A PROOF OF THE ADDITIVITY OF ROUGH INTEGRAL

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Dedicated to the memory of Professor María J. Garrido-Atienza

ABSTRACT. On the basis of fractional calculus, we introduce an explicit formulation of the integral of controlled paths along Hölder rough paths in terms of Lebesgue integrals for fractional derivatives. The additivity with respect to the interval of integration, a fundamental property of the integral, is not apparent under the formulation because the fractional derivatives depend heavily on the endpoints of the interval of integration. In this paper, we provide a proof of the additivity of the integral under the formulation. Our proof seems to be simpler than those provided in previous studies and is suitable for utilizing the fractional calculus approach to rough path analysis.

1. INTRODUCTION

Since Terry Lyons introduced rough path analysis in the seminal paper [18], several different approaches to the fundamental theory of rough path analysis have been proposed. One of them is based on fractional calculus, which was introduced by Hu and Nualart [11]. In [11], using basic formulas of fractional calculus and ideas from rough path analysis, they introduced an integral with respect to Hölder continuous functions of order $\beta \in (1/3, 1/2)$, and established a differential equation driven by the Hölder continuous functions by using the integral. The results of [11] have been applied to the study of stochastic calculus, particularly stochastic differential equations driven by fractional Brownian motions with Hurst parameter $H \in (1/3, 1/2)$, for example, [2, 3, 6, 7, 21]. The integral introduced in [11] is given explicitly by Lebesgue integrals for fractional derivatives, unlike the usual rough integral given by the limit of the compensated Riemann–Stieltjes sums. (The integrals with respect to rough paths are called rough integrals.) The usual rough integral is based on a discrete approximation argument from the Riemann–Stieltjes integration due to Young [23], called the Young integral, whereas the integral in [11] is derived from such an explicit definition via fractional calculus for Young integrals with respect to Hölder continuous functions provided in the integration by Zähle [24]. The author's previous study [15] provided a slight reformulation of the integral in [11] in the setting of controlled path theory [10] and showed that it is consistent with the usual rough integral of controlled paths along Hölder rough paths of order $\beta \in (1/3, 1/2]$. (For

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a study related to [15], we refer the reader to [14].) Such explicit definitions in terms of Lebesgue integrals for fractional derivatives enable us to provide direct quantitative estimates of rough integrals and solutions to differential equations driven by rough paths [1, 8, 9, 11, 17, 21].

The purpose of this paper is to further investigate the rough integral based on fractional calculus in [11, 15]. In particular, we are interested in finding a proof of additivity with respect to the interval of integration, namely, equality

$$\int_r^s Y_u\, dX_u + \int_s^t Y_u\, dX_u = \int_r^t Y_u\, dX_u \tag{1.1}$$

for $0 \leq r \leq s \leq t \leq T$. Here, T denotes a positive constant, and the integrals above stand for the rough integral of an X-controlled path (Y, Y') along a Hölder rough path $(X, \mathbb{X})$ of order $\beta \in (1/3, 1/2]$. Additivity is a fundamental property of rough integrals and is usually used to prove the well-posedness of differential equations driven by rough paths. However, it is not obvious for rough integrals based on fractional calculus because the left- and right-sided fractional derivatives in the above-mentioned explicit definition depend heavily on the left- and right-endpoints, respectively, of the interval of integration. (See Definitions 2.1 and 2.9 for further details.) One can verify the additivity of the rough integrals based on fractional calculus by using the following two types of argument: (a) one utilizes the consistency of the rough integrals with the Riemann–Stieltjes integrals for a smooth approximation of Hölder rough paths and continuity of the rough integrals with respect to a suitable rough path metric as in [11–13]; or (b) one utilizes consistency with the limit of compensated Riemann–Stieltjes sums, that is, the usual rough integral as in [14, 15]. Argument (a) corresponds to the case of geometric Hölder rough paths, and is not a strict limit for applications to stochastic calculus. Although the consistency shown in [14, 15] is not limited to geometric Hölder rough paths, (a) and (b) rely on the additivity of the Riemann–Stieltjes integrals and usual rough integrals, respectively. The argument of this paper for proof of the additivity of the rough integral seems to be simpler than both (a) and (b).

We briefly sketch the proof of equality (1.1) provided in this paper as follows: Let us consider the function h defined by

$$h(x) := \int_x^t Y_u\, dX_u - \int_x^s Y_u\, dX_u$$

for $x \in [0, s]$. It is clear that (1.1) is equivalent to $h(r) = h(s)$. Thus, it suffices to show that h is a constant function on $[0, s]$. By the definition of h, we have

$$h(y) - h(x) = \left(\int_x^s Y_u\, dX_u - \int_y^s Y_u\, dX_u \right) - \left(\int_x^t Y_u\, dX_u - \int_y^t Y_u\, dX_u \right)$$

for $0 \leq x < y \leq s$. As we will see in subsequent sections, the first and second terms on the right-hand side of the preceding equality are suitable for use of the rough integral in [11, 15] because the two integrals of each term possess the same right-endpoint. Indeed, by using the explicit definition and quantitative estimates of the rough integral, we can take $\theta > 1$ and easily show that the inequality $|h(y) - h(x)| \leq C(y-x)^\theta$ holds for $0 \leq x < y \leq s$. Here, C is a positive constant that is independent of x and y. Therefore, it follows from $\theta > 1$ that h is a constant function on $[0, s]$. (See the proofs of Propositions 2.7 and 2.15 for more details.) The argument of this paper described above is valid for general Hölder rough paths and makes the fundamental property (1.1) self-contained. Since it seems to be suitable for rough integrals based on fractional calculus, our proof may be useful for further developments of the fractional calculus approach to rough path analysis. We now comment on the author's previous study [16], which can be regarded as an extension of [15] to any Hölder exponent $\beta \in (0, 1]$. Combining the argument of this paper with [16], it is straightforward to show that the additivity of the rough integral in [16] holds for geometric β-Hölder rough paths in the case $\beta \in (0, 1/3]$. We restrict our attention to the case $\beta \in (1/3, 1]$ for the sake of simplicity.

The remainder of this paper is organized as follows. In Sections 2.1 and 2.2, we briefly define the notation and terminology, which includes the definition of fractional derivatives and their slight generalizations. Section 2.3 demonstrates that the additivity of the Young integral based on fractional calculus in [24] follows from the aforementioned argument. Section 2.4 focuses on the additivity of the rough integral in [11, 15].

2. Framework and Results

In this section, we briefly review both the Young integral and the rough integral based on fractional calculus in [24] and [11,15], respectively, and provide proofs of additivity with respect to the interval of integration for these integrals. We also briefly review concepts, such as fractional derivatives, Hölder rough paths, and controlled paths. We followed standard treatments for rough path analysis [4, 5, 10, 18–20] and fractional calculus [22, 24].

2.1. **Notation.** Let V and W be finite-dimensional normed spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. Although the fundamental theories of rough paths and controlled paths are valid for suitable infinite-dimensional Banach spaces, we consider only finite-dimensional cases in this paper to avoid technical difficulties that are not relevant to our theme. Let $L(V, W)$ denote the set of all linear maps from V to W. Let T denote a positive constant that is fixed throughout this paper. Simplex $\{(s,t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$ is denoted by $\triangle$ and is a closed subset of $\mathbb{R}^2$. Let $C([0,T], V)$ and $C(\triangle, V)$ denote the spaces of all V-valued continuous functions on the interval $[0,T]$ and $\triangle$, respectively. Let $a, b \in [0,T]$ with $a < b$. For $\psi \in C([0,T], V)$, we set $\|\psi\|_{\infty;[a,b]} := \sup_{a \leq t \leq b} \|\psi_t\|_V$. Let $\lambda \in (0,1]$. We set

$$\|\psi\|_{\lambda;[a,b]} := \sup_{a \leq s < t \leq b} \frac{\|\psi_t - \psi_s\|_V}{(t-s)^\lambda} \quad \text{and} \quad \|\Psi\|_{\lambda;[a,b]} := \sup_{a \leq s < t \leq b} \frac{\|\Psi_{s,t}\|_V}{(t-s)^\lambda}$$

for $\psi \in C([0,T], V)$ and $\Psi \in C(\triangle, V)$. We set $\mathcal{C}_1^\lambda(V) := \{\psi \in C([0,T], V) : \|\psi\|_{\lambda;[0,T]} < \infty\}$ and $\mathcal{C}_2^\lambda(V) := \{\Psi \in C(\triangle, V) : \|\Psi\|_{\lambda;[0,T]} < \infty\}$. Hereafter, d_1 and d_2 denote positive integers, E and F denote the Euclidean spaces $\mathbb{R}^{d_1}$ and

$\mathbb{R}^{d_2}$, respectively, and $|\cdot|$ denotes the Euclidean norms of E, F, and their tensor spaces. For $\alpha \in \mathbb{R}$, we set $(-1)^\alpha := e^{i\pi\alpha}$. Let 1_A denote the indicator function of a subset $A \subset \mathbb{R}$; in fact, A is taken as a half-open interval $(a, b]$.

2.2. **Fractional derivatives.** Let $a, b \in \mathbb{R}$ with $a < b$. For $p \in [1, \infty)$, let $L^p(a, b)$ denote the complex L^p-space on the interval $[a, b]$ with respect to the Lebesgue measure. Let $f \in L^1(a, b)$ and $\alpha \in (0, \infty)$. The left- and right-sided Riemann–Liouville fractional integrals of f of order α are defined for almost all $t \in (a, b)$ by

$$I^\alpha_{a+} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds$$

and

$$I^\alpha_{b-} f(t) := \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\, ds,$$

respectively, where Γ denotes the gamma function. For $p \in [1, \infty)$, let $I^\alpha_{a+}(L^p)$ and $I^\alpha_{b-}(L^p)$ denote the images of $L^p(a, b)$ by the operators I^α_{a+} and I^α_{b-}, respectively. Let $f \in I^\alpha_{a+}(L^1)$ with $0 < \alpha < 1$. The left-sided Weyl–Marchaud fractional derivative of f of order α is defined for almost all $t \in (a, b)$ by

$$D^\alpha_{a+} f(t) := \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^\alpha} + \alpha \int_a^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}}\, ds \right).$$

Similarly, let $f \in I^\alpha_{b-}(L^1)$ with $0 < \alpha < 1$. The right-sided Weyl–Marchaud fractional derivative of f of order α is defined for almost all $t \in (a, b)$ by

$$D^\alpha_{b-} f(t) := \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(b-t)^\alpha} + \alpha \int_t^b \frac{f(t) - f(s)}{(s-t)^{\alpha+1}}\, ds \right).$$

Here, the integrals above are well-defined for almost all $t \in (a, b)$. Let f be a real-valued Hölder continuous function of order $\lambda \in (0, 1]$ on the interval $[a, b]$ and $\alpha \in (0, \lambda)$. Then, $D^\alpha_{a+} f(t)$ and $D^\alpha_{b-} f(t)$ are well-defined for all $t \in (a, b]$ and $t \in [a, b)$, respectively. In addition, we define $D^\alpha_{a+}(f - f(a))(a) := 0$ and $D^\alpha_{b-}(f - f(b))(b) := 0$. For further details on the fractional integrals and derivatives, see [22, 24].

To describe our integration, we introduce slight generalizations of the fractional derivatives of Hölder continuous functions. Let $a \in [0, T)$, $b \in (0, T]$, $\lambda \in (0, 1]$, and $\alpha \in (0, \lambda)$. First, for $\Psi \in \mathcal{C}_2^\lambda(V)$, we define $\mathcal{D}^\alpha_{a+}\Psi$ and $\mathcal{D}^\alpha_{b-}\Psi$ by

$$\mathcal{D}^\alpha_{a+}\Psi(a) := 0,$$
$$\mathcal{D}^\alpha_{a+}\Psi(u) := \frac{1}{\Gamma(1-\alpha)} \left(\frac{\Psi_{a,u}}{(u-a)^\alpha} + \alpha \int_a^u \frac{\Psi_{v,u}}{(u-v)^{\alpha+1}}\, dv \right)$$

for $u \in (a,T]$ and

$$\mathcal{D}^{\alpha}_{b-}\Psi(b) := 0,$$
$$\mathcal{D}^{\alpha}_{b-}\Psi(u) := \frac{(-1)^{1+\alpha}}{\Gamma(1-\alpha)}\left(\frac{\Psi_{u,b}}{(b-u)^{\alpha}} + \alpha\int_u^b \frac{\Psi_{u,v}}{(v-u)^{\alpha+1}}\,dv\right)$$

for $u \in [0,b)$. It follows from a straightforward computation that

$$\|\mathcal{D}^{\alpha}_{a+}\Psi(u)\|_V \leq \frac{1}{\Gamma(1-\alpha)}\frac{\lambda}{\lambda-\alpha}\|\Psi\|_{\lambda;[a,u]}(u-a)^{\lambda-\alpha} \tag{2.1}$$

for $u \in [a,T]$ and

$$\|\mathcal{D}^{\alpha}_{b-}\Psi(u)\|_V \leq \frac{1}{\Gamma(1-\alpha)}\frac{\lambda}{\lambda-\alpha}\|\Psi\|_{\lambda;[u,b]}(b-u)^{\lambda-\alpha} \tag{2.2}$$

for $u \in [0,b]$. If $\Psi \in \mathcal{C}^{\lambda}_2(V)$ is of the form $\Psi_{s,t} = \psi_t - \psi_s$ for $(s,t) \in \triangle$ for some $\psi \in \mathcal{C}^{\lambda}_1(V)$, then $\mathcal{D}^{\alpha}_{a+}\Psi = D^{\alpha}_{a+}(\psi - \psi_a)$ and $\mathcal{D}^{\alpha}_{b-}\Psi = D^{\alpha}_{b-}(\psi - \psi_b)$ holds by definition. Next, we consider $\Psi \in C(\triangle, V)$ such that

$$\sup_{0\leq s<t\leq T}\frac{\|\Psi_{t,t} - \Psi_{s,t}\|_V}{(t-s)^{\lambda}} < \infty.$$

For $u \in (a,T]$, we define $D^{\alpha}_{a+}\Psi_{\cdot,u}(u) \in V$ by

$$D^{\alpha}_{a+}\Psi_{\cdot,u}(u) := \frac{1}{\Gamma(1-\alpha)}\left(\frac{\Psi_{u,u}}{(u-a)^{\alpha}} + \alpha\int_a^u \frac{\Psi_{u,u} - \Psi_{v,u}}{(u-v)^{\alpha+1}}\,dv\right).$$

Let $x, y \in [a,T]$ with $x < y$. For $u \in (a,T]$, we define $D^{\alpha}_{a+}(\Psi_{\cdot,u}1_{(x,y]})(u) \in V$ by

$$\begin{aligned}&D^{\alpha}_{a+}(\Psi_{\cdot,u}1_{(x,y]})(u)\\ &:= \frac{1}{\Gamma(1-\alpha)}\left(\frac{\Psi_{u,u}1_{(x,y]}(u)}{(u-a)^{\alpha}} + \alpha\int_a^u \frac{\Psi_{u,u}1_{(x,y]}(u) - \Psi_{v,u}1_{(x,y]}(v)}{(u-v)^{\alpha+1}}\,dv\right).\end{aligned}$$

By dividing the domain $(a,T]$ into $(a,x]$, $(x,y]$, and $(y,T]$, it is easy to see that $D^{\alpha}_{a+}(\Psi_{\cdot,u}1_{(x,y]})(u)$ is well-defined for $u \in (a,T]$. If Ψ is of the form $\Psi_{s,t} = \psi_s$ for $(s,t) \in \triangle$ for some $\psi \in \mathcal{C}^{\lambda}_1(V)$, then for $u \in (a,T]$, $D^{\alpha}_{a+}\Psi_{\cdot,u}(u) = D^{\alpha}_{a+}\psi(u)$ holds and $D^{\alpha}_{a+}(\Psi_{\cdot,u}1_{(x,y]})(u) = D^{\alpha}_{a+}(\psi 1_{(x,y]})(u)$ is well-defined. We refer to (2.8) and (2.9) for an example of $D^{\alpha}_{a+}\Psi_{\cdot,u}(u)$, which is fundamental to the argument in Section 2.4.

2.3. **Additivity of Young integral.** In Section 2.3, we introduce the Young integral based on fractional calculus in [24] (Definition 2.1 below) and provide a proof of the additivity with respect to the interval of integration for the Young integral (Theorem 2.8). Theorem 2.8 follows from Proposition 2.7. To prove Proposition 2.7, we introduce the following lemmas and propositions. The symbol $I_X(Y)_{s,t}$ in Definition 2.1 denotes the integral of Y along X on $[s,t]$.

Definition 2.1. Let $Y \in \mathcal{C}_1^\lambda(L(E,F))$ and $X \in \mathcal{C}_1^\mu(E)$ with $\lambda + \mu > 1$. Take $\alpha \in (1-\mu, \lambda)$. For $(s,t) \in \triangle$, we define $I_X(Y)_{s,t} \in F$ by

$$I_X(Y)_{s,t} := Y_s(X_t - X_s) + (-1)^\alpha \int_s^t D_{s+}^\alpha (Y - Y_s)(u) D_{t-}^{1-\alpha}(X - X_t)(u)\, du.$$

In the setting of Definition 2.1, $D_{s+}^\alpha(Y - Y_s)$ and $D_{t-}^{1-\alpha}(X - X_t)$ are well-defined from $\alpha < \lambda$ and $1 - \alpha < \mu$, respectively. In addition, from (2.1) and (2.2), we see that there exists a positive constant C, depending only on λ and α, such that

$$|I_X(Y)_{s,t} - Y_s(X_t - X_s)| \leq C \|Y\|_{\lambda;[s,t]} \|X\|_{\mu;[s,t]} (t-s)^{\lambda+\mu}$$

for $(s,t) \in \triangle$. Although the preceding inequality is not required in this paper, a similar inequality (2.7) is used in the proof of Proposition 2.7. As Lemma 2.2 and Proposition 2.3 are provided in [24], we omit these proofs.

Lemma 2.2. *Let $X \in \mathcal{C}_1^\mu(E)$ and $\alpha \in (1-\mu, 1)$. Fix $(s,t) \in \triangle$ with $s < t$. Then, for $x, y \in [s,t]$ with $x < y$,*

$$X_y - X_x = (-1)^\alpha \int_s^t D_{s+}^\alpha 1_{(x,y]}(u) D_{t-}^{1-\alpha}(X - X_t)(u)\, du.$$

For the indicator function $1_{(x,y]}$ of $(x,y] \subset [s,t]$, it is known that (1) $1_{(x,y]} \in I_{s+}^\alpha(L^p)$ if and only if $\alpha p < 1$; and (2) the equality

$$D_{s+}^\alpha 1_{(x,y]}(u) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{1_{(x,t]}(u)}{(u-x)^\alpha} - \frac{1_{(y,t]}(u)}{(u-y)^\alpha} \right)$$

holds for $u \in (s,t)$. For further details, see Proposition 2.2 in [24]. Proposition 2.3 follows from Definition 2.1 and Lemma 2.2 immediately.

Proposition 2.3. *Let $Y \in \mathcal{C}_1^\lambda(L(E,F))$ and $X \in \mathcal{C}_1^\mu(E)$ with $\lambda + \mu > 1$ and $\alpha \in (1-\mu, \lambda)$. Then, for $(s,t) \in \triangle$,*

$$I_X(Y)_{s,t} = (-1)^\alpha \int_s^t D_{s+}^\alpha Y(u) D_{t-}^{1-\alpha}(X - X_t)(u)\, du.$$

Although Lemma 2.4 is elementary, it plays a key role in this paper.

Lemma 2.4. *Let $Y \in \mathcal{C}_1^\lambda(F)$ and $\alpha \in (0, \lambda)$. Fix $s \in (0,T]$ and $x, y \in [0,s]$ with $x < y$. Then, for $u \in (x,s]$,*

$$D_{x+}^\alpha(Y 1_{(y,s]})(u) = D_{y+}^\alpha Y(u) 1_{(y,s]}(u).$$

Proof. From the definition of $D_{x+}^\alpha(Y 1_{(y,s]})(u)$,

$$D_{x+}^\alpha(Y 1_{(y,s]})(u) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{Y_u 1_{(y,s]}(u)}{(u-x)^\alpha} + \alpha \int_x^u \frac{Y_u 1_{(y,s]}(u) - Y_v 1_{(y,s]}(v)}{(u-v)^{\alpha+1}}\, dv \right)$$

for $u \in (x,s]$. Hence, for $u \in (y,s]$, we have

$$\begin{aligned}
\Gamma(1-\alpha) D_{x+}^\alpha(Y 1_{(y,s]})(u) &= \frac{Y_u}{(u-x)^\alpha} + \alpha \int_x^y \frac{Y_u}{(u-v)^{\alpha+1}}\, dv + \alpha \int_y^u \frac{Y_u - Y_v}{(u-v)^{\alpha+1}}\, dv \\
&= \frac{Y_u}{(u-y)^\alpha} + \alpha \int_y^u \frac{Y_u - Y_v}{(u-v)^{\alpha+1}}\, dv = \Gamma(1-\alpha) D_{y+}^\alpha Y(u).
\end{aligned}$$

Clearly, for $u \in (x, y]$, $D^\alpha_{x+}(Y 1_{(y,s]})(u) = 0$. Thus, the proof is complete. □

Using the lemmas and proposition above, we obtain the following equality.

Proposition 2.5. *Let $Y \in \mathcal{C}^\lambda_1(L(E, F))$ and $X \in \mathcal{C}^\mu_1(E)$ with $\lambda + \mu > 1$ and $\alpha \in (1 - \mu, \lambda)$. Fix $s \in (0, T]$. Then, for $x, y \in [0, s]$ with $x < y$,*

$$\begin{aligned} &I_X(Y)_{x,s} - I_X(Y)_{y,s} - Y_x(X_y - X_x) \\ &= (-1)^\alpha \int_x^s D^\alpha_{x+}((Y - Y_x)1_{(x,y]})(u) D^{1-\alpha}_{s-}(X - X_s)(u)\, du. \end{aligned}$$

Proof. By Proposition 2.3 and Lemma 2.4, we have

$$\begin{aligned} &I_X(Y)_{x,s} - I_X(Y)_{y,s} \\ &= I_X(Y)_{x,s} - (-1)^\alpha \int_x^s D^\alpha_{y+}Y(u)1_{(y,s]}(u) D^{1-\alpha}_{s-}(X - X_s)(u)\, du \\ &= (-1)^\alpha \int_x^s (D^\alpha_{x+}Y(u) - D^\alpha_{x+}(Y 1_{(y,s]})(u)) D^{1-\alpha}_{s-}(X - X_s)(u)\, du \\ &= (-1)^\alpha \int_x^s D^\alpha_{x+}(Y 1_{(x,y]})(u) D^{1-\alpha}_{s-}(X - X_s)(u)\, du. \end{aligned} \tag{2.3}$$

By Lemma 2.2, we have

$$Y_x(X_y - X_x) = (-1)^\alpha \int_x^s D^\alpha_{x+}(Y_x 1_{(x,y]})(u) D^{1-\alpha}_{s-}(X - X_s)(u)\, du. \tag{2.4}$$

Combining (2.3) and (2.4) yields the desired equality. □

To provide a quantitative estimate of the right-hand side of the equality in Proposition 2.5, we introduce Proposition 2.6.

Proposition 2.6. *Let $Y \in \mathcal{C}^\lambda_1(F)$ and $\alpha \in (0, \lambda)$. Fix $s \in (0, T]$. Then, there exists a positive constant C_1 that depends only on λ and α such that for $x, y \in [0, s]$ with $x < y$,*

$$\int_x^s |D^\alpha_{x+}((Y - Y_x)1_{(x,y]})(u)|\, du \le C_1 \|Y\|_{\lambda;[x,y]}(y - x)^{\lambda-\alpha+1}.$$

Proof. From the definition of $D^\alpha_{x+}((Y - Y_x)1_{(x,y]})(u)$,

$$\begin{aligned} D^\alpha_{x+}((Y - Y_x)1_{(x,y]})(u) = \frac{1}{\Gamma(1-\alpha)} \Bigg(&\frac{(Y_u - Y_x)1_{(x,y]}(u)}{(u - x)^\alpha} \\ &+ \alpha \int_x^u \frac{(Y_u - Y_x)1_{(x,y]}(u) - (Y_v - Y_x)1_{(x,y]}(v)}{(u - v)^{\alpha+1}}\, dv \Bigg) \end{aligned}$$

for $u \in (x, s]$. Hence, for $u \in (x, y]$, we have

$$D^\alpha_{x+}((Y - Y_x)1_{(x,y]})(u) = D^\alpha_{x+}(Y - Y_x)(u).$$

Thus, from (2.1),

$$\begin{aligned}
&\int_x^y |D^\alpha_{x+}((Y-Y_x)1_{(x,y]})(u)|\,du \\
&= \int_x^y |D^\alpha_{x+}(Y-Y_x)(u)|\,du \\
&\le \frac{1}{\Gamma(1-\alpha)}\frac{\lambda}{\lambda-\alpha}\|Y\|_{\lambda;[x,y]}\int_x^y (u-x)^{\lambda-\alpha}\,du \\
&= \frac{1}{\Gamma(1-\alpha)}\frac{\lambda}{\lambda-\alpha}\|Y\|_{\lambda;[x,y]}(\lambda-\alpha+1)^{-1}(y-x)^{\lambda-\alpha+1}. \qquad (2.5)
\end{aligned}$$

Additionally, for $u \in (y,s]$, we have

$$D^\alpha_{x+}((Y-Y_x)1_{(x,y]})(u) = \frac{-\alpha}{\Gamma(1-\alpha)}\int_x^y \frac{Y_v - Y_x}{(u-v)^{\alpha+1}}\,dv.$$

Thus, a straightforward computation yields

$$\begin{aligned}
&\int_y^s |D^\alpha_{x+}((Y-Y_x)1_{(x,y]})(u)|\,du \\
&\le \frac{\alpha}{\Gamma(1-\alpha)}\|Y\|_{\lambda;[x,y]}\int_y^s\int_x^y (v-x)^\lambda (u-v)^{-\alpha-1}\,dv\,du \\
&\le \frac{\alpha}{\Gamma(1-\alpha)}\|Y\|_{\lambda;[x,y]}(y-x)^\lambda \\
&\quad \times \alpha^{-1}(1-\alpha)^{-1}\{(s-y)^{1-\alpha}-(s-x)^{1-\alpha}+(y-x)^{1-\alpha}\} \\
&\le \frac{1}{\Gamma(1-\alpha)}\|Y\|_{\lambda;[x,y]}(1-\alpha)^{-1}(y-x)^{\lambda-\alpha+1}. \qquad (2.6)
\end{aligned}$$

Therefore, from (2.5) and (2.6), we have

$$\begin{aligned}
&\int_x^s |D^\alpha_{x+}((Y-Y_x)1_{(x,y]})(u)|\,du \\
&= \int_x^y |D^\alpha_{x+}((Y-Y_x)1_{(x,y]})(u)|\,du + \int_y^s |D^\alpha_{x+}((Y-Y_x)1_{(x,y]})(u)|\,du \\
&\le \frac{1}{\Gamma(1-\alpha)}\left(\frac{\lambda}{\lambda-\alpha}(\lambda-\alpha+1)^{-1} + (1-\alpha)^{-1}\right)\|Y\|_{\lambda;[x,y]}(y-x)^{\lambda-\alpha+1}.
\end{aligned}$$

This completes the proof. □

By using Propositions 2.5 and 2.6, we prove Proposition 2.7.

Proposition 2.7. *Let $Y \in \mathcal{C}^\lambda_1(L(E,F))$ and $X \in \mathcal{C}^\mu_1(E)$ with $\lambda+\mu>1$ and $\alpha \in (1-\mu,\lambda)$. Fix $s,t \in (0,T]$ with $s \le t$ and set $g(x) := I_X(Y)_{x,t} - I_X(Y)_{x,s}$ for $x \in [0,s]$. Then, g is a constant function on $[0,s]$.*

Proof. Suppose $s < t$ because $g(x) = 0$ obviously holds for $x \in [0, s]$ when $s = t$. Then, for $x, y \in [0, s]$ with $x < y$,

$$\begin{aligned} g(y) - g(x) &= (I_X(Y)_{y,t} - I_X(Y)_{y,s}) - (I_X(Y)_{x,t} - I_X(Y)_{x,s}) \\ &= (I_X(Y)_{x,s} - I_X(Y)_{y,s}) - (I_X(Y)_{x,t} - I_X(Y)_{y,t}) \\ &= (I_X(Y)_{x,s} - I_X(Y)_{y,s} - Y_x(X_y - X_x)) \\ &\quad - (I_X(Y)_{x,t} - I_X(Y)_{y,t} - Y_x(X_y - X_x)). \end{aligned}$$

By Propositions 2.5 and 2.6, we have

$$\begin{aligned} &|I_X(Y)_{x,b} - I_X(Y)_{y,b} - Y_x(X_y - X_x)| \\ &\leq \int_x^b |D_{x+}^{\alpha}((Y - Y_x)1_{(x,y]})(u)(-1)^{\alpha} D_{b-}^{1-\alpha}(X - X_b)(u)|\, du \\ &\leq C_1 \|Y\|_{\lambda;[x,y]}(y - x)^{\lambda-\alpha+1} \|D_{b-}^{1-\alpha}(X - X_b)\|_{\infty;[x,b]} \end{aligned} \tag{2.7}$$

for $b \in [s, t]$. Thus, by the triangle inequality and (2.7) with $b = s, t$, we have

$$|g(y) - g(x)| \leq C_1 \|Y\|_{\lambda;[0,s]} C_{s,t}^X (y - x)^{\lambda-\alpha+1},$$

where we write $C_{s,t}^X := \|D_{s-}^{1-\alpha}(X - X_s)\|_{\infty;[0,s]} + \|D_{t-}^{1-\alpha}(X - X_t)\|_{\infty;[0,t]}$. Therefore, it follows from $\lambda - \alpha + 1 > 1$ that g is a constant function on $[0, s]$. $\square$

We now prove the additivity of $I_X(Y)_{s,t}$.

Theorem 2.8. *Let $Y \in \mathcal{C}_1^{\lambda}(L(E, F))$ and $X \in \mathcal{C}_1^{\mu}(E)$ with $\lambda + \mu > 1$ and $\alpha \in (1 - \mu, \lambda)$. Then, the equality $I_X(Y)_{r,s} + I_X(Y)_{s,t} = I_X(Y)_{r,t}$ holds for $0 \leq r \leq s \leq t \leq T$.*

Proof. Suppose $s > 0$ because the equality obviously holds when $s = 0$. Then, it follows from Proposition 2.7 that

$$I_X(Y)_{r,t} - I_X(Y)_{r,s} = g(r) = g(s) = I_X(Y)_{s,t} - I_X(Y)_{s,s} = I_X(Y)_{s,t}.$$

Thus, the proof is complete. $\square$

2.4. **Additivity of rough integral.** In Section 2.4, following the steps in Section 2.3, we provide a proof of the additivity of the rough integral (Theorem 2.16). First, we recall the definitions of Hölder rough paths and controlled paths and introduce the rough integral based on fractional calculus in [11, 15] (Definition 2.9 below). Let β denote a real number with $1/3 < \beta \leq 1/2$. This number is fixed throughout Section 2.4. We say that pair $(X, \mathbb{X})$ is a β-Hölder rough path in E if $(X, \mathbb{X})$ satisfies the following two conditions:

(1) $X \in \mathcal{C}_1^{\beta}(E)$ and $\mathbb{X} \in \mathcal{C}_2^{2\beta}(E \otimes E)$;
(2) $\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = (X_u - X_s) \otimes (X_t - X_u)$ holds for $0 \leq s \leq u \leq t \leq T$.

The space of all β-Hölder rough paths in E is denoted by $\Omega_{\beta}(E)$. Let $X \in \mathcal{C}_1^{\beta}(E)$. We say that pair (Y, Y') is an X-controlled path with values in F if (Y, Y') satisfies the following two conditions:

(1) $Y \in \mathcal{C}_1^{\beta}(F)$ and $Y' \in \mathcal{C}_1^{\beta}(L(E, F))$;
(2) $R^Y \in \mathcal{C}_2^{2\beta}(F)$, where $R_{s,t}^Y := Y_t - Y_s - Y_s'(X_t - X_s)$ for $(s, t) \in \triangle$.

The space of all X-controlled paths with values in F is denoted by $\mathcal{Q}_X^\beta(F)$. For further details and examples of Hölder rough paths and controlled paths, see [4, 5, 10]. Let γ denote a real number with $(1-\beta)/2 < \gamma < \beta$. This number is fixed throughout Section 2.4. The symbol $I_{(X,\mathbb{X})}(Y, Y')_{s,t}$ in Definition 2.9 denotes the rough integral of (Y, Y') along $(X, \mathbb{X})$ on $[s,t]$ in this paper.

Definition 2.9. Let $(X, \mathbb{X}) \in \Omega_\beta(E)$ and $(Y, Y') \in \mathcal{Q}_X^\beta(L(E,F))$. For $(s,t) \in \triangle$, we define $I_{(X,\mathbb{X})}(Y, Y')_{s,t} \in F$ by

$$\begin{aligned} I_{(X,\mathbb{X})}(Y, Y')_{s,t} := {} & Y_s(X_t - X_s) + Y'_s \mathbb{X}_{s,t} \\ & + (-1)^{1-\gamma} \int_s^t \mathcal{D}_{s+}^{1-\gamma} R^Y(u) D_{t-}^\gamma (X - X_t)(u)\, du \\ & + (-1)^{1-2\gamma} \int_s^t D_{s+}^{1-2\gamma}(Y' - Y'_s)(u) D_{t-}^\gamma (\mathcal{D}_{t-}^\gamma \mathbb{X})(u)\, du. \end{aligned}$$

In the setting of Definition 2.9, $\mathcal{D}_{s+}^{1-\gamma} R^Y$, $D_{s+}^{1-2\gamma}(Y' - Y'_s)$, and $D_{t-}^\gamma(X - X_t)$ are well-defined from $1-\gamma < 2\beta$, $1-2\gamma < \beta$, and $\gamma < \beta$, respectively. In addition, from $\gamma < \beta$, $D_{t-}^\gamma(\mathcal{D}_{t-}^\gamma \mathbb{X})$ is well-defined on $[0,t]$ because $\mathcal{D}_{t-}^\gamma \mathbb{X}$ is β-Hölder continuous on $[0,t]$ and $\mathcal{D}_{t-}^\gamma \mathbb{X}(t) = 0$ holds by definition. For a proof of the Hölder continuity of $\mathcal{D}_{t-}^\gamma \mathbb{X}$, see, e.g., Lemma 6.3 in [11]. Furthermore, in the setting of Definition 2.9, there exists a positive constant C, depending only on β and γ, such that

$$\begin{aligned} & |I_{(X,\mathbb{X})}(Y, Y')_{s,t} - Y_s(X_t - X_s) - Y'_s \mathbb{X}_{s,t}| \\ & \leq C\{\|R^Y\|_{2\beta;[s,t]} \|X\|_{\beta;[s,t]} + \|Y'\|_{\beta;[s,t]}(\|\mathbb{X}\|_{2\beta;[s,t]} + \|X\|_{\beta;[s,t]}^2)\}(t-s)^{3\beta} \end{aligned}$$

for $(s,t) \in \triangle$. This easily follows from (2.1), (2.2), and Lemma 3.6 in [15]. We omit the proof of the preceding inequality because it is not required in this paper. However, a similar inequality (2.14) is used in the proof of Proposition 2.15.

Next, we prove the additivity of $I_{(X,\mathbb{X})}(Y, Y')_{s,t}$ (Theorem 2.16). This result follows from Proposition 2.15. To prove Proposition 2.15, we introduce the following lemmas and propositions. For proofs of Lemma 2.10 and Proposition 2.11, we refer to those of Lemmas 3.5 and 3.7 in [15], respectively.

Lemma 2.10. *Let $(X, \mathbb{X}) \in \Omega_\beta(E)$. Fix $(s,t) \in \triangle$ with $s < t$. Then, for $x, y \in [s,t]$ with $x < y$,*

$$\begin{aligned} \mathbb{X}_{x,y} = {} & (-1)^{1-\gamma} \int_s^t D_{s+}^{1-\gamma} 1_{(x,y]}(u)(X_u - X_x) \otimes D_{t-}^\gamma (X - X_t)(u)\, du \\ & + (-1)^{1-2\gamma} \int_s^t D_{s+}^{1-2\gamma} 1_{(x,y]}(u) D_{t-}^\gamma (\mathcal{D}_{t-}^\gamma \mathbb{X})(u)\, du. \end{aligned}$$

Let $X \in \mathcal{C}_1^\beta(E)$ and $(Y, Y') \in \mathcal{Q}_X^\beta(F)$. We define $\Phi \in C(\triangle, F)$ by

$$\Phi_{s,t} := Y_s + Y'_s(X_t - X_s) \tag{2.8}$$

for $(s,t) \in \triangle$. It is clear that $\Phi_{t,t} - \Phi_{s,t} = R_{s,t}^Y$ holds for $(s,t) \in \triangle$. Therefore, from $R^Y \in \mathcal{C}_2^{2\beta}(F)$, $D_{a+}^\alpha \Phi_{\cdot,u}(u)$ and $D_{a+}^\alpha(\Phi_{\cdot,u} 1_{(x,y]})(u)$ introduced in Section 2.2

are well-defined for $\alpha \in (0, 2\beta)$. We note that $D^{\alpha}_{a+}\Phi_{\cdot,u}(u)$ is consistent with the compensated fractional derivative introduced in [11]; namely, the equality

$$D^{\alpha}_{a+}\Phi_{\cdot,u}(u) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{Y_u}{(u-a)^{\alpha}} + \alpha\int_a^u \frac{R^Y_{v,u}}{(u-v)^{\alpha+1}}\,dv\right) \tag{2.9}$$

holds for $u \in (a, T]$. Using the above notation, we state Proposition 2.11, which can be regarded as a rough integral version of Proposition 2.3.

Proposition 2.11. *Let $(X, \mathbb{X}) \in \Omega_{\beta}(E)$ and $(Y, Y') \in \mathcal{Q}^{\beta}_X(L(E, F))$. Then, for $(s, t) \in \triangle$,*

$$\begin{aligned} I_{(X,\mathbb{X})}(Y, Y')_{s,t} = (-1)^{1-\gamma}&\int_s^t D^{1-\gamma}_{s+}\Phi_{\cdot,u}(u)D^{\gamma}_{t-}(X - X_t)(u)\,du \\ &+ (-1)^{1-2\gamma}\int_s^t D^{1-2\gamma}_{s+}Y'(u)D^{\gamma}_{t-}(\mathcal{D}^{\gamma}_{t-}\mathbb{X})(u)\,du. \end{aligned}$$

Lemma 2.12, Propositions 2.13, 2.14, and 2.15 can be regarded as rough integral versions of Lemma 2.4, Propositions 2.5, 2.6, and 2.7, respectively.

Lemma 2.12. *Let $X \in \mathcal{C}^{\beta}_1(E)$, $(Y, Y') \in \mathcal{Q}^{\beta}_X(F)$ and $\alpha \in (0, 2\beta)$. Fix $s \in (0, T]$ and $x, y \in [0, s]$ with $x < y$. Then, for $u \in (x, s]$,*

$$D^{\alpha}_{x+}(\Phi_{\cdot,u}1_{(y,s]})(u) = D^{\alpha}_{y+}\Phi_{\cdot,u}(u)1_{(y,s]}(u).$$

Proof. From the definition of $D^{\alpha}_{x+}(\Phi_{\cdot,u}1_{(y,s]})(u)$,

$$\begin{aligned} &D^{\alpha}_{x+}(\Phi_{\cdot,u}1_{(y,s]})(u) \\ &= \frac{1}{\Gamma(1-\alpha)}\left(\frac{\Phi_{u,u}1_{(y,s]}(u)}{(u-x)^{\alpha}} + \alpha\int_x^u \frac{\Phi_{u,u}1_{(y,s]}(u) - \Phi_{v,u}1_{(y,s]}(v)}{(u-v)^{\alpha+1}}\,dv\right) \end{aligned}$$

for $u \in (x, s]$. Hence, for $u \in (y, s]$, we have

$$\begin{aligned} &\Gamma(1-\alpha)D^{\alpha}_{x+}(\Phi_{\cdot,u}1_{(y,s]})(u) \\ &= \frac{\Phi_{u,u}}{(u-x)^{\alpha}} + \alpha\int_x^y \frac{\Phi_{u,u}}{(u-v)^{\alpha+1}}\,dv + \alpha\int_y^u \frac{\Phi_{u,u} - \Phi_{v,u}}{(u-v)^{\alpha+1}}\,dv \\ &= \frac{\Phi_{u,u}}{(u-y)^{\alpha}} + \alpha\int_y^u \frac{\Phi_{u,u} - \Phi_{v,u}}{(u-v)^{\alpha+1}}\,dv = \Gamma(1-\alpha)D^{\alpha}_{y+}\Phi_{\cdot,u}(u). \end{aligned}$$

Clearly, for $u \in (x, y]$, $D^{\alpha}_{x+}(\Phi_{\cdot,u}1_{(y,s]})(u) = 0$. Thus, the proof is complete. □

Proposition 2.13. *Let $(X, \mathbb{X}) \in \Omega_{\beta}(E)$ and $(Y, Y') \in \mathcal{Q}^{\beta}_X(L(E, F))$. Fix $s \in (0, T]$. Then, for $x, y \in [0, s]$ with $x < y$,*

$$\begin{aligned} &I_{(X,\mathbb{X})}(Y, Y')_{x,s} - I_{(X,\mathbb{X})}(Y, Y')_{y,s} - Y_x(X_y - X_x) - Y'_x\mathbb{X}_{x,y} \\ &= (-1)^{1-\gamma}\int_x^s D^{1-\gamma}_{x+}((\Phi_{\cdot,u} - \Phi_{x,u})1_{(x,y]})(u)D^{\gamma}_{s-}(X - X_s)(u)\,du \\ &\quad + (-1)^{1-2\gamma}\int_x^s D^{1-2\gamma}_{x+}((Y' - Y'_x)1_{(x,y]})(u)D^{\gamma}_{s-}(\mathcal{D}^{\gamma}_{s-}\mathbb{X})(u)\,du. \end{aligned}$$

Proof. By Proposition 2.11, Lemmas 2.12, and 2.4, we have

$$\begin{aligned}
&I_{(X,\mathbb{X})}(Y,Y')_{x,s} - I_{(X,\mathbb{X})}(Y,Y')_{y,s} \\
&= I_{(X,\mathbb{X})}(Y,Y')_{x,s} - (-1)^{1-\gamma}\int_x^s D_{y+}^{1-\gamma}\Phi_{\cdot,u}(u)1_{(y,s]}(u)D_{s-}^{\gamma}(X-X_s)(u)\,du \\
&\quad - (-1)^{1-2\gamma}\int_x^s D_{y+}^{1-2\gamma}Y'(u)1_{(y,s]}(u)D_{s-}^{\gamma}(\mathcal{D}_{s-}^{\gamma}\mathbb{X})(u)\,du \\
&= (-1)^{1-\gamma}\int_x^s (D_{x+}^{1-\gamma}\Phi_{\cdot,u}(u) - D_{x+}^{1-\gamma}(\Phi_{\cdot,u}1_{(y,s]})(u))D_{s-}^{\gamma}(X-X_s)(u)\,du \\
&\quad + (-1)^{1-2\gamma}\int_x^s (D_{x+}^{1-2\gamma}Y'(u) - D_{x+}^{1-2\gamma}(Y'1_{(y,s]})(u))D_{s-}^{\gamma}(\mathcal{D}_{s-}^{\gamma}\mathbb{X})(u)\,du \\
&= (-1)^{1-\gamma}\int_x^s D_{x+}^{1-\gamma}(\Phi_{\cdot,u}1_{(x,y]})(u)D_{s-}^{\gamma}(X-X_s)(u)\,du \\
&\quad + (-1)^{1-2\gamma}\int_x^s D_{x+}^{1-2\gamma}(Y'1_{(x,y]})(u)D_{s-}^{\gamma}(\mathcal{D}_{s-}^{\gamma}\mathbb{X})(u)\,du. \qquad (2.10)
\end{aligned}$$

By Lemmas 2.2 and 2.10, we have

$$\begin{aligned}
Y_x(X_y - X_x) + Y'_x\mathbb{X}_{x,y} &= (-1)^{1-\gamma}\int_x^s D_{x+}^{1-\gamma}(\Phi_{x,u}1_{(x,y]})(u)D_{s-}^{\gamma}(X-X_s)(u)\,du \\
&\quad + (-1)^{1-2\gamma}\int_x^s D_{x+}^{1-2\gamma}(Y'_x1_{(x,y]})(u)D_{s-}^{\gamma}(\mathcal{D}_{s-}^{\gamma}\mathbb{X})(u)\,du. \qquad (2.11)
\end{aligned}$$

Combining (2.10) and (2.11) yields the desired equality. □

Proposition 2.14. *Let $X \in \mathcal{C}_1^{\beta}(E)$ and $(Y,Y') \in \mathcal{Q}_X^{\beta}(F)$. Fix $s \in (0,T]$. Then, there exists a positive constant C_2 that depends only on β and γ such that for $x,y \in [0,s]$ with $x<y$,*

$$\begin{aligned}
&\int_x^s |D_{x+}^{1-\gamma}((\Phi_{\cdot,u} - \Phi_{x,u})1_{(x,y]})(u)|\,du \\
&\le C_2(\|R^Y\|_{2\beta;[x,y]} + \|Y'\|_{\beta;[x,y]}\|X\|_{\beta;[x,s]})(y-x)^{2\beta+\gamma}.
\end{aligned}$$

Proof. We set $\alpha := 1-\gamma$ and note that $\beta < \alpha < 2\beta$ holds. From the definition of $D_{x+}^{\alpha}((\Phi_{\cdot,u} - \Phi_{x,u})1_{(x,y]})(u)$,

$$\begin{aligned}
&D_{x+}^{\alpha}((\Phi_{\cdot,u} - \Phi_{x,u})1_{(x,y]})(u) \\
&= \frac{1}{\Gamma(1-\alpha)}\Bigg(\frac{(\Phi_{u,u} - \Phi_{x,u})1_{(x,y]}(u)}{(u-x)^{\alpha}} \\
&\quad + \alpha\int_x^u \frac{(\Phi_{u,u} - \Phi_{x,u})1_{(x,y]}(u) - (\Phi_{v,u} - \Phi_{x,u})1_{(x,y]}(v)}{(u-v)^{\alpha+1}}\,dv\Bigg)
\end{aligned}$$

for $u \in (x, s]$. Hence, for $u \in (x, y]$, we have

$$\begin{aligned}
D^\alpha_{x+}((\Phi_{\cdot,u} - \Phi_{x,u})1_{(x,y]})(u) &= \frac{1}{\Gamma(1-\alpha)}\left(\frac{\Phi_{u,u} - \Phi_{x,u}}{(u-x)^\alpha} + \alpha \int_x^u \frac{\Phi_{u,u} - \Phi_{v,u}}{(u-v)^{\alpha+1}}\, dv\right) \\
&= \mathcal{D}^\alpha_{x+} R^Y(u).
\end{aligned}$$

Thus, from (2.1),

$$\begin{aligned}
&\int_x^y |D^\alpha_{x+}((\Phi_{\cdot,u} - \Phi_{x,u})1_{(x,y]})(u)|\, du \\
&= \int_x^y |\mathcal{D}^\alpha_{x+} R^Y(u)|\, du \\
&\le \frac{1}{\Gamma(1-\alpha)} \frac{2\beta}{2\beta-\alpha} \|R^Y\|_{2\beta;[x,y]} \int_x^y (u-x)^{2\beta-\alpha}\, du \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{2\beta}{2\beta-\alpha} \|R^Y\|_{2\beta;[x,y]} (2\beta-\alpha+1)^{-1}(y-x)^{2\beta-\alpha+1}. \qquad (2.12)
\end{aligned}$$

Additionally, for $u \in (y, s]$, we have

$$\begin{aligned}
D^\alpha_{x+}((\Phi_{\cdot,u} - \Phi_{x,u})1_{(x,y]})(u) &= \frac{-\alpha}{\Gamma(1-\alpha)} \int_x^y \frac{\Phi_{v,u} - \Phi_{x,u}}{(u-v)^{\alpha+1}}\, dv \\
&= \frac{-\alpha}{\Gamma(1-\alpha)} \int_x^y \frac{R^Y_{x,v} + (Y'_v - Y'_x)(X_u - X_v)}{(u-v)^{\alpha+1}}\, dv.
\end{aligned}$$

Thus, a straightforward computation yields

$$\begin{aligned}
&\int_y^s |D^\alpha_{x+}((\Phi_{\cdot,u} - \Phi_{x,u})1_{(x,y]})(u)|\, du \\
&\le \frac{\alpha}{\Gamma(1-\alpha)} \|R^Y\|_{2\beta;[x,y]} \int_y^s \int_x^y (v-x)^{2\beta}(u-v)^{-\alpha-1}\, dv\, du \\
&\quad + \frac{\alpha}{\Gamma(1-\alpha)} \|Y'\|_{\beta;[x,y]} \|X\|_{\beta;[x,s]} \int_y^s \int_x^y (v-x)^{\beta}(u-v)^{\beta-\alpha-1}\, dv\, du \\
&\le \frac{\alpha}{\Gamma(1-\alpha)} \|R^Y\|_{2\beta;[x,y]} (y-x)^{2\beta} \\
&\quad \times \alpha^{-1}(1-\alpha)^{-1}\{(s-y)^{1-\alpha} - (s-x)^{1-\alpha} + (y-x)^{1-\alpha}\} \\
&\quad + \frac{\alpha}{\Gamma(1-\alpha)} \|Y'\|_{\beta;[x,y]} \|X\|_{\beta;[x,s]} (y-x)^{\beta} \\
&\qquad \times (\alpha-\beta)^{-1}(1-\alpha+\beta)^{-1}\{(s-y)^{1-\alpha+\beta} - (s-x)^{1-\alpha+\beta} + (y-x)^{1-\alpha+\beta}\} \\
&\le \frac{1}{\Gamma(1-\alpha)} \|R^Y\|_{2\beta;[x,y]} (1-\alpha)^{-1}(y-x)^{2\beta-\alpha+1} \\
&\quad + \frac{\alpha}{\Gamma(1-\alpha)} \|Y'\|_{\beta;[x,y]} \|X\|_{\beta;[x,s]} (\alpha-\beta)^{-1}(1-\alpha+\beta)^{-1}(y-x)^{2\beta-\alpha+1}. \qquad (2.13)
\end{aligned}$$

Therefore, from (2.12) and (2.13), we have

$$\begin{aligned}
&\int_x^s |D_{x+}^{\alpha}((\Phi_{\cdot,u}-\Phi_{x,u})1_{(x,y]})(u)|\,du \\
&= \int_x^y |D_{x+}^{\alpha}((\Phi_{\cdot,u}-\Phi_{x,u})1_{(x,y]})(u)|\,du + \int_y^s |D_{x+}^{\alpha}((\Phi_{\cdot,u}-\Phi_{x,u})1_{(x,y]})(u)|\,du \\
&\le \frac{1}{\Gamma(1-\alpha)}\left(\frac{2\beta}{2\beta-\alpha}(2\beta-\alpha+1)^{-1}+(1-\alpha)^{-1}+\alpha(\alpha-\beta)^{-1}(1-\alpha+\beta)^{-1}\right) \\
&\quad \times (\|R^Y\|_{2\beta;[x,y]}+\|Y'\|_{\beta;[x,y]}\|X\|_{\beta;[x,s]})(y-x)^{2\beta-\alpha+1}.
\end{aligned}$$

This completes the proof. □

Proposition 2.15. *Let $(X,\mathbb{X})\in\Omega_\beta(E)$ and $(Y,Y')\in\mathcal{Q}_X^\beta(L(E,F))$. Fix $s,t\in(0,T]$ with $s\le t$ and set $h(x):=I_{(X,\mathbb{X})}(Y,Y')_{x,t}-I_{(X,\mathbb{X})}(Y,Y')_{x,s}$ for $x\in[0,s]$. Then, h is a constant function on $[0,s]$.*

Proof. Suppose $s<t$ because $h(x)=0$ obviously holds for $x\in[0,s]$ when $s=t$. Then, for $x,y\in[0,s]$ with $x<y$,

$$\begin{aligned}
&h(y)-h(x) \\
&= (I_{(X,\mathbb{X})}(Y,Y')_{y,t}-I_{(X,\mathbb{X})}(Y,Y')_{y,s})-(I_{(X,\mathbb{X})}(Y,Y')_{x,t}-I_{(X,\mathbb{X})}(Y,Y')_{x,s}) \\
&= (I_{(X,\mathbb{X})}(Y,Y')_{x,s}-I_{(X,\mathbb{X})}(Y,Y')_{y,s})-(I_{(X,\mathbb{X})}(Y,Y')_{x,t}-I_{(X,\mathbb{X})}(Y,Y')_{y,t}) \\
&= (I_{(X,\mathbb{X})}(Y,Y')_{x,s}-I_{(X,\mathbb{X})}(Y,Y')_{y,s}-Y_x(X_y-X_x)-Y'_x\mathbb{X}_{x,y}) \\
&\quad -(I_{(X,\mathbb{X})}(Y,Y')_{x,t}-I_{(X,\mathbb{X})}(Y,Y')_{y,t}-Y_x(X_y-X_x)-Y'_x\mathbb{X}_{x,y}).
\end{aligned}$$

By Propositions 2.13, 2.14, and 2.6, we have

$$\begin{aligned}
&|I_{(X,\mathbb{X})}(Y,Y')_{x,b}-I_{(X,\mathbb{X})}(Y,Y')_{y,b}-Y_x(X_y-X_x)-Y'_x\mathbb{X}_{x,y}| \\
&\le \int_x^b |D_{x+}^{1-\gamma}((\Phi_{\cdot,u}-\Phi_{x,u})1_{(x,y]})(u)(-1)^{1-\gamma}D_{b-}^{\gamma}(X-X_b)(u)|\,du \\
&\quad + \int_x^b |D_{x+}^{1-2\gamma}((Y'-Y'_x)1_{(x,y]})(u)(-1)^{1-2\gamma}D_{b-}^{\gamma}(\mathcal{D}_{b-}^{\gamma}\mathbb{X})(u)|\,du \\
&\le C_2(\|R^Y\|_{2\beta;[x,y]}+\|Y'\|_{\beta;[x,y]}\|X\|_{\beta;[x,b]})(y-x)^{2\beta+\gamma}\|D_{b-}^{\gamma}(X-X_b)\|_{\infty;[x,b]} \\
&\quad + C_1\|Y'\|_{\beta;[x,y]}(y-x)^{\beta+2\gamma}\|D_{b-}^{\gamma}(\mathcal{D}_{b-}^{\gamma}\mathbb{X})\|_{\infty;[x,b]} \qquad (2.14)
\end{aligned}$$

for $b\in[s,t]$. Thus, by the triangle inequality and (2.14) with $b=s,t$, we have

$$\begin{aligned}
|h(y)-h(x)| \le \{&C_2(\|R^Y\|_{2\beta;[0,s]}+\|Y'\|_{\beta;[0,s]}\|X\|_{\beta;[0,t]})s^{\beta-\gamma}C_{s,t}^X \\
&+C_1\|Y'\|_{\beta;[0,s]}C_{s,t}^{\mathbb{X}}\}(y-x)^{\beta+2\gamma},
\end{aligned}$$

where we write $C_{s,t}^X:=\|D_{s-}^{\gamma}(X-X_s)\|_{\infty;[0,s]}+\|D_{t-}^{\gamma}(X-X_t)\|_{\infty;[0,t]}$ and $C_{s,t}^{\mathbb{X}}:=\|D_{s-}^{\gamma}(\mathcal{D}_{s-}^{\gamma}\mathbb{X})\|_{\infty;[0,s]}+\|D_{t-}^{\gamma}(\mathcal{D}_{t-}^{\gamma}\mathbb{X})\|_{\infty;[0,t]}$. Therefore, it follows from $\beta+2\gamma>1$ that h is a constant function on $[0,s]$. □

We can now prove the additivity of $I_{(X,\mathbb{X})}(Y,Y')_{s,t}$.

Theorem 2.16. *Let* $(X,\mathbb{X}) \in \Omega_\beta(E)$ *and* $(Y,Y') \in \mathcal{Q}_X^\beta(L(E,F))$*. Then, the equality* $I_{(X,\mathbb{X})}(Y,Y')_{r,s} + I_{(X,\mathbb{X})}(Y,Y')_{s,t} = I_{(X,\mathbb{X})}(Y,Y')_{r,t}$ *holds for* $0 \le r \le s \le t \le T$*.*

Proof. Suppose $s > 0$ because the equality obviously holds when $s = 0$. Then, it follows from Proposition 2.15 that

$$\begin{aligned}&I_{(X,\mathbb{X})}(Y,Y')_{r,t} - I_{(X,\mathbb{X})}(Y,Y')_{r,s}\\&= h(r) = h(s) = I_{(X,\mathbb{X})}(Y,Y')_{s,t} - I_{(X,\mathbb{X})}(Y,Y')_{s,s} = I_{(X,\mathbb{X})}(Y,Y')_{s,t}.\end{aligned}$$

Thus, the proof is complete. □

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