# Regularity of Solutions to Some Boundary Value Problem of the Stationary Transport Equation

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**Abstract**

We consider a boundary value problem of the stationary transport equation in a two-dimensional infinite strip domain. We prove the existence and the uniqueness of classical solutions to our boundary value problem in the case that two coefficients in the equation are constants and that both the boundary data and the scattering phase function are independent of a spatial coordinate. We prove infinite differentiability of the solution with respect to the spatial coordinate in this case, and we also prove continuous differentiability with respect to the angular coordinate under the assumption that both the boundary data and the scattering phase function are continuously differentiable.

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# 1 Introduction

We consider the stationary transport equation, which is the following integro-differential equation:

$$
\begin{aligned}
&-\xi\cdot\nabla_x I(x,\xi)-(\mu_a(x)+\mu_s(x))I(x,\xi)\\
&\qquad+\mu_s(x)\int_{S^1}p(x,\xi,\xi')I(x,\xi')\,d\sigma_{\xi'}=0,\quad (x,\xi)\in\Omega\times S^1, \qquad (1.1)
\end{aligned}
$$

where $\Omega=\mathbb{R}\times(0,1)$ is an infinite strip domain in $\mathbb{R}^2$. Here, the absorption coefficient $\mu_a(x)$ and the scattering coefficient $\mu_s(x)$ are nonnegative functions and the scattering phase function $p(x,\xi,\xi')$ is a nonnegative function which satisfies

$$\int_{S^1}p(x,\xi,\xi')\,d\sigma_{\xi'}=1,\quad (x,\xi)\in\Omega\times S^1.$$

We note that we call $\mu_a(x)+\mu_s(x)$ the attenuation coefficient and denote it by $\mu_t(x)$: $\mu_t(x)=\mu_a(x)+\mu_s(x)$. This equation is considered as a mathematical model of propagation of photons with absorption and scattering [4] [5], and its analysis has been very much paid attention to recently for its application.

We now pose a boundary condition to (1.1). Let us denote the boundary of $\Omega\times S^1$ by $\Gamma$; $\Gamma:=\mathbb{R}\times\{0,1\}\times S^1$. We introduce a parameter $\theta$ to identify $S^1$ with the interval $(-\pi,\pi]$ by the relation $\xi=(\cos\theta,\sin\theta)$ for $\xi\in S^1$. Let us also denote

$$
\begin{aligned}
\Gamma_+&:=\{(x_1,0,\xi)|x_1\in\mathbb{R},\,\theta\in(-\pi,0)\}\cup\{(x_1,1,\xi)|x_1\in\mathbb{R},\,\theta\in(0,\pi)\},\\
\Gamma_-&:=\{(x_1,0,\xi)|x_1\in\mathbb{R},\,\theta\in(0,\pi)\}\cup\{(x_1,1,\xi)|x_1\in\mathbb{R},\,\theta\in(-\pi,0)\},\\
\Gamma_0&:=\{(x_1,0,\xi)|x_1\in\mathbb{R},\,\theta\in\{0,\pi\}\}\cup\{(x_1,1,\xi)|x_1\in\mathbb{R},\,\theta\in\{0,\pi\}\},
\end{aligned}
$$

and $\Gamma=\Gamma_+\cup\Gamma_-\cup\Gamma_0$. We pose a boundary condition for the equation (1.1) as follows:

$$I(x,\xi)=I_0(x,\xi),\quad (x,\xi)\in\Gamma_-, \qquad (1.2)$$

where we later mention regularity of a given function $I_0(x,\xi)$.

Prior to the mathematical analysis, we should give definition of solutions to the boundary value problem (1.1) and (1.2). Let a function $\tau(x,\xi)$ be

$$\tau(x,\xi):=\frac{x_2}{\sin\theta},\qquad \theta\in(0,\pi)$$

and

$$\tau(x,\xi):=-\frac{1-x_2}{\sin\theta},\quad \theta\in(-\pi,0),$$

where $x_2$ represents the second component of $x\in\Omega$. We remark that $\tau$ means the distance between a point $x\in\Omega$ and the boundary point $x-\tau\xi\in\partial\Omega$. Since $\lim_{\theta\to0}\tau(x,\xi)=\lim_{\theta\to\pm\pi}\tau(x,\xi)=\infty$ for each $x\in\Omega$, we define $\tau(x,\xi)=\infty$ for $\theta\in\{0,\pi\}$.

Let $C_b\left(\left(\Omega \times S^1\right) \cup \Gamma_-\right)$ be the normed vector space consisting of all the bounded continuous functions on $\left(\Omega \times S^1\right) \cup \Gamma_-$ with a norm $\|\cdot\|_\infty$ defined by

$$\|I\|_\infty := \sup_{(x,\xi)\in(\Omega\times S^1)\cup\Gamma_-} |I(x,\xi)|,$$

and we remark that $C_b\left(\left(\Omega \times S^1\right) \cup \Gamma_-\right)$ is a Banach space. In a similar way, we also introduce a Banach space $C_b(\Gamma_-)$ with a norm denoted by $\|\cdot\|_{\Gamma_-}$.

Under these preparations, we give the following definition.

**Definition 1.1.** We say $I \in C_b\left(\left(\Omega \times S^1\right) \cup \Gamma_-\right)$ is a classical solution to the boundary value problem (1.1) and (1.2), if $I$ satisfies the following integral equation

$$\begin{aligned} I(x,\xi) = &\exp\left(-\int_0^\tau \mu_t(x - r\xi)\,dr\right) I_0(x - \tau\xi, \xi) \\ &+ \mu_s \int_0^\tau \exp\left(-\int_0^t \mu_t(x - r\xi)\,dr\right) \int_{S^1} p(x - t\xi, \xi, \xi') I(x - t\xi, \xi')\,d\sigma_{\xi'}\,dt, \\ &\qquad (x,\xi) \in (\Omega \times S^1) \cup \Gamma_-. \qquad (1.3) \end{aligned}$$

We remark that D. S. Anikonov et al. [1] proved the existence and the uniqueness of weak solutions to the boundary value problem (1.1) and (1.2) for a more general case where $\Omega$ is bounded and $\partial\Omega$ has $C^1$ regularity, but as far as the author knows, the existence and the uniqueness of classical solutions has not been explicitly proved. In this paper, we discuss the existence and the uniqueness of classical solutions, although we restrict ourselves for the case of the infinite strip domain. We also discuss regularity of the classical solution in the case, and it is necessary in discussion of a finite difference approach, aided by numerical computation, to the boundary value problem.

In order to state our result, we should pose some assumptions. We assume that our given boundary data $I_0$ does not depend on the first component $x_1$, that the coefficients $\mu_a$ and $\mu_s$ are nonnegative constants and also that the scattering phase function $p$ is independent of $x$. Under these assumptions, we remark that the solution $I$ also does not depend on $x_1$, and we can reduce the equation (1.1) to the following equation

$$\begin{aligned} -\sin\theta \frac{\partial}{\partial x_2} I(x_2,\xi) &- (\mu_a + \mu_s) I(x_2,\xi) \\ &+ \mu_s \int_{S^1} p(\xi,\xi') I(x_2,\xi')\,d\sigma_{\xi'} = 0, \quad (x_2,\xi) \in \tilde{\Omega} \times S^1, \end{aligned} \qquad (1.4)$$

where $\tilde{\Omega}$ is the interval $(0,1)$. We further suppose that $p$ is continuous on $S^1 \times S^1$.

Corresponding to (1.1), let us denote

$$\begin{aligned}
\tilde{\Gamma} :=& \left\{(0,\xi)|\xi\in S^1\right\}\cup\left\{(1,\xi)|\xi\in S^1\right\},\\
\tilde{\Gamma}_+ :=& \left\{(0,\xi)|\theta\in(-\pi,0)\right\}\cup\left\{(1,\xi)|\theta\in(0,\pi)\right\},\\
\tilde{\Gamma}_- :=& \left\{(0,\xi)|\theta\in(0,\pi)\right\}\cup\left\{(1,\xi)|\theta\in(-\pi,0)\right\},\\
\tilde{\Gamma}_0 :=& \left\{(0,\xi)|\theta\in\{0,\pi\}\right\}\cup\left\{(1,\xi)|\theta\in\{0,\pi\}\right\},
\end{aligned}$$

and the boundary condition can be reduced to

$$I(x_2,\xi)=\tilde{I}_0(x_2,\xi),\quad (x_2,\xi)\in\tilde{\Gamma}_-, \tag{1.5}$$

where $\tilde{I}_0$ is a given function corresponding to $I_0$ on $\Gamma_-$. For simplicity, we omit the tilde and the subscription of $x_2$ in the discussion below.

We rewrite the definition of classical solutions for our boundary value problem (1.4) and (1.5) as follows.

**Definition 1.2.** We say $I\in C_b\left(\left(\Omega\times S^1\right)\cup\Gamma_-\right)$ is a classical solution to the boundary value problem (1.4) and (1.5), if $I$ satisfies the following integral equations: for $x\in[0,1)$ and $\theta\in(0,\pi)$,

$$\begin{aligned}
I(x,\xi) =& \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}x\right)I_0(0,\xi)\\
&+\frac{\mu_s}{\sin\theta}\int_0^x\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)\int_{S^1}p(\xi,\xi')I(t,\xi')\,d\sigma_{\xi'}dt
\end{aligned}\tag{1.6}$$

and for $x\in(0,1]$ and $\theta\in(-\pi,0)$,

$$\begin{aligned}
I(x,\xi) =& \exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(1-x)\right)I_0(1,\xi)\\
&-\frac{\mu_s}{\sin\theta}\int_x^1\exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right)\int_{S^1}p(\xi,\xi')I(t,\xi')\,d\sigma_{\xi'}dt.
\end{aligned}\tag{1.7}$$

**Remark 1.3.** If the solution $I$ exists and belongs to $C_b\left(\left(\Omega\times S^1\right)\cup\Gamma_-\right)$, we differentiate the right hand side of (1.6) and (1.7) with respect to $x$ to obtain the following equations: for $x\in(0,1)$ and $\theta\in(0,\pi)$,

$$\begin{aligned}
\frac{\partial I}{\partial x}(x,\xi) =& -\frac{\mu_a+\mu_s}{\sin\theta}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}x\right)I_0(0,\xi)\\
&+\frac{\mu_s}{\sin\theta}\int_{S^1}p(\xi,\xi')I(x,\xi')\,d\sigma_{\xi'}\\
&-\frac{\mu_a+\mu_s}{\sin\theta}\frac{\mu_s}{\sin\theta}\int_0^x\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)\int_{S^1}p(\xi,\xi')I(t,\xi')\,d\sigma_{\xi'}dt\\
=& -\frac{\mu_a+\mu_s}{\sin\theta}I(x,\xi)+\frac{\mu_s}{\sin\theta}\int_{S^1}p(\xi,\xi')I(x,\xi')\,d\sigma_{\xi'}
\end{aligned}$$

and for $x \in (0,1)$ and $\theta \in (-\pi, 0)$,

$$\begin{aligned}\frac{\partial I}{\partial x}(x,\xi) =& -\frac{\mu_a+\mu_s}{\sin\theta}\exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(1-x)\right) I_0(1,\xi)\\ &+ \frac{\mu_s}{\sin\theta}\int_{S^1} p(\xi,\xi')I(x,\xi')\,d\sigma_{\xi'}\\ &+ \frac{\mu_a+\mu_s}{\sin\theta}\frac{\mu_s}{\sin\theta}\int_x^1 \exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right)\int_{S^1} p(\xi,\xi')I(t,\xi')\,d\sigma_{\xi'}dt\\ =& -\frac{\mu_a+\mu_s}{\sin\theta}I(x,\xi) + \frac{\mu_s}{\sin\theta}\int_{S^1} p(\xi,\xi')I(x,\xi')\,d\sigma_{\xi'}.\end{aligned}$$

Substituting $x = 0$ for the equation (1.6), we have $I(0,\xi) = I_0(x,\xi)$ for $\theta \in (0,\pi)$. Similary, substituting $x = 1$ for the equation (1.7), we have $I(1,\xi) = I_0(1,\xi)$ for $\theta \in (-\pi, 0)$. Consequently, the classical solution satisfies the equation (1.4) with the boundary condition (1.5) except for $\theta \in \{0, \pi\}$.

In this paper, we prove the following three theorems, which are the main results.

**Theorem 1.4.** *Suppose that $I_0 \in C_b(\Gamma_-)$ and $\mu_a > 0$. Then the equation* (1.4) *with the boundary condition* (1.5) *has a unique classical solution $I$.*

**Theorem 1.5.** *Under the same assumptions as Theorem* 1.4*, there exists $\frac{\partial I}{\partial x}$ in $C(\Omega \times S^1)$. Moreover, the classical solution $I$ is infinitely differentiable with respect to $x$ in $\Omega \times S^1$.*

**Theorem 1.6.** *In addition to the assumptions in Theorem* 1.4*, we suppose that $p$ is continuously differentiable and that $\frac{\partial I_0}{\partial \theta} \in C_b(\Gamma_-)$. Then, there exists $\frac{\partial I}{\partial \theta}$ in $C(\Omega \times S^1)$.*

# 2 Existence and Uniqueness of Solutions

We give a proof of Theorem 1.4. We firstly prove the uniqueness of classical solutions by some estimates and secondly prove the existence of a solution by a constructive method.

## 2.1 Uniqueness of Solutions

We start from the following proposition.

**Proposition 2.1.** *Suppose that $I_0 \in C_b(\Gamma_-)$ and $\mu_a > 0$. Then, the classical solution $I$ to the boundary value problem* (1.4) *and* (1.5) *is unique in $C_b\left((\Omega \times S^1) \cup \Gamma_-\right)$, if it exists.*

*Proof.* Let $I_1$ and $I_2$ be two classical solutions and let us denote by $I$ the difference of these two functions; $I := I_1 - I_2$. Then, by definition of classical solutions, $I$ belongs to $C_b\left((\Omega \times S^1) \cup \Gamma_-\right)$ and satisfies the following integral equations: for $x \in [0,1)$ and $\theta \in (0,\pi)$,

$$I(x,\xi) = \frac{\mu_s}{\sin\theta}\int_0^x \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)\int_{S^1} p(\xi,\xi')I(t,\xi')\,d\sigma_{\xi'}\,dt \tag{2.1}$$

and for $x \in (0,1]$ and $\theta \in (-\pi,0)$,

$$I(x,\xi) = -\frac{\mu_s}{\sin\theta}\int_x^1 \exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right)\int_{S^1} p(\xi,\xi')I(t,\xi')\,d\sigma_{\xi'}\,dt. \tag{2.2}$$

By the equation (2.1), we obtain, for $x \in [0,1)$ and $\theta \in (0,\pi)$,

$$\begin{aligned}
|I(x,\xi)| &\le \frac{\mu_s}{\sin\theta}\int_0^x \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)\int_{S^1} p(\xi,\xi')d\sigma_{\xi'}dt\|I\|_\infty \\
&= \frac{\mu_s}{\sin\theta}\int_0^x \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)dt\|I\|_\infty \\
&\le \frac{\mu_s}{\mu_a+\mu_s}\|I\|_\infty.
\end{aligned}$$

Similarly, by the equation (2.2), we obtain, for $x \in (0,1]$ and $\theta \in (-\pi,0)$,

$$\begin{aligned}
|I(x,\xi)| &\le -\frac{\mu_s}{\sin\theta}\int_x^1 \exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right)\int_{S^1} p(\xi,\xi')d\sigma_{\xi'}dt\|I\|_\infty \\
&= -\frac{\mu_s}{\sin\theta}\int_x^1 \exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right)dt\|I\|_\infty \\
&\le \frac{\mu_s}{\mu_a+\mu_s}\|I\|_\infty.
\end{aligned}$$

Thus, we have $\|I\|_\infty \le \dfrac{\mu_s}{\mu_a+\mu_s}\|I\|_\infty$, which leads us to the uniqueness of the classical solution. □

## 2.2 Existence of Solutions

We prove the existence of classical solutions to the boundary value problem (1.4) and (1.5), and, to this end, we define a sequence of functions $\{I^{(n)}\}_{n\ge 0}$ in $C_b\left((\Omega \times S^1) \cup \Gamma_-\right)$. We firstly define $I^{(0)} \in C_b\left((\Omega \times S^1) \cup \Gamma_-\right)$ as follows:

- For $x \in [0,1)$ and $\theta \in (0,\pi)$, $I^{(0)}(x,\xi) := \exp\left(-\dfrac{\mu_a+\mu_s}{\sin\theta}x\right) I_0(0,\xi)$.
- For $x \in (0,1]$ and $\theta \in (-\pi,0)$, $I^{(0)}(x,\xi) := \exp\left(\dfrac{\mu_a+\mu_s}{\sin\theta}(1-x)\right) I_0(1,\xi)$.
- For $x \in (0,1)$ and $\theta \in \{0,\pi\}$, $I^{(0)}(x,\xi) := 0$.

Once we suppose to define up to $I^{(n)}$, we define $I^{(n+1)}$ as follows:

- For $x \in [0,1)$ and $\theta \in (0,\pi)$,

$$I^{(n+1)}(x,\xi) := \frac{\mu_s}{\sin\theta}\int_0^x \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right) \times \int_{S^1} p(\xi,\xi')I^{(n)}(t,\xi')\,d\sigma_{\xi'}\,dt.$$

- For $x \in (0,1]$ and $\theta \in (-\pi,0)$,

$$I^{(n+1)}(x,\xi) := -\frac{\mu_s}{\sin\theta}\int_x^1 \exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right) \times \int_{S^1} p(\xi,\xi')I^{(n)}(t,\xi')\,d\sigma_{\xi'}\,dt.$$

- For $x \in (0,1)$ and $\theta \in \{0,\pi\}$,

$$I^{(n+1)}(x,\xi) := \frac{\mu_s}{\mu_a+\mu_s}\int_{S^1} p(\xi,\xi')I^{(n)}(x,\xi')\,d\sigma_{\xi'}.$$

For the sequence $\{I^{(n)}\}_{n\geq 0}$, we have the next proposition to conclude that the sequence is well-defined.

**Proposition 2.2.** *If $I_0 \in C_b(\Gamma_-)$, then $I^{(n)} \in C_b\left(\left(\Omega\times S^1\right)\cup\Gamma_-\right)$ for all the nonnegative integers $n$.*

*Proof.* We prove it by induction on $n$. We consider the case $n=0$. We remark that $I^{(0)}$ satisfies $\|I^{(0)}\|_\infty \leq \|I_0\|_{\Gamma_-}$. To prove continuity of $I^{(0)}$, we must check continuity at $(x,\xi)\in(0,1)\times\{\xi_0,\xi_\pi\}$, where we denote by $\xi_0$ and $\xi_\pi$ the points $(1,0)=(\cos 0,\sin 0)$ and $(-1,0)=(\cos\pi,\sin\pi)$, respectively. Continuity at the other points is obvious. Since $I_0$ is bounded, for $x\in(0,1)$,

$$\lim_{\theta\downarrow 0} I^{(0)}(x,\xi) = \lim_{\theta\downarrow 0}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}x\right)I_0(0,\xi) = 0$$

and

$$\lim_{\theta\uparrow 0} I^{(0)}(x,\xi) = \lim_{\theta\uparrow 0}\exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(1-x)\right)I_0(1,\xi) = 0.$$

In the same way, for $x\in(0,1)$,

$$\lim_{\theta\downarrow -\pi} I^{(0)}(x,\xi) = \lim_{\theta\downarrow -\pi}\exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(1-x)\right)I_0(1,\xi) = 0$$

and

$$\lim_{\theta\uparrow\pi} I^{(0)}(x,\xi) = \lim_{\theta\uparrow\pi} \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}x\right) I_0(0,\xi) = 0.$$

Thus, we show that $I^{(0)} \in C_b\left((\Omega\times S^1)\cup\Gamma_-\right)$.

Next, we suppose $I^{(n)} \in C_b\left((\Omega\times S^1)\cup\Gamma_-\right)$. We can prove the following lemma by straightforward estimates.

**Lemma 2.3.** *Suppose $I^{(n)} \in C_b\left((\Omega\times S^1)\cup\Gamma_-\right)$. Then, the following estimate holds:*

$$\|I^{(n+1)}\|_\infty \le \frac{\mu_s}{\mu_a+\mu_s}\|I^{(n)}\|_\infty. \tag{2.3}$$

From Lemma 2.3, $I^{(n+1)}$ is bounded on $(\Omega\times S^1)\cup\Gamma_-$. We prove continuity at $(x,\xi)\in(0,1)\times\{\xi_0,\xi_\pi\}$. For the same reason as the case $n=0$, this completes the proof. We consider continuity as $\theta\downarrow 0$ and as $\theta\uparrow\pi$. Using the following equality: for all $x\in[0,1)$ and $\theta\in(0,\pi)$,

$$\frac{\mu_s}{\mu_a+\mu_s} = \frac{\mu_s}{\sin\theta}\int_0^x \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)dt + \frac{\mu_s}{\mu_a+\mu_s}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}x\right), \tag{2.4}$$

we have, for $x\in(0,1)$ and $\theta\in(0,\pi)$,

$$\begin{aligned}
&|I^{(n+1)}(x,\xi_0) - I^{(n+1)}(x,\xi)|\\
&= \Bigg|\frac{\mu_s}{\mu_a+\mu_s}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}x\right)\int_{S^1} p(\xi_0,\xi')I^{(n)}(x,\xi')\,d\sigma_{\xi'}\\
&\quad+\frac{\mu_s}{\sin\theta}\int_0^x \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)dt\int_{S^1} p(\xi_0,\xi')I^{(n)}(x,\xi')\,d\sigma_{\xi'}\\
&\quad-\frac{\mu_s}{\sin\theta}\int_0^x \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)\int_{S^1} p(\xi,\xi')I^{(n)}(t,\xi')\,d\sigma_{\xi'}dt\Bigg|\\
&\le J_1+J_2+J_3,
\end{aligned}$$

where

$$J_1 := \frac{\mu_s}{\sin\theta}\int_0^x \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)dt \times\int_{S^1}|p(\xi_0,\xi')-p(\xi,\xi')||I^{(n)}(x,\xi')|\,d\sigma_{\xi'},$$

$$J_2 := \frac{\mu_s}{\sin\theta}\int_0^x \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right) \times\int_{S^1} p(\xi,\xi')|I^{(n)}(x,\xi')-I^{(n)}(t,\xi')|\,d\sigma_{\xi'}dt$$

and

$$J_3 := \frac{\mu_s}{\mu_a + \mu_s} \exp\left(-\frac{\mu_a + \mu_s}{\sin\theta} x\right) \int_{S^1} p(\xi_0, \xi') |I^{(n)}(x, \xi')| \, d\sigma_{\xi'}.$$

We estimate each term of the right hand side. By definition of $J_1$, we have

$$J_1 \leq \frac{\mu_a}{\mu_a + \mu_s} 2\pi \|I^{(n)}\|_\infty \max_{\xi' \in S^1} |p(\xi_0, \xi') - p(\xi, \xi')|.$$

Since $p$ is continuous, $\lim_{\theta \downarrow 0} \max_{\xi' \in S^1} |p(\xi_0, \xi') - p(\xi, \xi')| = 0$. Thus, we get $\lim_{\theta \downarrow 0} J_1 = 0$. Also, by definition of $J_3$, we have

$$J_3 \leq \frac{\mu_s}{\mu_a + \mu_s} \exp\left(-\frac{\mu_a + \mu_s}{\sin\theta} x\right) \|I^{(n)}\|_\infty.$$

From $\lim_{\theta \downarrow 0} \exp\left(-\frac{\mu_a + \mu_s}{\sin\theta} x\right) = 0$ for all $x \in (0, 1)$, we get $\lim_{\theta \downarrow 0} J_3 = 0$. Finally, for all $\delta$ with $0 < \delta < x$, we have

$$\begin{aligned}
J_2 =& \frac{\mu_s}{\sin\theta} \int_0^{x-\delta} \exp\left(-\frac{\mu_a + \mu_s}{\sin\theta}(x - t)\right) \\
&\quad \times \int_{S^1} p(\xi, \xi') |I^{(n)}(x, \xi') - I^{(n)}(t, \xi')| \, d\sigma_{\xi'} dt \\
&+ \frac{\mu_s}{\sin\theta} \int_{x-\delta}^{x} \exp\left(-\frac{\mu_a + \mu_s}{\sin\theta}(x - t)\right) \\
&\quad \times \int_{S^1} p(\xi, \xi') |I^{(n)}(x, \xi') - I^{(n)}(t, \xi')| \, d\sigma_{\xi'} dt \\
\leq& 2\|I^{(n)}\|_\infty \frac{\mu_s}{\mu_a + \mu_s} \exp\left(-\frac{\mu_a + \mu_s}{\sin\theta} \delta\right) \\
&+ \frac{\mu_s}{\mu_a + \mu_s} \sup_{t \in (x-\delta, x)} \left( \max_{\xi \in S^1} |I^{(n)}(x, \xi') - I^{(n)}(t, \xi')| \right).
\end{aligned}$$

Accordingly, we have

$$\lim_{\theta \downarrow 0} J_2 \leq \frac{\mu_s}{\mu_a + \mu_s} \sup_{t \in (x-\delta, x)} \left( \max_{\xi \in S^1} |I^{(n)}(x, \xi') - I^{(n)}(t, \xi')| \right).$$

Since $I^{(n)}$ is continuous by the induction hypothesis, the right hand side of this inequality becomes smaller as $\delta$ becomes smaller. As a result,

$$\lim_{\theta \downarrow 0} |I^{(n+1)}(x, \xi_0) - I^{(n+1)}(x, \xi)| = 0.$$

Similarly, we can prove that $\lim_{\theta \uparrow \pi} |I^{(n+1)}(x, \xi_\pi) - I^{(n+1)}(x, \xi)| = 0$. We secondly consider continuity as $\theta \uparrow 0$ and as $\theta \downarrow -\pi$. For $x \in (0, 1)$ and $\theta \in (-\pi, 0)$,

$$|I^{(n+1)}(x, \xi_0) - I^{(n+1)}(x, \xi)| \leq J_1' + J_2' + J_3',$$

where

$$J_1' := -\frac{\mu_s}{\sin\theta}\int_x^1 \exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right)\,dt \\ \times\int_{S^1} |p(\xi_0,\xi')-p(\xi,\xi')||I^{(n)}(x,\xi')|\,d\sigma_{\xi'},$$

$$J_2' := -\frac{\mu_s}{\sin\theta}\int_x^1 \exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right) \\ \times\int_{S^1} p(\xi,\xi')|I^{(n)}(x,\xi')-I^{(n)}(t,\xi')|\,d\sigma_{\xi'}dt$$

and

$$J_3' := \frac{\mu_s}{\mu_a+\mu_s}\exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(1-x)\right)\int_{S^1} p(\xi_0,\xi')|I^{(n)}(x,\xi')|\,d\sigma_{\xi'}.$$

Here, we used the following equality: for all $x \in (0,1]$ and $\theta \in (-\pi, 0)$,

$$\frac{\mu_s}{\mu_a+\mu_s} = -\frac{\mu_s}{\sin\theta}\int_x^1 \exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right)\,dt \\ -\frac{\mu_s}{\mu_a+\mu_s}\exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(1-x)\right). \quad (2.5)$$

In the same way as the case $\theta \in (0,\pi)$, we can prove that

$$\lim_{\theta\uparrow 0}|I^{(n+1)}(x,\xi_0)-I^{(n+1)}(x,\xi)| = 0$$

and

$$\lim_{\theta\downarrow -\pi}|I^{(n+1)}(x,\xi_\pi)-I^{(n+1)}(x,\xi)| = 0.$$

As a result, we conclude that $I^{(n+1)} \in C_b\left((\Omega\times S^1)\cup\Gamma_-\right)$. □

We define a sequence of functions $\{I_n\}_{n\geq 0}$ by

$$I_n(x,\xi) := \sum_{k=0}^n I^{(k)}(x,\xi).$$

By definition, $I_n$ belongs to $C_b\left((\Omega\times S^1)\cup\Gamma_-\right)$ for all the nonnegative integers $n$. Since the estimate (2.3) holds for all $n$, we obtain

$$\|I^{(n)}\|_\infty \leq \left(\frac{\mu_s}{\mu_a+\mu_s}\right)^n \|I_0\|_{\Gamma_-}.$$

Accordingly, we have

$$\|I_m - I_n\|_\infty = \left\|\sum_{k=n+1}^m I^{(k)}\right\|_\infty \leq \sum_{k=n+1}^m \|I^{(k)}\|_\infty \leq \sum_{k=n+1}^m \left(\frac{\mu_s}{\mu_a+\mu_s}\right)^k \|I_0\|_{\Gamma_-}$$

for all the nonnegative integer $m$ and $n$ with $m > n$. Since $\mu_a$ is positive, the right hand side of the inequality tends to 0 as $m$ and $n$ tend to $\infty$. For this reason, $\{I_n\}_{n\geq 0}$ is a Cauchy sequence in $C_b\left(\left(\Omega \times S^1\right) \cup \Gamma_-\right)$. Consequently, from the completeness of the space, we conclude that there exists $I = \lim_{n\to\infty} I_n = \sum_{n=0}^{\infty} I^{(n)}$ in $C_b\left(\left(\Omega \times S^1\right) \cup \Gamma_-\right)$.

At the end of this subsection, we check that the function $I$ above is indeed a solution to the boundary value problem (1.4) and (1.5). For $x \in [0,1)$ and $\theta \in (0,\pi)$,

$$
\begin{aligned}
I(x,\xi) =& \sum_{n=0}^{\infty} I^{(n)}(x,\xi) = I^{(0)}(x,\xi) + \sum_{n=1}^{\infty} I^{(n)}(x,\xi) \\
=& \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}x\right) I_0(0,\xi) \\
&+ \sum_{n=0}^{\infty} \frac{\mu_s}{\sin\theta} \int_0^x \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right) \int_{S^1} p(\xi,\xi') I^{(n)}(t,\xi')\, d\sigma_{\xi'}\, dt \\
=& \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}x\right) I_0(0,\xi) \\
&+ \frac{\mu_s}{\sin\theta} \int_0^x \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right) \int_{S^1} p(\xi,\xi') \sum_{n=0}^{\infty} I^{(n)}(t,\xi')\, d\sigma_{\xi'}\, dt \\
=& \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}x\right) I_0(0,\xi) \\
&+ \frac{\mu_s}{\sin\theta} \int_0^x \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right) \int_{S^1} p(\xi,\xi') I(t,\xi')\, d\sigma_{\xi'}\, dt,
\end{aligned}
$$

where we used the uniform convergence of $\sum_{n=0}^{\infty} I^{(n)}$ in the fourth equality. Similarly, for $x \in (0,1]$ and $\theta \in (-\pi,0)$,

$$
\begin{aligned}
I(x,\xi) =& \exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(1-x)\right) I_0(1,\xi) \\
&- \frac{\mu_s}{\sin\theta} \int_x^1 \exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right) \int_{S^1} p(\xi,\xi') I(t,\xi')\, d\sigma_{\xi'}\, dt.
\end{aligned}
$$

Therefore, $I$ is a solution to the boundary value problem (1.4) and (1.5).

# 3 Differentiability with respect to $x$

If the function $I$ is the unique solution to the boundary value problem (1.4) and (1.5), $I$ satisfies

$$
\frac{\partial I}{\partial x}(x,\xi) = -\frac{\mu_a+\mu_s}{\sin\theta} I(x,\xi) + \frac{\mu_s}{\sin\theta} \int_{S^1} p(\xi,\xi') I(x,\xi')\, d\sigma_{\xi'}
$$

except for $\theta \in \{0, \pi\}$. Since $I$ is continuous on $(\Omega \times S^1) \cup \Gamma_-$, the right hand side of this equation is also continuous except for $\theta \in \{0, \pi\}$. Now, it is not obvious whether $\dfrac{\partial I}{\partial x}(x, \xi)$ is defined and continuous at $\theta \in \{0, \pi\}$. In this section, we prove that $\dfrac{\partial I}{\partial x}$ belongs to $C(\Omega \times S^1)$.

## 3.1 Preliminaries

We prepare some notation to prove Theorem 1.5. We take a number $\delta$ with $0 < \delta < 1/2$ and denote by $K$ a closed interval $[\delta, 1-\delta]$ in $(0, 1)$. We also take a sequence of positive numbers $\{\delta_n\}_{n \geq 0}$ such that $\displaystyle\sum_{n=0}^{\infty} \delta_n = \delta$, and the corresponding closed intervals $\{K_n\}_{n \geq 0}$ such that $K_n := \left[\displaystyle\sum_{m=0}^{n} \delta_m, 1 - \sum_{m=0}^{n} \delta_m\right]$. We introduce the supremum norms $\| \cdot \|_K$ and $\| \cdot \|_n$ by

$$\|I\|_K := \sup_{K \times S^1} |I(x, \xi)|$$

and

$$\|I\|_n := \sup_{K_n \times S^1} |I(x, \xi)|,$$

respectively.

We differentiate each term of the series $\{I^{(n)}\}_{n \geq 0}$ with respect to $x$. For $x \in (0, 1)$ and $\theta \in (0, \pi)$, we have

$$\begin{aligned}
\frac{\partial I^{(0)}}{\partial x}(x, \xi) =& -\frac{\mu_a + \mu_s}{\sin\theta} \exp\left(-\frac{\mu_a + \mu_s}{\sin\theta} x\right) I_0(0, \xi), \\
\frac{\partial I^{(n+1)}}{\partial x}(x, \xi) =& \frac{\mu_s}{\sin\theta} \int_{S^1} p(\xi, \xi') I^{(n)}(x, \xi') \, d\sigma_{\xi'} \\
&- \frac{\mu_s(\mu_a + \mu_s)}{\sin^2\theta} \int_0^x \exp\left(-\frac{\mu_a + \mu_s}{\sin\theta}(x - t)\right) \\
&\qquad \times \int_{S^1} p(\xi, \xi') I^{(n)}(t, \xi') \, d\sigma_{\xi'} dt.
\end{aligned}$$

For $x \in (0, 1)$ and $\theta \in (-\pi, 0)$, we have

$$\begin{aligned}
\frac{\partial I^{(0)}}{\partial x}(x, \xi) =& -\frac{\mu_a + \mu_s}{\sin\theta} \exp\left(\frac{\mu_a + \mu_s}{\sin\theta}(1 - x)\right) I_0(1, \xi), \\
\frac{\partial I^{(n+1)}}{\partial x}(x, \xi) =& \frac{\mu_s}{\sin\theta} \int_{S^1} p(\xi, \xi') I^{(n)}(x, \xi') \, d\sigma_{\xi'} dt \\
&+ \frac{\mu_s(\mu_a + \mu_s)}{\sin^2\theta} \int_x^1 \exp\left(\frac{\mu_a + \mu_s}{\sin\theta}(t - x)\right) \\
&\qquad \times \int_{S^1} p(\xi, \xi') I^{(n)}(t, \xi') \, d\sigma_{\xi'} dt.
\end{aligned}$$

For $x \in (0,1)$ and $\theta \in \{0, \pi\}$, we have formally

$$\begin{aligned}\frac{\partial I^{(0)}}{\partial x}(x,\xi) =& 0,\\ \frac{\partial I^{(n+1)}}{\partial x}(x,\xi) =& \frac{\mu_s}{\mu_a+\mu_s}\int_{S^1} p(\xi,\xi')\frac{\partial I^{(n)}}{\partial x}(x,\xi')\, d\sigma_{\xi'}.\end{aligned}$$

Since $\dfrac{\partial I^{(0)}}{\partial x}$ is bounded on $K_0 \times S^1$, the following lemma guarantees that $\dfrac{\partial I^{(n)}}{\partial x}$ is bounded on $K_n \times S^1$ for all $n$.

**Lemma 3.1.** *If* $\dfrac{\partial I^{(n)}}{\partial x}$ *exists and is bounded on* $K_n \times S^1$*, then* $\dfrac{\partial I^{(n+1)}}{\partial x}$ *exists on* $K_{n+1} \times S^1$ *and the following estimate holds:*

$$\left\| \frac{\partial I^{(n+1)}}{\partial x} \right\|_{n+1} \le \frac{2\mu_s}{e(\mu_a+\mu_s)\delta_{n+1}} \|I^{(n)}\|_\infty + \frac{\mu_s}{\mu_a+\mu_s} \left\| \frac{\partial I^{(n)}}{\partial x} \right\|_n .$$

*Proof.* For $x \in K_{n+1}$ and $\theta \in (0,\pi)$, we have

$$\begin{aligned}\left|\frac{\partial I^{(n+1)}}{\partial x}(x,\xi)\right| = \Bigg| & \frac{\mu_s}{\sin\theta}\int_{S^1} p(\xi,\xi') I^{(n)}(x,\xi')\, d\sigma_{\xi'}\\ & - \frac{\mu_s(\mu_a+\mu_s)}{\sin^2\theta}\int_0^{x-\delta_{n+1}} \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)\\ & \qquad \times \int_{S^1} p(\xi,\xi') I^{(n)}(t,\xi')\, d\sigma_{\xi'}\, dt\\ & - \frac{\mu_s(\mu_a+\mu_s)}{\sin^2\theta}\int_{x-\delta_{n+1}}^{x} \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)\\ & \qquad \times \int_{S^1} p(\xi,\xi') I^{(n)}(t,\xi')\, d\sigma_{\xi'}\, dt\Bigg|\\ \le\ & J_4 + J_5 + J_6,\end{aligned}$$

where

$$\begin{aligned}J_4 :=& \frac{\mu_s(\mu_a+\mu_s)}{\sin^2\theta}\int_0^{x-\delta_{n+1}} \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)\int_{S^1} p(\xi,\xi')\left|I^{(n)}(t,\xi')\right| d\sigma_{\xi'}\, dt,\\ J_5 :=& \frac{\mu_s}{\sin\theta}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}\delta_{n+1}\right)\int_{S^1} p(\xi,\xi')\left|I^{(n)}(x-\delta_{n+1},\xi')\right| d\sigma_{\xi'}\end{aligned}$$

and

$$J_6 := \frac{\mu_s}{\sin\theta}\int_{x-\delta_{n+1}}^{x} \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)\int_{S^1} p(\xi,\xi')\left|\frac{\partial I^{(n)}}{\partial x}(t,\xi')\right| d\sigma_{\xi'}\, dt.$$

Here, we did integration by parts with respect to $t$ to obtain $J_6$. Since $[x-\delta_{n+1}, x] \subset K_n$, we obtain

$$\begin{aligned}\left|\frac{\partial I^{(n+1)}}{\partial x}(x,\xi)\right| &\le \frac{2\mu_s}{\sin\theta}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}\delta_{n+1}\right)\|I^{(n)}\|_\infty + \frac{\mu_s}{\mu_a+\mu_s}\left\|\frac{\partial I^{(n)}}{\partial x}\right\|_n \\ &\le \frac{2\mu_s}{e(\mu_a+\mu_s)\delta_{n+1}}\|I^{(n)}\|_\infty + \frac{\mu_s}{\mu_a+\mu_s}\left\|\frac{\partial I^{(n)}}{\partial x}\right\|_n.\end{aligned}$$

Here, in the last inequality we used the fact that the function

$$f(z) := 2\mu_s z \exp\left(-\left(\mu_a+\mu_s\right)\delta_{n+1} z\right), \quad z \in (0,\infty)$$

achieves its maximum value $\dfrac{2\mu_s}{e(\mu_a+\mu_s)\delta_{n+1}}$ at $z = \dfrac{1}{(\mu_a+\mu_s)\delta_{n+1}}$. Similarly, for $x \in K_{n+1}$ and $\theta \in (-\pi, 0)$, we have

$$\left|\frac{\partial I^{(n+1)}}{\partial x}(x,\xi)\right| \le J_4' + J_5' + J_6',$$

where

$$\begin{aligned}J_4' :=& \frac{\mu_s}{\sin\theta}\int_{x+\delta_{n+1}}^{1}\frac{\mu_a+\mu_s}{\sin\theta}\exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right)\int_{S^1} p(\xi,\xi')|I^{(n)}(t,\xi')|\,d\sigma_{\xi'}\,dt,\\ J_5' :=& \frac{\mu_s}{\sin\theta}\exp\left(\frac{\mu_a+\mu_s}{\sin\theta}\delta_{n+1}\right)\int_{S^1} p(\xi,\xi')|I^{(n)}(x+\delta_{n+1}),\xi')|\,d\sigma_{\xi'}\end{aligned}$$

and

$$J_6' := -\frac{\mu_s}{\sin\theta}\int_{x}^{x+\delta_{n+1}}\exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right)\int_{S^1} p(\xi,\xi')\left|\frac{\partial I^{(n)}}{\partial x}(t,\xi')\right|\,d\sigma_{\xi'}\,dt.$$

Since $[x, x+\delta_{n+1}] \subset K_n$, we obtain

$$\begin{aligned}\left|\frac{\partial I^{(n+1)}}{\partial x}(x,\xi)\right| &\le \frac{2\mu_s}{|\sin\theta|}\exp\left(-\frac{\mu_a+\mu_s}{|\sin\theta|}\delta_{n+1}\right)\|I^{(n)}\|_\infty + \frac{\mu_s}{\mu_a+\mu_s}\left\|\frac{\partial I^{(n)}}{\partial x}\right\|_n \\ &\le \frac{2\mu_s}{e(\mu_a+\mu_s)\delta_{n+1}}\|I^{(n)}\|_\infty + \frac{\mu_s}{\mu_a+\mu_s}\left\|\frac{\partial I^{(n)}}{\partial x}\right\|_n.\end{aligned}$$

For $x \in K_{n+1}$ and $\theta \in \{0, \pi\}$, we obviously have

$$\left|\frac{\partial I^{(n+1)}}{\partial x}(x,\xi)\right| \le \frac{\mu_s}{\mu_a+\mu_s}\left\|\frac{\partial I^{(n)}}{\partial x}\right\|_n.$$

Therefore, we obtain the statement of the lemma. □

### 3.2 Differentiability with respect to $x$

Let us denote $\tilde{\mu}_a := \dfrac{\mu_a}{2}$, $\delta_0 := \dfrac{\tilde{\mu}_a}{\tilde{\mu}_a + \mu_s}\delta$, and $\delta_{n+1} := \dfrac{\mu_s}{\tilde{\mu}_a + \mu_s}\delta_n$. Using Lemma 3.1, we get the following estimates:

$$\begin{aligned}
\left\| \frac{\partial I^{(n)}}{\partial x} \right\|_n \leq & \frac{\mu_s}{\mu_a + \mu_s} \left\| \frac{\partial I^{(n-1)}}{\partial x} \right\|_{n-1} + \frac{2\mu_s}{e(\mu_a + \mu_s)\delta_n} \|I^{(n-1)}\|_\infty \\
\leq & \left( \frac{\mu_s}{\mu_a + \mu_s} \right)^n \left\| \frac{\partial I^{(0)}}{\partial x} \right\|_0 \\
& + \frac{2\mu_s}{e(\mu_a + \mu_s)} \sum_{k=0}^{n-1} \frac{1}{\delta_{k+1}} \left( \frac{\mu_s}{\mu_a + \mu_s} \right)^{n-1-k} \|I^{(k)}\|_\infty .
\end{aligned}$$

Remembering that $\|I^{(k)}\|_\infty \leq \left( \dfrac{\mu_s}{\mu_a + \mu_s} \right)^k \|I_0\|_{\Gamma_-}$, we have

$$\begin{aligned}
& \frac{\mu_s}{\mu_a + \mu_s} \sum_{k=0}^{n-1} \frac{1}{\delta_{k+1}} \left( \frac{\mu_s}{\mu_a + \mu_s} \right)^{n-1-k} \|I^{(k)}\|_\infty \\
\leq & \sum_{k=0}^{n-1} \frac{1}{\delta_{k+1}} \left( \frac{\mu_s}{\mu_a + \mu_s} \right)^n \|I_0\|_{\Gamma_-} \\
= & \sum_{k=0}^{n-1} \frac{1}{\delta_0} \left( \frac{\tilde{\mu}_a + \mu_s}{\mu_s} \right)^{k+1} \left( \frac{\mu_s}{\mu_a + \mu_s} \right)^n \|I_0\|_{\Gamma_-} \\
\leq & \frac{\|I_0\|_{\Gamma_-}}{\delta_0} \sum_{k=0}^{n-1} \left( \frac{\tilde{\mu}_a + \mu_s}{\mu_a + \mu_s} \right)^n \\
= & \frac{\|I_0\|_{\Gamma_-}}{\delta_0} n \left( \frac{\tilde{\mu}_a + \mu_s}{\mu_a + \mu_s} \right)^n .
\end{aligned}$$

Consequently, we have

$$\left\| \frac{\partial I^{(n)}}{\partial x} \right\|_n \leq \left( \frac{\mu_s}{\mu_a + \mu_s} \right)^n \left\| \frac{\partial I^{(0)}}{\partial x} \right\|_0 + \frac{2}{e} n \left( \frac{\tilde{\mu}_a + \mu_s}{\mu_a + \mu_s} \right)^n \frac{\|I_0\|_{\Gamma_-}}{\delta_0} .$$

Hence, for all the nonnegative integer $N$,

$$\begin{aligned}
\sum_{n=0}^{N} \left\| \frac{\partial I^{(n)}}{\partial x} \right\|_K \leq & \sum_{n=0}^{N} \left\| \frac{\partial I^{(n)}}{\partial x} \right\|_n \\
\leq & \left\| \frac{\partial I^{(0)}}{\partial x} \right\|_0 \sum_{n=0}^{N} \left( \frac{\mu_s}{\mu_a + \mu_s} \right)^n + \frac{2\|I_0\|_{\Gamma_-}}{e\delta_0} \sum_{n=0}^{N} n \left( \frac{\tilde{\mu}_a + \mu_s}{\mu_a + \mu_s} \right)^n .
\end{aligned} \tag{3.1}$$

Since both $\dfrac{\mu_s}{\mu_a+\mu_s}$ and $\dfrac{\tilde{\mu}_a+\mu_s}{\mu_a+\mu_s}$ are less than unity, the right hand side of the equation (3.1) converges as $N$ tends to $\infty$. This implies that the series $\displaystyle\sum_{n=0}^{\infty}\frac{\partial I^{(n)}}{\partial x}$ converges absolutely and uniformly on $K\times S^1$. We can prove in a similar way as the case with $I^{(n)}$ that $\dfrac{\partial I^{(n)}}{\partial x}$ is continuous on $K_n\times S^1$ for all $n$. So, $\displaystyle\sum_{n=0}^{\infty}\frac{\partial I^{(n)}}{\partial x}$ is continuous on $K\times S^1$. Now, we can take any $\delta$ with $0<\delta<1/2$ and we conclude that $\dfrac{\partial I}{\partial x}=\displaystyle\sum_{n=0}^{\infty}\frac{\partial I^{(n)}}{\partial x}$ belongs to $C(\Omega\times S^1)$.

### 3.3 $m$ times Differentiability with respect to $x$

Let $\delta$ and $K$ be the positive number and the closed interval defined in subsection 3.1, respectively. We take a sequence of positive numbers $\{\delta_n\}_{n\geq 0}$ and the corresponding closed intervals $\{K_n\}_{n\geq 0}$ as in subsection 3.2, and suppose that $I^{(n)}$ is bounded on $K_n\times S^1$ up to $m$ times derivatives with respect to $x$. By the same discussion as in subsection 3.1, we obtain the following estimate.

**Lemma 3.2.** *For all the nonnegative integers $m$ and $n$,*

$$\begin{aligned}\left\|\frac{\partial^m I^{(n+1)}}{\partial x^m}\right\|_{n+1} \leq & \frac{\mu_s}{\mu_a+\mu_s}\left\|\frac{\partial^m I^{(n)}}{\partial x^m}\right\|_{n} \\ & + \frac{\mu_s}{\mu_a+\mu_s}\sum_{l=1}^{m-1}\left(\frac{m-l}{e\delta_{n+1}}\right)^{m-l}\left\|\frac{\partial^l I^{(n)}}{\partial x^l}\right\|_{n} \\ & + \frac{2\mu_s}{e(\mu_a+\mu_s)}\left(\frac{m}{e\delta_{n+1}}\right)^{m}\|I^{(n)}\|_\infty .\end{aligned}$$

*Proof.* For simplicity, we give a proof only the case $m=2$. Proof for the case

$m \geq 3$ is similar. For $x \in K_{n+1}$ and $\theta \in (0, \pi)$, we have

$$
\begin{aligned}
&\left|\frac{\partial^2 I^{(n+1)}}{\partial x^2}(x,\xi)\right| \\
&= \left|\frac{\mu_s}{\sin\theta}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}\delta_{n+1}\right)\int_{S^1} p(\xi,\xi')\frac{\partial I^{(n)}}{\partial x}(x-\delta_{n+1},\xi')\,d\sigma_{\xi'}\right. \\
&\quad - \frac{\mu_s(\mu_a+\mu_s)}{\sin^2\theta}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}\delta_{n+1}\right)\int_{S^1} p(\xi,\xi')I^{(n)}(x-\delta_{n+1},\xi')\,d\sigma_{\xi'} \\
&\quad + \frac{\mu_s}{\sin\theta}\int_{x-\delta_{n+1}}^{x}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)\int_{S^1} p(\xi,\xi')\frac{\partial^2 I^{(n)}}{\partial x^2}(t,\xi')\,d\sigma_{\xi'}\,dt \\
&\quad \left. + \frac{\mu_s(\mu_a+\mu_s)^2}{\sin^3\theta}\int_0^{x-\delta_{n+1}}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)\int_{S^1} p(\xi,\xi')I^{(n)}(t,\xi')\,d\sigma_{\xi'}\,dt\right| \\
&\leq \left(\frac{\mu_s}{\sin\theta}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}\delta_{n+1}\right)\right)\left\|\frac{\partial I^{(n)}}{\partial x}\right\|_n \\
&\quad + 2\left(\frac{\mu_s(\mu_a+\mu_s)}{\sin^2\theta}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}\delta_{n+1}\right)\right)\|I^{(n)}\|_\infty + \frac{\mu_s}{\mu_a+\mu_s}\left\|\frac{\partial^2 I^{(n)}}{\partial x^2}\right\|_n.
\end{aligned}
$$

Using the fact the function

$$f_m(z) := \mu_s z^m \exp\left(-\left(\mu_a+\mu_s\right)\delta_{n+1} z\right), \quad z \in (0,\infty)$$

achieves its maximum value $\dfrac{\mu_s m^m}{(e(\mu_a+\mu_s)\delta_{n+1})^m}$ at $z = \dfrac{m}{(\mu_a+\mu_s)\delta_{n+1}}$ for all $m \in \mathbb{N}$, we obtain the following estimate for $x \in K_{n+1}$ and $\theta \in (0, \pi)$:

$$
\begin{aligned}
\left|\frac{\partial^2 I^{(n+1)}}{\partial x^2}(x,\xi)\right| \leq & \frac{\mu_s}{\mu_a+\mu_s}\left\|\frac{\partial^2 I^{(n)}}{\partial x^2}\right\|_n + \frac{\mu_s}{\mu_a+\mu_s}\frac{1}{e\delta_{n+1}}\left\|\frac{\partial I^{(n)}}{\partial x}\right\|_n \\
& + \frac{\mu_s}{\mu_a+\mu_s}\frac{8}{e^2\delta_{n+1}^2}\|I^{(n)}\|_\infty.
\end{aligned}
$$

For the case $x \in K_{n+1}$ and $\theta \in (-\pi, 0)$, we obtain the same estimate as above. This completes the proof of Lemma 3.2. □

Now we take $\{\delta_n\}_{n\geq 0}$ as follows: for a fixed $m \in \mathbb{N}$,

$$\delta_0 = \delta_0^{(m)} := \left(1 - \left(\frac{\mu_s}{\tilde{\mu}_a+\mu_s}\right)^{\frac{1}{m}}\right)\delta$$

and

$$\delta_{n+1} = \delta_{n+1}^{(m)} := \left(\frac{\mu_s}{\tilde{\mu}_a+\mu_s}\right)^{\frac{1}{m}}\delta_n^{(m)}.$$

Here, we remark that $\delta_n^{(m)} < 1$ for all $m$, $n$ from the definition of $\delta$.

Then, we can prove that the series $\sum_{n=0}^{\infty} \frac{\partial^m I^{(n)}}{\partial x^m}$ converges absolutely and uniformly on $K \times S^1$. We show the convergence in the case $m = 2$. Applying Lemma 3.1 and Lemma 3.2, we obtain the following estimate:

$$
\begin{aligned}
\left\| \frac{\partial^2 I^{(n)}}{\partial x^2} \right\| \leq & \frac{\mu_s}{\mu_a + \mu_s} \left\| \frac{\partial^2 I^{(n-1)}}{\partial x^2} \right\|_{n-1} + \frac{\mu_s}{\mu_a + \mu_s} \frac{1}{e\delta_n} \left\| \frac{\partial I^{(n-1)}}{\partial x} \right\|_{n-1} \\
& + \frac{\mu_s}{\mu_a + \mu_s} \frac{8}{e^2 \delta_n^2} \| I^{(n-1)} \|_\infty \\
\leq & \left( \frac{\mu_s}{\mu_a + \mu_s} \right)^n \left\| \frac{\partial^2 I^{(0)}}{\partial x^2} \right\|_0 + \sum_{k=0}^{n-1} \frac{1}{e\delta_{k+1}} \left( \frac{\mu_s}{\mu_a + \mu_s} \right)^n \left\| \frac{\partial I^{(0)}}{\partial x} \right\|_0 \\
& + \sum_{k=1}^{n-1} \frac{1}{e\delta_{k+1}} \sum_{l=0}^{k-1} \frac{2\mu_s}{e\left(\mu_a + \mu_s\right) \delta_{k-l}} \left( \frac{\mu_s}{\mu_a + \mu_s} \right)^{n-1} \| I^{(0)} \|_\infty \\
& + 8 \sum_{k=0}^{n-1} \frac{1}{\left(e\delta_{k+1}\right)^2} \left( \frac{\mu_s}{\mu_a + \mu_s} \right)^n \| I^{(0)} \|_\infty .
\end{aligned}
$$

We now take $\delta_n^{(2)}$, defined above, as $\delta_n$. Since $\delta_n^{(2)} < 1$ for all $n$ and $\frac{\mu_s}{\tilde{\mu}_a + \mu_s} < 1$, the following estimate holds: for all $k$ with $0 \leq k \leq n$, $\frac{1}{\delta_k^{(2)}} \leq \frac{1}{\delta_n^{(2)}} \leq \frac{1}{(\delta_n^{(2)})^2}$. After all, we obtain the following estimate:

$$
\begin{aligned}
\left\| \frac{\partial^2 I^{(n)}}{\partial x^2} \right\| \leq & \left( \frac{\mu_s}{\mu_a + \mu_s} \right)^n \left\| \frac{\partial^2 I^{(0)}}{\partial x^2} \right\|_0 + \frac{n}{e(\delta_n^{(2)})^2} \left( \frac{\mu_s}{\mu_a + \mu_s} \right)^n \left\| \frac{\partial I^{(0)}}{\partial x} \right\|_0 \\
& + \frac{n(n-1)}{(e\delta_n^{(2)})^2} \left( \frac{\mu_s}{\mu_a + \mu_s} \right)^n \| I^{(0)} \|_\infty + \frac{8n}{(e\delta_n^{(2)})^2} \left( \frac{\mu_s}{\mu_a + \mu_s} \right)^n \| I^{(0)} \|_\infty \\
\leq & \left( \frac{\mu_s}{\mu_a + \mu_s} \right)^n \left\| \frac{\partial^2 I^{(0)}}{\partial x^2} \right\|_0 + \frac{n}{e(\delta_0^{(2)})^2} \left( \frac{\tilde{\mu}_a + \mu_s}{\mu_a + \mu_s} \right)^n \left\| \frac{\partial I^{(0)}}{\partial x} \right\|_0 \\
& + \frac{n(n-1)}{(e\delta_0^{(2)})^2} \left( \frac{\tilde{\mu}_a + \mu_s}{\mu_a + \mu_s} \right)^n \| I^{(0)} \|_\infty + \frac{8n}{(e\delta_0^{(2)})^2} \left( \frac{\tilde{\mu}_a + \mu_s}{\mu_a + \mu_s} \right)^n \| I^{(0)} \|_\infty .
\end{aligned}
$$

Therefore, for all $N$,

$$
\begin{aligned}
\sum_{n=0}^{N} \left\| \frac{\partial^2 I^{(n)}}{\partial x^2} \right\|_K \leq & \left\| \frac{\partial^2 I^{(0)}}{\partial x^2} \right\|_0 \sum_{n=0}^{N} \left( \frac{\mu_s}{\mu_a + \mu_s} \right)^n \\
& + \frac{1}{e(\delta_0^{(2)})^2} \left\| \frac{\partial I^{(0)}}{\partial x} \right\|_0 \sum_{n=0}^{N} n \left( \frac{\tilde{\mu}_a + \mu_s}{\mu_a + \mu_s} \right)^n \\
& + \frac{1}{(e\delta_0^{(2)})^2} \| I^{(0)} \|_\infty \sum_{n=0}^{N} n(n+7) \left( \frac{\tilde{\mu}_a + \mu_s}{\mu_a + \mu_s} \right)^n . \qquad (3.2)
\end{aligned}
$$

Since the right hand side of the equation (3.2) converges as $N$ tends to $\infty$, the series $\displaystyle\sum_{n=0}^{\infty} \frac{\partial^2 I^{(n)}}{\partial x^2}$ converges absolutely and uniformly on $K \times S^1$. We can also prove continuity of $\dfrac{\partial^m I^{(n)}}{\partial x^m}$ in the same way as $I^{(n)}$. Therefore, $\dfrac{\partial^m I}{\partial x^m} = \displaystyle\sum_{n=0}^{\infty} \frac{\partial^m I^{(n)}}{\partial x^m}$ is continuous on $K \times S^1$. Since we can take a compact set $K$ arbitrarily, we conclude that $\dfrac{\partial^m I}{\partial x^m}$ belongs to $C(\Omega \times S^1)$.

# 4 Differentiability with respect to $\theta$

We discuss differentiability of $I$ with respect to $\theta$ in this section.

## 4.1 Preliminaries

We now differentiate each term of the sequence $\{I^{(n)}\}_{n \geq 0}$ with respect to $\theta$. For $x \in (0,1)$ and $\theta \in (0, \pi)$, we have

$$
\begin{aligned}
\frac{\partial I^{(0)}}{\partial \theta}(x, \xi) =& \frac{\mu_a + \mu_s}{\sin^2 \theta} x \cos\theta \exp\left( -\frac{\mu_a + \mu_s}{\sin\theta} x \right) I_0(0, \xi) \\
&+ \exp\left( -\frac{\mu_a + \mu_s}{\sin\theta} x \right) \frac{\partial I_0}{\partial \theta}(0, \xi), \\
\frac{\partial I^{(n+1)}}{\partial \theta}(x, \xi) =& -\frac{\mu_s \cos\theta}{\sin^2\theta} \int_0^x \exp\left( -\frac{\mu_a + \mu_s}{\sin\theta}(x-t) \right) \int_{S^1} p(\xi, \xi') I^{(n)}(t, \xi')\, d\sigma_{\xi'} dt \\
&+ \frac{\mu_s(\mu_a + \mu_s)\cos\theta}{\sin^3\theta} \int_0^x (x-t) \exp\left( -\frac{\mu_a + \mu_s}{\sin\theta}(x-t) \right) \\
&\quad \times \int_{S^1} p(\xi, \xi') I^{(n)}(t, \xi')\, d\sigma_{\xi'} dt \\
&+ \frac{\mu_s}{\sin\theta} \int_0^x \exp\left( -\frac{\mu_a + \mu_s}{\sin\theta}(x-t) \right) \int_{S^1} \frac{\partial p}{\partial \theta}(\xi, \xi') I^{(n)}(t, \xi')\, d\sigma_{\xi'} dt.
\end{aligned}
$$

For $x \in (0,1)$ and $\theta \in (-\pi, 0)$, we have

$$\begin{aligned}
\frac{\partial I^{(0)}}{\partial\theta}(x,\xi) =& \frac{\mu_a+\mu_s}{\sin^2\theta}(1-x)\cos\theta \exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(1-x)\right) I_0(1,\xi) \\
&+ \exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(1-x)\right)\frac{\partial I_0}{\partial\theta}(1,\xi), \\
\frac{\partial I^{(n+1)}}{\partial\theta}(x,\xi) =& \frac{\mu_s\cos\theta}{\sin^2\theta}\int_x^1 \exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right)\int_{S^1} p(\xi,\xi')I^{(n)}(t,\xi')\,d\sigma_{\xi'}dt \\
&+ \frac{\mu_s(\mu_a+\mu_s)\cos\theta}{\sin^3\theta}\int_x^1 (t-x)\exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right) \\
&\times \int_{S^1} p(\xi,\xi')I^{(n)}(t,\xi')\,d\sigma_{\xi'}dt \\
&- \frac{\mu_s}{\sin\theta}\int_x^1 \exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right)\int_{S^1}\frac{\partial p}{\partial\theta}(\xi,\xi')I^{(n)}(t,\xi')\,d\sigma_{\xi'}dt.
\end{aligned}$$

In the case of $x \in (0,1)$ and $\theta \in \{0,\pi\}$, unlike the other cases, some further discussions are needed. We first introduce the following proposition.

**Proposition 4.1.** *For $x \in (0,1)$ and $\theta \in \{0,\pi\}$,*

$$\begin{aligned}
\frac{\partial I^{(0)}}{\partial\theta}(x,\xi) =& 0, \\
\frac{\partial I^{(n+1)}}{\partial\theta}(x,\xi) =& \frac{\mu_s}{\mu_a+\mu_s}\int_{S^1}\frac{\partial p}{\partial\theta}(\xi,\xi')I^{(n)}(x,\xi')\,d\sigma_{\xi'} \\
&- \frac{\mu_s}{(\mu_a+\mu_s)^2}\int_{S^1} p(\xi,\xi')\frac{\partial I^{(n)}}{\partial x}(x,\xi')\,d\sigma_{\xi'}.
\end{aligned}$$

*Proof.* For $n=0$ and $x \in (0,1)$, since $I_0$ is bounded on $\Gamma_-$, we have

$$\lim_{\theta\downarrow 0}\frac{I^{(0)}(x,\xi)-I^{(0)}(x,\xi_0)}{\theta} = \lim_{\theta\downarrow 0}\frac{1}{\theta}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}x\right)I_0(0,\xi) = 0$$

and

$$\lim_{\theta\uparrow 0}\frac{I^{(0)}(x,\xi)-I^{(0)}(x,\xi_0)}{\theta} = \lim_{\theta\uparrow 0}\frac{1}{\theta}\exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(1-x)\right)I_0(1,\xi) = 0.$$

Thus, we have for $x \in (0,1)$ and $\theta = 0$,

$$\frac{\partial I^{(0)}}{\partial\theta}(x,\xi) = 0.$$

Similarly, we have for $x \in (0,1)$ and $\theta = \pi$,

$$\frac{\partial I^{(0)}}{\partial\theta}(x,\xi) = 0.$$

For $I^{(n+1)}$, we consider differentiability only at $\theta = 0$. Differentiability at $\theta = \pi$ can be proved similarly. Using the equality (2.4), we have

$$
\begin{aligned}
&\lim_{\theta\downarrow 0} \frac{I^{(n+1)}(x,\xi) - I^{(n+1)}(x,\xi_0)}{\theta} \\
&= \lim_{\theta\downarrow 0} \left\{ \frac{\mu_s}{\theta \sin\theta} \int_0^x \exp\left( -\frac{\mu_a + \mu_s}{\sin\theta}(x-t) \right) \right. \\
&\qquad \times \int_{S^1} \left( p(\xi,\xi') I^{(n)}(t,\xi') - p(\xi_0,\xi') I^{(n)}(x,\xi') \right) d\sigma_{\xi'} dt \\
&\qquad \left. - \frac{\mu_s}{\theta(\mu_a+\mu_s)} \exp\left( -\frac{\mu_a+\mu_s}{\sin\theta} x \right) \int_{S^1} p(\xi_0,\xi') I^{(n)}(x,\xi') \, d\sigma_{\xi'} \right\} \\
&= \lim_{\theta\downarrow 0} (J_7 + J_8 + J_9),
\end{aligned}
$$

where

$$
\begin{aligned}
J_7 := \frac{\mu_s}{\theta\sin\theta} \int_0^x & \exp\left( -\frac{\mu_a+\mu_s}{\sin\theta}(x-t) \right) \\
& \times \int_{S^1} \left( p(\xi,\xi') - p(\xi_0,\xi') \right) I^{(n)}(x,\xi') \, d\sigma_{\xi'} dt,
\end{aligned}
$$

$$
\begin{aligned}
J_8 := \frac{\mu_s}{\theta\sin\theta} \int_0^x & \exp\left( -\frac{\mu_a+\mu_s}{\sin\theta}(x-t) \right) \\
& \times \int_{S^1} p(\xi,\xi') \left( I^{(n)}(t,\xi') - I^{(n)}(x,\xi') \right) d\sigma_{\xi'} dt
\end{aligned}
$$

and

$$
J_9 := -\frac{\mu_s}{\theta(\mu_a+\mu_s)} \exp\left( -\frac{\mu_a+\mu_s}{\sin\theta} x \right) \int_{S^1} p(\xi_0,\xi') I^{(n)}(x,\xi') \, d\sigma_{\xi'}.
$$

Now, we consider the limit of each term above. Since $x \in (0,1)$, we have $\lim_{\theta\downarrow 0} J_9 = 0$. We also have

$$
J_7 = \frac{\mu_s}{\mu_a+\mu_s} \left( 1 - \exp\left( -\frac{\mu_a+\mu_s}{\sin\theta} x \right) \right) \int_{S^1} \frac{p(\xi,\xi') - p(\xi_0,\xi')}{\theta} I^{(n)}(x,\xi') \, d\sigma_{\xi'}.
$$

Since $p$ is continuously differentiable with respect to $\theta$, we can change the order of integration and taking the limit. Hence, we have

$$
\lim_{\theta\downarrow 0} J_7 = \frac{\mu_s}{\mu_a+\mu_s} \int_{S^1} \frac{\partial p}{\partial\theta}(\xi_0,\xi') I^{(n)}(x,\xi') \, d\sigma_{\xi'}.
$$

We take $\delta'$ with $0 < \delta' < \delta_{n+1}$ and separate the interval of integration in $J_8$

into two parts:

$$J_8 = \frac{\mu_s}{\theta \sin\theta}\left(\int_0^{x-\delta'} + \int_{x-\delta'}^{x}\right)\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)$$
$$\times \int_{S^1} p(\xi,\xi')\left(I^{(n)}(t,\xi') - I^{(n)}(x,\xi')\right)\, d\sigma_{\xi'} dt.$$

The integral over $[0, x-\delta']$ above tends to 0 as $\theta$ tends to 0. We integrate the second term by parts with respect to $t$, and we have

$$\begin{aligned}
&\frac{\mu_s}{\theta\sin\theta}\int_{x-\delta'}^{x}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right) \\
&\qquad\qquad \times \int_{S^1} p(\xi,\xi')\left(I^{(n)}(t,\xi') - I^{(n)}(x,\xi')\right)\, d\sigma_{\xi'} dt \\
=&\frac{\mu_s}{\theta\sin\theta}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}\delta'\right)\int_{S^1} p(\xi,\xi')\left(I^{(n)}(x,\xi') - I^{(n)}(x-\delta',\xi')\right)\, d\sigma_{\xi'} \\
&- \frac{\mu_s}{\theta(\mu_a+\mu_s)}\int_{x-\delta'}^{x}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right) \\
&\times \int_{S^1} p(\xi,\xi')\left(\frac{\partial I^{(n)}}{\partial x}(t,\xi') - \frac{\partial I^{(n)}}{\partial x}(x,\xi')\right)\, d\sigma_{\xi'} dt \\
&- \frac{\mu_s}{\theta(\mu_a+\mu_s)}\int_{x-\delta'}^{x}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)\int_{S^1} p(\xi,\xi')\frac{\partial I^{(n)}}{\partial x}(x,\xi')\, d\sigma_{\xi'} dt.
\end{aligned}$$

The first term of the right hand side above tends to 0 as $\theta$ tends to 0. From the third term, we have

$$\begin{aligned}
&\lim_{\theta\downarrow 0} -\frac{\mu_s}{\theta(\mu_a+\mu_s)}\int_{x-\delta'}^{x}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right)\int_{S^1} p(\xi,\xi')\frac{\partial I^{(n)}}{\partial x}(x,\xi')\, d\sigma_{\xi'} dt \\
&= \lim_{\theta\downarrow 0} -\frac{\mu_s\sin\theta}{\theta(\mu_a+\mu_s)^2}\left(1 - \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}\delta\right)\right)\int_{S^1} p(\xi,\xi')\frac{\partial I^{(n)}}{\partial x}(x,\xi')\, d\sigma_{\xi'} \\
&= -\frac{\mu_s}{(\mu_a+\mu_s)^2}\int_{S^1} p(\xi_0,\xi')\frac{\partial I^{(n)}}{\partial x}(x,\xi')\, d\sigma_{\xi'}.
\end{aligned}$$

Finally, we estimate the second term:

$$
\begin{aligned}
&\left| -\frac{\mu_s}{\theta(\mu_a+\mu_s)} \int_{x-\delta'}^{x} \exp\left( -\frac{\mu_a+\mu_s}{\sin\theta}(x-t) \right) \right. \\
&\qquad\qquad \left. \times \int_{S^1} p(\xi,\xi') \left( \frac{\partial I^{(n)}}{\partial x}(t,\xi') - \frac{\partial I^{(n)}}{\partial x}(x,\xi') \right) d\sigma_{\xi'} dt \right| \\
&\le \frac{\mu_s}{\theta(\mu_a+\mu_s)} \int_{x-\delta'}^{x} \exp\left( -\frac{\mu_a+\mu_s}{\sin\theta}(x-t) \right) dt \\
&\qquad\qquad \times \sup_{t\in(x-\delta',x)} \left( \max_{\xi\in S^1} \left| \frac{\partial I^{(n)}}{\partial x}(t,\xi) - \frac{\partial I^{(n)}}{\partial x}(x,\xi) \right| \right) \\
&\le \frac{\mu_s \sin\theta}{\theta(\mu_a+\mu_s)^2} \sup_{t\in(x-\delta',x)} \left( \max_{\xi\in S^1} \left| \frac{\partial I^{(n)}}{\partial x}(t,\xi) - \frac{\partial I^{(n)}}{\partial x}(x,\xi) \right| \right).
\end{aligned}
$$

From the estimate above, we obtain

$$
\begin{aligned}
&\lim_{\theta\downarrow 0} \left| -\frac{\mu_s}{\theta(\mu_a+\mu_s)} \int_{x-\delta'}^{x} \exp\left( -\frac{\mu_a+\mu_s}{\sin\theta}(x-t) \right) \right. \\
&\qquad\qquad \left. \times \int_{S^1} p(\xi,\xi') \left( \frac{\partial I^{(n)}}{\partial x}(t,\xi') - \frac{\partial I^{(n)}}{\partial x}(x,\xi') \right) d\sigma_{\xi'} dt \right| \\
&\le \frac{\mu_s}{(\mu_a+\mu_s)^2} \sup_{t\in(x-\delta',x)} \left( \max_{\xi\in S^1} \left| \frac{\partial I^{(n)}}{\partial x}(t,\xi) - \frac{\partial I^{(n)}}{\partial x}(x,\xi) \right| \right).
\end{aligned}
$$

Since $\dfrac{\partial I^{(n)}}{\partial x}$ is continuous and bounded, we can make the right hand side of the inequality arbitrarily small by taking $\delta'$ sufficiently small. Therefore,

$$
\begin{aligned}
\lim_{\theta\downarrow 0} \frac{I^{(n+1)}(x,\xi) - I^{(n+1)}(x,\xi_0)}{\theta} =& \frac{\mu_s}{\mu_a+\mu_s} \int_{S^1} \frac{\partial p}{\partial\theta}(\xi_0,\xi') I^{(n)}(x,\xi')\, d\sigma_{\xi'} \\
&- \frac{\mu_s}{(\mu_a+\mu_s)^2} \int_{S^1} p(\xi_0,\xi') \frac{\partial I^{(n)}}{\partial x}(x,\xi')\, d\sigma_{\xi'}.
\end{aligned}
$$

Using the equality (2.5), we have

$$
\lim_{\theta\uparrow 0} \frac{I^{(n+1)}(x,\xi) - I^{(n+1)}(x,\xi_0)}{\theta} = \lim_{\theta\uparrow 0} (J_7' + J_8' + J_9'),
$$

where

$$
\begin{aligned}
J_7' :=& -\frac{\mu_s}{\theta\sin\theta} \int_{x}^{1} \exp\left( \frac{\mu_a+\mu_s}{\sin\theta}(t-x) \right) \\
&\qquad\qquad \times \int_{S^1} \left( p(\xi,\xi') - p(\xi_0,\xi') \right) I^{(n)}(x,\xi')\, d\sigma_{\xi'} dt,
\end{aligned}
$$

$$J_8' := -\frac{\mu_s}{\theta \sin\theta} \int_x^1 \exp\left(\frac{\mu_a + \mu_s}{\sin\theta}(t-x)\right) \times \int_{S^1} p(\xi,\xi') \left(I^{(n)}(t,\xi') - I^{(n)}(x,\xi')\right) d\sigma_{\xi'} dt$$

and

$$J_9' := \frac{\mu_s}{\theta(\mu_a+\mu_s)} \exp\left(\frac{\mu_a + \mu_s}{\sin\theta}(1-x)\right) \int_{S^1} p(\xi_0,\xi') I^{(n)}(x,\xi')\, d\sigma_{\xi'}.$$

In the same way as the case $\theta \downarrow 0$, we have

$$\begin{aligned}
\lim_{\theta\uparrow 0} J_7' =& \frac{\mu_s}{\mu_a+\mu_s} \int_{S^1} \frac{\partial p}{\partial\theta}(\xi_0,\xi') I^{(n)}(x,\xi')\, d\sigma_{\xi'}, \\
\lim_{\theta\uparrow 0} J_8' =& -\frac{\mu_s}{(\mu_a+\mu_s)^2} \int_{S^1} p(\xi_0,\xi') \frac{\partial I^{(n)}}{\partial x}(x,\xi')\, d\sigma_{\xi'}
\end{aligned}$$

and

$$\lim_{\theta\uparrow 0} J_9' = 0,$$

that is,

$$\begin{aligned}
\lim_{\theta\uparrow 0} \frac{I^{(n+1)}(x,\xi) - I^{(n+1)}(x,\xi_0)}{\theta} =& \frac{\mu_s}{\mu_a+\mu_s} \int_{S^1} \frac{\partial p}{\partial\theta}(\xi_0,\xi') I^{(n)}(x,\xi')\, d\sigma_{\xi'} \\
& - \frac{\mu_s}{(\mu_a+\mu_s)^2} \int_{S^1} p(\xi_0,\xi') \frac{\partial I^{(n)}}{\partial x}(x,\xi')\, d\sigma_{\xi'}.
\end{aligned}$$

Thus, we obtain the statement of the proposition. □

With a similar calculation, we can prove that

$$\lim_{\theta\to 0} \frac{\partial I^{(n+1)}}{\partial\theta}(x,\xi) = \frac{\partial I^{(n+1)}}{\partial\theta}(x,\xi_0)$$

and

$$\lim_{\theta\to \pm\pi} \frac{\partial I^{(n+1)}}{\partial\theta}(x,\xi) = \frac{\partial I^{(n+1)}}{\partial\theta}(x,\xi_\pi)$$

for $x \in (0,1)$, that is, $\dfrac{\partial I^{(n+1)}}{\partial\theta}$ is continuous on $\Omega \times S^1$.

## 4.2 Differentiability with respect to $\theta$

We prove the following proposition.

**Proposition 4.2.** *For every compact set $K$ in $(0,1)$, $\displaystyle\sum_{n=0}^{\infty} \frac{\partial I^{(n)}}{\partial\theta}(x,\xi)$ converges absolutely and uniformly on $K \times S^1$.*

We take a closed interval $K$, a sequence of positive numbers $\{\delta_n\}_{n\geq 0}$ and the corresponding closed intervals $\{K_n\}_{n\geq 0}$ as in subsection 3.1. Then, the following lemma holds.

**Lemma 4.3.** *If $\dfrac{\partial I^{(n)}}{\partial x}$ exists and bounded on $K_n \times S^1$, then the following estimate holds:*

$$\left\| \frac{\partial I^{(n+1)}}{\partial\theta} \right\|_{n+1} \leq C \frac{\mu_s}{(\mu_a+\mu_s)^2 \delta_{n+1}} \|I^{(n)}\|_\infty + \frac{\mu_s}{(\mu_a+\mu_s)^2} \left\| \frac{\partial I^{(n)}}{\partial x} \right\|_n + \frac{2\pi\mu_s}{\mu_a+\mu_s} \left\| \frac{\partial p}{\partial\theta} \right\|_\infty \|I^{(n)}\|_\infty$$

*with some constant $C > 0$.*

*Proof.* For $x \in K_{n+1}$ and $\theta \in (0,\pi)$, separating the interval of integration into two parts and integrating by parts with respect to $t$, we obtain

$$\frac{\partial I^{(n+1)}}{\partial\theta}(x,\xi) = J_{10} + J_{11} + J_{12} + J_{13} + J_{14},$$

where

$$J_{10} := -\frac{\mu_s \cos\theta}{\sin^2\theta} \int_0^{x-\delta_{n+1}} \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right) \int_{S^1} p(\xi,\xi') I^{(n)}(t,\xi')\, d\sigma_{\xi'} dt,$$

$$J_{11} := -\frac{\mu_s \cos\theta}{\sin^2\theta} \delta_{n+1} \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}\delta_{n+1}\right) \int_{S^1} p(\xi,\xi') I^{(n)}(x-\delta_{n+1},\xi')\, d\sigma_{\xi'},$$

$$J_{12} := -\frac{\mu_s \cos\theta}{\sin^2\theta} \int_{x-\delta_{n+1}}^{x} (x-t) \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right) \times \int_{S^1} p(\xi,\xi') \frac{\partial I^{(n)}}{\partial x}(t,\xi')\, d\sigma_{\xi'} dt,$$

$$J_{13} := \frac{\mu_s}{\sin\theta} \int_0^{x-\delta_{n+1}} \frac{\mu_a+\mu_s}{\sin^2\theta}(x-t)\cos\theta \times \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right) \int_{S^1} p(\xi,\xi') I^{(n)}(t,\xi')\, d\sigma_{\xi'} dt$$

and

$$J_{14} := \frac{\mu_s}{\sin\theta} \int_0^{x} \exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(x-t)\right) \int_{S^1} \frac{\partial p}{\partial\theta}(\xi,\xi') I^{(n)}(t,\xi')\, d\sigma_{\xi'} dt.$$

Here, we obtain the following estimates:

$$\begin{aligned}|J_{10}| &\le \frac{\mu_s}{(\mu_a+\mu_s)^2\sin\theta}\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}\delta_{n+1}\right)\|I^{(n)}\|_\infty\\ &\le \frac{\mu_s}{(\mu_a+\mu_s)^2 e\delta_{n+1}}\|I^{(n)}\|_\infty,\\ |J_{11}| &\le \frac{\mu_s}{4(\mu_a+\mu_s)^2 e^2\delta_{n+1}}\|I^{(n)}\|_\infty,\\ |J_{12}| &\le \frac{\mu_s}{(\mu_a+\mu_s)^2}\left\|\frac{\partial I^{(n)}}{\partial x}\right\|_n,\\ |J_{13}| &\le \frac{\mu_s}{4(\mu_a+\mu_s)^2 e^2\delta_{n+1}}\|I^{(n)}\|_\infty,\end{aligned}$$

and

$$|J_{14}| \le \frac{2\pi\mu_s}{\mu_a+\mu_s}\left\|\frac{\partial p}{\partial\theta}\right\|_\infty \|I^{(n)}\|_\infty.$$

Thus, we have

$$\begin{aligned}\left|\frac{\partial I^{(n+1)}}{\partial\theta}(x,\xi)\right| \le& \frac{\mu_s}{(\mu_a+\mu_s)^2 e\delta_{n+1}}\|I^{(n)}\|_\infty + \frac{\mu_s}{2(\mu_a+\mu_s)^2 e^2\delta_{n+1}}\|I^{(n)}\|_\infty\\ &+\frac{\mu_s}{(\mu_a+\mu_s)^2}\left\|\frac{\partial I^{(n)}}{\partial x}\right\|_n + \frac{2\pi\mu_s}{\mu_a+\mu_s}\left\|\frac{\partial p}{\partial\theta}\right\|_\infty \|I^{(n)}\|_\infty \qquad (4.1)\end{aligned}$$

for $x \in K_{n+1}$ and $\theta \in (0,\pi)$.

In a similar way, we have, for $x \in K_{n+1}$ and $\theta \in (-\pi, 0)$,

$$\frac{\partial I^{(n+1)}}{\partial\theta}(x,\xi) = J'_{10} + J'_{11} + J'_{12} + J'_{13} + J'_{14},$$

where

$$\begin{aligned}J'_{10} :=& \frac{\mu_s\cos\theta}{\sin^2\theta}\int_{x+\delta_{n+1}}^1 \exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right)\int_{S^1} p(\xi,\xi')I^{(n)}(t,\xi')\,d\sigma_{\xi'}\,dt,\\ J'_{11} :=& \frac{\mu_s\cos\theta}{\sin^2\theta}\delta_{n+1}\exp\left(\frac{\mu_a+\mu_s}{\sin\theta}\delta_{n+1}\right)\int_{S^1} p(\xi,\xi')I^{(n)}(x+\delta_{n+1},\xi')\,d\sigma_{\xi'},\\ J'_{12} :=& -\frac{\mu_s\cos\theta}{\sin^2\theta}\int_x^{x+\delta_{n+1}}(t-x)\exp\left(-\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right)\\ &\times\int_{S^1} p(\xi,\xi')\frac{\partial I^{(n)}}{\partial x}(t,\xi')\,d\sigma_{\xi'}\,dt,\\ J'_{13} :=& \frac{\mu_s}{\sin\theta}\int_{x-\delta_{n+1}}^1 \frac{\mu_a+\mu_s}{\sin^2\theta}(t-x)\cos\theta\\ &\times\exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right)\int_{S^1} p(\xi,\xi')I^{(n)}(t,\xi')\,d\sigma_{\xi'}\,dt\end{aligned}$$

and

$$J'_{14} := -\frac{\mu_s}{\sin\theta}\int_x^1 \exp\left(\frac{\mu_a+\mu_s}{\sin\theta}(t-x)\right)\int_{S^1}\frac{\partial p}{\partial\theta}(\xi,\xi')I^{(n)}(t,\xi')\,d\sigma_{\xi'}\,dt.$$

Since $J'$s above satisfy the same estimates as $J$s', the estimate (4.1) also holds for $x \in K_{n+1}$ and $\theta \in (-\pi, 0)$. It is obvious that the estimate (4.1) holds for $x \in K_{n+1}$ and $\theta \in \{0, \pi\}$. If we take a positive constant $C$ such that $C > \frac{1}{e} + \frac{1}{2e^2}$, we obtain the estimation in the statement of this lemma. □

We now take $\{\delta_n^{(1)}\}_{n\geq 0}$, defined in subsection 3.2, as $\{\delta_n\}_{n\geq 0}$. From Lemma 4.3, we obtain, for all $N$,

$$\begin{aligned}\sum_{n=0}^{N}\left\|\frac{\partial I^{(n)}}{\partial\theta}\right\|_K &\leq \sum_{n=0}^{N}\left\|\frac{\partial I^{(n)}}{\partial\theta}\right\|_n \\ &\leq \left\|\frac{\partial I^{(0)}}{\partial\theta}\right\|_0 + \frac{2\pi\mu_s}{\mu_a+\mu_s}\left\|\frac{\partial p}{\partial\theta}\right\|_\infty \sum_{n=0}^{N-1}\|I^{(n)}\|_\infty \\ &\quad + C\frac{\mu_s}{(\mu_a+\mu_s)^2 e}\sum_{n=0}^{N-1}\frac{1}{\delta_{n+1}}\|I^{(n)}\|_\infty + \frac{\mu_s}{(\mu_a+\mu_s)^2}\sum_{n=0}^{N-1}\left\|\frac{\partial I^{(n)}}{\partial x}\right\|_n.\end{aligned}$$

Obviously, the first term of the right hand side above is finite. The other terms converge as $N$ tends to $\infty$ as we have seen in subsection 3.2. Thus the series $\sum_{n=0}^{\infty}\frac{\partial I^{(n)}}{\partial\theta}$ converges absolutely and uniformly on $K \times S^1$. Since $\frac{\partial I^{(n)}}{\partial\theta}$ is continuous on $K \times S^1$, $\frac{\partial I}{\partial\theta} = \sum_{n=0}^{\infty}\frac{\partial I^{(n)}}{\partial\theta}$ is also continuous on $K \times S^1$. Now we take $K$ arbitrarily to conclude that $\frac{\partial I}{\partial\theta} \in C(\Omega \times S^1)$.

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