# Resolution of sigma-fields for multiparticle finite-state action evolutions with infinite past 

Yu It $\xlongequal[(1)(2)]{(2)}$, Toru Sere $\xlongequal{(3)(4)}$ and Kouji Yand ${ }^{(5)(6)}$

August 31, 2020


#### Abstract

For multiparticle finite-state action evolutions, we prove that the observation $\sigma$-field admits a resolution involving a third noise which is generated by a random variable with uniform law. The Rees decomposition from the semigroup theory and the theory of infinite convolutions are utilized in our proofs.


## 1 Introduction

Let us consider the stochastic recursive equation

$$
\begin{equation*}
X_{k}=N_{k} X_{k-1} \quad \text { a.s. for } k \in \mathbb{Z}, \tag{1.1}
\end{equation*}
$$

which we call the action evolution, where the observation $X=\left(X_{k}\right)_{k \in \mathbb{Z}}$ taking values in a measurable space $V$ evolves from $X_{k-1}$ to $X_{k}$ at each time $k$ being acted by a random map $N_{k}$ of $V$. Here we mean by $N_{k} X_{k-1}$ the evaluation $N_{k}\left(X_{k-1}\right)$ of a random mapping $N_{k}$ at $X_{k-1}$; we always abbreviate the parentheses to write $f v$ simply for the evaluation $f(v)$. As our processes are indexed by $\mathbb{Z}$, the state $X_{k}$ we observe at time $k$ is a result after a long time has passed.

We would like to clarify the structure of the observation noise $\mathcal{F}_{k}^{X}=\sigma\left(X_{j}: j \leq k\right)$. For familes of events, we write $\mathcal{A} \vee \mathcal{B}:=\sigma(\mathcal{A} \bigcup \mathcal{B})$. For $\sigma$-fields, we say $\mathcal{F} \subset \mathcal{G}$ a.s. (resp. $\mathcal{F}=\mathcal{G}$ a.s.) if $\mathcal{F} \subset \mathcal{G} \vee \mathcal{N}$ (resp. $\mathcal{F} \vee \mathcal{N}=\mathcal{G} \vee \mathcal{N}$ ) with $\mathcal{N}$ being the family of null events. By iterating the equation (1.1), we have $X_{k}=N_{k} N_{k-1} \cdots N_{j+1} X_{j}$ a.s. for $j<k$. One may then expect that, for any $k \in \mathbb{Z}$,

$$
\begin{equation*}
\mathcal{F}_{k}^{X} \subset \bigcap_{j<k}\left(\mathcal{F}_{k}^{N} \vee \mathcal{F}_{j}^{X}\right) \stackrel{?}{\subset} \mathcal{F}_{k}^{N} \vee\left(\bigcap_{j<k} \mathcal{F}_{j}^{X}\right)=\mathcal{F}_{k}^{N} \vee \mathcal{F}_{-\infty}^{X} \quad \text { a.s. } \tag{1.2}
\end{equation*}
$$

[^0]and may conclude that the observation $\mathcal{F}_{k}^{X}$ can be known by the driving noise $\mathcal{F}_{k}^{N}:=$ $\sigma\left(N_{j}: j \leq k\right)$ together with the remote past noise $\mathcal{F}_{-\infty}^{X}:=\bigcap_{k} \mathcal{F}_{k}^{X}$, which plays the role of the initial noise at time $-\infty$. But the a.s. inclusion $\stackrel{?}{\subset}$ in (1.2) is false in general; see [10, (1) of Remark 1.4] for erroneous discussions by Kolmogorov and Wiener. We must refer to [1, Section 2.5] for careful treatment of exchanging the order of supremum and intersection between $\sigma$-fields.

### 1.1 Action evolutions and resolution of the observation

We would like to reveal the extra noise hidden in the observation noise. To this end let us introduce some terminology.

Definition 1.1. Let $\mu$ be a probability on a measurable space $\Sigma$ of mappings of $V$ into itself and call it the mapping law.

- A $\mu$-evolution is a pair $(X, N)$ of a $V$-valued process $X=\left(X_{k}\right)_{k \in \mathbb{Z}}$ and an iid $\Sigma$-valued process $N=\left(N_{k}\right)_{k \in \mathbb{Z}}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the following hold for each $k \in \mathbb{Z}$ :
(i) $X_{k}=N_{k} X_{k-1}$ holds a.s.;
(ii) $N_{k}$ is independent of $\mathcal{F}_{k-1}^{X, N}:=\sigma\left(X_{j}, N_{j}: j \leq k-1\right)$;
(iii) $N_{k}$ has law $\mu$.
- For a mapping $f: V \rightarrow V$ and a vector $\boldsymbol{x}=\left(x^{1}, \ldots, x^{m}\right) \in V^{m}$, we understand that $f$ operates $\boldsymbol{x}$ componentwise, i.e., $f \boldsymbol{x}=\left(f x^{1}, \ldots, f x^{m}\right)$. An m-particle $\mu$-evolution is a $\mu$-evolution $(\mathbb{X}, N)$ with $\mathbb{X}=\left(\mathbb{X}_{k}\right)_{k \in \mathbb{Z}}$ taking values in $V^{m}$; precisely, the following hold for each $k \in \mathbb{Z}$ :
(i) $\mathbb{X}_{k}=N_{k} \mathbb{X}_{k-1}$ holds a.s., i.e., $X_{k}^{i}=N_{k} X_{k-1}^{i}$ holds a.s. for $i=1, \ldots, m$;
(ii) $N_{k}$ is independent of $\mathcal{F}_{k-1}^{\mathbb{X}, N}:=\sigma\left(\mathbb{X}_{j}, N_{j}: j \leq k-1\right)$;
(iii) $N_{k}$ has law $\mu$.
- For a $\mu$-evolution, a third noise is a sequence of random variables $\left(U_{k}\right)_{k \in \mathbb{Z}}$ such that the following hold for each $k \in \mathbb{Z}$ :
(i) the inclusion $\mathcal{F}_{k}^{X} \subset \mathcal{F}_{k}^{N} \vee \mathcal{F}_{-\infty}^{X} \vee \sigma\left(U_{k}\right)$ holds a.s.;
(ii) $\sigma\left(U_{k}\right) \subset \mathcal{F}_{k}^{X, N}$ holds a.s.;
(iii) the three $\sigma$-fields $\mathcal{F}_{k}^{N}, \mathcal{F}_{-\infty}^{X}$ and $\sigma\left(U_{k}\right)$ are independent.
- For a $\mu$-evolution, a reduced driving noise is a sequence of $\sigma$-fields $\left(\mathcal{G}_{k}^{N}\right)_{k \in \mathbb{Z}}$ accompanying with a sequence of random variables $\left(U_{k}\right)_{k \in \mathbb{Z}}$ such that the following hold for each $k \in \mathbb{Z}$ :
(i) the identity $\mathcal{F}_{k}^{X}=\mathcal{G}_{k}^{N} \vee \mathcal{F}_{-\infty}^{X} \vee \sigma\left(U_{k}\right)$ holds a.s.;
(ii) $\mathcal{G}_{k}^{N} \subset \mathcal{F}_{k}^{N}$ holds a.s.;
(iii) the three $\sigma$-fields $\mathcal{F}_{k}^{N}, \mathcal{F}_{-\infty}^{X}$ and $\sigma\left(U_{k}\right)$ are independent.

The identity in Condition (i) will be called the resolution of the observation. Note that $\left(U_{k}\right)_{k \in \mathbb{Z}}$ is necessarily a third noise.

It is easy to see that $(X, N)$ is a $\mu$-evolution if and only if the Markov property

$$
\begin{equation*}
\mathbb{P}\left(\left(X_{k}, N_{k}\right) \in B \mid \mathcal{F}_{k-1}^{X, N}\right)=Q_{\mu}\left(X_{k-1} ; B\right), \quad k \in \mathbb{Z}, B \subset V \times \Sigma \tag{1.3}
\end{equation*}
$$

holds with the joint transition probability:

$$
\begin{equation*}
Q_{\mu}(x ; B)=\mu\{f:(f x, f) \in B\}, \quad x \in V, B \subset V \times \Sigma \tag{1.4}
\end{equation*}
$$

If $(X, N)$ is a $\mu$-evolution, then the marginal process $X$ satisfies the Markov property

$$
\begin{equation*}
\mathbb{P}\left(X_{k} \in A \mid \mathcal{F}_{k-1}^{X}\right)=P_{\mu}\left(X_{k-1} ; A\right), \quad k \in \mathbb{Z}, A \subset V \tag{1.5}
\end{equation*}
$$

with the marginal transition probability:

$$
\begin{equation*}
P_{\mu}(x ; A)=\mu\{f: f x \in A\}, \quad A \subset V . \tag{1.6}
\end{equation*}
$$

It is also easy to see that, if two $\mu$-evolutions $(X, N)$ and ( $X^{\prime}, N^{\prime}$ ) satisfy $X_{k} \stackrel{\mathrm{~d}}{=} X_{k}^{\prime}$ a.s. for $k \in \mathbb{Z}$, then $(X, N) \stackrel{\text { d }}{=}\left(X^{\prime}, N^{\prime}\right)$.

In this paper, we shall give a general result of resolution of the observation for multiparticle action evolutions when the state space $V$ is a finite set.

### 1.2 Infinite convolutions on finite semigroups

For our purpose we need several known facts from the theory of semigroups, which we recall without proofs. We may consult [5 for the details.

In what follows we assume $S$ be a finite semigroup and we denote the set of all idempotents in $S$ by $E(S)=\left\{f \in S: f^{2}=f\right\}$. For $A, B \subset S$ and $f \in S$, we write $A B=\{a b: a \in A, b \in B\}$ and $A f=\{a f: a \in A\}$, etc. We say that $S$ is completely simple if $S$ has no proper ideal, i.e., $\emptyset \neq I S \cup S I \subset I \subset S$ implies $I=S$, and if there exists $e \in E(S)$ which is primitive, i.e., ef $=f e=f \in E(S)$ implies $f=e$.

Proposition 1.2. A finite semigroup $S$ has a unique minimal ideal, which will be called the kernel of S. In addition, the kernel is completely simple.

The proof of Proposition 1.2 can be found, e.g., in [5, Proposition 1.7].
Proposition 1.3 (Rees decomposition). Suppose $S$ be a completely simple finite semigroup with a primitive idempotent $e$. Set $\widetilde{L}=S e, G=e S e, \widetilde{R}=e S, L=E(\widetilde{L})$ and $R=E(\widetilde{R})$. Then the following hold:
(i) $G$ is a group whose unit is $e$.
(ii) $e L=R e=\{e\}$.
(iii) $S=L G R$.
(iv) The product mapping $\psi: L \times G \times R \ni(f, g, h) \mapsto f g h \in S$ is bijective and its inverse is given as

$$
\begin{equation*}
\psi^{-1}(z)\left(=:\left(z^{L}, z^{G}, z^{R}\right)\right)=\left(z e(e z e)^{-1}, \text { eze },(e z e)^{-1} e z\right) . \tag{1.7}
\end{equation*}
$$

The proof of Proposition 1.3 can be found, e.g., in [5, Theorem 1.1]. The product decomposition $S=L G R$ will be called the Rees decomposition of $S$ at $e$, and $G$ will be called the group factor.

Note by definition that $R L \subset \widetilde{R} \widetilde{L}=e S S e \subset e S e=G$ and by the product bijectivity that $\psi^{-1}\left((f g h)\left(f^{\prime} g^{\prime} h^{\prime}\right)\right)=\left(f, g h f^{\prime} g^{\prime}, h^{\prime}\right)$. It is obvious that the product $z=f g h \in S$ is idempotent if and only if $h f=g^{-1}$. It is also obvious that all idempotents of $S$ are primitive; in fact, if $e^{\prime}=f^{\prime} g^{\prime} h^{\prime} \in E(S)$ and $z=f g h \in S$ satisfies $e^{\prime} z=z e^{\prime}=z \in$ $E(S)$, then we have $f^{\prime}=f$ and $h^{\prime}=h$ by the product bijectivity and thus we have $g^{\prime}=\left(h^{\prime} f^{\prime}\right)^{-1}=(h f)^{-1}=g$, which shows $e^{\prime}=z$ so that $e^{\prime}$ is also primitive.

Proposition 1.3 is foundamental in the theory of infinite convolutions. Let $\mathcal{P}(S)$ denote the set of probability measures on a finite semigroup $S$ and write $\mu \nu$ for the convolution of $\mu$ and $\nu$ in $\mathcal{P}(S)$ :

$$
\begin{equation*}
(\mu \nu)(A)=\sum_{f, g \in S} 1_{A}(f g) \mu\{f\} \nu\{g\}, \quad A \subset S \tag{1.8}
\end{equation*}
$$

We write $\mathcal{S}(\mu)=\{f \in S: \mu\{f\}>0\}$ for the support of $\mu$. It is easy to see that $\mathcal{S}(\mu \nu)=\mathcal{S}(\mu) \mathcal{S}(\nu)$ for $\mu, \nu \in \mathcal{P}(S)$. We write $\omega_{G}$ for the normalized Haar measure of a finite group $G$, or the uniform law on $G$.

Proposition 1.4 (Convolution idempotents). Suppose $\mu^{2}=\mu \in \mathcal{P}(S)$. Then $\mathcal{S}(\mu)$ is a completely simple subsemigroup of $S$ and $\mu$ has a factorization

$$
\begin{equation*}
\mu=\mu^{L} \omega_{G} \mu^{R} \tag{1.9}
\end{equation*}
$$

where we fix $e \in E(\mathcal{S}(\mu))$, take $L=E(\mathcal{S}(\mu) e), G=e \mathcal{S}(\mu) e$ and $R=E(e \mathcal{S}(\mu))$ so that $\mathcal{S}(\mu)=L G R$ gives the Rees decomposition of $\mathcal{S}(\mu)$ at e, and write $\mu^{L}(\cdot)=\mu\{z \in \mathcal{S}(\mu)$ : $\left.z^{L} \in \cdot\right\}$ and $\mu^{R}(\cdot)=\mu\left\{z: z^{R} \in \cdot\right\}$. Consequently, if $Z$ is a random variable whose law is $\mu$, then the projections $Z^{L}, Z^{G}$ and $Z^{R}$ are independent and $Z^{G}$ is uniform on $G$.

The proof of Proposition 1.4 can be found, e.g., in [5, Theorem 2.2].
The following proposition plays a key role in our analysis.
Proposition 1.5 (Infinite convolutions). Let $\mu \in \mathcal{P}(S)$ and suppose that $S$ coincide with $\bigcup_{n=1}^{\infty} \mathcal{S}(\mu)^{n}$, the semigroup generated by $\mathcal{S}(\mu)$. Then the following hold:
(i) The set of subsequential limits of $\left\{\mu^{n}\right\}$ is a finite cyclic group of the form

$$
\begin{equation*}
\mathcal{K}:=\left\{\eta, \mu \eta, \ldots, \mu^{p-1} \eta\right\} \tag{1.10}
\end{equation*}
$$

for some $p \in \mathbb{N}$, where $\eta$ is the unit of $\mathcal{K}$ (so that $\eta^{2}=\eta$ ) and $\mu^{p} \eta=\eta$. The support $\mathcal{S}(\eta)$ is a completely simple subsemigroup of $S$ (but not in general an ideal of $S$.)
(ii) It holds that $\frac{1}{n} \sum_{k=1}^{n} \mu^{k} \underset{n \rightarrow \infty}{\longrightarrow} \nu:=\frac{1}{p} \sum_{k=0}^{p-1} \mu^{k} \eta$ (so that $\nu^{2}=\nu$ ). The support $\mathcal{S}(\nu)$ is the kernel of $S$.
(iii) Let $e \in E(\mathcal{S}(\eta))$ be fixed. Then the Rees decompositions at $e$ of $\mathcal{S}(\nu)$ and of $\mathcal{S}(\eta)$ are given as

$$
\begin{equation*}
\mathcal{S}(\nu)=L G R \quad \text { and } \quad \mathcal{S}(\eta)=L H R, \tag{1.11}
\end{equation*}
$$

respectively, where $L=E(\mathcal{S}(\eta) e), R=E(e \mathcal{S}(\eta)), H=e \mathcal{S}(\eta) e$ and $G=e \mathcal{S}(\nu) e$. Moreover, the group factor $H$ of $\mathcal{S}(\eta)$ is a normal subgroup of the group factor $G$ of $\mathcal{S}(\nu)$, and the convolution factorizations of $\nu$ and $\eta$ are given as

$$
\begin{equation*}
\nu=\eta^{L} \omega_{G} \eta^{R} \quad \text { and } \quad \eta=\eta^{L} \omega_{H} \eta^{R}, \tag{1.12}
\end{equation*}
$$

respectively, where $\eta^{L}(\cdot)=\eta\left\{z: z^{L} \in \cdot\right\}$ and $\eta^{R}(\cdot)=\eta\left\{z: z^{R} \in \cdot\right\}$.
(iv) There exists $\gamma \in G$ such that $G / H=\left\{H, \gamma H, \ldots, \gamma^{p-1} H\right\}, \gamma^{p}=e$ and

$$
\begin{equation*}
\mu^{k} \eta=\eta^{L} \gamma^{k} \omega_{H} \eta^{R}, \quad k=0,1, \ldots, p-1, \tag{1.13}
\end{equation*}
$$

where we identify an element of $S$ with the Dirac mass at it. (We write $C=$ $\left\{e, \gamma, \ldots, \gamma^{p-1}\right\}$ so that $C H=\bigcup G / H=G$.)

The proof of Proposition 1.5 can be found, e.g., in [5, Theorem 2.7].

### 1.3 Main result

Let $V$ be a non-empty finite set and let $\Sigma$ denote the set of mappings of $V$ into itself. Note that $\Sigma$ is also a finite set. For $\mu \in \mathcal{P}(\Sigma)$ and $\Lambda \in \mathcal{P}\left(V^{m}\right)$, we define $\mu \Lambda \in \mathcal{P}\left(V^{m}\right)$ as

$$
\begin{equation*}
(\mu \Lambda)(A)=\sum_{f \in \Sigma} \sum_{\boldsymbol{x} \in V^{m}} 1_{A}(f \boldsymbol{x}) \mu\{f\} \Lambda\{\boldsymbol{x}\}, \quad A \subset V^{m}, \tag{1.14}
\end{equation*}
$$

where we understand $f \boldsymbol{x}=\left(f x^{1}, \ldots, f x^{m}\right)$ for $\boldsymbol{x}=\left(x^{1}, \ldots, x^{m}\right) \in V^{m}$. Denote

$$
\begin{equation*}
V_{\times}^{m}=\left\{\boldsymbol{x}=\left(x^{1}, \ldots, x^{m}\right) \in V^{m}: x^{1}, \ldots, x^{m} \text { are distinct }\right\} . \tag{1.15}
\end{equation*}
$$

Proposition 1.6. Let $\mu \in \mathcal{P}(\Sigma)$ and set $S=\bigcup_{n=1}^{\infty} \mathcal{S}(\mu)^{n}$. We apply Proposition 1.5 and adopt its notation. Denote

$$
\begin{equation*}
m_{\mu}=\min \{\#(g V): g \in S\}, \tag{1.16}
\end{equation*}
$$

where $\#(A)$ denotes the number of elements of a set $A$. Define

$$
\begin{equation*}
W_{\mu}=\left\{\boldsymbol{x} \in V_{\times}^{m_{\mu}}: f \boldsymbol{x} \in V_{\times}^{m_{\mu}} \text { for all } f \in S\right\} \tag{1.17}
\end{equation*}
$$

Then there exists a subset $W$ of $\mathrm{e} W_{\mu}$ such that the following hold:
(i) $W_{\mu}=L G W$. Consequently, $e W_{\mu}=G W$.
(ii) The product mapping $L \times G \times W \ni(f, g, \boldsymbol{w}) \mapsto f g \boldsymbol{w} \in W_{\mu}$ is bijective. Its inverse will be denoted by $\boldsymbol{x} \mapsto\left(\boldsymbol{x}^{L}, \boldsymbol{x}^{G}, \boldsymbol{x}^{W}\right)$.
(iii) $\Lambda \in \mathcal{P}\left(V_{\times}^{m_{\mu}}\right)$ is $\mu$-invariant, i.e., $\Lambda=\mu \Lambda$, if and only if $\Lambda=\eta^{L} \omega_{G} \Lambda_{W}$ for some $\Lambda_{W} \in \mathcal{P}(W)$.

The proof of Proposition 1.6 will be given in Section 3 ,
If an $m$-particle $\mu$-evolution $(\mathbb{X}, N)$ is stationary, i.e., $\left(\mathbb{X}{ }_{+1}, N_{+1}\right) \stackrel{\mathrm{d}}{=}(\mathbb{X}, N)$, then the sequence $\mathbb{X}$ has a common law which is $\mu$-invariant. Conversely, if $\Lambda \in \mathcal{P}\left(V^{m}\right)$ is $\mu$-invariant, then there exists a stationary $m$-particle $\mu$-evolution ( $\mathbb{X}, N$ ) such that the sequence $\mathbb{X}$ has $\Lambda$ as its common law. We now state our main theorem, which will be proved in Section 4.
Theorem 1.7. Suppose the same assumptions of Proposition 1.6 be satisfied. Suppose that $\Lambda \in \mathcal{P}\left(V_{\times}^{m_{\mu}}\right)$ be $\mu$-invariant and let $(\mathbb{X}, N)$ be a stationary $m_{\mu}$-particle $\mu$-evolution such that the sequence $\mathbb{X}$ has $\Lambda$ as its common law. Then the following hold:
(i) $\mathbb{X}_{k} \in L G W$ a.s. and $\mathbb{X}_{k}^{L} \stackrel{\mathrm{~d}}{=} \eta^{L}$ for all $k \in \mathbb{Z}$.
(ii) $\mathbb{X}_{k}^{G}=\left(\gamma^{k} Y_{C}\right) U_{k}^{H}$ a.s. for $k \in \mathbb{Z}$ for some $C$-valued random variable $Y_{C}$ and some $H$-valued random variables $U_{k}^{H}$ such that $U_{k}^{H}$ is uniform on $H$.
(iii) $\mathbb{X}_{k}^{W}=\mathbb{Z}_{W}$ a.s. for $k \in \mathbb{Z}$ for some $W$-valued random variable $\mathbb{Z}_{W}$.
(iv) If we write $M_{j}^{G}:=\mathbb{X}_{j}^{G}\left(\mathbb{X}_{j-1}^{G}\right)^{-1}$ for $j \in \mathbb{Z}$ and $M_{k, j}^{G}:=\mathbb{X}_{k}^{G}\left(\mathbb{X}_{j}^{G}\right)^{-1}=M_{k}^{G} M_{k-1}^{G} \cdots M_{j+1}^{G}$ for $j \leq k$, we have the following factorization:

$$
\begin{equation*}
\mathbb{X}_{j}=\mathbb{X}_{j}^{L}\left(M_{k, j}^{G}\right)^{-1}\left(\gamma^{k} Y_{C}\right) U_{k}^{H} \mathbb{Z}_{W} \quad \text { a.s. for } j \leq k \tag{1.18}
\end{equation*}
$$

(v) A resolution of the observation holds in the sense that

$$
\begin{equation*}
\mathcal{F}_{k}^{\mathbb{X}}=\mathcal{G}_{k}^{N} \vee \mathcal{F}_{-\infty}^{\mathbb{X}} \vee \sigma\left(U_{k}^{H}\right) \quad \text { a.s. } \tag{1.19}
\end{equation*}
$$

where

$$
\begin{gather*}
\mathcal{G}_{k}^{N}=\sigma\left(\mathbb{X}_{j}^{L}, M_{j}^{G}: j \leq k\right) \subset \mathcal{F}_{k}^{N} \quad \text { a.s. }  \tag{1.20}\\
\text { the three } \sigma \text {-fields } \mathcal{F}_{k}^{N}\left(\supset \mathcal{G}_{k}^{N}\right), \mathcal{F}_{-\infty}^{\mathbb{X}} \text { and } \sigma\left(U_{k}^{H}\right) \text { are independent } \tag{1.21}
\end{gather*}
$$

and

$$
\begin{equation*}
\mathcal{F}_{-\infty}^{\mathbb{X}}=\sigma\left(Y_{C}, \mathbb{Z}_{W}\right) \quad \text { a.s. } \tag{1.22}
\end{equation*}
$$

(vi) $Y_{C} \stackrel{\mathrm{~d}}{=} \omega_{C}$ and $\mathbb{Z}_{W} \stackrel{\mathrm{~d}}{=} \Lambda_{W}$, where $\omega_{C}$ denotes the Haar probability on the cyclic group $C$. It holds that $Y_{C}$ and $\mathbb{Z}_{W}$ are independent.

We shall show in Section 5 that the non-stationary case can be reduced to the stationary case and satisfies Properties (i)-(v) of Theorem 1.7.

### 1.4 Historical remarks

Inspired by Tsirelson [2] of a stochastic differential equation, Yor [13] has made a thorough study of the action evolution $X_{k}=N_{k} X_{k-1}$ when both $X$ and $N$ take values in the torus $\mathbb{T}=\{z \in \mathbb{C}:|z|=1\}$ and $N$ is not necessarily idd, where we understand $N_{k} X_{k-1}$ as the usual product between two complex values. He obtained a general result of the resolution of the observation. Hirayama and Yano [4] generalized Yor's results for the state space being a compact group. In these results the third noise is generated by a random variable with uniform law on a subgroup of the state space group. See also [12 for a survey of this topic.

Yano [11] studied mono-particle action evolution on a finite set. He proved existence of a non-trivial third noise when $m_{\mu} \geq 2$. He utilized several notion from the road coloring theory; for the details see Trahtman [9] and the references therein.

The theories of Rees decomposition, convolution idempotents and infinite convolutions for finite semigroups are very old results and have nowadays been generalized to topological semigroups; see the textbook [5, Chapters 1 and 2] for the details. In particular, Proposition 1.5, which plays a fundamental tool for our results, dates back to Rosenblatt [7], Collins [3] and Schwarz [8].

### 1.5 Organization

The organization of this paper is as follows. In Section2 we discuss an example. In Section 3 we prove Proposition 1.6 and discuss characterization of stationary probabilities. Section 44 is devoted to the proof of our main theorem, Theorem 1.7. In Section 5 we discuss the non-stationary case.

## 2 Example

Let us investigate an example which was discussed in [11, Subsection 3.3] for mono-particle $\mu$-evolution. We look at it from the viewpoint of multiparticle $\mu$-evolution. See [6] for other examples.

Let $V=\{1,2,3,4,5\}$. We write $f=\left[y^{1}, y^{2}, y^{3}, y^{4}, y^{5}\right]$ if $f: V \rightarrow V$ is such that $f 1=y^{1}, \ldots, f 5=y^{5}$. Consider the two mappings

$$
\begin{equation*}
f=[2,3,4,1,5] \quad \text { and } \quad g=[2,5,5,2,4] . \tag{2.1}
\end{equation*}
$$

Let $\mu=\left(\delta_{f}+\delta_{g}\right) / 2$ be the uniform law on $\{f, g\}$, where $\delta_{f}$ stands for the Dirac mass at
$f$. The marginal transition probability $P_{\mu}$ of (1.6) is given as

$$
\left(\begin{array}{cccc}
P_{\mu}(1,\{1\}) & P_{\mu}(1,\{2\}) & \cdots & P_{\mu}(1,\{5\})  \tag{2.2}\\
P_{\mu}(2,\{1\}) & P_{\mu}(2,\{2\}) & \cdots & P_{\mu}(2,\{5\}) \\
\vdots & \vdots & \ddots & \vdots \\
P_{\mu}(5,\{1\}) & P_{\mu}(5,\{2\}) & \cdots & P_{\mu}(5,\{5\})
\end{array}\right)=\frac{1}{2}\left(\begin{array}{ccccc}
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1
\end{array}\right) .
$$

It is obvious that $\mu \lambda=\lambda$ if and only if $\lambda P_{\mu}=\lambda$, and it is easy to see that there exists a unique $\mu$-invariant probability measure given as

$$
\begin{equation*}
\lambda=\frac{1}{9} \delta_{1}+\frac{2}{9} \delta_{2}+\frac{1}{9} \delta_{3}+\frac{2}{9} \delta_{4}+\frac{3}{9} \delta_{5} . \tag{2.3}
\end{equation*}
$$

In [11, Theorem 1], for a stationary mono-particle $\mu$-evolution ( $X, N$ ) with $X$ having $\lambda$ as its common law, it was proved that there exists a third noise $\left(U_{k}\right)_{k \in \mathbb{Z}}$ such that $\sigma\left(U_{k}\right) \subset \mathcal{F}_{k}^{X, N}$ a.s. for $k \in \mathbb{Z}$ and

$$
\begin{equation*}
\mathcal{F}_{k}^{X} \subset \mathcal{F}_{k}^{N} \vee \sigma\left(U_{k}\right) \quad \text { a.s. for } k \in \mathbb{Z} \tag{2.4}
\end{equation*}
$$

with $\mathcal{F}_{-\infty}^{X}$ being trivial a.s. and $\sigma\left(U_{k}\right)$ being independent of $\mathcal{F}_{k}^{N}$.
Set $S=\bigcup_{n=1}^{\infty}\{f, g\}^{n}$ and apply Propositions (1.5 and 1.6. Let us prove that

$$
\begin{equation*}
L=\{e, f e\}, \quad G=\left\{e, g, g^{2}, h, g h, g^{2} h\right\}, \quad R=\{e, e f\} \tag{2.5}
\end{equation*}
$$

where $e:=g^{3}=[4,2,2,4,5] \in E(S)$ and $h:=f^{2} e=e f^{2}=e f^{2} e=[2,4,4,2,5]$. We temporarily set $L^{\prime}=\{e, f e\}, G^{\prime}=\left\{e, g, g^{2}, h, g h, g^{2} h\right\}, R^{\prime}=\{e, e f\}$ and $K=L^{\prime} G^{\prime} R^{\prime}$. Since $g^{3}=h^{2}=e, h g=g^{2} h$ and $h g^{2}=g h$, we see that $G^{\prime}$ is a group. Since efe $=e$, $f^{2} e=e f^{2}=h$ and $g f e=e f g=g$, we see that $S K \cup K S \subset K$ and hence that $K$ is an ideal of $S$. For any $k \in K$, we have eke $\in G^{\prime}$ since $e f e=e$ and then we see that $S k S \supset G^{\prime} \ni e$, so that $K$ is the kernel of $S$, which shows $K=\mathcal{S}(\nu)$. We now see that $G=e K e=G^{\prime}, L=E(K e)=E\left(L^{\prime} G^{\prime}\right)=L^{\prime}$ and $R=E(e K)=E\left(G^{\prime} R^{\prime}\right)=R^{\prime}$.

Let $H=e \mathcal{S}(\eta) e$ be the subgroup of $G$ in Proposition 1.5. Then we have $\mu \eta^{L} \omega_{H} \eta^{R}=$ $\eta^{L} \gamma \omega_{H} \eta^{R}$ so that $\mu \eta^{L} \omega_{H}=\eta^{L} \gamma \omega_{H}$, since $\eta^{R} e=\delta_{e}$. Let $\eta^{L}=p \delta_{e}+q \delta_{f e}$ for some $p, q>0$ with $p+q=1$. Since $g f e=g e f e=g e=g$, we have

$$
\begin{equation*}
\mu \eta^{L}=\left(\frac{1}{2} \delta_{f}+\frac{1}{2} \delta_{g}\right)\left(p \delta_{e}+q \delta_{f e}\right)=\frac{p}{2} \delta_{f e}+\frac{1}{2} \delta_{g}+\frac{q}{2} \delta_{h} . \tag{2.6}
\end{equation*}
$$

Since $f e \in \mathcal{S}\left(\mu \eta^{L} \omega_{H}\right)=\mathcal{S}\left(\eta^{L} \gamma \omega_{H}\right)=L \gamma H$, we see that $H=G$. We now have

$$
\begin{equation*}
\left(p \delta_{e}+q \delta_{f e}\right) \omega_{G}=\eta^{L} \omega_{G}=\mu \eta^{L} \omega_{G}=\left(\frac{p}{2} \delta_{f e}+\frac{1+q}{2} \delta_{e}\right) \omega_{G} \tag{2.7}
\end{equation*}
$$

which yields $p=2 / 3$ and $q=1 / 3$, that is,

$$
\begin{equation*}
\eta^{L}=\frac{2}{3} \delta_{e}+\frac{1}{3} \delta_{f e} . \tag{2.8}
\end{equation*}
$$

In the same way we have $\eta^{R}=\frac{2}{3} \delta_{e}+\frac{1}{3} \delta_{e f}$, and thus we have obtained that

$$
\begin{equation*}
\mu^{n} \rightarrow \eta=\nu=\eta^{L} \omega_{G} \eta^{R} . \tag{2.9}
\end{equation*}
$$

Note that $f e=[1,3,3,1,5]$ and $e f=[2,2,4,4,5]$. For $(a, b, c) \in L \times G \times R$, we have

$$
\left\{\begin{array}{l}
a=e \quad \Longleftrightarrow a V=\{2,4,5\}  \tag{2.10}\\
a=f e \Longleftrightarrow a V=\{1,3,5\}
\end{array}, \quad\left\{\begin{array}{l}
c=e \quad \Longleftrightarrow c 1=c 4, c 2=c 3 \\
c=e f \quad \Longleftrightarrow c 1=c 2, c 3=c 4
\end{array}\right.\right.
$$

We note that elements of $G$ act as permutations over $\{2,4,5\}$ :

$$
\begin{equation*}
e(2,4,5)=(2,4,5), \quad g(2,4,5)=(5,2,4), \quad h(2,4,5)=(4,2,5) . \tag{2.11}
\end{equation*}
$$

It is easy to see that $m_{\mu}=3$ and

$$
\begin{equation*}
W_{\mu}=\{(x, y, z): \text { a permutation of }(2,4,5) \text { or }(1,3,5)\} . \tag{2.12}
\end{equation*}
$$

We may take a set $W$ of Proposition 1.6 as

$$
\begin{equation*}
W=\{(2,4,5)\} \tag{2.13}
\end{equation*}
$$

For example, for $\boldsymbol{x}=(3,5,1) \in W_{\mu}$, we see that $\boldsymbol{x}^{L}=f e, \boldsymbol{x}^{G}=g h$ and $\boldsymbol{x}^{W}=(2,4,5)$.
By (iii) of Proposition 1.6, we see that $\Lambda=\eta^{L} \omega_{G}(2,4,5)$ is the unique $\mu$-invariant probability measure on $V_{\times}^{3}$. Let $(\mathbb{X}, N)$ be a stationary tri-particle $\mu$-evolution such that $\mathbb{X}$ has $\Lambda$ as its common law. Then we have the factorization

$$
\begin{equation*}
\mathbb{X}_{j}=\mathbb{X}_{j}^{L}\left(M_{k, j}^{G}\right)^{-1} U_{k}^{G}(2,4,5) \quad \text { a.s. for } j \leq k \tag{2.14}
\end{equation*}
$$

with $U_{k}^{G}=\mathbb{X}_{k}^{G}, M_{j}^{G}=\mathbb{X}_{j}^{G}\left(\mathbb{X}_{j-1}^{G}\right)^{-1}$ and $M_{k, j}^{G}=M_{k}^{G} M_{k-1}^{G} \cdots M_{j+1}^{G}$, and consequently, we obtain the resolution

$$
\begin{equation*}
\mathcal{F}_{k}^{\mathbb{X}}=\mathcal{G}_{k}^{N} \vee \sigma\left(U_{k}^{G}\right) \quad \text { a.s. } \quad \text { with } \mathcal{G}_{k}^{N}=\sigma\left(\mathbb{X}_{j}^{L}, M_{j}^{G}: j \leq k\right) \tag{2.15}
\end{equation*}
$$

where the two $\sigma$-fields $\mathcal{F}_{k}^{N}\left(\supset \mathcal{G}_{k}^{N}\right)$ and $\sigma\left(U_{k}^{G}\right)$ are independent.
Note that the first component $\left(X^{1}, N\right)$ is a mono-particle $\mu$-evolution such that $X^{1}$ has a common law

$$
\begin{equation*}
\eta^{L} \omega_{G} 2=\left(\frac{2}{3} \delta_{e}+\frac{1}{3} \delta_{f e}\right) \omega_{\{2,4,5\}}=\frac{2}{3} \omega_{\{2,4,5\}}+\frac{1}{3} \omega_{\{3,1,5\}}=\lambda, \tag{2.16}
\end{equation*}
$$

where $\omega_{A}$ stands for the uniform law on a finite set $A$. We note that $X_{k}^{1}=\mathbb{X}_{k}^{L} U_{k}^{G} 2$ and we easily see that

$$
\begin{align*}
& \left\{X_{k}^{1}=1\right\}=\left\{\mathbb{X}_{k}^{L}=f e\right\} \cap\left\{U_{k}^{G} 2=4\right\}  \tag{2.17}\\
& \left\{X_{k}^{1}=2\right\}=\left\{\mathbb{X}_{k}^{L}=e\right\} \cap\left\{U_{k}^{G} 2=2\right\}  \tag{2.18}\\
& \left\{X_{k}^{1}=3\right\}=\left\{\mathbb{X}_{k}^{L}=f e\right\} \cap\left\{U_{k}^{G} 2=2\right\}  \tag{2.19}\\
& \left\{X_{k}^{1}=4\right\}=\left\{\mathbb{X}_{k}^{L}=e\right\} \cap\left\{U_{k}^{G} 2=4\right\}  \tag{2.20}\\
& \left\{X_{k}^{1}=5\right\}=\left\{U_{k}^{G} 2=5\right\} . \tag{2.21}
\end{align*}
$$

This shows that $\sigma\left(U_{k}^{G} 2\right) \subset \sigma\left(X_{k}^{1}\right)$ a.s. We thus conclude that $\left(U_{k}^{G} 2\right)_{k \in \mathbb{Z}}$ is a third noise for $\left(X^{1}, N\right)$ since

$$
\begin{equation*}
\mathcal{F}_{k}^{X^{1}} \subset \mathcal{G}_{k}^{N} \vee \sigma\left(U_{k}^{G} 2\right) \quad \text { a.s. for } k \in \mathbb{Z}, \tag{2.22}
\end{equation*}
$$

where $U_{k}^{G} 2$ is independent of $\mathcal{F}_{k}^{N}\left(\supset \mathcal{G}_{k}^{N}\right)$.

## 3 F-cliques and stationary probabilities

Throughout this section we suppose all the assumptions of Proposition 1.6 be satisfied.
We borrow several notation from the road coloring theory. A pair $\{x, y\}$ from $V$ will be called a deadlock if $g x \neq g y$ for all $g \in S:=\bigcup_{n=1}^{\infty} \mathcal{S}(\mu)^{n}$, or in other words, $f_{n} f_{n-1} \cdots f_{1} x \neq f_{n} f_{n-1} \cdots f_{1} y$ for all $n \in \mathbb{N}$ and $f_{1}, \ldots, f_{n} \in \mathcal{S}(\mu)$. A subset $F$ of $V$ will be called an $F$-clique if every pair from $F$ is a deadlock and if every set $F \cup\{x\}$ with $x \notin F$ contains a pair which is not a deadlock. In other words, an F-clique $F$ is a maximal subset of $V$ every pair from which is a deadlock.

The F-cliques can be characterized as follows.
Lemma 3.1. For $g \in S$, the set $g V$ is an $F$-clique if and only if $\#(g V)=m_{\mu}$. In addition, it holds that

$$
\begin{equation*}
\mathcal{S}(\nu)=\{g \in S: g V \text { is an F-clique }\}=\left\{g \in S: \#(g V)=m_{\mu}\right\} . \tag{3.1}
\end{equation*}
$$

Proof. If $\#(g V)=m_{\mu}$, for any $f \in S$ we have $m_{\mu} \leq \#(f g V) \leq \#(g V)=m_{\mu}$ so that $\#(f g V)=m_{\mu}$, which implies that $g V$ is an F-clique. Conversely, if $g V$ is an F-clique, then $\#(f V) \geq \#(f g V)=\#(g V) \geq m_{\mu}$ for any $f \in S$ so that $\#(g V)=m_{\mu}$.

To prove (3.1), it suffices to show that $K:=\left\{g \in S: \#(g V)=m_{\mu}\right\}$ is a minimal ideal of $S$, because $\mathcal{S}(\nu)$ is the unique minimal ideal of $S$. It is obvious by definition that $K$ is an ideal. Suppose $\emptyset \neq I S \cup S I \subset I \subset K$. Let $f \in I$ and $g \in K$. Since $\left.g f\right|_{g V}: g V \rightarrow g V$ is bijective, the mapping $\left(\left.g f\right|_{g V}\right)^{p}$ is identity for some $p \in \mathbb{N}$ so that $(g f)^{p} g=g$. Hence $g=(g f)^{p-1} g f g \in S I S \subset I$, which shows $I=K$.

By Lemma 3.1, the set $W_{\mu}$ defined in (1.17) can be represented as

$$
\begin{equation*}
W_{\mu}=\left\{\boldsymbol{x}=\left(x^{1}, \ldots, x^{m_{\mu}}\right):\left\{x^{1}, \ldots, x^{m_{\mu}}\right\} \text { is an F-clique }\right\} . \tag{3.2}
\end{equation*}
$$

Lemma 3.2. For any $\boldsymbol{x}, \boldsymbol{x}^{\prime} \in e W_{\mu}$, the two measures $\eta^{L} \omega_{G} \boldsymbol{x}$ and $\eta^{L} \omega_{G} \boldsymbol{x}^{\prime}$ either coincide or have disjoint supports.

Proof. Suppose $\mathcal{S}\left(\eta^{L} \omega_{G} \boldsymbol{x}\right)(=L G \boldsymbol{x})$ and $\mathcal{S}\left(\eta^{L} \omega_{G} \boldsymbol{x}^{\prime}\right)\left(=L G \boldsymbol{x}^{\prime}\right)$ have a common element $f g \boldsymbol{x}=f^{\prime} g^{\prime} \boldsymbol{x}^{\prime}$ for some $f, f^{\prime} \in L$ and $g, g^{\prime} \in G$. Since $e f=e f^{\prime}=e$, we have $g \boldsymbol{x}=g^{\prime} \boldsymbol{x}^{\prime}$. We thus obtain that

$$
\begin{equation*}
\eta^{L} \omega_{G} \boldsymbol{x}=\eta^{L} \omega_{G} g \boldsymbol{x}=\eta^{L} \omega_{G} g^{\prime} \boldsymbol{x}^{\prime}=\eta^{L} \omega_{G} \boldsymbol{x}^{\prime} . \tag{3.3}
\end{equation*}
$$

The proof is complete.

We now prove Proposition 1.6.

Proof of Proposition [1.6. (i) By Lemma [3.2, we can find a subset $W$ of $e W_{\mu}$ such that the family $\left\{\eta^{L} \omega_{G} \boldsymbol{w}: \boldsymbol{w} \in W\right\}$ consists of measures with distinct supports and exhausts $\left\{\eta^{L} \omega_{G} \boldsymbol{x}: \boldsymbol{x} \in e W_{\mu}\right\}$. By (3.1) and (3.2), we have $W_{\mu}=\mathcal{S}(\nu) W_{\mu}=L G R W_{\mu}$. It is easy to see that $R W_{\mu}=G W_{\mu}=e W_{\mu}$ and that $L W_{\mu}=W_{\mu}$. Hence we obtain

$$
\begin{equation*}
W_{\mu}=L G e W_{\mu}=\mathcal{S}\left(\eta^{L} \omega_{G}\right) e W_{\mu}=\bigcup_{\boldsymbol{x} \in e W_{\mu}} \mathcal{S}\left(\eta^{L} \omega_{G} \boldsymbol{x}\right)=\bigcup_{\boldsymbol{w} \in W} \mathcal{S}\left(\eta^{L} \omega_{G} \boldsymbol{w}\right)=L G W \tag{3.4}
\end{equation*}
$$

(ii) We have only to prove injectivity of the product $L \times G \times W \ni(f, g, \boldsymbol{w}) \mapsto f g \boldsymbol{w} \in$ $W_{\mu}$. Suppose $f g \boldsymbol{w}=f^{\prime} g^{\prime} \boldsymbol{w}^{\prime}$. Since $e L=\{e\} \subset G$, we have

$$
\begin{equation*}
\eta^{L} \omega_{G} f g \boldsymbol{w}=\eta^{L} \omega_{G}(e f) g \boldsymbol{w}=\eta^{L} \omega_{G} \boldsymbol{w} . \tag{3.5}
\end{equation*}
$$

We thus obtain $\eta^{L} \omega_{G} \boldsymbol{w}=\eta^{L} \omega_{G} f g \boldsymbol{w}=\eta^{L} \omega_{G} f^{\prime} g^{\prime} \boldsymbol{w}^{\prime}=\eta^{L} \omega_{G} \boldsymbol{w}^{\prime}$, which implies $\boldsymbol{w}=\boldsymbol{w}^{\prime}$ by definition of $W$.

If we write $\boldsymbol{w}=\left(w^{1}, \ldots, w^{m_{\mu}}\right)$, then $\left\{w^{1}, \ldots, w^{m_{\mu}}\right\}=e V$ because $\#(e V)=m_{\mu}$ by (3.1). Hence the identity $f g \boldsymbol{w}=f^{\prime} g^{\prime} \boldsymbol{w}$ implies that $f g=f^{\prime} g^{\prime}$ on eV . Since $g=g e$ and $g^{\prime}=g^{\prime} e$, we see that $f g=f^{\prime} g^{\prime}$ on $V$, which implies $f=f^{\prime}$ and $g=g^{\prime}$.
(iii) Since $\eta^{R} e \Lambda_{W}=\Lambda_{W}$, we see that $\mu\left(\eta^{L} \omega_{G} \Lambda_{W}\right)=\mu \nu e \Lambda_{W}=\nu e \Lambda_{W}=\eta^{L} \omega_{G} \Lambda_{W}$.

Suppose $\Lambda \in \mathcal{P}\left(V_{\times}^{m_{\mu}}\right)$ be $\mu$-invariant. Since $\Lambda=\mu \Lambda$, we have $\Lambda=\eta \Lambda$ and hence $\Lambda=\nu \Lambda=\eta^{L} \omega_{G} \eta^{R} \Lambda$. By (3.2), we have $\mathcal{S}(\Lambda)=\mathcal{S}(\nu \Lambda) \subset W_{\mu}$. We then have $\mathcal{S}\left(\eta^{R} \Lambda\right)=$ $\mathcal{S}\left(\eta^{R} \nu \Lambda\right)=\mathcal{S}\left(\omega_{G} \eta^{R} \Lambda\right) \subset G R W_{\mu}=e W_{\mu}=G W$. Hence

$$
\begin{align*}
\Lambda=\left(\eta^{L} \omega_{G}\right)\left(\eta^{R} \Lambda\right) & =\sum_{\boldsymbol{x} \in G W}\left(\eta^{L} \omega_{G} \boldsymbol{x}\right)\left(\eta^{R} \Lambda\right)\{\boldsymbol{x}\}  \tag{3.6}\\
& =\sum_{\boldsymbol{x} \in G W}\left(\eta^{L} \omega_{G} \boldsymbol{x}^{W}\right)\left(\eta^{R} \Lambda\right)\{\boldsymbol{x}\}=\eta^{L} \omega_{G} \Lambda_{W}, \tag{3.7}
\end{align*}
$$

where we take $\Lambda_{W}:=\left(\eta^{R} \Lambda\right)\left\{\boldsymbol{x}^{W}: \boldsymbol{x} \in G W\right\}$. The proof is now complete.

## 4 Proof of our main theorem

Throughout this section we suppose all the assumptions of Theorem 1.7 be satisfied. We divide the proof of Theorem 1.7 into several steps.

### 4.1 Factorizing $\mathbb{X}_{k}$ into $L G$ - and $W$-factors

By Proposition 1.6, we have $\mathbb{X}_{k} \in L G W$ a.s. and $\mathbb{X}_{k}^{L} \stackrel{\text { d }}{=} \eta^{L}$ for all $k \in \mathbb{Z}$, and so we have shown Claim (i) of Theorem 1.7. Let us focus on the factorization $\mathbb{X}_{k}=\left(\mathbb{X}_{k}^{L} \mathbb{X}_{k}^{G}\right) \mathbb{X}_{k}^{W}$ for $k \in \mathbb{Z}$.

Proposition 4.1. Set $Y_{k}=\mathbb{X}_{k}^{L} \mathbb{X}_{k}^{G}$ for $k \in \mathbb{Z}$ and $Y=\left(Y_{k}\right)_{k \in \mathbb{Z}}$. Then the following hold:
(i) $(Y, N)$ is a $\mu$-evolution such that the sequence $Y$ has a common law $\eta^{L} \omega_{G}$.
(ii) There exists a $W$-valued random variable $\mathbb{Z}_{W}$ such that $\mathbb{X}_{k}^{W}=\mathbb{Z}_{W}$ a.s. for $k \in \mathbb{Z}$.
(iii) $(Y, N)$ and $\mathbb{Z}_{W}$ are independent.

Proof. Note that

$$
\begin{equation*}
Y_{k} \mathbb{X}_{k}^{W}=\mathbb{X}_{k}=N_{k} \mathbb{X}_{k-1}=\left(N_{k} Y_{k-1}\right) \mathbb{X}_{k-1}^{W} \quad \text { a.s. } \tag{4.1}
\end{equation*}
$$

Since $S L G=S \mathcal{S}(\nu) e \subset \mathcal{S}(\nu) e=L G$ and by Proposition 1.6, we have

$$
\begin{equation*}
Y_{k}=N_{k} Y_{k-1} \quad \text { and } \quad \mathbb{X}_{k}^{W}=\mathbb{X}_{k-1}^{W} \quad \text { a.s. } \tag{4.2}
\end{equation*}
$$

We now obtain Claim (ii) of Proposition 4.1 (and consequently we have shown Claim (iii) of Theorem (1.7).

Since $N_{k}$ is independent of $\mathcal{F}_{k-1}^{Y}\left(\subset \mathcal{F}_{k-1}^{\mathbb{X}}\right)$, we see that $(Y, N)$ is a $\mu$-evolution. Since $\mathbb{X}_{k} \stackrel{\mathrm{~d}}{=} \Lambda=\eta^{L} \omega_{G} \Lambda_{W}$ and by Proposition 1.6, we see that, for each fixed $k \in \mathbb{Z}$, the three random variables $\mathbb{X}_{k}^{L}, \mathbb{X}_{k}^{G}$ and $\mathbb{X}_{k}^{W}$ are independent and have law $\eta^{L}, \omega_{G}$ and $\Lambda_{W}$, respectively. We now obtain Claim (i).

Let $k \in \mathbb{Z}$ be fixed. By the above argument, we see that $Y_{k}=\mathbb{X}_{k}^{L} \mathbb{X}_{k}^{G}$ is independent of $\mathbb{Z}_{W}$. Since $\left\{N_{j}: j>k\right\}$ is independent of $\left\{Y_{k}, \mathbb{Z}_{W}\right\}$ and since $Y_{j}=N_{j} N_{j-1} \cdots N_{k+1} Y_{k}$ for $j>k$, we see that $\left\{\left(Y_{j}, N_{j}\right): j>k\right\}$ is independent of $\mathbb{Z}_{W}$. Since $k \in \mathbb{Z}$ is arbitrary, we obtain Claim (iii). The proof is complete.

### 4.2 Factorizing $\mathbb{X}_{k}^{G}$ into $C$ - and $H$-factors

By definition of $C$ and $H$ in Proposition 1.5, we see that the product mapping $C \times$ $H \ni\left(\gamma^{j}, h\right) \mapsto \gamma^{j} h \in G$ is bijective. Its inverse will be denoted by $g \mapsto\left(g^{C}, g^{H}\right)$. For $f \in \mathcal{S}(\nu)=L G R$, we write $f^{C}=\left(f^{G}\right)^{C}$ and $f^{H}=\left(f^{G}\right)^{H}$. For $\boldsymbol{x} \in L G W$, we write $\boldsymbol{x}^{C}=\left(\boldsymbol{x}^{G}\right)^{C}$ and $\boldsymbol{x}^{H}=\left(\boldsymbol{x}^{G}\right)^{H}$.

Since $H$ is a normal subgroup of $G$ and since $C$ is a cyclic group, we have

$$
\begin{equation*}
\left(g_{1} g_{2}\right)^{C} H=\left(g_{1} g_{2}\right) H=\left(g_{1} H\right)\left(g_{2} H\right)=\left(g_{1}^{C} H\right)\left(g_{2}^{C} H\right)=\left(g_{1}^{C} g_{2}^{C}\right) H \tag{4.3}
\end{equation*}
$$

so that $\left(g_{1} g_{2}\right)^{C}=g_{1}^{C} g_{2}^{C}$.
We proceed to prove part of Theorem 1.7.
Proposition 4.2. Claim (1.20) of Theorem 1.7 holds and it holds that

$$
\begin{equation*}
\mathbb{X}_{k}^{C}=\gamma^{k} Y_{C} \text { a.s. for } k \in \mathbb{Z} \text { for some } C \text {-valued random variable } Y_{C} . \tag{4.4}
\end{equation*}
$$

Proof. Set $N_{k, k}=e$ for $k \in \mathbb{Z}$ and set

$$
\begin{equation*}
N_{k, l}:=N_{k} N_{k-1} \cdots N_{l+1}, \quad k>l . \tag{4.5}
\end{equation*}
$$

Since $e \in S=\bigcup_{n=1}^{\infty} \mathcal{S}(\mu)^{n}$, we can find $f_{1}, f_{2}, \ldots, f_{n} \in \mathcal{S}(\mu)$ such that $f_{n} f_{n-1} \cdots f_{1}=e$, and hence we have

$$
\begin{equation*}
T_{k}^{e}:=\sup \left\{l<k-n: N_{l+n, l}=e\right\}>-\infty \quad \text { a.s. } \tag{4.6}
\end{equation*}
$$

Since $S e \subset S L G \subset L G=L G e \subset S e$, we see that $N_{k, T_{k}^{e}}=N_{k, T_{k}^{e}+n} N_{T_{k}^{e}+n, T_{k}^{e}}=N_{k, T_{k}^{e}+n} e \in$ $S e=L G$.

Let us prove Claim (1.20). Since $\mathbb{X}_{k}=N_{k, T_{k}^{e}} \mathbb{X}_{T_{k}^{e}}$, we have

$$
\begin{equation*}
\mathbb{X}_{k}^{L}=\left(N_{k, T_{k}^{e}}\right)^{L}, \quad \mathbb{X}_{k}^{G}=\left(N_{k, T_{k}^{e}}\right)^{G} \mathbb{X}_{T_{k}^{e}}^{G} \quad \text { a.s. } \tag{4.7}
\end{equation*}
$$

Hence we obtain $\mathbb{X}_{k}^{L} \in \mathcal{F}_{k}^{N}$ a.s. Since $\mathbb{X}_{k}=N_{k} \mathbb{X}_{k-1}^{L} \mathbb{X}_{k-1}^{G} \mathbb{X}_{k-1}^{W}$ and $S L=S L e \subset S e=L G$, we have

$$
\begin{equation*}
\mathbb{X}_{k}^{L}=\left(N_{k} \mathbb{X}_{k-1}^{L}\right)^{L}, \quad \mathbb{X}_{k}^{G}=\left(N_{k} \mathbb{X}_{k-1}^{L}\right)^{G} \mathbb{X}_{k-1}^{G} \quad \text { a.s. } \tag{4.8}
\end{equation*}
$$

Hence we obtain $M_{k}^{G}=\mathbb{X}_{k}^{G}\left(\mathbb{X}_{k-1}^{G}\right)^{-1}=\left(N_{k} \mathbb{X}_{k-1}^{L}\right)^{G} \in \mathcal{F}_{k}^{N}$ a.s. We thus obtain Claim (1.20).

Let $\xi$ be a random variable such that $\xi \stackrel{\mathrm{d}}{=} \omega_{H}$ and $\xi$ is independent of $(\mathbb{X}, N)$. Let $k \in \mathbb{Z}$. By $N_{k} \mathbb{X}_{k-1}^{L} \in L G$, we have

$$
\begin{equation*}
M_{k}^{G} \xi=\left(N_{k} \mathbb{X}_{k-1}^{L}\right)^{G} \xi=\left(N_{k} \mathbb{X}_{k-1}^{L} \xi\right)^{G} \tag{4.9}
\end{equation*}
$$

Since

$$
\begin{equation*}
N_{k} \mathbb{X}_{k-1}^{L} \xi \stackrel{\mathrm{~d}}{=} \mu \eta^{L} \omega_{H}=\mu \eta e=\eta^{L} \gamma \omega_{H} \eta^{R} e=\eta^{L} \gamma \omega_{H} \tag{4.10}
\end{equation*}
$$

we have $M_{k}^{G} \xi \stackrel{\mathrm{~d}}{=} \gamma \omega_{H} \stackrel{\mathrm{~d}}{=} \gamma \xi$, which shows $\left(M_{k}^{G}\right)^{C}=\gamma$ a.s. for $k \in \mathbb{Z}$. We now see that

$$
\begin{equation*}
\mathbb{X}_{k}^{C}=\left(\mathbb{X}_{k}^{G}\right)^{C}=\left(M_{k}^{G} \mathbb{X}_{k-1}^{G}\right)^{C}=\left(M_{k}^{G}\right)^{C}\left(\mathbb{X}_{k-1}^{G}\right)^{C}=\gamma \mathbb{X}_{k-1}^{C} \quad \text { a.s. for } k \in \mathbb{Z} \tag{4.11}
\end{equation*}
$$

which yields (4.4). The proof is now complete.

### 4.3 Finding the third noise

The following lemma plays a key role.
Lemma 4.3. For any deterministic sequences $\left\{f_{n}\right\}$ and $\left\{h_{n}\right\}$ from $\mathcal{S}(\nu)$, it holds that

$$
\begin{equation*}
\left(f_{n} N_{1} N_{2} \cdots N_{n} h_{n}\right)^{H} \xrightarrow{\mathrm{~d}} \omega_{H} . \tag{4.12}
\end{equation*}
$$

Proof. Let $\{n(m)\}$ be a subsequence of $\mathbb{N}$. We can extract a further subsequence $\left\{n^{\prime}(m)\right\}$ from $\{n(m)\}$ such that $f_{n^{\prime}(m)} \rightarrow f_{0}$ and $h_{n^{\prime}(m)} \rightarrow h_{0}$ for some $f_{0}, h_{0} \in \mathcal{S}(\nu)$ and

$$
\begin{equation*}
\mu_{n^{\prime}(m)} \rightarrow \mu^{k} \eta=\eta^{L} \gamma^{k} \omega_{H} \eta^{R} \tag{4.13}
\end{equation*}
$$

for some $k=0,1, \ldots, p-1$. Hence we have

$$
\begin{equation*}
\left(f_{n^{\prime}(m)} N_{1} N_{2} \cdots N_{n^{\prime}(m)} h_{n^{\prime}(m)}\right)^{H} \xrightarrow{\mathrm{~d}}\left(f_{0} \eta^{L} \gamma^{k} \omega_{H} \eta^{R} h_{0}\right)^{H} . \tag{4.14}
\end{equation*}
$$

Since $R L \subset H$ and $g^{-1} H g=H$ for $g \in G$, we have

$$
\begin{align*}
\left(f_{0} \eta^{L} \gamma^{k} \omega_{H} \eta^{R} h_{0}\right)^{G} & =f_{0}^{C} \gamma^{k}\left(\gamma^{-k} f_{0}^{H} f_{0}^{R} \eta^{L} \gamma^{k}\right) \omega_{H}\left(\eta^{R} h_{0}^{L}\right)\left(h_{0}^{C} h_{0}^{H}\left(h_{0}^{C}\right)^{-1}\right) h_{0}^{C}  \tag{4.15}\\
& =f_{0}^{C} \gamma^{k} \omega_{H} h_{0}^{C}=f_{0}^{C} \gamma^{k} h_{0}^{C} \omega_{H}, \tag{4.16}
\end{align*}
$$

which yields $\left(f_{0} \eta^{L} \gamma^{k} \omega_{H} \eta^{R} h_{0}\right)^{H}=\omega_{H}$. We thus obtain (4.12).
We proceed to prove part of Theorem 1.7 ,
Proposition 4.4. For $k \in \mathbb{Z}$, set $U_{k}^{H}:=\mathbb{X}_{k}^{H}=Y_{k}^{H}$. Then $U_{k}^{H} \stackrel{\text { d }}{=} \omega_{H}$ and the three $\sigma$-fields $\mathcal{F}_{k}^{N}, \mathcal{F}_{-\infty}^{\mathbb{X}}$ and $\sigma\left(U_{k}^{H}\right)$ are independent. (Consequently Claims (ii) and (1.21) of Theorem 1.7 hold.)

Proof. Set

$$
\begin{equation*}
\mathcal{F}_{k, l}^{N}=\sigma\left(N_{k}, N_{k-1}, \ldots, N_{l+1}\right), \quad k>l . \tag{4.17}
\end{equation*}
$$

Let $k \in \mathbb{Z}$ and let $\varphi: H \rightarrow \mathbb{R}$ be a test function. Let $l<k, n \in \mathbb{N}, A \in \mathcal{F}_{k, l}^{N}$ and $B \in \mathcal{F}_{-\infty}^{\mathbb{X}}$. Note that the three $\sigma$-fields $\sigma\left(N_{k, T_{l}^{e}}, 1_{A}\right), \sigma\left(N_{T_{l}^{e}, T_{l}^{e}-n}\right)$ and $\sigma\left(Y_{T_{l}^{e}-n}, 1_{B}\right)$ are independent, where $T_{k}^{e}$ has been introduced in the proof of Proposition 4.2. We thus have

$$
\begin{align*}
\mathbb{E}\left[\varphi\left(U_{k}^{H}\right) 1_{A} 1_{B}\right] & =\mathbb{E}\left[\varphi\left(Y_{k}^{H}\right) 1_{A} 1_{B}\right]  \tag{4.18}\\
& =\mathbb{E}\left[\varphi\left(\left(N_{k, T_{l}^{e}} N_{T_{l}^{e}, T_{l}^{e}-n} Y_{T_{l}^{e}-n}\right)^{H}\right) 1_{A} 1_{B}\right]  \tag{4.19}\\
& =\mathbb{E} \mathbb{E}^{\prime}\left[\varphi\left(\left(N_{k, T_{l}^{e}} N_{1}^{\prime} N_{2}^{\prime} \cdots N_{n}^{\prime} Y_{T_{l}^{e}-n}\right)^{H}\right) 1_{A} 1_{B}\right]  \tag{4.20}\\
& =\mathbb{E}\left[\left.\mathbb{E}^{\prime}\left[\varphi\left(\left(f N_{1}^{\prime} N_{2}^{\prime} \cdots N_{n}^{\prime} h_{n}\right)^{H}\right)\right]\right|_{\substack{\begin{subarray}{c}{ \\
h_{n}=N_{k, T} e \\
T_{l}} }}  \tag{4.21}\\
{1_{A}-n}\end{subarray}} 1_{B}\right],
\end{align*}
$$

where $\left\{N_{1}^{\prime}, N_{2}^{\prime}, \ldots\right\}$ is an iid sequence with a common law $\mu$ which is independent of $(\mathbb{X}, N)$, and $\mathbb{E}^{\prime}$ denotes the expectation with respect to $\left\{N_{1}^{\prime}, N_{2}^{\prime}, \ldots\right\}$. Noting that $N_{k, T_{l}^{e}} \in \mathcal{S}(\nu)$ (see the proof of Proposition 4.2) and $Y_{T_{l}^{e}-n} \in \mathcal{S}(\nu)$, we apply Lemma 4.3 to see that

$$
\begin{equation*}
\text { (4.21) } \underset{n \rightarrow \infty}{\longrightarrow} \int \varphi \mathrm{~d} \omega_{H} \cdot \mathbb{E}\left[1_{A} 1_{B}\right]=\int \varphi \mathrm{d} \omega_{H} \cdot \mathbb{P}(A) \mathbb{P}(B) . \tag{4.22}
\end{equation*}
$$

Since $l<k$ is arbitrary, we obtain

$$
\begin{equation*}
\mathbb{E}\left[\varphi\left(U_{k}^{H}\right) 1_{A} 1_{B}\right]=\int \varphi \mathrm{d} \omega_{H} \cdot \mathbb{P}(A) \mathbb{P}(B), \quad A \in \mathcal{F}_{k}^{N}, B \in \mathcal{F}_{-\infty}^{\mathbb{X}} \tag{4.23}
\end{equation*}
$$

which leads to the desired result.

### 4.4 Determining the remote past noise

We need the following lemma.
Lemma 4.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be three sub- $\sigma$-fields of $\mathcal{F}$. Suppose that $\mathcal{A} \subset \mathcal{B} \vee \mathcal{C}$ a.s. and that $\mathcal{A} \vee \mathcal{B}$ be independent of $\mathcal{C}$. Then $\mathcal{A} \subset \mathcal{B}$ a.s.

The proof of Lemma 4.5 can be found in [1, Section 2.2], and so we omit it.
We shall now complete the proof of Theorem 1.7 .

Proof of Theorem 1.7. What remains unproved are Claims (iv), (v) and (vi).
We have shown that $\mathbb{X}_{k}^{C}=\gamma^{k} Y_{C}, \mathbb{X}_{k}^{H}=U_{k}^{H}$ and $\mathbb{X}_{k}^{W}=\mathbb{Z}_{W}$. Let $j \leq k$. Since $\mathbb{X}_{k}^{G}=M_{k}^{G} \mathbb{X}_{k-1}^{G}$ by definition of $M_{k}^{G}$, we have $\mathbb{X}_{k}^{G}=M_{k, j}^{G} \mathbb{X}_{j}^{G}$. Hence we obtain

$$
\begin{equation*}
\mathbb{X}_{j}=\mathbb{X}_{j}^{L} \mathbb{X}_{j}^{G} \mathbb{Z}_{W}=\mathbb{X}_{j}^{L}\left(M_{k, j}^{G}\right)^{-1} \mathbb{X}_{k}^{G} \mathbb{Z}_{W}=\mathbb{X}_{j}^{L}\left(M_{k, j}^{G}\right)^{-1} \gamma^{k} Y_{C} U_{k}^{H} \mathbb{Z}_{W} \quad \text { a.s. } \tag{4.24}
\end{equation*}
$$

which shows Claim (iv) and leads to

$$
\begin{equation*}
\mathcal{F}_{k}^{\mathbb{X}}=\mathcal{G}_{k}^{N} \vee \sigma\left(Y_{C}, \mathbb{Z}_{W}\right) \vee \sigma\left(U_{k}^{H}\right) \quad \text { a.s. } \tag{4.25}
\end{equation*}
$$

Since $\sigma\left(Y_{C}, \mathbb{Z}_{W}\right) \subset \mathcal{F}_{-\infty}^{\mathbb{X}}$ a.s., which is obvious by definition, and by (1.21), we can apply Lemma 4.5 for $\mathcal{A}=\mathcal{F}_{-\infty}^{\mathbb{X}}, \mathcal{B}=\sigma\left(Y_{C}, \mathbb{Z}_{W}\right)$ and $\mathcal{C}=\mathcal{F}_{k}^{N} \vee \sigma\left(U_{k}^{H}\right)$, and hence we obtain $\mathcal{F}_{-\infty}^{\mathbb{X}} \subset \sigma\left(Y_{C}, \mathbb{Z}_{W}\right)$ a.s. We thus obtain (1.22). Combining (4.25) and (1.22), we obtain (1.19), which shows Claim (v).

Since $\mathbb{X}_{k}=\mathbb{X}_{k}^{L} \gamma^{k} Y_{C} U_{k}^{H} \mathbb{Z}_{W}$ and since $\Lambda=\eta^{L} \omega_{G} \Lambda_{W}$, we see that $\gamma^{k} Y_{C} U_{k}^{H}$ and $\mathbb{Z}_{W}$ are independent and that $\gamma^{k} Y_{C} U_{k}^{H} \stackrel{\stackrel{\mathrm{~d}}{=}}{=} \omega_{G}$ and $\mathbb{Z}_{W} \stackrel{\mathrm{~d}}{=} \Lambda_{W}$. Since $\omega_{G}=\omega_{C} \omega_{H}$, we see that $Y_{C}$ and $U_{k}^{H}$ are independent, $Y_{C} \stackrel{\mathrm{~d}}{=} \omega_{C}$ and $U_{k}^{H} \stackrel{\mathrm{~d}}{=} \omega_{H}$. We now obtain Claim (vi).

The proof of Theorem 1.7 is therefore complete.

## 5 The non-stationary case

Throughout this section we adopt the settings of Subsection 1.3.
Proposition 5.1. For a sequence $\left(\Lambda_{k}\right)_{k \in \mathbb{Z}}$ from $\mathcal{P}\left(V_{\times}^{m_{\mu}}\right)$, the following are equivalent:
(i) $\Lambda_{k}=\mu \Lambda_{k-1}, k \in \mathbb{Z}$.
(ii) There exist $\Lambda_{W}^{0}, \ldots, \Lambda_{W}^{p-1} \in \mathcal{P}(W)$ and constants $c_{0}, \ldots, c_{p-1} \geq 0$ such that $c_{0}+$ $\cdots+c_{p-1}=1$ such that

$$
\begin{equation*}
\Lambda_{k}=\sum_{i=0}^{p-1} c_{i} \eta^{L} \gamma^{k+i} \omega_{H} \Lambda_{W}^{i}, \quad k \in \mathbb{Z} \tag{5.1}
\end{equation*}
$$

Proof. Since $\mu\left(\eta^{L} \gamma^{k+i} \omega_{H}\right)=\eta^{L} \gamma^{k+i+1} \omega_{H}$, it is obvious that Condition (ii) implies Condition (i).

Suppose Condition (i) be satisfied. Take a subsequence $\{n(m)\}$ of $\mathbb{N}$ such that $\mu^{n(m)} \rightarrow$ $\eta$ and $\Lambda_{-n(m)} \rightarrow \Lambda_{*}$ for some $\Lambda_{*} \in \mathcal{P}\left(V_{\times}^{m_{\mu}}\right)$. We then have $\Lambda_{0}=\mu^{n(m)} \Lambda_{-n(m)} \rightarrow \eta \Lambda_{*}$, which shows that $\Lambda_{0}=\eta \Lambda_{*}=\eta^{L} \omega_{H} \eta^{R} \Lambda_{*}$. Let $\mathbb{X}_{*}$ be a random variable whose law is $\eta^{R} \Lambda_{*}$. Since $\eta^{L} \omega_{H} \mathbb{X}_{*} \in \mathcal{P}\left(V_{\times}^{m_{\mu}}\right)$ and $\mathbb{X}_{*} \in R V_{\times}^{m_{\mu}}=e V_{\times}^{m_{\mu}}$ a.s., we have $\mathbb{X}_{*} \in e W_{\mu}$ a.s. Since $e W_{\mu}=G W=C H W$, we have $\eta^{L} \omega_{H} \mathbb{X}_{*}=\eta^{L} \omega_{H} \mathbb{X}_{*}^{C} \mathbb{X}_{*}^{H} \mathbb{X}_{*}^{W} \stackrel{\mathrm{~d}}{=} \eta^{L} \mathbb{X}_{*}^{H} \omega_{H} \mathbb{X}_{*}^{W}$, where we wrote $\boldsymbol{x}^{C}=\left(\boldsymbol{x}^{G}\right)^{C}$ and $\boldsymbol{x}^{H}=\left(\boldsymbol{x}^{G}\right)^{H}$. For $i=0,1, \ldots, p-1$, set

$$
\begin{equation*}
c_{i}=\mathbb{P}\left(\mathbb{X}_{*}^{C}=\gamma^{i}\right) \quad \text { and } \quad \Lambda_{W}^{i}(\cdot)=\mathbb{P}\left(\mathbb{X}_{*}^{W} \in \cdot \mid \mathbb{X}_{*}^{C}=\gamma^{i}\right) \tag{5.2}
\end{equation*}
$$

We then obtain (5.1) for $k=0$. Since $\mu\left(\eta^{L} \gamma^{k+i} \omega_{H}\right)=\eta^{L} \gamma^{k+i+1} \omega_{H}$, we have (5.1) also for $k \geq 1$.

Let us prove (5.1) for $k \leq-1$ by induction. Suppose (5.1) for $k \leq 0$ hold true. We want to prove (5.1) for $k-1$. By the same argument as for $\Lambda_{0}$, we have

$$
\begin{equation*}
\Lambda_{k-1}=\sum_{i=0}^{p-1} \widetilde{c}_{i} \eta^{L} \gamma^{k-1+i} \omega_{H} \widetilde{\Lambda}_{W}^{i}, \quad k \in \mathbb{Z} \tag{5.3}
\end{equation*}
$$

for some $\widetilde{\Lambda}_{W}^{0}, \ldots, \widetilde{\Lambda}_{W}^{p-1} \in \mathcal{P}(W)$ and some constants $\widetilde{c}_{0}, \ldots, \widetilde{c}_{p-1} \geq 0$ such that $\widetilde{c}_{0}+\cdots+$ $\widetilde{c}_{p-1}=1$. We then have

$$
\begin{equation*}
\Lambda_{k}=\mu \Lambda_{k-1}=\sum_{i=0}^{p-1} \widetilde{c}_{i} \eta^{L} \gamma^{k+i} \omega_{H} \widetilde{\Lambda}_{W}^{i}, \quad k \in \mathbb{Z} \tag{5.4}
\end{equation*}
$$

Comparing this identity with (5.1) and using Proposition 1.6, we obtain $c_{i}=\widetilde{c}_{i}$ and $\Lambda_{W}^{i}=\widetilde{\Lambda}_{W}^{i}$ for $i=0,1, \ldots, p-1$. We thus obtain (5.1) for $k-1$. We have proved (5.1) for $k \leq-1$ by induction.

We now deal with the non-stationary case by reducing it to the stationary case.
Theorem 5.2. Suppose the same assumptions of Proposition 1.6 be satisfied. Let $(\mathbb{X}, N)$ be an $m_{\mu}$-particle $\mu$-evolution such that $\left\{X^{1}, \ldots, X^{m_{\mu}}\right\}$ is distinct a.s. Set $\Lambda_{k}(\cdot):=\mathbb{P}\left(\mathbb{X}_{k} \in\right.$ .) for $k \in \mathbb{Z}$. Then it holds that $\left(\Lambda_{k}\right)_{k \in \mathbb{Z}}$ satisfies the equivalent conditions of Proposition 5.1, that Claims (i)-(v) of Theorem 1.7 are satisfied, and that

$$
\begin{equation*}
\mathbb{P}\left(Y_{C}=\gamma^{i}, \mathbb{Z}_{W}=\boldsymbol{w}\right)=c_{i} \Lambda_{W}^{i}\{\boldsymbol{w}\} \quad \text { for } i=0, \ldots, p-1 \text { and } \boldsymbol{w} \in W \tag{5.5}
\end{equation*}
$$

Proof. We have $\mathbb{X}_{k}=N_{k, T_{k}^{e}} \mathbb{X}_{T_{k}^{e}} \in L G V^{m_{\mu}}$, where $T_{k}^{e}$ has been introduced in the proof of Proposition 4.2. This shows that, for each $k \in \mathbb{Z}$, every distinct pair from $\left\{X_{k}^{1}, \ldots, X_{k}^{m_{\mu}}\right\}$ is a deadlock. Hence we see that the number of distinct elements of $\left\{X_{k}^{1}, \ldots, X_{k}^{m_{\mu}}\right\}$ is constant in $k \in \mathbb{Z}$ a.s. By the assumption that $\left\{X^{1}, \ldots, X^{m_{\mu}}\right\}$ is distinct a.s., we see that $\mathbb{X}_{k} \in V_{\times}^{m_{\mu}}$ a.s., which shows $\Lambda_{k} \in \mathcal{P}\left(V_{\times}^{m_{\mu}}\right)$.

By definition of $\mu$-evolution, we see that Condition (i) of Proposition 5.1 is satisfied. Hence we have a representation (5.1).

We write $\omega_{W}$ for the uniform probability on $W$ and write $\widetilde{\Lambda}=\eta^{L} \omega_{G} \omega_{W}$, which is a $\mu$-invariant probability whose support is $W_{\mu}$. Let $(\widetilde{\mathbb{X}}, \widetilde{N})$ under $\widetilde{\mathbb{P}}$ be a stationary $m_{\mu^{-}}$ particle $\mu$-evolution such that $\widetilde{\mathbb{X}}$ has $\widetilde{\Lambda}$ as its common law. By (vi) of Theorem 1.7, we know that

$$
\begin{equation*}
\widetilde{\mathbb{P}}\left(\widetilde{Y}_{C}=\gamma^{i}, \widetilde{\mathbb{Z}}_{W}=\boldsymbol{w}\right)=\frac{1}{p} \cdot \frac{1}{\#(W)}>0 \tag{5.6}
\end{equation*}
$$

and so the conditional probability

$$
\begin{equation*}
\widetilde{\mathbb{P}}_{\gamma^{i}, \boldsymbol{w}}(\cdot):=\widetilde{\mathbb{P}}\left(\cdot \mid \widetilde{Y}_{C}=\gamma^{i}, \widetilde{\mathbb{Z}}_{W}=\boldsymbol{w}\right) \tag{5.7}
\end{equation*}
$$

is well-defined. We then see that $(\widetilde{\mathbb{X}}, \widetilde{N})$ under $\widetilde{\mathbb{P}}_{\gamma^{i}}, \boldsymbol{w}$ is a (non-stationary) $m_{\mu}$-particle $\mu$-evolution; in fact, since $\widetilde{Y}_{C}, \widetilde{\mathbb{Z}}_{W} \in \mathcal{F}_{-\infty}^{\widetilde{\mathbb{X}}}$ a.s., we can verify the Markov property (1.3). Note that, for each $k \in \mathbb{Z}$, the law of $\widetilde{\mathbb{X}}_{k}$ under $\widetilde{\mathbb{P}}_{\gamma^{i}}, \boldsymbol{w}$ equals to $\eta^{L} \gamma^{k+i} \omega_{H} \boldsymbol{w}$. Moreover, by (1.18), we obtain the following factorization:

$$
\begin{equation*}
\widetilde{\mathbb{X}}_{j}=\widetilde{\mathbb{X}}_{j}^{L}\left(\widetilde{M}_{k, j}^{G}\right)^{-1} \gamma^{k+i} \widetilde{U}_{k}^{H} \boldsymbol{w} \quad \widetilde{\mathbb{P}}_{\gamma^{i}, \boldsymbol{w}^{- \text {a.s. for }} j \leq k,} \tag{5.8}
\end{equation*}
$$

where $\widetilde{M}_{k, j}^{G}$ and $\widetilde{U}_{k}^{H}$ are defined in the same way as in Theorem 1.7. We then see that Claims (i)-(v) of Theorem 1.7 are satisfied for $(\widetilde{\mathbb{X}}, \widetilde{N})$ under $\widetilde{\mathbb{P}}_{\gamma^{i}}, \boldsymbol{w}$.

Define

$$
\begin{equation*}
\widetilde{\mathbb{Q}}=\sum_{i=0}^{p-1} c_{i} \sum_{\boldsymbol{w} \in W} \Lambda_{W}^{i}\{\boldsymbol{w}\} \widetilde{\mathbb{P}}_{\gamma^{i}, \boldsymbol{w}} \tag{5.9}
\end{equation*}
$$

We then see that the joint law of $(\mathbb{X}, N)$ under $\mathbb{P}$ equals to that of $(\widetilde{\mathbb{X}}, \widetilde{N})$ under $\widetilde{\mathbb{Q}}$; in fact, they are $\mu$-evolutions and

$$
\begin{align*}
\widetilde{\mathbb{Q}}\left(\widetilde{\mathbb{X}}_{k} \in \cdot\right) & =\sum_{i=0}^{p-1} c_{i} \sum_{\boldsymbol{w} \in W} \Lambda_{W}^{i}\{\boldsymbol{w}\}\left(\eta^{L} \gamma^{k+i} \omega_{H} \boldsymbol{w}\right)  \tag{5.10}\\
& =\sum_{i=0}^{p-1} c_{i} \eta^{L} \gamma^{k+i} \omega_{H} \Lambda_{W}^{i}=\Lambda_{k}=\mathbb{P}\left(\mathbb{X}_{k} \in \cdot\right) . \tag{5.11}
\end{align*}
$$

We thus derive from (5.8) the following factorization:

$$
\begin{equation*}
\widetilde{\mathbb{X}}_{j}=\widetilde{\mathbb{X}}_{j}^{L}\left(\widetilde{M}_{k, j}^{G}\right)^{-1} \gamma^{k} \widetilde{Y}_{C} \widetilde{U}_{k}^{H} \widetilde{\mathbb{Z}}_{W} \quad \widetilde{\mathbb{Q}} \text {-a.s. for } j \leq k, \tag{5.12}
\end{equation*}
$$

where $\widetilde{Y}_{C}$ and $\widetilde{\mathbb{Z}}_{W}$ are defined in the same way as in Theorem 1.7. We therefore obtain the desired result.

## References

[1] L. Chaumont and M. Yor. Exercises in probability. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, second edition, 2012. A guided tour from measure theory to random processes, via conditioning.
[2] B. S. Cirel'son. An example of a stochastic differential equation that has no strong solution. Teor. Verojatnost. i Primenen., 20(2):427-430, 1975.
[3] H. S. Collins. Convergence of convolution iterates of measures. Duke Math. J., 29:259-264, 1962.
[4] T. Hirayama and K. Yano. Extremal solutions for stochastic equations indexed by negative integers and taking values in compact groups. Stochastic Process. Appl., 120(8):1404-1423, 2010.
[5] G. Högnäs and A. Mukherjea. Probability measures on semigroups. Probability and its Applications (New York). Springer, New York, second edition, 2011. Convolution products, random walks, and random matrices.
[6] Y. Ito, T. Sera, and K. Yano. Examples of third noise problems for action evolutions with infinite past. RIMS Kōkyūroku, Proceedings of Research on the Theory of Random Dynamical Systems and Fractal Geometry in 2019, to appear.
[7] M. Rosenblatt. Limits of convolution sequences of measures on a compact topological semigroup. J. Math. Mech., 9:293-305, 1960.
[8] Š. Schwarz. Convolution semigroup of measures on compact noncommutative semigroups. Czechoslovak Math. J., 14 (89):95-115, 1964.
[9] A. N. Trahtman. The road coloring problem. Israel J. Math., 172:51-60, 2009.
[10] R. van Handel. On the exchange of intersection and supremum of $\sigma$-fields in filtering theory. Israel J. Math., 192(2):763-784, 2012.
[11] K. Yano. Random walk in a finite directed graph subject to a road coloring. $J$. Theoret. Probab., 26(1):259-283, 2013.
[12] K. Yano and M. Yor. Around Tsirelson's equation, or: The evolution process may not explain everything. Probab. Surv., 12:1-12, 2015.
[13] M. Yor. Tsirel'son's equation in discrete time. Probab. Theory Related Fields, 91(2):135-152, 1992.


[^0]:    ${ }^{(1)}$ Kyoto Sangyo University, Kyoto, JAPAN.
    ${ }^{(2)}$ The research of this author was supported by JSPS KAKENHI Grant Number JP18K13431
    ${ }^{(3)}$ Graduate School of Science, Kyoto University, Kyoto, JAPAN. Research Fellow of Japan Society for the Promotion of Science.
    ${ }^{(4)}$ The research of this author was supported by JSPS KAKENHI Grant Number JP19J11798
    ${ }^{(5)}$ Graduate School of Science, Kyoto University, Kyoto, JAPAN.
    ${ }^{(6)}$ The research of this author was supported by JSPS KAKENHI grant no.'s JP19H01791, JP19K21834 and JP18K03441 and by JSPS Open Partnership Joint Research Projects grant no. JPJSBP120209921.

