1 \mathbf{P}\left(\limsup_{r\to 0} g(r)^{-1} \int_0^1 \mathbf{1}_{\{W_t^{(U)} < r\}} dt < \infty \middle| U = s\right) ds \\
&= \mathbf{P}\left(\limsup_{r\to 0} g(r)^{-1} \int_0^1 \mathbf{1}_{\{W_t^{(U)} < r\}} dt < \infty\right) \\
&= \mathbf{P}\left(\limsup_{r\to 0} g(r)^{-1} \int_0^1 \mathbf{1}_{\{W_t < r\}} dt < \infty\right) \\
&= 1,
\end{aligned}$$

where we use the random re-rooting of Lemma 3.5.1 for the penultimate equality and the result at the root, (3.15), for the final one. The upper bound follows. $\square$

Proposition 3.7.2 **P**-*a.s., we have*

$$0 < \liminf_{r\to 0} \frac{\mu(B(\sigma,r))}{r^2\left(\ln\ln r^{-1}\right)^{-1}}, \qquad \liminf_{r\to 0} \frac{\mu(B(\sigma,r))}{r^2} < \infty \tag{3.16}$$

for μ-a.e. $\sigma \in \mathcal{T}$.

Proof: The proof of the left hand inequality is essentially the same as the proof of the lower bound of the previous proposition, with only a few minor changes needed. The key observation is that, if $s \in [0,1]$ satisfies

$$\limsup_{\delta\to 0} \frac{|W_{s+\delta} - W_s|}{\sqrt{\delta \ln\ln \delta^{-1}}} < \infty, \tag{3.17}$$

then, again using an argument similar to that of Proposition 3.6.1, it may be deduced that

$$\liminf_{r\to 0} \frac{\mu(B(\sigma_s,r))}{r^2\left(\ln\ln r^{-1}\right)^{-1}} > 0.$$

To complete the proof it is then enough to note that, by the local modulus of continuity result of Lemma 3.4.3, (3.17) holds for a subset of [0,1] with Lebesgue measure 1, **P**-a.s.

We now prove the right hand inequality. By Fatou's lemma and the expression for the expected volume of a ball around the root of $\mathcal{T}$, Theorem 3.1.1, we have

$$\mathbf{E}\left(\liminf_{r\to 0}\frac{\mu(B(\rho,r))}{r^2}\right)\leq \liminf_{r\to 0}\frac{\mathbf{E}(\mu(B(\rho,r)))}{r^2}=2.$$

Hence

$$\liminf_{r\to 0}\frac{\mu(B(\rho,r))}{r^2}<\infty,\qquad \mathbf{P}\text{-a.s.},$$

which is the result at the root. The proof may be completed by applying random re-rooting, as in the proof of the upper bound of Proposition 3.7.1. □

Proof of Theorem 3.1.2: Propositions 3.5.3 and 3.6.2 contain the upper and lower bounds for $\sup\mu(B(\sigma,r))$ respectively. The lower bound for $\inf\mu(B(\sigma,r))$ was proved in Proposition 3.6.1; the upper bound follows easily from the local upper bound at (3.14). □

3.8 Brownian excursion upcrossings

To complete the proofs of the heat kernel bounds in Sections 3.9 and 3.10, as well as the volume estimates we have already obtained, we need some extra information about the local structure of the CRT. This will follow from the the results about the upcrossings of a normalised Brownian excursion that we prove in this section.

Henceforth, we define, for $f\in U$,

$$N_a^b(f):=\#\{\text{upcrossings of } [a,b] \text{ by } f\}$$

and $N_a^b:=N_a^b(W)$ to be the *upcrossings* of $[a,b]$ by the normalised Brownian excursion. Also, for $f\in U$, define $h(f):=\sup\{f(t):\ t\geq 0\}$, the height of the excursion function. It is well known, (see [51], Chapter XII) that the tail of the "distribution" of $h(f)$ under n is simply given by

$$n(h(f)\geq x)=\frac{1}{x},\qquad \forall x>0. \tag{3.18}$$

We now calculate the generating function of $N_\delta^{2\delta}(f)$ under the probability measure $n(\cdot|\,h(f)\geq\delta)$. Notice that the expression we obtain does not depend on δ.

Lemma 3.8.1 *For $z < 2$, $\delta > 0$,*

$$n(z^{N_\delta^{2\delta}(f)}|\, h(f) \geq \delta) = \frac{1}{2-z}.$$

Proof: Suppose $X = (X_t)_{t\geq 0}$ is a process with law $n(\cdot|\, h(f) \geq \delta)$, then by Neveu and Pitman's result ([49], Theorem 1.1), about the branching process in a Brownian excursion, the process $(N_a^{a+\delta}(X))_{a\geq 0}$ is a continuous time birth and death process, starting from 1, with stationary transition intensities from i to $i \pm 1$ of i/δ. Standard branching process arguments (see [8], Section 3) allow us to deduce from this observation that

$$\mathbf{E}(z^{N_x^{x+\delta}(X)}) = \frac{\delta z - x(z-1)}{\delta - x(z-1)}, \qquad \forall z < \frac{x+\delta}{\delta}.$$

The result follows on setting $x = \delta$. □

In the proof of the following result about the tail of the distribution of $N_\delta^{2\delta}(f)$, we shall use a result of Le Gall and Duquesne that states that the set of excursions above a fixed level form a certain Poisson process. We outline briefly the result here, full details may be found in [21], Section 3.

Fix $a > 0$ and denote by (α_j, β_j), $j \in \mathcal{J}$, the connected components of the open set $\{s \geq 0 : \ f(s) > a\}$. For any $j \in \mathcal{J}$, denote by f^j the corresponding excursion of f defined by:

$$f^j(s) := f((\alpha_j + s) \wedge \beta_j) - a, \qquad s \geq 0,$$

and let $\tilde{f}^a$ represent the evolution of f below the level a. Applying the Poisson mapping theorem to [21], Corollary 3.2, we find that under the probability measure $n(\cdot|\, h(f) > a)$ and conditionally on $\tilde{f}^a$, the point measure

$$\sum_{j\in\mathcal{J}} \delta_{f^j}(df')$$

is distributed as a Poisson random measure on U with intensity given by a multiple of $n(df')$. The relevant scaling factor is given by the local time of f, which we shall denote by l^a. Note that this is a measurable function of $\tilde{f}^a$.

Lemma 3.8.2 *For $z \in [1, 2)$, $\varepsilon > 0$, $\delta \in (0, \varepsilon/2)$,*

$$n(N_\delta^{2\delta}(f) \geq \lambda,\ h(f) \geq \varepsilon) \leq \frac{2z^{1-\lambda}}{(2-z)^2\varepsilon}, \qquad \forall \lambda \geq 0.$$

Proof: For brevity, during this lemma we shall write $h = h(f)$ and $h^j = h(f^j)$, where f^j, $j \in \mathcal{J}$ are the excursions above the level δ. Note that it is elementary to show that the quantity $N_\delta^{2\delta}(f)$ also counts the number of excursions of f above the level δ which hit the level 2δ. Thus

$$\begin{aligned} N_\delta^{2\delta}(f) &= \#\{j \in \mathcal{J} : h^j \geq \delta\} \\ &= \#\{j \in \mathcal{J} : h^j \in [\delta, \varepsilon - \delta)\} + \#\{j \in \mathcal{J} : h^j \geq \varepsilon - \delta\}. \end{aligned}$$

We shall denote these two summands N_1 and N_2, respectively. From the observation preceding this lemma that the excursions above the level δ form a Poisson process on U and the fact that the sets $\{h \in [\delta, \varepsilon - \delta)\}$ and $\{h \geq \varepsilon - \delta\}$ are disjoint, we can conclude that the N_1 and N_2 are independent under the measure $n(\cdot \,|\, h \geq \delta)$ and conditional on $\tilde{f}^\delta$. Furthermore, we note that $h \geq \varepsilon$ if and only if $N_2 \geq 1$. Hence, for $z \in (0, 1]$,

$$\begin{aligned} & n(z^{N_\delta^{2\delta}(f)} \mathbf{1}_{\{h \geq \varepsilon\}} \,|\, h \geq \delta) \\ &= n(n(z^{N_1 + N_2} \mathbf{1}_{\{N_2 \geq 1\}} \,|\, \tilde{f}^\delta, h \geq \delta) \,|\, h \geq \delta) \\ &= n(n(z^{N_1} \,|\, \tilde{f}^\delta, h \geq \delta) n(z^{N_2} \mathbf{1}_{\{N_2 \geq 1\}} \,|\, \tilde{f}^\delta, h \geq \delta) \,|\, h \geq \delta). \end{aligned} \tag{3.19}$$

Since on the set $\{h \geq \delta\}$, and conditional on $\tilde{f}^\delta$, N_1 and N_2 are Poisson random variables with means $l^\delta n(h \in [\delta, \varepsilon - \delta))$ and $l^\delta n(h \geq \varepsilon - \delta)$ respectively, it is elementary to conclude that

$$n(z^{N_1} \,|\, \tilde{f}^\delta, h \geq \delta) = e^{-l^\delta n(h \in [\delta, \varepsilon - \delta))(1-z)},$$

and also

$$n(z^{N_2} \mathbf{1}_{\{N_2 \geq 1\}} \,|\, \tilde{f}^\delta, h \geq \delta) = e^{-l^\delta n(h \geq \varepsilon - \delta)(1-z)} (1 - e^{-l^\delta n(h \geq \varepsilon - \delta) z}).$$

Substituting these expressions back into (3.19) and applying the formula for the excursion height distribution that was stated at (3.18), it follows that

$$\begin{aligned} n(z^{N_\delta^{2\delta}(f)} \mathbf{1}_{\{h \geq \varepsilon\}} \,|\, h \geq \delta) &= n(e^{-l^\delta (1-z)\delta^{-1}} \,|\, h \geq \delta) \\ & \quad - n(e^{-l^\delta [(1-z)\delta^{-1} + z(\varepsilon - \delta)^{-1}]} \,|\, h \geq \delta). \end{aligned} \tag{3.20}$$

Now, by [21], equation (13), l^δ satisfies $n(1 - e^{-\lambda l^\delta}) = \lambda (1 + \lambda \delta)^{-1}$, for $\lambda \geq 0$. Thus

$$n(e^{-\lambda l^\delta} \,|\, h \geq \delta) = \frac{1}{1 + \lambda \delta},$$

which confirms the well-known fact that under $n(\cdot\,|\,h \geq \delta)$, l^δ behaves as an exponential, mean δ, random variable. Returning to (3.20), this fact implies that

$$n(z^{N_\delta^{2\delta}(f)}\mathbf{1}_{\{h\geq\varepsilon\}}\,|\,h\geq\delta)=\frac{1}{2-z}-\frac{1}{2-z+\delta z(\varepsilon-\delta)^{-1}}.$$

By a simple analytic continuation argument, we may extend the range of z for which this holds to [0,2). Finally, for $z\in[1,2)$, we have that, because $\delta<\varepsilon/2$,

$$\begin{aligned}
n(N_\delta^{2\delta}(f)\geq\lambda,\ h(f)\geq\varepsilon) &\leq n(z^{N_\delta^{2\delta}(f)}\mathbf{1}_{\{h\geq\varepsilon\}})z^{-\lambda}\\
&= n(z^{N_\delta^{2\delta}(f)}\mathbf{1}_{\{h\geq\varepsilon\}}\,|\,h\geq\delta)n(h\geq\delta)z^{-\lambda}\\
&= \frac{1}{\delta z^\lambda}\left(\frac{1}{2-z}-\frac{1}{2-z+\delta z(\varepsilon-\delta)^{-1}}\right)\\
&\leq \frac{2z^{1-\lambda}}{(2-z)^2\varepsilon},
\end{aligned}$$

which completes the proof. $\square$

We now reach the main results of this section, which give us an upper bound on the upcrossings of $[\delta,4\delta]$ by the normalised Brownian excursion for small δ and also a distribution tail estimate.

Proposition 3.8.3 **P**-*a.s.*,

$$\limsup_{\delta\to0}\frac{N_\delta^{4\delta}}{\ln\ln\delta^{-1}}<\infty.$$

Proof: Fix $z\in(1,2)$, $\varepsilon\in(0,1)$ and let λ be a constant. Then, taking m_0 suitably large, we have by the previous lemma that

$$\sum_{m=m_0}^\infty n(N_{2^{-m}}^{2^{1-m}}(f)\geq\lambda\ln m,\ h(f)\geq\varepsilon)\leq\sum_{m=m_0}^\infty\frac{2m^{(1-\lambda)\ln z}}{(2-z)^2\varepsilon}<\infty,$$

for λ chosen suitably large. Hence an application of Borel-Cantelli implies that, on $\{h(f)>\varepsilon\}$,

$$\limsup_{m\to\infty}\frac{N_{2^{-m}}^{2^{1-m}}(f)}{\ln m}<\infty,\qquad n\text{-a.s.}$$

Now for $\delta\in[2^{-(m+1)},2^{-m}]$, we have $[2^{-m},2^{1-m}]\subseteq[\delta,4\delta]$, and so $N_\delta^{4\delta}(f)\leq N_{2^{-m}}^{2^{1-m}}(f)$. Thus, on $\{h(f)>\varepsilon\}$,

$$\limsup_{\delta\to0}\frac{N_\delta^{4\delta}(f)}{\ln\ln\delta^{-1}}\leq\limsup_{m\to\infty}\frac{N_{2^{-m}}^{2^{1-m}}(f)}{\ln m}<\infty,\qquad n\text{-a.s.}$$

By σ-additivity, this is easily extended to hold on U, n-a.s. A simple rescaling argument allows us to obtain the same result $n^{(1)}$-a.s. on $U^{(1)}$. Since $n^{(1)}$ is the law of W, we are done. $\square$

Proposition 3.8.4 *There exist constants $c_{3.32}, c_{3.33}$ such that*

$$\mathbf{P}\left(N_\delta^{4\delta} \geq \lambda\right) \leq c_{3.32} e^{-c_{3.33}\lambda}, \qquad \forall \delta > 0, \lambda \geq 1.$$

Proof: Using the scaling property, $n(A) = \frac{1}{\sqrt{c}} n(\Lambda_c(A))$, of the Ito measure, it is possible to deduce the following alternative characterisation of $n^{(1)}$. For measurable $A \subseteq U^{(1)}$, $n^{(1)}(A) = n(\Lambda_\tau(f) \in A |\, \tau \in [1,2])$. Hence

$$\mathbf{P}(N_\delta^{4\delta} \geq \lambda) = n^{(1)}(N_\delta^{4\delta}(f) \geq \lambda) = c_{3.34} n(N_{\delta\sqrt{\tau}}^{4\delta\sqrt{\tau}}(f) \geq \lambda,\ \tau \in [1,2]).$$

However, for $\tau \in [1,2]$ we have that $[\sqrt{2}\delta, 2\sqrt{2}\delta] \subseteq [\delta\sqrt{\tau}, 4\delta\sqrt{\tau}]$, and so $N_{\delta\sqrt{\tau}}^{4\delta\sqrt{\tau}}(f) \leq N_{\sqrt{2}\delta}^{2\sqrt{2}\delta}(f)$. Hence

$$\mathbf{P}(N_\delta^{4\delta} \geq \lambda) \leq c_{3.34} n(N_{\sqrt{2}\delta}^{2\sqrt{2}\delta}(f) \geq \lambda,\ \tau \in [1,2]). \tag{3.21}$$

We now consider the cases $\lambda \leq \delta^{-1}$ and $\lambda \geq \delta^{-1}$ separately. First, suppose $\lambda \geq \delta^{-1}$. Since $\lambda \geq 1$, if $N_{\sqrt{2}\delta}^{2\sqrt{2}\delta}(f) \geq \lambda$, then $h(f) \geq \sqrt{2}\delta$. Thus

$$\begin{aligned}
\mathbf{P}(N_\delta^{4\delta} \geq \lambda) &\leq c_{3.34} n(N_{\sqrt{2}\delta}^{2\sqrt{2}\delta}(f) \geq \lambda,\ h \geq \sqrt{2}\delta) \\
&= \frac{c_{3.34}}{\sqrt{2}\delta} n\left(N_{\sqrt{2}\delta}^{2\sqrt{2}\delta}(f) \geq \lambda \,\middle|\, h \geq \sqrt{2}\delta\right) \\
&\leq \frac{c_{3.34} z^{-\lambda}}{\sqrt{2}\delta} n\left(z^{N_{\sqrt{2}\delta}^{2\sqrt{2}\delta}(f)} \,\middle|\, h \geq \sqrt{2}\delta\right) \\
&\leq \frac{c_{3.35}\lambda z^{-\lambda}}{2-z},
\end{aligned}$$

for $z \in (1,2)$, by Lemma 3.8.1. By fixing $z \in (1,2)$, this is clearly bounded above by $c_{3.36} e^{-c_{3.37}\lambda}$ for suitable choice of $c_{3.36}, c_{3.37}$.

The case $\lambda \leq \delta^{-1}$ requires a little more work. We first break the upper bound of (3.21) in two parts. For $\varepsilon > 0$,

$$\mathbf{P}(N_\delta^{4\delta} \geq \lambda) \leq c_{3.34} n(N_{\sqrt{2}\delta}^{2\sqrt{2}\delta}(f) \geq \lambda,\ h \geq \varepsilon) + c_{3.34} n(\tau \in [1,2],\ h < \varepsilon). \tag{3.22}$$

An upper bound for the first term of the form $c_{3.38} e^{-c_{3.39}\lambda}\varepsilon^{-1}$ is given by Lemma 3.8.2 when $\delta \leq \varepsilon/2\sqrt{2}$. For the second term, we have the following decomposition, see [51], Chapter XII,

$$n(\tau \in [1,2],\ h < \varepsilon) = \int_1^2 n^{(s)}(h < \varepsilon) \frac{ds}{\sqrt{2\pi s^3}},$$

where $n^{(s)}$ is a probability measure that satisfies $n^{(s)}(A) = n^{(1)}(\Lambda_s(A))$. However, this scaling property implies that $n^{(s)}(h < \varepsilon)$ is decreasing in s. Consequently, we have that $n(\tau \in [1,2],\ h < \varepsilon) \leq (2\pi)^{-1/2} n^{(1)}(h < \varepsilon)$. The right hand side of this expression represents the distribution of the maximum of a normalised Brownian excursion, which is known exactly, see [18], Theorem 7. In particular, we have that

$$n^{(1)}(h < \varepsilon) = c_{3.40}\varepsilon^{-3} \sum_{m=1}^{\infty} m^2 e^{-\frac{m^2\pi^2}{2\varepsilon^2}},$$

and some elementary analysis shows this is bounded above by $c_{3.41}\varepsilon^{-3}e^{-\frac{c_{3.42}}{\varepsilon}}$. By taking $\varepsilon = 2\sqrt{2}\lambda^{-1}$, we can use these bounds on the first and second terms of (3.22) to obtain the desired result. $\square$

3.9 Quenched heat kernel bounds

The heat kernel estimates deduced in this section are a straightforward application of two main ideas. Firstly, we apply results of [38], by Kigami, to construct a natural Dirichlet form on $(\mathcal{T}, d_{\mathcal{T}}, \mu)$. Associated with such a form is a Laplacian, $\Delta_{\mathcal{T}}$, on $\mathcal{T}$. Secondly, we use the results of Chapter 2, which show that the volume bounds we have already obtained are sufficient to deduce the existence of a heat kernel for $\Delta_{\mathcal{T}}$ and bounds upon it.

The first key result is proved by Kigami, and is stated in Chapter 1 as Theorem 1.4.11. It explains how to build a Dirichlet form on an arbitrary dendrite equipped with a shortest path metric. We shall not explain here how to construct the finite resistance form associated with a shortest path metric on a dendrite, but we will note that it may be done using a finite vertex approximation procedure, similar to the ideas of Chapter 1. Full details are given in Section 3 of [38].

Now, consider $f \in U^{(1)}$. As remarked in Section 3.2.2, $\mathcal{T}_f$ is a dendrite and $d_{\mathcal{T}_f}$ a shortest path metric on $\mathcal{T}_f$. Using the fact that f is a continuous function on a closed bounded interval, and hence uniformly continuous, elementary analysis allows it to be deduced that $(\mathcal{T}_f, d_{\mathcal{T}_f})$ is compact and hence it is complete and locally compact. (We note that compactness of $\mathcal{T}$ has already been proved in [1], Theorem 3). Finally, using simple path properties of the Brownian excursion, it is easy to check that μ_f satisfies the measure conditions of the Theorem 1.4.11 for $f \in \tilde{U}$, where $\tilde{U} \subseteq U^{(1)}$ is a set which satisfies $\mathbf{P}(W \in \tilde{U}) = 1$. Hence we can use this result to define a finite resistance form $(\mathcal{E}_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}})$ associated with $(\mathcal{T}, d_{\mathcal{T}})$ such that $(\mathcal{E}_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}} \cap L^2(\mathcal{T}, \mu))$ is a local, regular Dirichlet form on $\mathcal{T}$, $\mathbf{P}$-a.s.

In fact, it is also proved in [38] that the correspondence between shortest path metrics on dendrites and resistance forms is one-to-one in a certain sense. Specifically, if $W \in \tilde{U}$ and so $(\mathcal{E}_\mathcal{T}, \mathcal{F}_\mathcal{T})$ exists, define the resistance function, as in Chapter 2, by

$$R(A,B)^{-1} := \inf \left\{ \mathcal{E}_\mathcal{T}(u,u) : \; u \in \mathcal{F}_\mathcal{T}, u|_A = 1, u|_B = 0 \right\}, \tag{3.23}$$

for disjoint subsets A, B of $\mathcal{T}$. We can recover $d_\mathcal{T}$ by taking, for $\sigma, \sigma' \in \mathcal{T}$, $\sigma \neq \sigma'$, $d_\mathcal{T}(\sigma,\sigma') = R(\{\sigma\},\{\sigma'\})$ and $d_\mathcal{T}(\sigma,\sigma) = 0$. This means that the the metric $d_\mathcal{T}$ is the effective resistance metric associated with $(\mathcal{E}_\mathcal{T}, \mathcal{F}_\mathcal{T})$, see Corollary 3.4 of [38] for a proof of this. As in the proof of Theorem 1.4.12, a consequence of this is that $\mathcal{F}_\mathcal{T} \subseteq L^2(\mathcal{T},\mu)$, which implies the Dirichlet form of interest in this section is simply $(\mathcal{E}_\mathcal{T}, \mathcal{F}_\mathcal{T})$.

We now recall, using the notation of this chapter, the construction of a diffusion from a Dirichlet form that was introduced in Chapter 2. Given the Dirichlet form $(\mathcal{E}_\mathcal{T}, \mathcal{F}_\mathcal{T})$, we can use the standard association to define a non-negative self-adjoint operator, $-\Delta_\mathcal{T}$, which has domain dense in $L^2(\mathcal{T},\mu)$ and satisfies

$$\mathcal{E}_\mathcal{T}(u,v) = -\int_\mathcal{T} u \Delta_\mathcal{T} v d\mu, \qquad \forall u \in \mathcal{F}_\mathcal{T}, v \in \mathcal{D}(\Delta_\mathcal{T}).$$

We can use this to define a reversible strong Markov process,

$$X = ((X_t)_{t\geq 0}, \mathbf{P}_\sigma^\mathcal{T}, \sigma \in \mathcal{T}),$$

with semi-group given by $P_t := e^{t\Delta_\mathcal{T}}$. In fact, the locality of our Dirichlet form ensures that the process X is a diffusion on $\mathcal{T}$.

As we observed in the previous chapter, a key factor in the description of the transition density of X, if it exists, is the volume growth of the space with respect to the resistance metric, which in this case is simply $d_\mathcal{T}$. The volume bounds we have already obtained for $\mathcal{T}$ mean that we can directly apply the bounds obtained there. The only condition which has not already been checked is the separability of $(\mathcal{T}, d_\mathcal{T})$, but for a metric space this follows easily from compactness, and so $(\mathcal{T}, d_\mathcal{T})$ is separable, $\mathbf{P}$-a.s. We are now able to state the main result of this section, which is an application of the volume bounds of earlier sections of this chapter and Theorem 2.4.1.

Theorem 3.9.1 $\mathbf{P}$-*a.s., there is a reversible strong Markov diffusion X on $\mathcal{T}$ with invariant measure μ and transition density $(p_t(\sigma,\sigma'))_{\sigma,\sigma'\in\mathcal{T},t>0}$, that satisfies the bounds at (3.2) and (3.3).*

Since, if it exists, the transition density of the process X is a heat kernel of $\Delta_{\mathcal{T}}$, we can state the previous result in the following alternative form. For full definitions of these two objects, see Section 2.5, and note that for an arbitrary heat kernel of $\Delta_{\mathcal{T}}$, the bounds we have proved will hold only μ-a.e.

Corollary 3.9.2 **P**-*a.s., there exists a local, regular Dirichlet form associated with* $(\mathcal{T}, d_{\mathcal{T}}, \mu)$. *The related non-positive self-adjoint operator,* $\Delta_{\mathcal{T}}$, *admits a heat kernel* $(p_t(\sigma,\sigma'))_{\sigma,\sigma'\in\mathcal{T},t>0}$, *that satisfies the bounds at (3.2) and (3.3).*

The proof of the remaining quenched heat kernel bounds also employ the techniques used in Chapter 2. However, as well as the volume bounds, we need to apply the following extra fact about the asymptotics of the resistance from the centre of a ball to its surface, as $r \to 0$.

Lemma 3.9.3 **P**-*a.s., for* μ-*a.e.* $\sigma \in \mathcal{T}$, *there exist constants* $c_{3.43}, r_2 > 0$ *such that*

$$c_{3.43} r \left(\ln\ln r^{-1}\right)^{-1} \leq R(\{\sigma\}, B(\sigma,r)^c) \leq r, \qquad \forall r \in (0, r_2).$$

Proof: Choosing r_2 to be small enough so that σ is connected to $B(\sigma,r)^c$ by a path of length r immediately implies the upper bound. For the lower bound, following the argument of [14], Lemma 4.4 we obtain

$$R(\{\sigma\}, B(\sigma,r)^c)^{-1} \leq \frac{8M(\sigma,r)}{r},$$

where $M(\sigma,r)$ is defined to be the smallest number such that there exists a set $A = \{\sigma_1, \dots, \sigma_{M(\sigma,r)}\}$ with $d_{\mathcal{T}}(\sigma,\sigma_i) = r/4$ for each i, such that any path from σ to $B(\sigma,r)^c$ must pass through the set A. For the continuum random tree it is elementary to deduce that $M(\rho,r) \leq N^r_{r/4}$, where $N^r_{r/4}$ is the number of upcrossings of $[r/4, r]$ by W, as defined in Section 3.8. Thus, applying Proposition 3.8.3, the result holds at the root. This may be extended to hold μ-a.e. using the random re-rooting of Lemma 3.5.1. $\square$

Proof of Theorem 3.1.6: As we noted in Chapter 2, on measure-metric spaces equipped with a resistance form, an upper bound for the on-diagonal part of the heat kernel of the form

$$p_{2r\mu(B(\sigma,r))}(\sigma,\sigma) \leq \frac{2}{\mu(B(\sigma,r))} \tag{3.24}$$

follows from a relatively simple analytic argument from the lower bound on the volume growth. Applying this, the two upper bounds of this result follow easily from the lower local volume bounds of Theorem 3.1.3 and so we omit their proof here.

It remains to prove the lower local heat kernel bound. Again, the proof of this result is standard and so we shall only outline it briefly here. First, the resistance result of Lemma 3.9.3 and the local volume results of Theorem 3.1.3 allow us to deduce that $\mathbf{P}$-a.s. for μ-a.e. $\sigma \in \mathcal{T}$, there exist constants $c_{3.44}, c_{3.45}, r_3 > 0$ such that, for $r \in (0, r_3)$,

$$\mathbf{E}_{\sigma'}^{\mathcal{T}} T_{B(\sigma,r)} \leq c_{3.44} r^3 \ln\ln r^{-1}, \qquad \forall \sigma' \in B(\sigma, r),$$
$$\mathbf{E}_{\sigma}^{\mathcal{T}} T_{B(\sigma,r)} \geq c_{3.45} r^3 \left(\ln\ln r^{-1}\right)^{-4},$$

by applying an argument similar to the proof of Proposition 2.6.6. Here, $T_{B(\sigma,r)}$ is the exit time of the process X from the ball $B(\sigma, r)$. Applying the Markov property for X as at (2.26) and substituting the above bounds for the expected exit times of balls yields

$$\mathbf{P}_{\sigma}^{\mathcal{T}} \left(T_{B(\sigma,r)} > t\right) \geq \frac{c_{3.45}}{c_{3.44}} \left(\ln\ln r^{-1}\right)^{-5} - \frac{t}{c_{3.44} r^3 \ln\ln r^{-1}}. \tag{3.25}$$

Also, recall from (2.28) that the Cauchy-Schwarz inequality implies

$$\mu(B(\sigma, r)) p_{2t}(\sigma, \sigma) \geq \mathbf{P}_{\sigma}^{\mathcal{T}} \left(T_{B(\sigma,r)} > t\right)^2.$$

Furthermore, by the volume asymptotics of Theorem 3.1.3, if we choose r_3 small enough, there exists a constant $c_{3.46}$ such that $\mu(B(\sigma, r)) \leq c_{3.46} r^2 \ln\ln r^{-1}$, for $r \in (0, r_3)$. Set $t_3 = \frac{c_{3.45}}{2} r_3^3 (\ln\ln r_3^{-1})^{-4}$. For $t \in (0, t_3)$, we can choose $r \in (0, r_3)$ to satisfy the equality $t = \frac{c_{3.45}}{2} r^3 (\ln\ln r^{-1})^{-4}$. Hence the lower bound for the tail of the exit time distribution at (3.25) implies that

$$p_{2t}(\sigma, \sigma) \geq c_{3.47} r^{-2} \left(\ln\ln r^{-1}\right)^{-11} \geq c_{3.48} t^{-2/3} \left(\ln\ln t^{-1}\right)^{-14}.$$

□

Proof of Theorem 3.1.5: The upper bound of (3.4) and the lower bound of (3.5) are contained in Theorem 3.9.1. The lower bound of (3.4) is a simple consequence of the local lower bound on the heat kernel of Theorem 3.1.6. The remaining inequality is proved using the analytic bound of (3.24); the volume bound we need to utilise in this case being the lower bound for $\sup_{\sigma} \mu(B(\sigma, r))$ appearing in Theorem 3.1.2. □

3.10 Annealed heat kernel bounds

Rather than $\mathbf{P}$-a.s. results about $\mu(B(\rho,r))$ and $R(\{\rho\},B(\rho,r)^c)$, we need to apply estimates on the tails of their distributions to obtain annealed heat kernel bounds. We have already proved one of the necessary bounds in Lemma 3.5.2; the remaining two bounds are proved in the following lemma. To complete the proof of Theorem 3.1.7, we employ a similar argument to the proof of [14], Theorem 1.4.

Lemma 3.10.1 *There exist constants $c_{3.49},\dots,c_{3.52}$ such that, for all $r>0,\lambda\geq 1$,*

$$\mathbf{P}\left(R(\{\rho\},B(\rho,r)^c)\leq r\lambda^{-1}\right)\leq c_{3.49}e^{-c_{3.50}\lambda},$$

and when $r^2\lambda^{-1}\leq \frac{1}{4}$,

$$\mathbf{P}\left(\mu(B(\rho,r))<r^2\lambda^{-1}\right)\leq c_{3.51}e^{-c_{3.52}\lambda}.$$

Proof: Let $r>0,\lambda\geq 1$. In the proof of Lemma 3.9.3 it was noted that $8r^{-1}N^r_{r/4}$ is an upper bound for $R(\{\rho\},B(\rho,r)^c)^{-1}$. Thus, by Proposition 3.8.4,

$$\begin{aligned}\mathbf{P}\left(R(\{\rho\},B(\rho,r)^c)\leq r\lambda^{-1}\right) &\leq \mathbf{P}\left(8N^r_{r/4}\geq\lambda\right)\\ &\leq c_{3.32}e^{-\frac{c_{3.33}\lambda}{8}},\end{aligned}$$

which proves the first inequality.

For the second inequality, suppose $r^2\lambda^{-1}\leq\frac{1}{4}$. Observe that if $\mu(B(\rho,r))$ is strictly less than $r^2\lambda^{-1}$, then the normalised Brownian excursion must hit the level r before time $r^2\lambda^{-1}$. Thus

$$\mathbf{P}\left(\mu(B(\rho,r))<r^2\lambda^{-1}\right)\leq\mathbf{P}\left(\sup_{0\leq t\leq r^2\lambda^{-1}}W_t\geq r\right).$$

The explicit distribution of the maximum of the Brownian excursion up to a fixed time is known and we can use the formula given in [22], Section 3, to show that the right hand side of this inequality is equal to

$$1-\sqrt{\frac{2\lambda^3}{\pi r^6(1-r^2\lambda^{-1})^3}}\sum_{m=-\infty}^{\infty}e^{-2m^2r^2}\int_0^r y(2mr+y)e^{-\frac{(y+2mr(1-r^2\lambda^{-1}))^2}{2r^2\lambda^{-1}(1-r^2\lambda^{-1})}}dy.$$

We can neglect the terms with $m>0$ as removing them only increases this expression. By changing variables in the integral, it is possible to show that the $m=0$ term is equal to

$$\sqrt{\frac{2}{\pi}}\int_0^{\sqrt{\frac{\lambda}{1-r^2\lambda^{-1}}}}u^2e^{-\frac{u^2}{2}}du.$$

Integrating by parts and applying standard bounds for the error function, it is elementary to obtain that this is bounded below by $1 - c_{3.53}\sqrt{\lambda}e^{-\frac{\lambda}{2}}$. For the remaining terms we have

$$
\begin{aligned}
&-\sqrt{\frac{2\lambda^3}{\pi r^6(1-r^2\lambda^{-1})^3}}\sum_{m=-\infty}^{-1} e^{-2m^2r^2}\int_0^r y(2mr+y)e^{-\frac{(y+2mr(1-r^2\lambda^{-1}))^2}{2r^2\lambda^{-1}(1-r^2\lambda^{-1})}}dy \\
\leq\ & \sqrt{\frac{2\lambda^3}{\pi r^6(1-r^2\lambda^{-1})^3}}\sum_{m=1}^{\infty}\int_0^r 2mr^2 e^{-\frac{(y-2mr(1-r^2\lambda^{-1}))^2}{2r^2\lambda^{-1}(1-r^2\lambda^{-1})}}dy \\
\leq\ & c_{3.54}\lambda^{3/2}\sum_{m=1}^{\infty} me^{-\frac{\lambda(3m-2)^2}{8}}.
\end{aligned}
$$

The sum in this expression may be bounded above by $c_{3.55}e^{-\frac{\lambda}{8}}$. Thus

$$
\mathbf{P}\left(\mu(B(\rho,r)) < r^2\lambda^{-1}\right) \leq c_{3.53}\sqrt{\lambda}e^{-\frac{\lambda}{2}} + c_{3.54}c_{3.55}\lambda^{3/2}e^{-\frac{\lambda}{8}} \leq c_{3.56}e^{-c_{3.57}\lambda},
$$

for suitable choice of $c_{3.56}, c_{3.57}$. □

Proof of Theorem 3.1.7: Let $t \in (0,1)$. For $\lambda \geq 2$, define r by $t = 2\lambda r^3$ and $A_{\lambda,t} := \{r^2\lambda^{-1} \leq \mu(B(\rho,r)) \leq r^2\lambda\}$. On $A_{\lambda,t}$, we can use the inequality at (3.24) to show that

$$
p_t(\rho,\rho) \leq \frac{2^{5/3}\lambda^{5/3}}{t^{2/3}}.
$$

Define $\Lambda_t := \inf\{\lambda \geq 2 :\ A_{\lambda,t} \text{ occurs}\}$, then $\mathbf{E}p_t(\rho,\rho) \leq 2^{5/3}t^{-2/3}\mathbf{E}\Lambda_t^{5/3}$. However, for $\lambda \geq 2$,

$$
\begin{aligned}
\mathbf{P}(\Lambda_t \geq \lambda) &\leq \mathbf{P}(A^c_{\lambda,t}) \\
&\leq \mathbf{P}(\mu(B(\rho,r)) > r^2\lambda) + \mathbf{P}(\mu(B(\rho,r)) < r^2\lambda^{-1}).
\end{aligned}
$$

Since $r^2\lambda^{-1} = t^{2/3}2^{-2/3}\lambda^{-5/3} \leq \frac{1}{4}$, we can apply the tail bounds of Lemma 3.10.1 to obtain that $\mathbf{P}(\Lambda_t \geq \lambda) \leq c_{3.58}e^{-c_{3.59}\lambda}$, uniformly in $t \in (0,1)$. Thus $\mathbf{E}\Lambda_t^{5/3} \leq c_{3.60} < \infty$, uniformly in $t \in (0,1)$, which proves the upper bound.

For the lower bound we need a slightly different scaling. Let $t \in (0,1)$, $\lambda \geq 64$ and define r by $t = r^3/64\lambda^4$, and

$$
B_{\lambda,t} := \left\{\mu(B(\rho,r)) \leq r^2\lambda,\ R(\{\rho\},B(\rho,r)^c) \geq r\lambda^{-1},\ \mu(B(\rho,\frac{r}{4\lambda})) \geq \frac{r^2}{16\lambda^3}\right\}.
$$

On $B_{\lambda,t}$, by following a similar argument to that used for the proof of Theorem 3.1.6, we find that $p_t(\rho,\rho) \geq c_{3.61}t^{-2/3}\lambda^{-14}$. Now,

$$
\begin{aligned}
\mathbf{P}(B^c_{\lambda,t}) \leq\ & \mathbf{P}(\mu(B(\rho,r)) > r^2\lambda) + \mathbf{P}(R(\{\rho\},B(\rho,r)^c) < r\lambda^{-1}) \\
&+\mathbf{P}(\mu(B(\rho,\frac{r}{4\lambda})) < \frac{r^2}{16\lambda^3}).
\end{aligned}
$$

Since $r^2/16\lambda^3 = t^{2/3}\lambda^{-1/3} \leq \frac{1}{4}$, again we can apply the bounds of Lemma 3.10.1 to find that $\mathbf{P}(B^c_{\lambda,t}) \leq c_{3.62}e^{-c_{3.63}\lambda}$, uniformly in $t \in (0,1)$. Hence we can find a $\lambda_0 \in [64,\infty)$ such that $\mathbf{P}(B^c_{\lambda_0,t}) \leq \frac{1}{2}$ for all $t \in (0,1)$. Thus

$$\mathbf{E}p_t(\rho,\rho) \geq \mathbf{P}(B_{\lambda_0,t})\frac{c_{3.61}}{\lambda_0^{14}t^{2/3}} \geq c_{3.64}t^{-2/3}, \qquad \forall t \in (0,1),$$

for some $c_{3.64} > 0$. □

3.11 Brownian motion on the CRT

To complete the proof of Theorem 3.1.4, it remains to show that the Markov process with infinitesimal generator $\Delta_{\mathcal{T}}$ is Brownian motion on $\mathcal{T}$. Brownian motion on $\mathcal{T}_f$ is defined to be a $\mathcal{T}_f$-valued process, $X^f = ((X^f_t)_{t\geq 0}, \mathbf{P}^{\mathcal{T}_f}_\sigma, \sigma \in \mathcal{T}_f)$, with the following properties.

i) Continuous sample paths.

ii) Strong Markov.

iii) Reversible with respect to its invariant measure μ_f.

iv) For $\sigma^1, \sigma^2 \in \mathcal{T}_f$, $\sigma^1 \neq \sigma^2$, we have

$$\mathbf{P}^{\mathcal{T}_f}_\sigma(T_{\sigma^1} < T_{\sigma^2}) = \frac{d_{\mathcal{T}_f}(b(\sigma,\sigma^1,\sigma^2),\sigma^2)}{d_{\mathcal{T}_f}(\sigma^1,\sigma^2)}, \qquad \sigma \in \mathcal{T}_f,$$

where $T_\sigma := \inf\{t \geq 0 : X^f_t = \sigma\}$ and $b(\sigma,\sigma^1,\sigma^2)$ is the branch point of $\sigma, \sigma^1, \sigma^2$ in $\mathcal{T}$, as defined in Section 1.2.

v) For $\sigma^1, \sigma^2 \in \mathcal{T}_f$, the mean occupation measure for the process started at σ^1 and killed on hitting σ^2 has density

$$d_{\mathcal{T}_f}(b(\sigma,\sigma^1,\sigma^2),\sigma^2)\mu(d\sigma), \qquad \sigma \in \mathcal{T}_f.$$

As remarked in [2], Section 5.2, these properties are enough for uniqueness of Brownian motion on $\mathcal{T}_f$. Note that the definition given by Aldous has an extra factor of 2 in property v). This is a result of Aldous' description of the continuum random tree being based on the random function $2W$. By Theorem 3.9.1, we already have that properties i), ii) and iii) hold for the process X on $\mathcal{T}$, $\mathbf{P}$-a.s. Before proceeding with demonstrating that X satisfies the remaining properties, we first need to prove the following technical result on the capacity of sets of $\mathcal{T}$, where we use the notation $\tilde{U}$ to represent the set of excursions on which a finite resistance form is defined on $\mathcal{T}_f$, as in Section 3.9.

Lemma 3.11.1 *For $f \in \tilde{U}$, all non-empty subsets of $\mathcal{T}_f$ have strictly positive capacity.*

Proof: This result may be deduced by applying the same argument as in the proof of Proposition 1.5.3(b). □

The above result allows us to define local times for X, and in the following lemma we use these to define a time-changed process on a finite subset of $\mathcal{T}$. By considering the hitting probabilities for the time-changed process, we are able to deduce that X satisfies property iv) of the Brownian motion definition. An important tool for the proof of this and property v) will be the trace operator for Dirichlet forms, as defined at (1.5). In particular, we will use the fact that the quadratic form corresponding to our time-changed process is simply the trace of $\mathcal{E}$ on the same finite subset.

Lemma 3.11.2 **P**-*a.s., the process X of Theorem 3.9.1 satisfies property iv) of the definition of Brownian motion on $\mathcal{T}$.*

Proof: Suppose $W \in \tilde{U}$, so that the resistance form, $\mathcal{E}_\mathcal{T}$, and process, X, are defined for $\mathcal{T}$. Fix $\sigma, \sigma^1, \sigma^2 \in \mathcal{T}$, $\sigma^1 \neq \sigma^2$, and set $b = b(\sigma, \sigma^1, \sigma^2)$. Write $V_1 = \{\sigma, \sigma^1, \sigma^2, b\}$ and $\mathcal{E}_1 = \mathrm{Tr}(\mathcal{E}_\mathcal{T}|V_1)$. Using simple properties of resistance forms, the following explicit expression for $\mathcal{E}_1$ can be calculated:

$$\mathcal{E}_1(u,u) = \sum_{\sigma' \in \{\sigma, \sigma^1, \sigma^2\}} \frac{(u(b) - u(\sigma'))^2}{d_\mathcal{T}(b, \sigma')}, \qquad u \in C(V_1), \tag{3.26}$$

where, if $b = \sigma'$, the relevant term is defined to be 0.

By the previous lemma, $\{\sigma'\}$ has strictly positive capacity for each $\sigma' \in \mathcal{T}$. As outlined in Section 4 of [9], a result of this is that X has jointly measurable local times $(L_t^{\sigma'}, \sigma' \in \mathcal{T}, t \geq 0)$ such that

$$\int_0^t u(X_s)ds = \int_\mathcal{T} u(\sigma') L_t^{\sigma'} \mu(d\sigma'), \qquad u \in L^2(\mathcal{T}, \mu).$$

Now, denote $\nu := \frac{1}{|V_1|} \sum_{\sigma' \in V_1} \delta_{\sigma'}$, the uniform distribution on V_1 and define

$$A_t := \int_\mathcal{T} L_t^{\sigma'} \nu(d\sigma'), \qquad \tau_t := \inf\{s : \ A_s > t\}.$$

Consider the process $\tilde{X} = (\tilde{X}, \mathbf{P}^{\mathcal{T}}_{\sigma'}, \sigma' \in V_1)$, defined by $\tilde{X}_t := X_{\tau_t}$. As described in [9], $\tilde{X}$ is a ν-symmetric Hunt process and has associated regular Dirichlet form $(\mathcal{E}_1, C(V_1))$. Using elementary theory for continuous time Markov chains on a finite state space, we obtain the following result for $\tilde{X}$

$$\mathbf{P}^{\mathcal{T}}_{\sigma}\left(\tilde{T}_{\sigma^1} < \tilde{T}_{\sigma^2}\right) = \frac{d_{\mathcal{T}}(b, \sigma^2)}{d_{\mathcal{T}}(\sigma^1, \sigma^2)},$$

where $\tilde{T}_{\sigma'} := \inf\{t \geq 0 : \tilde{X}_t = \sigma'\}$. Since the hitting distribution is unaffected by the time change from X to $\tilde{X}$, this implies

$$\mathbf{P}^{\mathcal{T}}_{\sigma}\left(T_{\sigma^1} < T_{\sigma^2}\right) = \frac{d_{\mathcal{T}}(b, \sigma^2)}{d_{\mathcal{T}}(\sigma^1, \sigma^2)},$$

and so, if $W \in \tilde{U}$, the process X satisfies property iv) of the Brownian motion definition. Since $W \in \tilde{U}$, $\mathbf{P}$-a.s., this completes the proof. □

A result that will be useful in proving that X satisfies property v) is the following uniqueness result, which is proved in [38], Lemma 3.5.

Lemma 3.11.3 *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a set K and V be a finite subset of K. Then for any $v \in C(V)$, there exists a unique $u \in \mathcal{F}$ such that*

$$\mathcal{E}(u, u) = \mathrm{Tr}(\mathcal{E}|V)(v, v), \qquad u|_V = v.$$

Lemma 3.11.4 $\mathbf{P}$-*a.s., the process X of Theorem 3.9.1 satisfies property v) of the definition of Brownian motion on $\mathcal{T}$.*

Proof: First, assume $W \in \tilde{U}$, so that the resistance form, $\mathcal{E}_{\mathcal{T}}$, and process, X, are defined for $\mathcal{T}$. Fix $\sigma^1, \sigma^2 \in \mathcal{T}$, $\sigma^1 \neq \sigma^2$, and define $D = D(\sigma^1, \sigma^2)$ to be the path-connected component of $\mathcal{T} \backslash \{\sigma^2\}$ containing σ^1. Using the same argument as in [41], Proposition 4.2, we can deduce the existence of a Green kernel $g^D(\cdot, \cdot)$ for the process killed on exiting D which satisfies

$$\mathcal{E}_{\mathcal{T}}(g^D(\sigma, \cdot), f) = f(\sigma), \qquad \forall \sigma \in \mathcal{T}, f \in \mathcal{F}_D, \tag{3.27}$$

where $\mathcal{F}_D := \{f \in \mathcal{F}_{\mathcal{T}} : f|_{D^c} = 0\}$. By standard arguments, this implies that $g^D(\sigma^1, \sigma^1) > 0$;

$$\mathcal{E}_{\mathcal{T}}(\tilde{g}, \tilde{g}) = \inf\{\mathcal{E}_{\mathcal{T}}(u, u) : u(\sigma^1) = 1, u(\sigma^2) = 0\}, \tag{3.28}$$

where $\tilde{g}(\cdot) := g^D(\sigma^1, \cdot)/g^D(\sigma^1, \sigma^1)$; and for μ-measurable f,

$$\mathbf{E}^{\sigma^1} \int_0^{T_{\sigma^2}} f(X_s)ds = \int_{\mathcal{T}} g^D(\sigma^1, \sigma) f(\sigma) \mu(d\sigma).$$

This means that $g^D(\sigma^1, \sigma)\mu(d\sigma)$ is the mean occupation density of the process started at σ^1 and killed on hitting σ^2. Furthermore, note that combining (3.27), (3.28) and the characterisation of $d_{\mathcal{T}}$ at (3.23), we can deduce that $g^D(\sigma^1, \sigma^1) = d_{\mathcal{T}}(\sigma^1, \sigma^2)$.

Now, fix $\sigma \in \mathcal{T}$, and define $b := b(\sigma, \sigma^1, \sigma^2)$,

$$V_1 := \{\sigma, \sigma^1, \sigma^2, b\}, \qquad \mathcal{E}_1 := \mathrm{Tr}(\mathcal{E}_{\mathcal{T}} | V_1),$$

$$V_0 := \{\sigma^1, \sigma^2\}, \qquad \mathcal{E}_0 := \mathrm{Tr}(\mathcal{E}_1 | V_0).$$

Let $f_0 \in C(V_0)$ be defined by $f_0(\sigma^1) = 1$, $f_0(\sigma^2) = 0$; f_1 be the unique (by Lemma 3.11.3) function in $C(V_1)$ that satisfies $\mathcal{E}_1(f_1, f_1) = \mathcal{E}_0(f_0, f_0)$ and $f_1|_{V_0} = f_0$; and f_2 be the unique function in $\mathcal{F}$ such that $\mathcal{E}_{\mathcal{T}}(f_2, f_2) = \mathcal{E}_1(f_1, f_1)$ and $f_2|_{V_1} = f_1$. Applying the tower property for the trace operator, $\mathcal{E}_0 = \mathrm{Tr}(\mathrm{Tr}(\mathcal{E}_{\mathcal{T}}|V_1)|V_0) = \mathrm{Tr}(\mathcal{E}_{\mathcal{T}}|V_0)$, we have that f_2 is the unique function that satisfies

$$\mathcal{E}_{\mathcal{T}}(f_2, f_2) = \mathrm{Tr}(\mathcal{E}_{\mathcal{T}}|V_0)(f_0, f_0), \qquad f_2|_{V_0} = f_0.$$

However, we have from (3.28) that $\tilde{g}$ also has these properties and so it follows from the uniqueness of Lemma 3.11.3 that $\tilde{g} = f_2$. Thus $\tilde{g}|_{V_1} = f_1$. Recall the explicit expression for $\mathcal{E}_1$ given at (3.26). A simple minimisation of this quadratic polynomial allows us to determine the function f_1. In particular, we have $\tilde{g}(\sigma) = f_1(\sigma) = d_{\mathcal{T}}(b, \sigma^2) d_{\mathcal{T}}(\sigma^1, \sigma^2)^{-1}$. Hence the mean occupation density of the process started at σ^1 and killed on hitting σ^2 is

$$g^D(\sigma^1, \sigma)\mu(d\sigma) = g^D(\sigma^1, \sigma^1)\tilde{g}(\sigma)\mu(d\sigma) = d_{\mathcal{T}}(b, \sigma^2)\mu(d\sigma).$$

Thus, if $W \in \tilde{U}$, the process X satisfies property v) of the Brownian motion definition. Since $W \in \tilde{U}$, $\mathbf{P}$-a.s., this completes the proof. □

Combining the results of Theorem 3.9.1 and Lemmas 3.11.2 and 3.11.4 we immediately have the following.

Corollary 3.11.5 $\mathbf{P}$-*a.s., the process X of Theorem 3.9.1 is Brownian motion on $\mathcal{T}$.*

Appendix A

Exact self-similarity of the continuum random tree

In this appendix, we demonstrate that the continuum random tree is precisely a random p.c.f.s.s. dendrite of the type constructed in Chapter 1. This characterisation links the continuum random tree, and all of its representations, with yet another area, namely analysis on self-similar fractals. The description we provide here is of particular interest as it allows the continuum random tree to be be built upon a highly structured, deterministic subset of $\mathbb{R}^2$, which is a striking contrast to some of its abstract tree formulations. The main idea that we will apply is the recursive self-similarity for the continuum random tree, which was proved by Aldous in [5], and is stated here as Lemma A.1.1.

A.1 Decomposition of the continuum random tree

To make precise the decomposition of the continuum random tree that we shall apply, we use the excursion description of the set, as introduced in Section 3.2.2. This allows us to prove rigorously the independence properties that are important to our argument. However, as with the random re-rooting of Lemma 3.5.1, it may not be immediately obvious exactly what the excursion picture is telling us about the continuum random tree, and so, after Lemma A.1.1, we present a more heuristic discussion of the procedure we use in terms of the related dendrites.

The initial object of consideration is an independent triple (W, U, V), where W is a normalised Brownian excursion, and U and V are $U[0,1]$ random variables. From this triple, it is possible to define three independent Brownian excursions. The following decomposition is rather awkward to write down, but is made clearer by Figure A.1.

First, suppose $U<V$. On this set, it is $\mathbf{P}$-a.s. possible to define $H\in[0,1]$ by

$$\{H\}:=\{t\in[U,V]:\ W_t=\inf_{s\in[U,V]}W_s\}. \tag{A.1}$$

We also define

$$H_-:=\sup\{t<U:\ W_t=W_H\},\qquad H_+:=\inf\{t>V:\ W_t=W_H\}, \tag{A.2}$$

$$\begin{gathered}\Delta_1:=1+H_--H_+,\qquad \Delta_2:=H-H_-,\qquad \Delta_3:=H_+-H,\\ \tilde{U}_1:=\frac{H_-}{\Delta_1},\qquad U_2:=\frac{U-H_-}{\Delta_2},\qquad U_3:=\frac{V-H}{\Delta_3},\end{gathered}$$

and for $t\in[0,1]$,

$$\begin{aligned}\tilde{W}^1_t&:=\Delta_1^{-1/2}(W_{t\Delta_1}\mathbf{1}_{\{t\leq\tilde{U}_1\}}+W_{H_++(t-\tilde{U}_1)\Delta_1}\mathbf{1}_{\{t>\tilde{U}_1\}}),\\ W^2_t&:=\Delta_2^{-1/2}(W_{H_-+t\Delta_2}-W_H),\\ W^3_t&:=\Delta_3^{-1/2}(W_{H+t\Delta_3}-W_H).\end{aligned}$$

Finally, define W^1 to be $\tilde{W}^1$ shifted by $\tilde{U}_1$, via the formula given in Section 3.7, and set $U_1:=1-\tilde{U}_1$. If $U>V$, the definition of these quantities is similar, with W^1 again being the rescaled, re-rooted excursion containing $t=0$, W^2 being the rescaled excursion containing $t=U$, and W^3 being the rescaled excursion containing $t=V$. A minor adaptation of [5], Corollary 3, using the invariance under random re-rooting of the continuum random tree (Lemma 3.5.1), then gives us the following result, which we state without proof.

Lemma A.1.1 *The quantities W^1,W^2,W^3,U_1,U_2,U_3 and $(\Delta_1,\Delta_2,\Delta_3)$ are independent. Each W^i is a normalised Brownian excursion, each U_i is $U[0,1]$, and the triple $(\Delta_1,\Delta_2,\Delta_3)$ has the Dirichlet $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ distribution.*

Describing the result in terms of the corresponding trees gives a much clearer picture of what the above decomposition does. Using the notation of Chapter 3, let $(\mathcal{T},d_{\mathcal{T}},\mu)$ be the continuum random tree associated with W, and $\rho=[0]$ its root. Here, we use $[t]$, for $t\in[0,1]$, to represent the equivalence classes of $[0,1]$ under the equivalence relation defined at (3.7). If we define $Z^1:=[U]$ and $Z^2:=[V]$, then Z^1 and Z^2 are two independent μ-random vertices of $\mathcal{T}$. We now split the tree $\mathcal{T}$ at the branch point $b(\rho,Z^1,Z^2)$, which may be checked to be equal to $[H]$, and denote by $\mathcal{T}^1$, $\mathcal{T}^2$ and $\mathcal{T}^3$ the components of $\mathcal{T}$ containing ρ, Z^1 and Z^2 respectively. Choose the root of each subtree to be equal to $b(\rho,Z^1,Z^2)$ and, for $i=1,2,3$, let μ^i be the

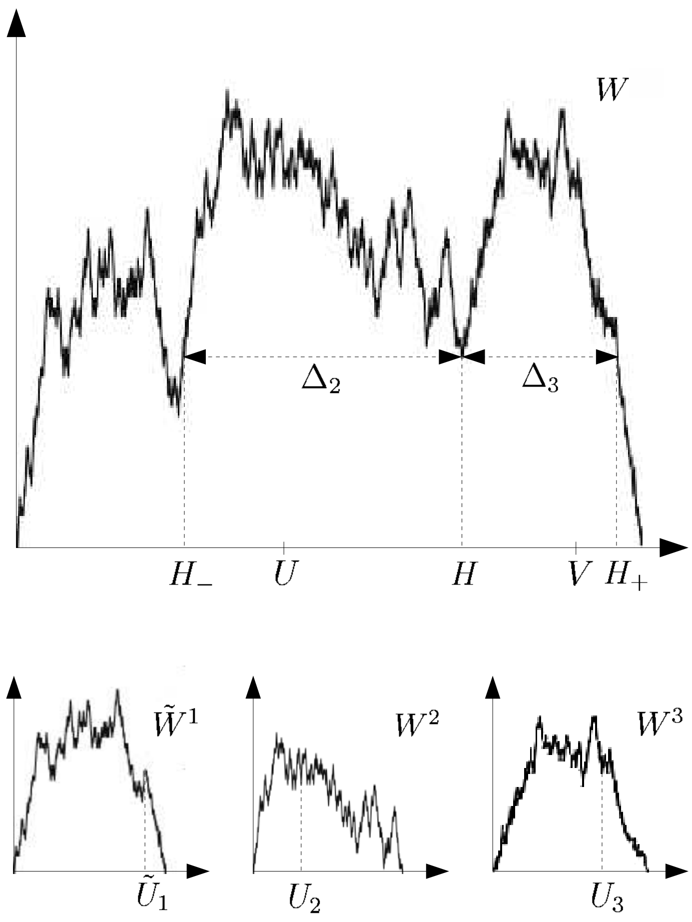

Figure A.1: Brownian excursion decomposition.

probability measure on $\mathcal{T}^i$ defined by $\mu^i(A) = \mu(A)/\Delta_i$, for measurable $A \subseteq \mathcal{T}^i$, where $\Delta_i := \mu(\mathcal{T}^i)$. The previous result tells us precisely that $(\mathcal{T}^i, \Delta_i^{-1/2} d_{\mathcal{T}}, \mu^i)$, $i = 1, 2, 3$, are three independent copies of $(\mathcal{T}, d_{\mathcal{T}}, \mu)$. Furthermore, if $Z_i := \rho, Z^1, Z^2$ for $i = 1, 2, 3$, respectively, then Z_i is a μ^i-random variable in $\mathcal{T}^i$. Finally, all these quantities are independent of the masses $(\mu(\mathcal{T}^1), \mu(\mathcal{T}^2), \mu(\mathcal{T}^3))$, which form a Dirichlet $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ triple. Although it is possible to deal with the subtrees directly using conditional definitions of the random variables to decompose the continuum random tree in this way, the excursion description allows us to keep track of exactly what is independent more easily, and it is to this setting that we return. However, we shall not completely neglect the tree description of the algorithm we now introduce, and a summary in this vein appears below Figure A.2.

We start by applying inductively the decomposition map from $U^{(1)} \times [0, 1]^2$ to ${U^{(1)}}^3 \times [0, 1]^3 \times \Delta$ (where Δ is the standard 2-simplex) that takes the triple (W, U, V) to the collection $(W^1, W^2, W^3, U_1, U_2, U_3, (\Delta_1, \Delta_2, \Delta_3))$ of excursions and uniform and

Dirichlet random variables. We shall denote the decomposition map by Υ and index the random variables by Σ_*, the address space introduced in Chapter 1, where in this case $S = \{1, 2, 3\}$. First, suppose we are given an independent collection $(W, U, (V_i)_{i\in\Sigma_*})$, where W is a normalised Brownian excursion, U is $U[0,1]$, and $(V_i)_{i\in\Sigma_*}$ is a family of independent $U[0,1]$ random variables. Set $(W^\emptyset, U_\emptyset) := (W, U)$. Given (W^i, U_i), define

$$(W^{i1}, W^{i2}, W^{i3}, U_{i1}, U_{i2}, U_{i3}, (\Delta_{i1}, \Delta_{i2}, \Delta_{i3})) := \Upsilon(W^i, U_i, V_i),$$

and denote the filtration associated with $(\Delta_i)_{i\in\Sigma_*\backslash\{\emptyset\}}$ by $(\mathcal{F}_n)_{n\geq 0}$. In particular, $\mathcal{F}_n := \sigma(\Delta_i : |i| \leq n)$. The subsequent result is easily deduced by applying the previous lemma repeatedly.

Theorem A.1.2 *For each n, $((W^i, U_i, V_i))_{i\in\Sigma_n}$ is an independent collection of independent triples consisting of a normalised Brownian excursion and two $U[0,1]$ random variables, and moreover, the entire family of random variables is independent of $\mathcal{F}_n$.*

From this result, it is clear that $(\Delta_i)_{i\in\Sigma_*\backslash\{\emptyset\}}$ forms a multiplicative cascade in the sense of Section 1.3, with related filtration $(\mathcal{F}_n)_{n\geq 0}$. Furthermore, Lemma A.1.1 implies that each triple of the form $(\Delta_{i1}, \Delta_{i2}, \Delta_{i3})$ has the Dirichlet $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ distribution. We shall also be interested in the collection $(w(i))_{i\in\Sigma_*\backslash\{\emptyset\}}$, where for each i, we define

$$w(i) := \Delta_i^{1/2}.$$

Note that this is also a multiplicative cascade, with the same associated filtration, $(\mathcal{F}_n)_{n\geq 0}$. We shall write $l(i)$ to represent the product $w(i|1)w(i|2)\dots w(i||i|)$, as in Chapter 1. The reason for considering such families is that, in our decomposition of the continuum random tree, $(\Delta_i)_{i\in\Sigma_*\backslash\{\emptyset\}}$ and $(w(i))_{i\in\Sigma_*\backslash\{\emptyset\}}$ represent the mass and length scaling factors respectively.

By viewing the inductive procedure for decomposing excursions as the repeated splitting of trees in the way described after Lemma A.1.1, it is possible to use the above algorithm to break the continuum random tree into smaller components, with the subtrees in the nth level of construction being described by the excursions $(W^i)_{i\in\Sigma_n}$. The maps we now introduce will make this idea precise. For the remainder of this section, the arguments that we give hold $\mathbf{P}$-a.s. First, denote by H^i, H^i_- and H^i_+ the random variables in $[0,1]$ associated with (W^i, U_i, V_i) by the formulae at (A.1) and (A.2). Let $i \in \Sigma_*$. Define, for $t \in [0,1]$,

$$\phi_{i1}(t) := (H^i_+ + t\Delta_{i1})\mathbf{1}_{\{t<U_{i1}\}} + (t - U_{i1})\Delta_{i1}\mathbf{1}_{\{t\geq U_{i1}\}},$$

and if $U_i < V_i$, define ϕ_{i2} and ϕ_{i3} to be the linear contractions from $[0,1]$ to $[H^i_-, H^i]$ and $[H^i, H^i_+]$ respectively. If $U_i > V_i$, the images of ϕ_{i2} and ϕ_{i3} are reversed. Note that, for each i, the map ϕ_i satisfies, for any measurable $A \subseteq [0,1]$,

$$\lambda(\phi_i(A)) = \Delta_i \lambda(A), \tag{A.3}$$

where λ is the usual Lebesgue measure on $[0,1]$. Importantly, these maps also satisfy a certain distance scaling property. In particular, it is elementary to check from the definitions of the excursions that, for any $i \in \Sigma_*$, $j \in S$,

$$d_{W^i}(\phi_{ij}(s), \phi_{ij}(t)) = w(ij) d_{W^{ij}}(s,t), \qquad \forall s,t \in [0,1], \tag{A.4}$$

where d_{W^i} is the distance on $[0,1]$ associated with W^i by the definition at (3.6). This equality allows us to define a map on the trees related to the excursions. Let $(\tilde{\mathcal{T}}_i, d_{\tilde{\mathcal{T}}_i})$ be the metric space dendrite determined from W^i by the equivalence relation given at (3.7). Denote the corresponding equivalence classes $[t]_i$ for $t \in [0,1]$. Now define, for $i \in \Sigma_*$, $j \in S$,

$$\begin{aligned} \tilde{\phi}_{ij} : \tilde{\mathcal{T}}_{ij} &\to \tilde{\mathcal{T}}_i \\ [t]_{ij} &\mapsto [\phi_{ij}(t)]_i. \end{aligned}$$

The following result demonstrates that this is a well-defined map satisfying a distance scaling property related to (A.4).

Lemma A.1.3 **P**-*a.s., for every* $i \in \Sigma_*$, $j \in S$, $\tilde{\phi}_{ij}$ *is well-defined and moreover,*

$$d_{\tilde{\mathcal{T}}_i}(\tilde{\phi}_{ij}(x), \tilde{\phi}_{ij}(y)) = w(ij) d_{\tilde{\mathcal{T}}_{ij}}(x,y), \qquad \forall x,y \in \tilde{\mathcal{T}}_{ij}.$$

Proof: First, choose $s,t \in [0,1]$. Applying the distance scaling property of (A.4), we find that

$$d_{\tilde{\mathcal{T}}_i}([\phi_{ij}(s)]_i, [\phi_{ij}(t)]_i) = d_{W^i}(\phi_{ij}(s), \phi_{ij}(t)) = w(ij) d_{W^{ij}}(s,t) = w(ij) d_{\tilde{\mathcal{T}}^{ij}}([s]_{ij}, [t]_{ij}).$$

Thus, if $[s]_{ij} = [t]_{ij}$, the right hand side of this equation is 0, and it follows that $[\phi_{ij}(s)]_i = [\phi_{ij}(t)]_i$. Hence $\tilde{\phi}_{ij}$ is indeed a well-defined map. The desired distance scaling property is also an immediate consequence of the above equation. $\square$

By iterating the functions $(\tilde{\phi}_i)_{i\in\Sigma_*\backslash\{\emptyset\}}$, we can map any $\tilde{\mathcal{T}}_i$ to the original continuum random tree, $\mathcal{T}\equiv\tilde{\mathcal{T}}_\emptyset$, which is the object of interest. We will denote the map from $\tilde{\mathcal{T}}_i$ to $\mathcal{T}$ by $\tilde{\phi}_{*i}:=\tilde{\phi}_{i|1}\circ\tilde{\phi}_{i|2}\circ\dots\circ\tilde{\phi}_i$, and its image by

$$\mathcal{T}_i:=\tilde{\phi}_{*i}(\tilde{\mathcal{T}}_i).$$

It is these sets that form the basis of our decomposition of $\mathcal{T}$. We will also have cause to refer to the following points in $\mathcal{T}_i$:

$$\rho_i:=\tilde{\phi}_{*i}([0]_i),\qquad Z_i^1:=\tilde{\phi}_{*i}([U_i]_i),\qquad Z_i^2:=\tilde{\phi}_{*i}([V_i]_i).$$

Although it has been quite hard work arriving at the definition of $(\mathcal{T}_i)_{i\in\Sigma_*}$, the properties of this family of sets that we will need are derived without too many difficulties from the construction. The proposition we now prove includes the following results: the sets $(\mathcal{T}_i)_{i\in\Sigma_n}$ cover $\mathcal{T}$; $\mathcal{T}_i$ is simply a rescaled copy of $\tilde{\mathcal{T}}_i$ with μ-measure $l(i)^2$; the overlaps of sets in the collection $(\mathcal{T}_i)_{i\in\Sigma_n}$ are small; and also describes various relationships between points of the form ρ_i, Z_i^1 and Z_i^2.

Proposition A.1.4 **P**-*a.s., for every* $i\in\Sigma_*$,
(a) $\mathcal{T}_i=\cup_{j\in\Sigma_n}\mathcal{T}_{ij}$, *for all* $n\geq 0$.
(b) $(\mathcal{T}_i,d_{\mathcal{T}})$ *and* $(\tilde{\mathcal{T}}_i,l(i)d_{\tilde{\mathcal{T}}_i})$ *are isometric.*
(c) $\rho_{i1}=\rho_{i2}=\rho_{i3}=b(\rho_i,Z_i^1,Z_i^2)$.
(d) $Z_{ij}^1=\rho_i,Z_i^1,Z_i^2$, *for* $j=1,2,3$ *respectively.*
(e) $\rho_i\notin\mathcal{T}_{i2}\cup\mathcal{T}_{i3}$, $Z_i^1\notin\mathcal{T}_{i1}\cup\mathcal{T}_{i3}$ *and* $Z_i^2\notin\mathcal{T}_{i1}\cup\mathcal{T}_{i2}$.
(f) if $|j|=|i|$, *but* $j\neq i$, *then* $\mathcal{T}_i\cap\mathcal{T}_j\subseteq\{\rho_i,Z_i^1\}$.
(g) $\mu(\mathcal{T}_i)=l(i)^2$.
Proof: By induction, it suffices to show that (a) holds for $n=1$. By definition, we have $\cup_{j\in S}\phi_{ij}([0,1])=[0,1)$, and so

$$\begin{aligned}\tilde{\mathcal{T}}_i &= \{[t]_i:\ t\in[0,1)\}\\ &= \cup_{j\in S}\{[t]_i:\ t\in\phi_{ij}([0,1])\}\\ &= \cup_{j\in S}\{[\phi_{ij}(t)]_i:\ t\in[0,1]\}\\ &= \cup_{j\in S}\tilde{\phi}_{ij}(\tilde{\mathcal{T}}_{ij}),\end{aligned}$$

where we apply the definition of $\tilde{\phi}_{ij}$ for the final equality. Hence

$$\mathcal{T}_i=\tilde{\phi}_{*i}(\tilde{\mathcal{T}}_i)=\cup_{j\in S}\tilde{\phi}_{*ij}(\tilde{\mathcal{T}}_{ij})=\cup_{j\in S}\mathcal{T}_{ij},$$

which completes the proof of (a). Part (b) is an immediate consequence of the definition of $\mathcal{T}_i$ and the distance scaling property of $\tilde{\phi}_{*i}$, which follows from Lemma A.1.3.

Analogous to the remark made after Lemma A.1.1, the point $[H^i]_i$ represents the branch point of $[0]_i$, $[U_i]_i$ and $[V_i]_i$ in $\tilde{\mathcal{T}}_i$. Thus, since $\tilde{\phi}_{*i}$ is simply a rescaling map, we have that

$$b(\rho_i, Z_i^1, Z_i^2) = b(\tilde{\phi}_{*i}([0]_i), \tilde{\phi}_{*i}([U_i]_i), \tilde{\phi}_{*i}([V_i]_i)) = \tilde{\phi}_{*i}([H^i]_i).$$

Now, note that for any $j \in S$, we have by definition that $\phi_{ij}(0) \in \{H^i, H^i_-, H^i_+\}$, and so $[\phi_{ij}(0)]_i = [H^i]_i$. Consequently,

$$\tilde{\phi}_{*i}([H^i]_i) = \tilde{\phi}_{*i}([\phi_{ij}(0)]_i) = \tilde{\phi}_{*ij}([0]_{ij}) = \rho_{ij}, \tag{A.5}$$

which proves (c). Part (d) and (e) are proved using similar ideas. First, observe that $\phi_{ij}(U_{ij}) = 0, U_i, V_i$, for $j = 1, 2, 3$ respectively, then take equivalence classes and apply $\tilde{\phi}_{*i}$ to obtain (d). Secondly, it is easy to check from the construction that $[0]_i \notin \tilde{\phi}_{ij}(\tilde{\mathcal{T}}_{ij})$ for $j = 2, 3$; $[U_i]_i \notin \tilde{\phi}_{ij}(\tilde{\mathcal{T}}_{ij})$ for $j = 1, 3$; and $[V_i]_i \notin \tilde{\phi}_{ij}(\tilde{\mathcal{T}}_{ij})$ for $j = 1, 2$. Applying $\tilde{\phi}_{*i}$ to these results yields (e).

Now note that, for $k \in \Sigma_*$, the decomposition of the excursions, and the fact that the local minima of a Brownian excursion are distinct, implies that for $j_1, j_2 \in S$, $j_1 \neq j_2$, we have $\tilde{\phi}_{kj_1}(\tilde{\mathcal{T}}_{kj_1}) \cap \tilde{\phi}_{kj_2}(\tilde{\mathcal{T}}_{kj_2}) = \{[H^k]_k\}$. Applying the injection $\tilde{\phi}_{*k}$ to this equation yields

$$\mathcal{T}_{kj_1} \cap \mathcal{T}_{kj_2} = \{\tilde{\phi}_{*k}([H^k]_k)\} = \{\rho_{k1}\}, \tag{A.6}$$

with the second equality following from (A.5). This fact will allow us to prove (f) by induction on the length of i. Obviously, there is nothing to prove for $|i| = 0$. Suppose now that $|i| \geq 1$ and the desired result holds for any index of length strictly less than $|i|$. Suppose $|j| = |i|$, but $j \neq i$, and define $k := i|(|i| - 1)$. If $j|(|j| - 1) \neq k$, then the inductive hypothesis implies that

$$\mathcal{T}_i \cap \mathcal{T}_j \subseteq \mathcal{T}_k \cap \mathcal{T}_{j|(|j|-1)} \subseteq \{\rho_k, Z_k^1\},$$

where we apply part (a) to obtain the first inclusion. Using parts (d) and (e) of the proposition it is straightforward to deduce from this that $\mathcal{T}_i \cap \mathcal{T}_j \subseteq \{Z_i^1\}$ in this case. If $j|(|j| - 1) = k$, then we can apply the equality at (A.6) to obtain that $\mathcal{T}_i \cap \mathcal{T}_j = \{\rho_{k1}\} = \{\rho_i\}$, which completes the proof of part (f).

Finally, μ is non-atomic and so $\mu(\mathcal{T}_i) = \mu(\mathcal{T}_i \backslash \{\rho_i, Z_i^1\})$. Hence, by the disjointness of the sets and the fact that μ is a probability measure, we have that

$$1 \geq \sum_{i \in \Sigma_n} \mu(\mathcal{T}_i \backslash \{\rho_i, Z_i^1\}) = \sum_{i \in \Sigma_n} \mu(\mathcal{T}_i).$$

Now, by definition, for each i,

$$\begin{aligned} \mathcal{T}_i &= \{\tilde{\phi}_{*i}([t]_i) : t \in [0,1]\} \\ &= \{[t] : t \in \phi_{i|1} \circ \phi_{i|2} \circ \cdots \circ \phi_i([0,1])\}. \end{aligned}$$

Thus, since μ is the projection of Lebesgue measure, this implies that $\mu(\mathcal{T}_i)$ is no smaller than $\lambda(\phi_{i|1} \circ \phi_{i|2} \circ \cdots \circ \phi_i([0,1]))$. By repeated application of (A.3), this lower bound is equal to $\Delta_{i|1}\Delta_{i|2} \dots \Delta_i = l(i)^2$. Now observe that, because $(\Delta_{i1}, \Delta_{i2}, \Delta_{i3})$ are Dirichlet $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ random variables, we have $\Delta_{i1} + \Delta_{i2} + \Delta_{i3} = 1$ for every $i \in \Sigma_*$, and from this it is simple to show that $\sum_{i \in \Sigma_n} l(i)^2 = 1$. Hence

$$\sum_{i \in \Sigma_n} \mu(\mathcal{T}_i) \geq \sum_{i \in \Sigma_n} l(i)^2 = 1.$$

Thus $\sum_{i \in \Sigma_n} \mu(\mathcal{T}_i)$ is actually equal to 1, and moreover, (g) must hold. $\square$

This result is summarised in Figure A.2. Note that the fact that sets from $(\mathcal{T}_{ij})_{j \in S}$ only intersect at ρ_{i1} was shown at (A.6), and so the the diagram is representative of the set structure of the decomposition. Furthermore, it is clear that the sets $\mathcal{T}_i$ are all compact dendrites, because they are simply rescaled versions of the compact dendrites $\tilde{\mathcal{T}}_i$.

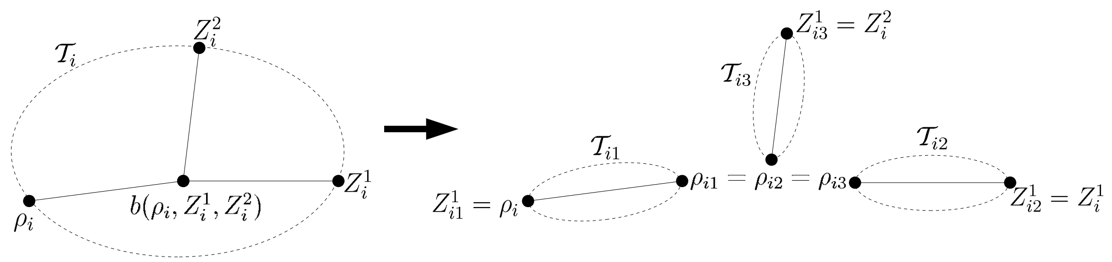

Figure A.2: Continuum random tree decomposition.

The tree description of the inductive algorithm runs as follows. Suppose that the triples $((\mathcal{T}_i, l(i)^{-1}d_{\mathcal{T}}, \mu^i))_{i \in \Sigma_n}$ are independent copies of $(\mathcal{T}, d_{\mathcal{T}}, \mu)$, independent of $\mathcal{F}_n$, where $\mu^i(A) := \mu(A)/\mu(\mathcal{T}_i)$ for measurable $A \subseteq \mathcal{T}_i$. Furthermore, suppose $\mathcal{T}_i$ has root ρ_i, and Z_i^1 and Z_i^2 are two μ^i-random variables in $\mathcal{T}_i$. For $j = 1, 2, 3$, define $\mathcal{T}_{ij}$ to be the component of $\mathcal{T}_i$ (when split at $b(\rho_i, Z_i^1, Z_i^2)$) containing ρ_i, Z_i^1, Z_i^2 respectively.

Define $\Delta_{ij} := \mu^i(\mathcal{T}_{ij})$, and equip the sets with the metrics $\Delta_{ij}^{-1/2} l(i)^{-1} d_{\mathcal{T}} = l(ij)^{-1} d_{\mathcal{T}}$ and measures μ^{ij}, defined by

$$\mu^{ij}(A) := \frac{\mu^i(A)}{\Delta_{ij}} = \frac{\mu(A)}{\mu(\mathcal{T}_{ij})}.$$

Then the triples $((\mathcal{T}_i, l(i)^{-1} d_{\mathcal{T}}, \mu^i))_{i \in \Sigma_{n+1}}$ are independent copies of the continuum random tree, independent of $\mathcal{F}_{n+1}$. Moreover, for $i \in \Sigma_{n+1}$, the algorithm gives us the root ρ_i of $\mathcal{T}_i$ and also a μ^i-random vertex, Z_i^1. To continue the algorithm, we pick independently for each $i \in \Sigma_{n+1}$ a second μ^i-random vertex, Z_i^2. Note that picking this extra μ^i-random vertex is the equivalent of picking the $U[0,1]$ random variable V_i in the excursion picture.

To complete this section, we introduce one further family of variables associated with the decomposition of the continuum random tree. From Proposition A.1.4(f), observe that the sets in $(\mathcal{T}_i)_{i \in \Sigma_n}$ only intersect at points of the form ρ_i or Z_i^1, and so, because of this, it is possible to consider the two point set $\{\rho_i, Z_i^1\}$ to be the boundary of $\mathcal{T}_i$. Denote the renormalised distance between boundary points by, for $i \in \Sigma_*$,

$$D_i := l(i)^{-1} d_{\mathcal{T}}(\rho_i, Z_i^1). \tag{A.7}$$

By construction, we have that

$$\begin{aligned} d_{\mathcal{T}}(\rho_i, Z_i^1) &= d_{\mathcal{T}}(\tilde{\phi}_{*i}([0]_i), \tilde{\phi}_{*i}([U_i]_i)) \\ &= d_W(\phi_{i|1} \circ \phi_{i|2} \circ \cdots \circ \phi_i(0), \phi_{i|1} \circ \phi_{i|2} \circ \cdots \circ \phi_i(U_i)) \\ &= l(i) d_{W^i}(0, U_i). \end{aligned}$$

Hence we can also write $D_i = d_{W^i}(0, U_i)$, and so, for each n, $(D_i)_{i \in \Sigma_n}$ is a collection of independent random variables, independent of $\mathcal{F}_n$. Moreover, the random variables $(D_i)_{i \in \Sigma_*}$ are identically distributed as $D_\emptyset$, which represents the height of a μ-random vertex in $\mathcal{T}$. Finally, we have the following recursive relationship

$$\begin{aligned} D_i &= l(i)^{-1} d_{\mathcal{T}}(\rho_i, Z_i^1) \\ &= l(i)^{-1} \left(d_{\mathcal{T}}(\rho_i, b(\rho_i, Z_i^1, Z_i^2)) + d_{\mathcal{T}}(b(\rho_i, Z_i^1, Z_i^2), Z_i^1) \right) \\ &= l(i)^{-1} \left(d_{\mathcal{T}}(\rho_{i1}, Z_{i1}^1) + d_{\mathcal{T}}(\rho_{i2}, Z_{i2}^1) \right) \\ &= w(i1) D_{i1} + w(i2) D_{i2}, \end{aligned} \tag{A.8}$$

where we use parts (c) and (d) of Proposition A.1.4 to deduce the third equality.

A.2 Self-similar dendrite in $\mathbb{R}^2$

The scaling factors $(w(i))_{i\in\Sigma_*\backslash\{\emptyset\}}$ defined from the continuum random tree allow us to build a random self-similar dendrite, and we now detail the set on which this is based. Using the terminology and notation of Chapter 1, the underlying metric space we consider is $\mathbb{R}^2$, equipped with the usual Euclidean metric. The contractions of interest, $(F_i)_{i=1}^3$, will be those defined at (1.63), and the resulting set, T, is that shown in Figure 1.1. It is readily checked that the scaling factors satisfy the conditions (W1) and (W2) and hence, by applying the results of Chapter 1, we are able to construct the associated resistance metric, R, $\mathbf{P}$-a.s. Furthermore, the resistance perturbations $(R_i)_{i\in\Sigma_*}$ are well-defined $(0,\infty)$ random variables, $\mathbf{P}$-a.s., satisfying $R_i=\lim_{n\to\infty}R_i(n)$, where

$$R_i(n):=\sum_{j\in\{1,2\}^n}\frac{l(ij)}{l(i)}. \tag{A.9}$$

Note that, since there is only one edge in the set $\tilde{E}^0$, we have dropped the superscript e from the resistance perturbations. Finally, to achieve the correct scaling, we take

$$H_e:=\sqrt{\frac{8}{\pi}}.$$

To complete this section, we provide an alternative characterisation of the resistance perturbations using the random variables $(D_i)_{i\in\Sigma_*}$ defined at (A.7). First, by iterating the identity of (A.8), we have

$$D_i=\sum_{j\in\{1,2\}^n}\frac{l(ij)}{l(i)}D_{ij}. \tag{A.10}$$

As remarked in the previous section, the distribution of $D_\emptyset$ is the same as the distribution the height of a μ-random vertex in $\mathcal{T}$. The explicit distribution of this is known (see [2]), and has mean $\sqrt{\pi/8}$ and finite variance. Using these facts and a comparison of the formulae at (A.9) and (A.10), it is possible to deduce that the collection of random variables $(R_i)_{i\in\Sigma_*}$ is simply a rescaled version of the collection $(D_i)_{i\in\Sigma_*}$. This result will be extremely important for establishing the $\mathbf{P}$-a.s. existence of an isometry between $(\mathcal{T},d_\mathcal{T})$ and (T,R) in the next section.

Lemma A.2.1 $\mathbf{P}$*-a.s., we have that*

$$(R_i)_{i\in\Sigma_*}=(\tilde{D}_i)_{i\in\Sigma_*},$$

where $\tilde{D}_i:=\sqrt{8/\pi}D_i$ *for* $i\in\Sigma_*$.

Proof: By the countability of Σ_*, it suffices to show that $R_i(n)\to\tilde{D}_i$, $\mathbf{P}$-a.s., for each $i \in \Sigma_*$. Assume now that $i \in \Sigma_*$ is fixed. Conditioning on $\mathcal{F}_{|i|+n}$, we obtain, for $\lambda > 0$,

$$\begin{aligned} \mathbf{P}\left(|\tilde{D}_i - R_i(n)| > \lambda\right) &= \mathbf{E}\left(\mathbf{P}\left(|\tilde{D}_i - R_i(n)| > \lambda \middle| \mathcal{F}_{|i|+n}\right)\right) \\ &\leq \lambda^{-2}\mathbf{E}\left(\mathbf{E}\left(|\tilde{D}_i - R_i(n)|^2 \middle| \mathcal{F}_{|i|+n}\right)\right). \end{aligned} \tag{A.11}$$

Now, since

$$\mathbf{E}\left(\tilde{D}_i - R_i(n) \middle| \mathcal{F}_{|i|+n}\right) = \sum_{j \in \{1,2\}^n} \frac{l(ij)}{l(i)} \mathbf{E}(\tilde{D}_{ij} - 1) = 0,$$

we are able to deduce that

$$\begin{aligned} \mathbf{E}\left(|\tilde{D}_i - R_i(n)|^2 \middle| \mathcal{F}_{|i|+n}\right) &= \mathrm{Var}\left(\sum_{j \in \{1,2\}^n} \frac{l(ij)}{l(i)}(\tilde{D}_{ij} - 1) \middle| \mathcal{F}_{|i|+n}\right) \\ &= \sum_{j \in \{1,2\}^n} \frac{l(ij)^2}{l(i)^2} \mathrm{Var}(\tilde{D}_\emptyset), \end{aligned}$$

where we have used the independence properties of the $(D_i)_{i \in \Sigma_*}$ to obtain the second equality. Recalling the definition of the $w(i)$ as the square roots of the Dirichlet random variables, Δ_i, we can use this conditional expectation and the inequality of (A.11) to deduce that

$$\mathbf{P}\left(|\tilde{D}_i - R_i(n)| > \lambda\right) \leq \lambda^{-2}\mathbf{E}(\Delta_1 + \Delta_2)^n \mathrm{Var}(\tilde{D}_\emptyset).$$

As we noted prior to this lemma, the random variable $D_\emptyset$ has finite variance. Furthermore, a simple symmetry argument yields that the expectation in the right hand side of the above bound is precisely $2/3$. Hence the sum of probabilities over n is finite, and applying a simple Borel-Cantelli argument yields the result. □

A.3 Isometry between $(\mathcal{T}, d_\mathcal{T})$ and (T, R)

In this section, we demonstrate how the decomposition of the continuum random tree presented in Section A.1 allows us to define an isometry from the continuum random tree to the random self-similar dendrite, (T, R), described in the previous section. An important consequence of the decomposition is that it allows us to label points in $\mathcal{T}$ using the shift space of infinite sequences, $\Sigma := \{1, 2, 3\}^\mathbb{N}$. The following lemma defines the projection $\pi_\mathcal{T} : \Sigma \to \mathcal{T}$ that we will use. This is analogous to the result that was stated as Theorem 1.1.1 for self-similar dendrites, and we shall denote by π_T the corresponding projection from Σ onto T.

Lemma A.3.1 **P**-*a.s., there exists a map* $\pi_{\mathcal{T}} : \Sigma \to \mathcal{T}$ *such that*

$$\pi_{\mathcal{T}} \circ \sigma_i(\Sigma) = \mathcal{T}_i, \qquad \forall i \in \Sigma_*,$$

where $\sigma_i : \Sigma \to \Sigma$ *is defined by* $\sigma_i(j) = ij$ *for* $j \in \Sigma$. *Furthermore, this map is continuous, surjective and unique.*

Proof: **P**-a.s., for each $i \in \Sigma$, the sets in the collection $(\mathcal{T}_{i|n})_{n\geq 0}$ are compact, non-empty subsets of $(\mathcal{T}, d_{\mathcal{T}})$, and by Proposition A.1.4(a), the sequence is decreasing. Hence, to show that $\cap_{n\geq 0}\mathcal{T}_{i|n}$ contains exactly one point for each $i \in \Sigma$, **P**-a.s., it will suffice to demonstrate that, **P**-a.s.,

$$\sup_{i\in\Sigma_n} \mathrm{diam}_{d_{\mathcal{T}}} \mathcal{T}_i \to 0, \qquad \text{as } n \to \infty. \tag{A.12}$$

From Proposition A.1.4(b), we have that $\mathrm{diam}_{d_{\mathcal{T}}}\mathcal{T}_i = l(i)\mathrm{diam}_{d_{\tilde{\mathcal{T}}_i}}\tilde{\mathcal{T}}_i$, and by definition, $\mathrm{diam}_{d_{\tilde{\mathcal{T}}_i}}\tilde{\mathcal{T}}_i = \sup_{s,t\in[0,1]} d_{W^i}(s,t)$, which is a measurable function of W^i. Hence, by Theorem A.1.2, the collection $(\mathrm{diam}_{d_{\tilde{\mathcal{T}}_i}}\tilde{\mathcal{T}}_i)_{i\in\Sigma_n}$ is a family of independent, identically distributed random variables, and is independent of $\mathcal{F}_n$. Furthermore, it is also clear that $\mathrm{diam}_{d_{\tilde{\mathcal{T}}_i}}\tilde{\mathcal{T}}_i \leq 2\sup_{t\in[0,1]} W^i_t$. The upper bound here is simply twice the maximum of a normalised Brownian excursion, a random variable whose explicit distribution is known (see [2], for example) and has positive moments of all orders. Thus, for all $\theta > 0$,

$$\mathbf{E}\left((\mathrm{diam}_{d_{\tilde{\mathcal{T}}_i}}\tilde{\mathcal{T}}_i)^\theta\right) < \infty.$$

Consequently, we can apply Lemma 1.3.1(ii) to deduce that the limit result at (A.12) does indeed hold.

Using the result of the previous paragraph, it is **P**-a.s. possible to define a map $\pi_{\mathcal{T}} : \Sigma \to \mathcal{T}$ such that, for $i \in \Sigma$,

$$\{\pi_{\mathcal{T}}(i)\} = \bigcap_{n\geq 0} \mathcal{T}_{i|n}.$$

That $\pi_{\mathcal{T}}$ satisfies the claims of the lemma, and is the unique map to do so, may be proved in exactly the same way as in the self-similar fractal case (see [9], Lemma 5.10, or [39], Theorem 1.2.3). $\square$

Heuristically, the isometry that we will define between the two dendrites under consideration can be thought of as simply "$\varphi = \pi_T \circ \pi_{\mathcal{T}}^{-1}$". However, to introduce the map rigorously, so that it is well-defined, we first need to prove some simple, but fundamental, results about the geometry of the sets and the maps π_T and $\pi_{\mathcal{T}}$.

Lemma A.3.2 **P**-*a.s.,*
(a) $\mathcal{T}_{ki} \cap \mathcal{T}_{kj} = \{\rho_{k1}\}$*, for all* $k \in \Sigma_*$*, and* $i, j \in S$*,* $i \neq j$*.*
(b) $\pi_{\mathcal{T}}^{-1}(\rho_{k1}) = \{k11\dot{2}, k21\dot{2}, k31\dot{2}\}$*, for all* $k \in \Sigma_*$*.*
(c) For every $i, j \in \Sigma$*,*

$$\pi_{\mathcal{T}}(i) = \pi_{\mathcal{T}}(j) \quad \Leftrightarrow \quad \pi_T(i) = \pi_T(j).$$

Proof: The proof we give holds on the **P**-a.s. set for which the decomposition of $\mathcal{T}$ and the definition of $\pi_{\mathcal{T}}$ is possible. Part (a) was proved in the course of the proof of Proposition A.1.4 at (A.6). Recall that $\rho_{k1} = b(\rho_k, Z_k^1, Z_k^2)$. For this branch point to equal ρ_k or Z_k^1, we would require at least two of its arguments to be equal, which happens with zero probability. Thus $\rho_{k1} \in \mathcal{T}_k \backslash \{\rho_k, Z_k^1\}$, and so Proposition A.1.4(f) implies that if $\pi_{\mathcal{T}}(i) = \rho_{k1}$ for some $i \in \Sigma$, then $i||k| = k$. Given this fact, it is elementary to apply the defining property of $\pi_{\mathcal{T}}$ and the results about ρ_i and Z_i^1 that were deduced in Proposition A.1.4 to deduce that part (b) of this lemma also holds. It now remains to prove part (c).

Fix $i, j \in \Sigma$, $i \neq j$, and let m be the unique integer satisfying $i|m = j|m$ and $i_{m+1} \neq j_{m+1}$. Furthermore, define $k = i_1 \dots i_m \in \Sigma_*$. Now by standard arguments for p.c.f.s.s. fractals (see [39], Proposition 1.2.5 and the subsequent remark) we have that $\pi_T(i) = \pi_T(j)$ implies that $\sigma^m(i), \sigma^m(j) \in \mathcal{C}$, where $\mathcal{C}$ is the critical set for the self-similar structure, T, as defined in Section 1.1. Here, we use the notation σ to represent the shift map, also introduced in Section 1.1. Note that it is elementary to calculate that $\mathcal{C} = \{11\dot{2}, 21\dot{2}, 31\dot{2}\}$ for this structure. Thus $i, j \in \{k11\dot{2}, k21\dot{2}, k31\dot{2}\}$, and so, by part (b), $\pi_{\mathcal{T}}(i) = \rho_{k1} = \pi_{\mathcal{T}}(j)$, which completes one implication of the desired result.

Now suppose $\pi_{\mathcal{T}}(i) = \pi_{\mathcal{T}}(j)$. From the definition of $\pi_{\mathcal{T}}$, we have that $\pi_{\mathcal{T}}(i) \in \mathcal{T}_{ki_{m+1}}$ and also $\pi_{\mathcal{T}}(j) \in \mathcal{T}_{kj_{m+1}}$. Hence

$$\pi_{\mathcal{T}}(i), \pi_{\mathcal{T}}(j) \in \mathcal{T}_{ki_{m+1}} \cap \mathcal{T}_{kj_{m+1}} = \{\rho_{k1}\},$$

where we use part (a) to deduce the above equality. In particular, this allows us to apply part (b) to deduce that $i, j \in \{k11\dot{2}, k21\dot{2}, k31\dot{2}\}$. Applying the shift map to this m times yields $\sigma^m(i), \sigma^m(j) \in \mathcal{C}$. It is easy to check that $\pi_T(\mathcal{C})$ contains only the single point $(\frac{1}{2}, 0)$. Thus $\pi_T(i) = F_k \circ \pi_T(\sigma^m(i)) = F_k \circ \pi_T(\sigma^m(j)) = \pi_T(j)$, which completes the proof. $\square$

We are now able to define the map φ precisely on a **P**-a.s. set by

$$\begin{aligned}\varphi : \mathcal{T} &\to T \\ x &\mapsto \pi_T(i), \qquad \text{for any } i \in \Sigma \text{ with } \pi_{\mathcal{T}}(i) = x.\end{aligned}$$

By part (c) of the previous lemma, this is a well-defined injection. Furthermore, since π_T is surjective, so is φ. Hence we have constructed a bijection from $\mathcal{T}$ to T and it remains to show that it is also an isometry. We start by checking that φ is continuous, which will enable us to deduce that it maps geodesic paths in $\mathcal{T}$ to geodesic paths in T. However, before we proceed with the lemma, we introduce the following notation for $x \in \mathcal{T}$, $n \geq 0$,

$$\mathcal{T}_n(x) := \bigcup \{\mathcal{T}_i : \ i \in \Sigma_n, \ x \in \mathcal{T}_i\}.$$

Note that this is analogous to the definition of $(T_n(x))_{x \in T, n \geq 0}$ that was first used at (1.33). Also, from the properties $\pi_T(i\Sigma) = T_i$, $\pi_{\mathcal{T}}(i\Sigma) = \mathcal{T}_i$, and the definition of φ, it is straightforward to deduce that

$$\varphi(\mathcal{T}_i) = T_i, \qquad \forall i \in \Sigma_*, \tag{A.13}$$

on the **P**-a.s. set that we can define all the relevant objects.

Lemma A.3.3 **P**-*a.s.,* φ *is a continuous map from* $(\mathcal{T}, d_{\mathcal{T}})$ *to* (T, R).

Proof: Recall from Lemma 1.4.7 that for each $x \in T$, the collection $(T_n(x))_{n \geq 0}$ is a base of neighbourhoods of x with respect to the Euclidean metric on $\mathbb{R}^2$. Since, by Proposition 1.4.8, R is topologically equivalent to this metric, **P**-a.s., then the same is true when we consider the collections of neighbourhoods with respect to the metric R, **P**-a.s. Similarly, we may use the fact that $\sup_{i \in \Sigma_n} \operatorname{diam} \mathcal{T}_i \to 0$, **P**-a.s., from (A.12) to imitate the proofs of these results to deduce that **P**-a.s., for each $x \in \mathcal{T}$, the collection $(\mathcal{T}_n(x))_{n \geq 0}$ is a base of neighbourhoods of x with respect to $d_{\mathcal{T}}$.

The remaining argument applies **P**-a.s. Let U be an open subset of (T, R) and $x \in \varphi^{-1}(U)$. Define $y = \varphi(x) \in U$. Now, since U is open, there exists an n such that $T_n(y) \subseteq U$. Also, by (A.13), for each $i \in \Sigma_n$, we have that $x \in \mathcal{T}_i$ implies that $y \in T_i$. Hence

$$\varphi(\mathcal{T}_n(x)) = \varphi\left(\cup_{i \in \Sigma_n, \, x \in \mathcal{T}_i} \mathcal{T}_i\right) \subseteq \cup_{i \in \Sigma_n, \, y \in T_i} T_i = T_n(y) \subseteq U.$$

Consequently, $\mathcal{T}_n(x) \subseteq \varphi^{-1}(U)$. Since $\mathcal{T}_n(x)$ is a $d_{\mathcal{T}}$-neighbourhood of x it follows that $\varphi^{-1}(U)$ is open in $(\mathcal{T}, d_{\mathcal{T}})$. The lemma follows. $\square$

We are now ready to proceed with the main result of this section. In the proof, we will use the notation $\gamma^{\mathcal{T}}_{xy} : [0,1] \to \mathcal{T}$ to denote a geodesic path from x to y, where x and y are points in the dendrite $\mathcal{T}$. Clearly, because φ is a continuous injection, $\varphi \circ \gamma^{\mathcal{T}}_{xy}$ describes a geodesic path from $\varphi(x)$ to $\varphi(y)$ in T.

Theorem A.3.4 $\mathbf{P}$*-a.s., the map* φ *is an isometry, and the metric spaces* $(\mathcal{T}, d_{\mathcal{T}})$ *and* (T, R) *are isometric.*

Proof: Obviously, the second statement of the theorem is an immediate consequence of the first. The following argument, in which we demonstrate that φ is indeed an isometry, holds $\mathbf{P}$-a.s. Given $\varepsilon > 0$, by Lemma 1.5.1(c) and (A.12), we can choose an $n \geq 1$ such that

$$\sup_{i \in \Sigma_n} \mathrm{diam}_{d_{\mathcal{T}}} \mathcal{T}_i, \ \sup_{i \in \Sigma_n} \mathrm{diam}_R T_i < \frac{\varepsilon}{4}.$$

Now, fix $x, y \in \mathcal{T}$, define $t_0 := 0$ and set

$$t_{m+1} := \inf\{t > t_m \ : \ \gamma^{\mathcal{T}}_{xy}(t) \notin \mathcal{T}_n(\gamma^{\mathcal{T}}_{xy}(t_m))\},$$

where $\inf \emptyset := 1$. We will also denote $x_m := \gamma^{\mathcal{T}}_{xy}(t_m)$. Since, for each $x' \in \mathcal{T}$, the collection $(\mathcal{T}_n(x'))_{n \geq 0}$ forms a base of neighbourhoods of x', we must have that $t_{m-1} < t_m$ whenever $t_{m-1} < 1$. We now claim that for any m with $t_{m-1} < 1$ there exists a unique $i(m) \in \Sigma_n$ such that

$$\gamma^{\mathcal{T}}_{xy}(t) \in \mathcal{T}_{i(m)}, \qquad t_{m-1} \leq t \leq t_m. \tag{A.14}$$

Let m be such that $t_{m-1} < 1$. By the continuity of $\gamma^{\mathcal{T}}_{xy}$, we have that $x_m \in \mathcal{T}_n(x_{m-1})$, and hence there exists an $i(m) \in \Sigma_n$ such that $x_{m-1}, x_m \in \mathcal{T}_{i(m)}$. Clearly, the image of $\gamma^{\mathcal{T}}_{xy}$ restricted to $t \in [t_{m-1}, t_m]$ is the same as the image of $\gamma^{\mathcal{T}}_{x_{m-1}x_m}$, which describes the unique path in $\mathcal{T}$ from x_{m-1} to x_m. Note also that $\mathcal{T}_{i(m)}$ is a path-connected subset of $\mathcal{T}$, and so the path from x_{m-1} to x_m lies in $\mathcal{T}_{i(m)}$. Consequently, the set $\gamma_{xy}([t_{m-1}, t_m])$ is contained in $\mathcal{T}_{i(m)}$. Thus to prove the claim at (A.14), it remains to show that $i(m)$ is unique. Suppose that there exists $j \in \Sigma_n$, $j \neq i(m)$ for which the inclusion at (A.14) holds. Then the uncountable set $\gamma^{\mathcal{T}}_{xy}([t_{m-1}, t_m])$ is contained in $\mathcal{T}_{i(m)} \cap \mathcal{T}_j$, which, by Proposition A.1.4(f), contains at most two points. Hence no such j can exist.

Now assume that $m_1 < m_2$ and that $t_{m_2 - 1} < 1$. Suppose that $i(m_1) = i(m_2)$, then $x_{m_1 - 1}, x_{m_2} \in \mathcal{T}_{i(m_1)}$ By a similar argument to the previous paragraph, it follows that $\gamma^{\mathcal{T}}_{xy}([t_{m_1 - 1}, t_{m_2}]) \subseteq \mathcal{T}_{i(m_1)}$. By definition, this implies that $t_{m_1} \geq t_{m_2}$, which cannot be true. Consequently, we must have that $i(m_1) \neq i(m_2)$. Since Σ_n is a finite set, it follows from this observation that

$$N := \inf\{m : t_m = 1\}$$

is finite, and moreover, the elements of $(i(m))_{m=1}^N$ are distinct.

The conclusion of the previous paragraph provides us with a useful decomposition of the path from x to y, which we will be able to use to complete the proof. The fact that $d_{\mathcal{T}}$ is a shortest path metric allows us to write

$$d_{\mathcal{T}}(x,y) = \sum_{m=1}^{N} d_{\mathcal{T}}(x_{m-1}, x_m).$$

For $m \in \{2, \dots, N-1\}$, we have that $i(m) \neq i(m+1)$, and so by applying Proposition A.1.4(f), we can deduce that $x_m \in \mathcal{T}_{i(m)} \cap \mathcal{T}_{i(m+1)} \subseteq \{\rho_{i(m)}, Z^1_{i(m)}\}$. Similarly, we have $x_{m-1} \in \mathcal{T}_{i(m-1)} \cap \mathcal{T}_{i(m)} \subseteq \{\rho_{i(m)}, Z^1_{i(m)}\}$. Thus, by the injectivity of $\gamma^{\mathcal{T}}_{xy}$, we must have that $\{x_{m-1}, x_m\} = \{\rho_{i(m)}, Z^1_{i(m)}\}$, which implies $d_{\mathcal{T}}(x_{m-1}, x_m) = d_{\mathcal{T}}(\rho_{i(m)}, Z_{i(m)}) = l(i(m))D_{i(m)}$. Hence we can conclude that

$$d_{\mathcal{T}}(x,y) - \sum_{m=2}^{N-1} l(i(m))D_{i(m)} = d_{\mathcal{T}}(x_0, x_1) + d_{\mathcal{T}}(x_{N-1}, x_N). \tag{A.15}$$

As remarked before this lemma, $\varphi \circ \gamma^{\mathcal{T}}_{xy}$ is a geodesic path from $\varphi(x)$ to $\varphi(y)$. Thus the shortest path property of R allows us to write

$$R(\varphi(x), \varphi(y)) = \sum_{m=1}^{N} R(\varphi(x_{m-1}), \varphi(x_m)). \tag{A.16}$$

Let $m \in \{2, \dots, N-1\}$. By applying φ to the expression for $\{x_{m-1}, x_m\}$ that was deduced above, we obtain that $\{\varphi(x_{m-1}), \varphi(x_m)\} = \{\varphi(\rho_{i(m)}), \varphi(Z^1_{i(m)})\}$. Now, part (b) of Lemma A.3.2 implies that

$$\varphi(\rho_{i(m)}) = \pi_T(k11\dot{2}) = F_k(\pi_T(11\dot{2})) = F_k((\frac{1}{2}, 0)) = F_{i(m)}((0,0)),$$

where $k := i(m)|(|i(m)| - 1)$. In Proposition A.1.4(d) it was shown that $Z^1_i = Z^1_{i2}$, for every $i \in \Sigma_*$. It follows that $i(m)\dot{2} \in \pi_{\mathcal{T}}^{-1}(Z^1_{i(m)})$, and so

$$\varphi(Z^1_{i(m)}) = \pi_T(i(m)\dot{2}) = F_{i(m)}(\pi_T(\dot{2})) = F_{i(m)}((1,0)).$$

Thus $R(\varphi(x_{m-1}), \varphi(x_m)) = R(F_{i(m)}((0,0)), F_{i(m)}((1,0)))$. However, $\{(0,0),(1,0)\}$ is the only edge in $\tilde{E}^0$, and so from the expression at (1.31), we can deduce that

$$R(\varphi(x_{m-1}), \varphi(x_m)) = \sqrt{\pi/8}\, l(i(m))R_{i(m)} = l(i(m))D_{i(m)},$$

where we have used Lemma A.2.1 to obtain the second equality. Substituting this into (A.16), and combining the resulting equation with the equality at (A.15) yields

$$|d_{\mathcal{T}}(x,y) - R(\varphi(x),\varphi(y))| \leq \sum_{m\in\{1,N\}} \left(d_{\mathcal{T}}(x_{m-1},x_m) + R(\varphi(x_{m-1}),\varphi(x_m))\right).$$

Now, x_0 and x_1 are both contained in $\mathcal{T}_{i(1)}$, and so the choice of n implies that $d_{\mathcal{T}}(x_0,x_1) < \varepsilon/4$. Furthermore, $\varphi(x_0)$ and $\varphi(x_1)$ are both contained in $\varphi(\mathcal{T}_{i(1)}) = T_{i(1)}$, and so we also have $R(\varphi(x_0),\varphi(x_1)) < \varepsilon/4$. Thus the summand with $m = 1$ is bounded by $\varepsilon/2$. Similarly for $m = N$. Hence

$$|d_{\mathcal{T}}(x,y) - R(\varphi(x),\varphi(y))| < \varepsilon.$$

Since the choice of x, y and ε was arbitrary, the proof is complete. □

The final result that we present in this section completes the proof of the fact that $(\mathcal{T}, d_{\mathcal{T}}, \mu)$ and (T, R, μ^T) are equivalent measure-metric spaces, where we use the notation μ^T to represent the self-similar measure on (T, R), as defined in Section 1.8. Note that, since $w(1)^2+w(2)^2+w(3)^2 = \Delta_1+\Delta_2+\Delta_3 = 1$, **P**-a.s., the appropriate exponent for the measure μ^T is $\alpha = 2$ and moreover, there are no tail fluctuations in the μ^T-measure of sets of the form T_i. In particular, the equation at (1.51) becomes, **P**-a.s.,

$$\mu^T(T_i) = l(i)^2, \qquad \forall i \in \Sigma_*. \tag{A.17}$$

Theorem A.3.5 **P**-*a.s., the probability measures μ and $\mu^T \circ \varphi$ agree on the Borel σ-algebra of $(\mathcal{T}, d_{\mathcal{T}})$.*

Proof: Again, this argument holds **P**-a.s. First, note that μ^T is a non-atomic Borel probability measure on (T, R). Thus, since φ is an isometry, $\mu^T \circ \varphi$ is a non-atomic Borel probability measure on $(\mathcal{T}, d_{\mathcal{T}})$. Secondly, μ is a Borel probability measure on $(\mathcal{T}, d_{\mathcal{T}})$ by construction. Now, let $\mathcal{A}$ be the collection of sets of the form $\{\rho_i\}$, $\{Z_i^1\}$, $\mathcal{T}_i$, for $i \in \Sigma_*$, and the empty set. From Proposition A.1.4, we have that $\mathcal{A}$ is a π-system. Furthermore, because for each $x \in \mathcal{T}$, $(\mathcal{T}_n(x))_{n\geq 0}$ is a base of neighbourhoods of x and contains sets in $\sigma(\mathcal{A})$, it is the case that $\mathcal{A}$ generates the Borel σ-algebra of $(\mathcal{T}, d_{\mathcal{T}})$. Thus, by standard measure theory (see [17], Theorem 3.3), to deduce the result, it is sufficient to check that the measures μ and $\mu^T \circ \varphi$ agree on $\mathcal{A}$. Since both measures are non-atomic, we are left with showing that they agree on $(\mathcal{T}_i)_{i\in\Sigma_*}$. Recall from Proposition A.1.4(g) that $\mu(\mathcal{T}_i) = l(i)^2$. Applying this and the identities of (A.13) and (A.17), we have

$$\mu^T \circ \varphi(\mathcal{T}_i) = \mu^T(T_i) = l(i)^2 = \mu(\mathcal{T}_i),$$

which completes the proof. □

Appendix B

Perron-Frobenius eigenvalue derivative

In this appendix, we give a proof of the fact that the derivative of the Perron-Frobenius eigenvalue of $\overline{M}(\theta)$, as defined in Section 1.6, is strictly negative whenever $\overline{M}(1)$ is regular and (W1) holds. This condition guarantees that, if the branching random walk model is applicable to the resistance perturbations, $R_i^e(n)$ converges in mean to R_i^e, ([42], Theorem 1). In fact, that the assumption (R1) holds under the conditions of the proposition is a simple extension of this result.

Proposition B.0.1 *Suppose $\overline{M}(1)$ is regular and assume (W1). Let $\rho(\theta)$ denote the Perron-Frobenius (maximum positive) eigenvalue of $\overline{M}(\theta)$ for $\theta > 0$, then*

$$\rho'(1) < 0.$$

Proof: We start by rewriting $\overline{M}(\theta)$ so that it does not depend on $(H_e)_{e\in\tilde{E}^0}$. Define $N(\theta) = (n_{ee'}(\theta))_{e,e'\in\tilde{E}^0}$ by

$$N(\theta) := \mathrm{diag}(H^{-\theta})\overline{M}(\theta)\mathrm{diag}(H^{\theta}),$$

where entry in the ee' position of $\mathrm{diag}(H^{\pm\theta})$ is $H_e^{\pm\theta}\mathbf{1}_{\{e=e'\}}$. Since these diagonal matrices are invertible, $N(\theta)$ has the same eigenvalues as $\overline{M}(\theta)$ and

$$n_{ee'}(\theta) = \mathbf{E}\left(\sum_{i\in S} w(i)^{\theta}\mathbf{1}_{\{F_i(G_{e'})\subseteq G_e\}}\right).$$

Since $\overline{M}(1)$ is positive regular, then so is $N(1)$. Consequently, the same is true for $N(\theta)$, $\theta > 0$. Furthermore, $n_{ee'}(\theta)$ is analytic in $\mathbb{C}$ for $\theta = x + iy$ with $x > 0$. It follows from [16], Theorem 1, that for $\theta \in \mathbb{R}$, $\theta > 0$, the Perron-Frobenius eigenvalue of $N(\theta)$, $\rho(\theta)$ is differentiable. Moreover, if $u(\theta) = (u_e(\theta))_{e\in\tilde{E}^0}$, $v(\theta) = (v_e(\theta))_{e\in\tilde{E}^0}$ are the left, right eigenvectors of $N(\theta)$ corresponding to $\rho(\theta)$ and normalised so that $\sum_e u_e(\theta)u_e(\theta) = 1$, $\sum_e u_e(\theta)v_e(\theta) = 1$, then $u(\theta)$, $v(\theta)$ are differentiable and $u_e(\theta) > 0$, $v_e(\theta) > 0$, for all $e \in \tilde{E}^0$. We can now follow a similar argument to [54], Proposition 2.2. First, we can differentiate $u(\theta)N(\theta) = \rho(\theta)u(\theta)$ to obtain

$$u'(\theta)N(\theta) + u(\theta)N'(\theta) = \rho'(\theta)u(\theta) + \rho(\theta)u'(\theta).$$

Multiplying on the right by $v(\theta)$ yields $\rho'(\theta) = u(\theta)N'(\theta)v(\theta)$. In particular,

$$\rho'(1) = \sum_{e,e'\in\tilde{E}^0} u_e(1)v_{e'}(1)\mathbf{E}\left(\sum_{i\in S} w(i)\ln w(i)\mathbf{1}_{\{F_i(G_{e'})\subseteq G_e\}}\right).$$

Since (W1) holds, $\mathbf{E}(w(i)\ln w(i)) < 0$, for every $i \in S$. The result follows. $\square$

Appendix C

Resistance and Dirichlet forms

The aim of this appendix is to explain the connection between Dirichlet and resistance forms, at least within the framework of Chapter 2. In particular, we suppose that (X, d, μ) is a measure-metric space satisfying the conditions of Chapter 2, and $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet on $L^2(X, \mu)$ for which there exists a resistance form $(\mathcal{E}, \mathcal{F}')$ such that $\mathcal{F}' \supseteq \mathcal{F}$. Moreover, we assume that the topology induced upon X by the associated resistance metric, R, is compatible with the original topology of (X, d). In our description of the connection between the two quadratic forms we use the idea of an extended Dirichlet space, which we shall denote $\mathcal{F}_e$. We follow [28] in defining this to be the collection of μ-measurable functions f on X such that $|f| < \infty$, μ-a.e., and there exists an $\mathcal{E}$-Cauchy sequence $(f_n)_{n\geq 0}$ in $\mathcal{F}$ such that $f_n(x) \to f(x)$, μ-a.e. Note that in this section we do not need the locality assumption on our Dirichlet form $(\mathcal{E}, \mathcal{F})$ or assume that its extended Dirichlet space $(\mathcal{E}, \mathcal{F}_e)$ is a resistance form. Our main result is the following.

Proposition C.0.2
(a) Define $\tilde{\mathcal{F}} := \mathcal{F}' \cap L^2(X, \mu)$, then $(\mathcal{E}, \tilde{\mathcal{F}})$ is a regular Dirichlet form.
(b) Suppose $(\mathcal{E}, \mathcal{F})$ is recurrent, then $(\mathcal{E}, \mathcal{F}_e)$ is a resistance form. The associated resistance metric $R^{\mathcal{F}}$ is bounded above by R, and for $x \neq y$, we have

$$R^{\mathcal{F}}(x, y)^{-1} = \inf\{\mathcal{E}(f, f) : f \in \mathcal{F},\ f(x) = 1,\ f(y) = 0\}. \tag{C.1}$$

Moreover, $R^{\mathcal{F}} = R$ if and only if $\mathcal{F}_e = \mathcal{F}'$.

Proof: The proof of (a) is straightforward. Clearly $(\mathcal{E}, \tilde{\mathcal{F}})$ is a symmetric, Markov form. That it is closed is demonstrated in [39], Theorem 2.4.1. The denseness of its domain in $L^2(X, \mu)$ and regularity follow from the fact that $\tilde{\mathcal{F}} \supseteq \mathcal{F}$.

We now prove (b). First note by the definition of $\mathcal{F}_e$, and the fact that $\mathcal{F}'$ is complete with respect to $\mathcal{E}$, we must have that $\mathcal{F}_e \subseteq \mathcal{F}'$, so $\mathcal{E}$ is well-defined on $\mathcal{F}_e$.

By construction, $\mathcal{F}_e$ is a linear subspace. Also observe that if $(\mathcal{E}, \mathcal{F})$ is recurrent, then $1 \in \mathcal{F}_e$ and $\mathcal{E}(1,1) = 0$, ([28], Theorem 1.6.2). Hence, $(\mathcal{E}, \mathcal{F}_e)$ satisfies the first property of a resistance form. That $\mathcal{E}(f,f) = 0$ if and only if f is a constant, as well as the fourth and fifth properties of a resistance form are easily checked. Thus it remains to show that $(\mathcal{F}_e/\sim, \mathcal{E})$ is a Hilbert space, where $f \sim g$ if and only if $f - g$ is constant, and also that we can extend any function defined on a finite set of points to a function in $\mathcal{F}_e$. This final property is a simple consequence of the regularity of $(\mathcal{E}, \mathcal{F})$, and we omit its proof.

Suppose $(f_n)_{n\geq 0}$ is a Cauchy sequence in $(\mathcal{F}_e, \mathcal{E})$. Fix x_0 and define $g_n(x) := f_n(x) - f_n(x_0) \in \mathcal{F}_e$. By the inequality at (2.17) for the resistance form $(\mathcal{E}, \mathcal{F}')$, we have that $(g_n(x))_{n\geq 0}$ is Cauchy, and consequently convergent, for each $x \in X$. Define $g(x) := \lim_{n\to\infty} g_n(x)$, and note that $(g_n)_{n\geq 0}$ is $\mathcal{E}$-Cauchy. It follows that $g \in \mathcal{F}_e$, and $\mathcal{E}(f_n - g, f_n - g) \to 0$, which completes the proof of completeness.

We have thus proved that $(\mathcal{E}, \mathcal{F}_e)$ is a resistance form. That the associated resistance metric $R^{\mathcal{F}}$ is bounded above by R is clear from the definition of a resistance metric after recalling that $\mathcal{F}_e \subseteq \mathcal{F}'$. The expression for $R^{\mathcal{F}}$ at (C.1) follows from the fact that $\mathcal{F}$ is dense in $(\mathcal{F}_e, \mathcal{E})$. The final claim is a consequence of the one-to-one correspondence between resistance metrics and resistance forms (see [39], Theorem 2.3.4 and Theorem 2.3.6). $\square$

A question left open by this result is whether $\tilde{\mathcal{F}}_e = \mathcal{F}'$. In the case of (X,d) compact, we shall show that this is true. This means that there is a natural correspondence between the resistance form and a Dirichlet form, and thus also a connection between the resistance form and a Markov process. However, in the general non-compact case, this problem means that we are unable to say whether it is possible to construct a Markov process that contains all the information about our original resistance form $(\mathcal{E}, \mathcal{F}')$.

In the case of (X,d) compact the previous result may be stated more neatly. This is mainly due to the fact that the regularity of a Dirichlet form implies $1 \in \mathcal{F}$, which immediately implies the recurrence of the related semi-group in our setting. Specifically, we have the following result.

Proposition C.0.3 *If (X,d) is compact, then*
(a) $(\mathcal{E}, \mathcal{F}')$ is a regular Dirichlet form.
(b) $(\mathcal{E}, \mathcal{F})$ is a resistance form. The associated resistance metric $R^{\mathcal{F}}$ is bounded above by R, and moreover, $R^{\mathcal{F}} = R$ if and only if $\mathcal{F} = \mathcal{F}'$.
Proof: To prove (a) it suffices to show that $\mathcal{F}'$ is contained within $L^2(X,\mu)$. However,

the inequality at (2.17) for the resistance form $(\mathcal{E}, \mathcal{F}')$ and the assumption that the topology of R and d are compatible implies that $\mathcal{F}' \subseteq C(X)$. Consequently, because X is compact and μ is a finite Borel measure, we must have that $\mathcal{F}' \subseteq L^2(X, \mu)$ as desired.

We start the proof of (b) by showing that $(\mathcal{E}, \mathcal{F})$ is recurrent. As remarked before the proposition, that $1 \in \mathcal{F}$ is a simple consequence of the regularity of $(\mathcal{E}, \mathcal{F})$. Using the first property of a resistance form, we must necessarily have $\mathcal{E}(1, 1) = 0$, and hence $(\mathcal{E}, \mathcal{F})$ is recurrent ([28], Theorem 1.6.2). Consequently, the result will follow from Proposition C.0.2 if we can show that $\mathcal{F}_e = \mathcal{F}$. However, by [28], Theorem 1.5.2, we have that $\mathcal{F} = \mathcal{F}_e \cap L^2(X, \mu)$. Note now that $\mathcal{F}_e \subseteq \mathcal{F}' \subseteq L^2(X, \mu)$. Thus $\mathcal{F}_e \cap L^2(X, \mu) = \mathcal{F}_e$, and the proof is complete. $\square$

Finally, note that this result implies the existence of proper resistance form subspaces when we have regular Dirichlet form subspaces. A resistance form subspace of a resistance form $(\mathcal{E}, \mathcal{F}')$ is a set $\mathcal{F}'' \subseteq \mathcal{F}'$ such that $(\mathcal{E}, \mathcal{F}'')$ is also a resistance form, and it is proper if $\mathcal{F}'' \neq \mathcal{F}'$. The definition of a regular Dirichlet form subspace is similar. For example, suppose that (X, d) is compact, so that $(\mathcal{E}, \mathcal{F}')$ is a regular Dirichlet form. Now if $\mathcal{F}$ is a proper regular Dirichlet subspace of $\mathcal{F}'$, then by the previous proposition we have a proper resistance form subspace. Examples of when this occurs appear in [26] for the case of $X = [0, 1]$. Note that the regular Dirichlet form associated with reflecting Brownian motion on the interval is also a resistance form. Hence all the proper regular Dirichlet subspaces of this form that are exhibited in [26] are also proper resistance form subspaces.

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