Geometry of metric spaces via weighted partitions by trees

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Abstract

Successive division of a compact metric space, called a partition, and weight functions of pieces of a division are the main interest of this paper. Successive division means; let X be a compact metric space of interest. Divide X into finite number of compact subsets $X_1, \ldots, X_N$. Next each compact subset X_i in the division is again divided into $X_{i1}, X_{i2}, \ldots, X_{in_i}$. Then X_{ij} is divided into finite number of subsets and repeat this again and again infinitely many times. Such a successive division appears naturally in the construction of self-similar sets, Markov partition of hyperbolic dynamical systems, dyadic cubes associated with a doubling metric space and so on. A weight function assigns a value between 0 and 1 to each piece of the division. For example, for given metric, the correspondence of pieces to their (normalized) diameters is an example of a weight function. The main purpose of this paper is to study relation between a weight function and a geometry of the original set X. In the course of our study, the notions like bi-Lipschitz equivalence, Ahlfors regularity, the volume doubling property and quasisymmetry will be shown to be equivalent to certain properties of weight functions.

1 Introduction

Successive division of a space has played important roles in many area of mathematics. One of the simplest examples is the binary division of the unit interval $[0, 1]$ as is shown in Figure 1, i.e. let $K_\phi = [0, 1]$. Then divide K_ϕ in halves as $K_0 = [0, \frac{1}{2}]$ and $K_1 = [\frac{1}{2}, 1]$. Next, K_0 and K_1 are divided in halves again and yield K_{ij} for each $(i, j) \in \{0, 1\}^2$. Successively,

$$K_{i_1 \ldots i_m} = K_{i_1 \ldots i_m 0} \cup K_{i_1 \ldots i_m 1} \tag{1.1}$$

for any $m \geq 0$ and any $i_1 \ldots i_m \in \{0, 1\}^m$. In this well-known example, we pay attention to the following two properties.

The first one is the role of the (infinite) binary tree

$$T_b = \{\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\} = \bigcup_{m \geq 0} \{0, 1\}^m,$$

where $\{0, 1\}^0 = \{\phi\}$. The vertex ϕ is called the root or the reference point and T_b is called the tree with the root (or the reference point) ϕ. Note that the correspondence $i_1 \ldots i_m \to K_{i_1 \ldots i_m}$ determines a map from the binary tree to the collection of compact subsets of $[0, 1]$ with the property (1.1).

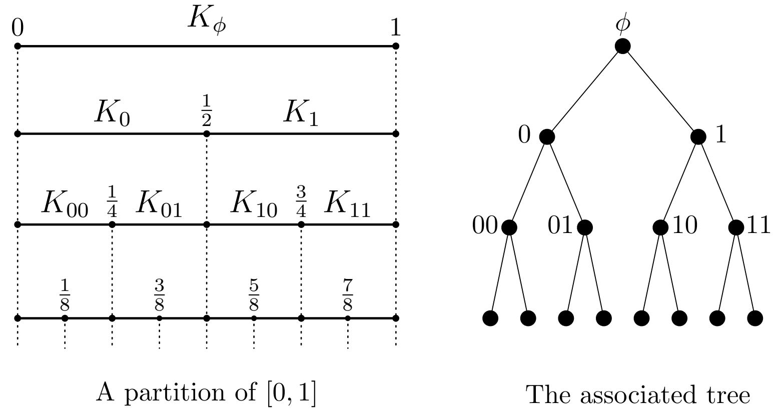


Figure 1: A partition of the unit interval $[0, 1]$ and the associated tree

Secondly, note that $K_{i_1} \supseteq K_{i_1 i_2} \supseteq K_{i_1 i_2 i_3} \supseteq \dots$ by (1.1) and

$$\bigcap_{m \geq 1} K_{i_1 \dots i_m} \text{ is a single point} \tag{1.2}$$

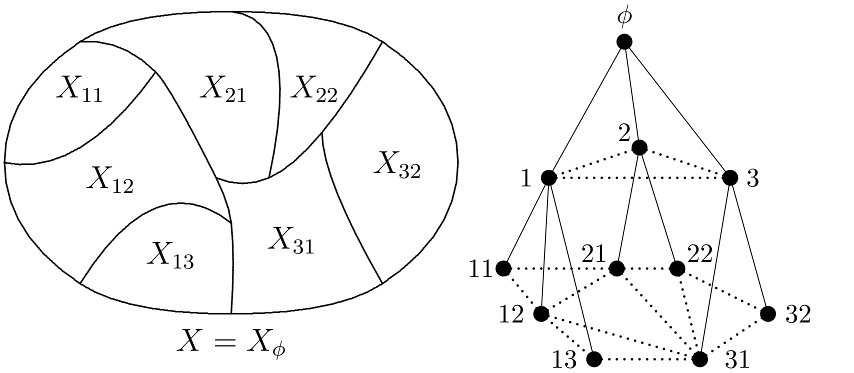
for any infinite sequence $i_1 i_2 \dots$. (Of course, this is the binary expansion and hence the single point is $\sum_{m \geq 1} \frac{i_m}{2^m}$.) In other words, there is a natural map $\sigma : \{0,1\}^{\mathbb{N}} \to [0,1]$ given by

$$\sigma(i_1 i_2 \dots) = \bigcap_{m \geq 1} K_{i_1 \dots i_m}.$$

Such a successive division of a compact metric space, which may not be as simple as the last one, appears various situation. One of the typical examples is a self-similar set in fractal geometry. A self-similar set is a union of finite number of contracted copies of itself. Then each contracted copy is again a union of contracted copies and so forth. Another example is the Markov partition associated with hyperbolic dynamical systems. See [1] for details. Also the division of a metric measure space having the volume doubling property by dyadic cubes can be thought of as another example of such a division of a space. See Christ[5] for example.

Let X be the compact metric space in question. The common properties of the above examples are;
(i) There exists a tree T (i.e. a connected graph without loops) with the root ϕ.
(ii) For any vertex p of T, there is a corresponding nonempty compact subset of X denoted by X_p and $X = X_\phi$.
(iii) Every vertex p of T except ϕ has unique predecessor $\pi(p) \in T$ and

$$X_q = \bigcup_{p \in \{p' | \pi(p') = q\}} X_p \tag{1.3}$$



A partition of X

The associated graphs T and T^{+h}
Solid lines are edges of the tree T
Dotted lines are horizontal edges B^h

Figure 2: A partition and the associated graphs (up to the 2nd stage)

(iv) The totality of edges of T is $\{(\pi(q), q)|q \in T\backslash\{\phi\}\}$.
(v) For any infinite sequence $(p_0, p_1, p_2, \ldots)$ of vertices of X satisfying $p_0 = \phi$ and $\pi(p_{i+1}) = p_i$ for any $i \geq 1$,

$$\bigcap_{i\geq 1} X_{p_i} \text{ is a single point.} \tag{1.4}$$

See Figure 2 for an illustration of the idea. Note that the properties (1.3) and (1.4) corresponds to (1.1) and (1.2) respectively. In this paper such $\{X_p\}_{p\in T}$ is called a partition of X parametrized by the tree T. (We will give the precise definition in Section 4.)

For a metric d which produces the original topology of X and a Radon measure μ on X, the diameter of X_p with repsect to d, $\mathrm{diam}(X_p, d)$, and the measure of X_p, $\mu(X_p)$, are associated natural weights of X_p for $p \in T$. In both cases, if $\rho_d(p) = \mathrm{diam}(X_p, d)/\mathrm{diam}(X, d)$ and $\rho_\mu(p) = \mu(X_p)/\mu(X)$ for any $p \in T$, then the function $\rho_\# : T \to (0, 1]$ for $\# = d, \mu$ satisfies

$$\rho_\#(\pi(p)) \geq \rho_\#(p) \tag{1.5}$$

for any $p \in T\backslash\{\phi\}$ and

$$\lim_{i\to\infty} \rho_\#(p_i) = 0 \tag{1.6}$$

if $\pi(p_{i+1}) = p_i$ for any $i \geq 1$. (To have the second property (1.6) in case of $\# = \mu$, we must assume that the measure μ is non-atomic, i.e. $\mu(\{x\}) = 0$ for any $x \in X$.)

As we have seen above, we ordinarily begin with a metric space with some structure like self-similarity, dynamical system or a measure with the volume doubling property, construct a partition in association with the structure and obtain a weight function from a metric or a measure. In this paper, we are interested in the opposite way. Namely, given a partition of a compact metrizable space parametrized by a tree T, we define the notion of weight functions as the collection of functions from T to $(0, 1]$ satisfying the same properties as (1.5) and (1.6) of $\rho_{\#}$. Then our main object of interest is the space of weight functions which includes those coming form metrics and measures. Naively we believe that a partition and a weight function essentially determine "geometry and/or analysis" of the original set no matter where the weight function comes from. It may come from a metric, a measure or else. Keeping this intuition in mind, we are going to study the structure of the collection of weight functions from the following two viewpoints.

The first question is when is a weight function naturally associated with a (power of) metric? The phrase "naturally associated" is rather vague. The most strict usage would mean that there exist a metric d and α such that $\rho(p) = \mathrm{diam}(X_0, d)^\alpha$. This is, however, too restrictive. As a reasonable alternative, we will define the "visual pre-metric" δ^ρ_M from the weight function ρ, where $M \geq 1$ is a parameter, in Section 5. If a metric d is bi-Lipschitz equivalent to the visual pre-metric δ^ρ_M, we consider that the weight function ρ is "naturally associated with" (formally, we use the terminology "adapted to" instead of "naturally associated with" in the following sections) the metric d. The notion of visual metric has appeared as natural metrics on the boundaries of Gromov-hyperbolic spaces. See [7] for example. The notion of visual pre-metric is a kind of generalization of visual metric. We will give more detailed accounts in Section 5. Anyway, in Theorem 6.11, we are going to give conditions for a weight function to be adapted to a power of a metric.

The second question is about the relationship of various relations between weight functions, metrics and measures. For examples, Ahlfors regularity and the volume property are relations between measures and metrics. A measure μ is Ahlfors α-regular with respect to a metric d for some $\alpha > 0$ if and only if there exist $c_1, c_2 > 0$ such that

$$c_1 r^\alpha \leq \mu(B_d(x, r)) \leq c_2 r^\alpha,$$

where $B_d(x, r) = \{y | y \in X, d(x, y) < r\}$, for any $r \in (0, \mathrm{diam}(X, d)]$ and $x \in X$. See Definition 9.3 for the precise definition of the volume doubling property. On the other hand, bi-Lipschitz equivalence and quasisymmetry are (equivalence) relations between two metrics. (The precise definitions of bi-Lipschitz equivalence and quasisymmetry are given in Definitions 7.9 and 12.1 respectively.) About those relations, two of our claims in this paper are

$$\text{bi-Lipschitz} = \text{Ahlfors regularity} = \text{being adapted} \tag{1.7}$$

and

$$\text{the volume doubling property} = \text{quasisymmetry}. \tag{1.8}$$

in the framework of weight functions. To illustrate the first claim more explicitly, let us introduce the notion of bi-Lipschitz equivalence of weight functions. Two weight functions ρ_1 and ρ_2 are said to be bi-Lipschitz equivalent if and only if there exist $c_1, c_2 > 0$ such that

$$c_1 \rho_1(p) \leq \rho_2(p) \leq c_2 \rho_1(p)$$

for any $p \in T$. Now the first claim can be explained as follows: consider two weight functions ρ_1 and ρ_2, if both ρ_1 and ρ_2 come from metrics d_1 and d_2 respectively, then bi-Lipschitz equivalence of ρ_1 and ρ_2 is the bi-Lipschitz equivalence of metrics d_1 and d_2, if ρ_1 comes from a metric d and ρ_2 comes from a measure μ, then it is Ahlfors regularity of μ with respect to d, and if ρ_1 comes from a metric d, then ρ_1 and ρ_2 are bi-Lipschitz equivalent if and only if ρ_2 is adapted to the metric d. One can find the precise statement in Theorem 2.11 in the case of partitions of S^2. The second claim is rationalized in the same manner. See Theorem 2.12 for the exact statement in the case of S^2 for example.

One of the ideas behind this study is to approximate a space by a series of graphs. It is well-known that such an idea has already been explored. For example, if a compact metric space is doubling and uniformly perfect, Bourdon and Pajot [4] have constructed a infinite graph whose hyperbolic boundary is homeomorphic to the original compact metric space. Their method is first construct a series of coverings of the space, which is a counterpart of our partition, and construct a graph from the series. In [13], Carrasco Piaggio has utilized this series of coverings to study Ahlfors regular conformal dimension of the space. His notion of "relative radius" essentially corresponds to our weight function, although the objectives of his study and ours are not the same. In our case, a counterpart of Bourdin-Pajot's graph can be obtained by adding "horizontal edges" to the tree T associated with a partition. "Horizontal edges" represent the intersections of X_p and X_q in the same level of the division. Specifically, the collection of horizontal edges is given by

$$B^h = \{(p,q)|p,q \in T, p \neq q, p \text{ and } q \text{ have the same graph distance from } \phi, \\ X_p \cap X_q \neq \emptyset\}.$$

See Figure 2. Adding B^h to the original edges of the tree T, we obtain a new graph T^{+h}, which will be called the resolution graph. Since we do not assume doubling property of the space in general, T^{+h} many not be Gromov-hyperbolic but if this is the case, the original space is homeomorphic to the hyperbolic boundary of T^{+h}. See the discussion after Proposition 4.8 for details. In other words, the original space is the hyperbolic filling of T^{+h}. (See [3] for the notion of hyperbolic fillings.) In this point of view, our study in this paper may be thought of as a theory of weighted hyperbolic fillings.

The organization of this paper is as follows. In Section 2, we give a summary of the main results of this paper in the case of the 2 dimensional sphere as a showcase of the full theory. In Section 3, we give basic definitions and notations on trees. Section 4 is devoted to the introduction of partitions and related notions. In Section 5, we define the notion of weight function and the associated "visual pre-metric". We study our first question mentioned above, namely, when a weight function is naturally associated with a (power of) metric in Section 6. Section 7 is devoted to justifying the statement (1.7). In Sections 8, 9, 11 and 12, we will study the rationalized version of (1.8) as mathematical statement. In particular, in Section 9, we introduce the key notion of being "gentle". In

Section 10, we apply our general theory to certain class of subsets of the square and obtain concrete (counter) examples. Finally in Section 14, we present the whereabouts of definitions, notations and conditions appearing in this paper for reader's sake.

2 Summary of the main results; the case of 2-dim. sphere

In this section, we summarize our main results in this paper in the case of 2-dimensional sphere S^2 (or the Riemann sphere in other words), which is denoted by X in what follows. We think that X is equipped with the standard geodesic metric d_S on the sphere. Set

$$\mathcal{U} = \{A | A \subseteq X, \text{closed}, \text{int}(A) \neq \emptyset, \partial A \text{ is homeomorphic to the circle } S^1.\}$$

First we divide X as a union of finite number of subsets $X_1, \ldots, X_{N_0}$ belonging to $\mathcal{U}$. We assume that $X_i \cap X_j = \partial X_i \cap \partial X_j$ if $i \neq j$. Next we divide each X_i as a union of finite number of its subsets $X_{i1}, X_{i2}, \ldots, X_{iN_i} \in \mathcal{U}$ in the same manner as before. We repeat this process, i.e. for any $i_1 \ldots i_k$, $X_{i_1 \ldots i_k} \in \mathcal{U}$,

$$X_{i_1\ldots i_k} = \bigcup_{j=1,\ldots,N_{i_1\ldots i_k}} X_{i_1\ldots i_k j} \tag{2.1}$$

and if $i_1 \ldots i_k \neq j_1 \ldots j_k$, then

$$X_{i_1\ldots i_k} \cap X_{j_1\ldots j_k} = \partial X_{i_1\ldots i_k} \cap \partial X_{j_1\ldots j_k}. \tag{2.2}$$

Note that (2.1) is a counterpart of (1.3). Next define

$$T_k = \{i_1 \ldots i_k | i_j \in \{1, \ldots, N_{i_1\ldots i_{j-1}}\} \text{ for any } j = 1, \ldots, k-1\}$$

for any $k = 0, 1, \ldots$, where T_0 is a one point set $\{\phi\}$. Let $T = \cup_{k\geq 0} T_k$. Then T is naturally though of as a (non-directed) tree where the edges are in the form of $(i_1 \ldots i_k, i_1 \ldots i_k i_{k+1})$. We regard the correspondence $w \in T$ to $X_w \in \mathcal{U}$ as a map from T to $\mathcal{U}$, which is denoted by $\mathcal{X}$. Namely, $\mathcal{X}(w) = X_w$ for any $w \in T$. Note that $\mathcal{X}(\phi) = X$. Define

$$\Sigma = \{i_1 i_2 \ldots | i_1 \ldots i_k \in T_k \text{ for any } k \geq 0\},$$

which is the "boundary" of the infinite tree T.

Furthermore we assume that for any $i_1 i_2 \ldots \in \Sigma$

$$\bigcap_{k=1,2,\ldots} X_{i_1\ldots i_k}$$

is a single point, which is denoted by $\sigma(i_1 i_2 \ldots)$. Note that σ is a map from Σ to X. This assumption corresponds to (1.4) and hence the map $\mathcal{X}$ is a partition of X parametrized by the tree T. Since $X = \cup_{w \in T_k} X_w$ for any $k \geq 0$, this map σ is surjective.

In [2, Chapter 5], the authors have constructed "cell decomposition" associated with an expanding Thurston map. This "cell decomposition" is, in fact, an example of a partition formulated above.

Throughout this section, for simplicity, we assume the following conditions (SF) and (TH), where (SF) is called strong finiteness in Definition 4.4 and (TH) is the condition (TH1) appearing in Theorem 8.3:
(SF)

$$\#(\sigma^{-1}(x)) < +\infty, \tag{2.3}$$

where $\#(A)$ is the number of elements in a set A.
(TH) There exists $m \geq 1$ such that for any $w = i_1 \ldots i_n \in T$, there exists $v = i_1 \ldots i_n i_{n+1} \ldots i_{n+m} \in T$ such that $X_v \subseteq \mathrm{int}(X_w)$.

The main purpose of this paper is to describe geometry and measures of X from given weight assigned to each piece X_w of the partition $\mathcal{X}$.

Definition 2.1. A map $g : T \to (0, 1]$ is called a weight function if and only if it satisfies the following conditions (G1), (G2) and (G3).
(G1) $g(\phi) = 1$
(G2) $g(i_1 \ldots i_k) \geq g(i_1 \ldots i_k i_{k+1})$ for any $i_1 \ldots i_k \in T$ and $i_1 \ldots i_k i_{k+1} \in T$.
(G3)

$$\lim_{m \to 0} \sup_{w \in T_k} g(w) = 0.$$

Moreover, in this section, we assume that following conditions (SpE) and (SbE), which represents "super-exponential" and "sub-exponential" respectively:
(SpE) There exists $\lambda \in (0, 1)$ such that

$$g(i_1 \ldots i_k i_{k+1}) \geq \gamma g(i_1 \ldots i_k)$$

for any $i_1 \ldots i_k \in T$ and $i_1 \ldots i_k i_{k+1} \in T$.
(SbE) There exist $p \in \mathbb{N}$ and $\gamma \in (0, 1)$ such that

$$g(i_1 \ldots i_k i_{k+1} \ldots i_{k+p}) \leq \gamma g(i_1 \ldots i_k)$$

for any $i_1 \ldots i_k \in T$ and $i_1 \ldots i_k i_{k+1} \ldots i_{k+p} \in T$.

Set

$$\mathcal{G}_e(T) = \{g | g : T \in (0, 1] \text{ is a weight function satisfying (SpE) and (SbE).}\}.$$

Metrics and measures on X naturally have associated weight functions.

Definition 2.2. Set

$$\mathcal{D}(X) = \{d | d \text{ is a metric on } X \text{ which produces the original topology of } X, \\ \text{and } \mathrm{diam}(X, d) = 1\}$$

and

$$\mathcal{M}(X) = \{\mu|\mu \text{ is a Borel regular probability measure on } X,\ \mu(\{x\}) = 0 \\ \text{for any } x \in T \text{ and } \mu(O) > 0 \text{ for any non-empty open set } O \subseteq X\}$$

For any $d \in \mathcal{D}(X)$, define $g_d : T \to (0,1]$ by $g_d(i_1 \ldots i_k) = \mathrm{diam}(X_{i_1\ldots i_k}, d)$ and for any $\mu \in \mathcal{M}(X)$, define $g_\mu : T \to (0,1]$ by $g_\mu(w) = \mu(X_w)$ for any $w \in T$.

From Proposition 5.5, we have the following fact.

Proposition 2.3. *If $d \in \mathcal{D}(X)$ and $\mu \in \mathcal{M}(X)$, then g_d and g_μ are weight functions.*

So a metric $d \in \mathcal{D}(X)$ has associated weight function g_d. How about the converse direction, i.e. is there a metric whose associated weight function is given weight function g? To make this question more rigorous and flexible, we define the notion of "visual pre-metric" $\delta_M^g(\cdot,\cdot)$ associated with a weight function g.

Definition 2.4. Let $g \in \mathcal{G}_e(T)$. Define

$$\Lambda_s^g = \{i_1 \ldots i_k | i_1 \ldots i_k \in T, g(i_1 \ldots i_{k-1}) > s \geq g(i_1 \ldots i_k)\}$$

for any $s \in (0,1]$ and

$$\delta_M^g(x,y) = \inf\{s|\text{there exist } w(1),\ldots,w(M+1) \in \Lambda_s^g \text{ such that} \\ x \in K_{w(1)}, y \in K_{w(M+1)} \text{ and } X_{w(j)} \cap X_{w(j+1)} \neq \emptyset \text{ for any } j = 1,\ldots,M\}$$

for any $x, y \in X$. A weight function is called uniformly finite if and only if

$$\sup_{s\in(0,1],w\in\Lambda_s^g} \#(\{v|w \in \Lambda_s^g, X_w \cap X_v \neq \emptyset\}) < +\infty.$$

Although $\delta_M^g(x,y) \geq 0$, $\delta_M^g(x,y) = 0$ if and only if $x = y$ and $\delta_M^g(x,y) = \delta_M^g(y,x)$, the quantity δ_M^g does not satisfy the triangle inequality in general. The visual pre-metric $\delta_M^g(x,y)$ is a counterpart of the visual metric defined in [2]. See Section 5 for details.

If the pre-metric $\delta_M^g(\cdot,\cdot)$ is bi-Lipschitz equivalent to a metric d, we consider d as the metric which is naturally associated with the weight function g.

Definition 2.5. Let $M \geq 1$
(1) A metric $d \in \mathcal{D}(X)$ is said to be M-adapted to a weight function $g \in \mathcal{G}_e(X)$ if and only if there exist $c_1, c_2 > 0$ such that

$$c_1 d(x,y) \leq \delta_M^g(x,y) \leq c_2 d(x,y)$$

for any $x, y \in X$.
(2) A metric d is said to be M-adapted if and only if it is adapted to g_d and said to be adapted if it is M-adapted for some $M \geq 1$.
(3) Define

$$\mathcal{D}_{A,e}(X) = \{d|d \in \mathcal{D}(X),\ g_d \in \mathcal{G}_e(T) \text{ and } d \text{ is adapted.}\}$$
$$\mathcal{M}_e(X) = \{\mu|\mu \in \mathcal{M}(X), g_\mu \in \mathcal{G}_e(T)\}$$

The value M really matters. See Example 10.9 for an example.

The following definition is used to describe an equivalent condition for the existence of an adapted metric in Theorem 2.7.

Definition 2.6. Let $g \in \mathcal{G}_e(T)$. For $w, v \in T$, (w, v) said to be M-separated in Λ_s^g if and only if whenever $w(1), \ldots, w(k) \in \Lambda_s^g$ and $X_w \cap X_{w(1)} \neq \emptyset$, $X_{w(k)} \cap X_v \neq \emptyset$ and $X_{w(i)} \cap X_{w(i+1)} \neq \emptyset$ for any $i = 1, \ldots, k-1$, it follows that $k \geq M$.

The following theorem is a spacial case of Theorem 6.11.

Theorem 2.7. *Let $g \in \mathcal{G}_e(X)$ and let $M \geq 1$. There exists a metric $d \in \mathcal{D}(X)$ which is M-adapted to g^α for some $\alpha > 0$ if and only if the following condition* $(\mathrm{EV})_\mathrm{M}$ *is satisfied;*
$(\mathrm{EV})_\mathrm{M}$ *There exists $\gamma \in (0, 1)$ such that if (w, v) is M-separated in Λ_s^g, then it is $(M+1)$-separated in $\Lambda_{\gamma s}^g$.*

If there exists $r \in (0, 1)$ such that $g(i_1 \ldots i_m) = r^m$ for any $i_1 \ldots i_m \in T$, then the metric d which is 1-adapted to g^α is (bi-Lipschitz equivalent to) the visual metric in [2, Chapter 8]. More precisely, let $m_{f,\mathcal{C}}(x, y)$ be the number defined in [2, Section 8.1]. Under the condition $(\mathrm{EV})_1$, there exist $c_1, c_2 > 0$ such that

$$c_1 \delta_1^g(x, y) \leq r^{-m_{f,\mathcal{C}}(x,y)} \leq c_2 \delta_1^g(x, y)$$

for any $x, y \in X$. Indeed, a counterpart of $(\mathrm{EV})_1$ has been shown in the proof of [2, Lemma 8.6] as a step to prove the existence of a visual metric. In this sense, a metric adapted to a weight function is a generalization of the notion of visual metric.

Next, we define two equivalent relations $\underset{\mathrm{BL}}{\sim}$ and $\underset{\mathrm{GE}}{\sim}$ on the collection of exponential weight functions. Those equivalent relations will be revealed to have different aliases according to classes of weight functions.

Definition 2.8. For $g, h \in \mathcal{G}_e(T)$, g and h are said to be bi-Lipschitz equivalent if and only if there exists $c_1, c_2 > 0$ such that

$$c_1 g(w) \leq h(w) \leq c_2 g(w).$$

for any $w \in T$. We write $g \underset{\mathrm{BL}}{\sim} h$ if g and h are bi-Lipschitz equivalent.
For $g, h \in \mathcal{G}_e(T)$, h is said to be gentle to g if and only there exists $\gamma > 0$ such that if $w, v \in \Lambda_s^g$ and $X_w \cap X_v \neq \emptyset$, then $h(w) \leq \gamma h(v)$. We write $g \underset{\mathrm{GE}}{\sim} h$ if h is gentle to g.

It is immediate to see that $\underset{\mathrm{BL}}{\sim}$ is an equivalence relation. On the other hand, that fact that $\underset{\mathrm{GE}}{\sim}$ is an equivalence relation is not quite obvious and going to be shown in Theorem 11.2.

Proposition 2.9. *The relations $\underset{BL}{\sim}$ and $\underset{\mathrm{GE}}{\sim}$ are equivalent relations in $\mathcal{G}_e(T)$. Moreover, if $g \underset{BL}{\sim} h$, then $g \underset{\mathrm{GE}}{\sim} h$.*

Some of the properties of a weight function is invariant under the equivalence relation $\underset{\mathrm{GE}}{\sim}$ as follows.

Proposition 2.10. (1) *Being uniformly finite is invariant under the equivalence relation* $\underset{\mathrm{GE}}{\sim}$, *i.e. if* $g \in \mathcal{G}_e(T)$ *is uniformly finite,* $h \in \mathcal{G}_e(T)$ *and* $g \underset{\mathrm{GE}}{\sim} h$, *then* h *is uniformly finite.*
(2) *The condition* $(\mathrm{EV})_{\mathrm{M}}$ *appearing in Theorem 2.7 is invariant under the equivalence relation* $\underset{\mathrm{GE}}{\sim}$.

The statements (1) and (2) of the above theorem are the special cases of Theorem 11.7 and Theorem 11.9 respectively.

The next theorem shows that bi-Lipschitz equivalence of weight functions has several aliases according as types of involved weight functions

Theorem 2.11. (1) *For* $d, \rho \in \mathcal{D}_{A,e}(X)$, $g_d \underset{BL}{\sim} g_\rho$ *if and only if* d *and* g *are bi-Lipschitz equivalent as metrics.*
(2) *For* $\mu, \nu \in \mathcal{M}(X)$, $g_\mu \underset{BL}{\sim} g_\nu$ *if and only if there exist* $c_1, c_2 > 0$ *such that*

$$c_1\mu(A) \leq \nu(A) \leq c_2\mu(A)$$

for any Borel set $A \subseteq X$.
(3) *For* $g \in \mathcal{G}_e(X)$ *and* $d \in \mathcal{D}_{A,e}(X)$, $g \underset{BL}{\sim} g_d$ *if and only if* d *is* M*-adapted to* g *for some* $M \geq 1$.
(4) *For* $d \in \mathcal{D}_{A,e}(X)$ *and* $\mu \in \mathcal{M}(X)$, $(g_d)^\alpha \underset{BL}{\sim} g_\mu$ *and* g_d *is uniformly finite if and only if* μ *is Ahlfors* α*-regular with respect to* d *for some* $\alpha > 0$, *i.e. there exist* $c_1, c_2 > 0$ *such that*

$$c_1 r^\alpha \leq \mu(B_d(x, r)) \leq c_2 r^\alpha$$

for any $r > 0$ *and* $x \in X$.

The statements (1), (2), (3) and (4) of the above theorem follows from Corollary 7.10, Theorem 7.4, Corollary 7.11 and Theorem 7.21 respectively.

The gentle equivalence relation is called "quasisymmetry" between metrics and "volume doubling" property in case of metrics versus measures.

Theorem 2.12. (1) *Let* $d \in \mathcal{D}_{A,e}(X)$ *and* $\mu \in \mathcal{M}(X)$. *Then* $g_\mu \in \mathcal{G}_e(T)$, $g_d \underset{\mathrm{GE}}{\sim} g_\mu$ *and* g_d *is uniformly finite if and only if* μ *has the volume doubling property with respect to* d, *i.e. there exists* $C > 0$ *such that*

$$\mu(B_d(x, 2r)) \leq C\mu(B_d(x, r))$$

for any $r > 0$ *and* $x \in X$.
(2) *For* $d \in \mathcal{D}_{A,e}(X)$ *and* $\rho \in \mathcal{D}(X)$, d *is quasisymmetric with respect to* ρ *if and only if* $\rho \in \mathcal{D}_{A,e}(X)$ *and* $g_d \underset{\mathrm{GE}}{\sim} g_\rho$.

The statement (1) of the above theorem follows from Proposition 9.5 and Theorem 9.8-(2). Note that the assumption of g_d being thick is satisfied since the condition (TH) implies every exponential weight function is thick by Theorem 8.3. The statement (2) is immediate from Corollary 12.7.

In [2, Section 17], they have shown that the visual metric is quasisymmetric to the chordal metric which is bi-Lipschitz equivalent to the standard geodesic metric d_S on S^2 in the case of certain class of expanding Thurston maps. In view of their proof, they have essentially shown a counterpart of the condition given in Theorem 2.12-(2).

3 Tree with a reference point

In this section, we review basic notions and notations on a tree with a reference point.

Definition 3.1. Let T be a countably infinite set and let $\mathcal{A} : T \times T \to \{0, 1\}$ which satisfies $\mathcal{A}(w, v) = \mathcal{A}(v, w)$ and $\mathcal{A}(w, w) = 0$ for any $w, v \in T$. We call the pair $(T, \mathcal{A})$ a (non-directed) graph with the vertices T and the adjacent matrix $\mathcal{A}$.
(1) Define $V(w) = \{v|\mathcal{A}(w, v) = 1\}$ and call it the neighborhood of w. $(T, \mathcal{A})$ is said to be locally finite if $V(w)$ is a finite set for any $w \in T$.
(2) For $w_0, \ldots, w_n \in T$, $(w_0, w_1, \ldots, w_n)$ is called a path between w_0 and w_n if $\mathcal{A}(w_i, w_{i+1}) = 1$ for any $i = 0, 1, \ldots n - 1$. A path $(w_0, w_1, \ldots, w_n)$ is called simple if and only if $w_i \neq w_i$ for any i, j with $0 \leq i < j \leq n$ and $(i, j) \neq (0, n)$.
(3) $(T, \mathcal{A})$ is called a (non-directed) tree if and only if there exists a unique simple path between w and v for any $w, v \in T$ with $w \neq v$. For a tree $(T, \mathcal{A})$, the unique simple path between two vertices w and v is called the geodesic between w and v and denoted by $\overline{wv}$. We write $u \in \overline{wv}$ if $\overline{wv} = (w_0, w_1, \ldots, w_n)$ and $u = w_i$ for some i.

In this paper, we always fix a point in a tree as the root of the tree and call the point the reference point.

Definition 3.2. Let $(T, \mathcal{A})$ be a tree and let $\phi \in T$. The triple $(T, \mathcal{A}, \phi)$ is called a tree with a reference point ϕ.
(1) Define $\pi : T \to T$ by

$$\pi(w) = \begin{cases} w_{n-1} & \text{if } w \neq \phi \text{ and } \overline{\phi y} = (w_0, w_1, \ldots, w_{n-1}, w_n), \\ \phi & \text{if } w = \phi \end{cases}$$

and set $S(w) = V(w) \backslash \{\pi(w)\}$.
(2) For $w \in T$, we define $|w| = n$ if and only if $\overline{\phi w} = (w_0, w_1, \ldots, w_n)$. Moreover, we set $(T)_m = \{w|w \in T, |w| = m\}$.
(4) An infinite sequence of vertices $(w_0, w_1, \ldots)$ is called an infinite geodesic ray originated from w_0 if and only if $(w_0, \ldots, w_n) = \overline{w_0 w_n}$ for any $n \geq 0$. Two infinite geodesic rays $(w_0, w_1, \ldots)$ and $(v_0, v_1, \ldots)$ are equivalent if and only if

there exists $k \in \mathbb{Z}$ such that $w_{n+k} = v_n$ for sufficiently large n. An equivalent class of infinite geodesic rays is called an end of T. We use Σ to denote the collection of ends of T.
(5) Define Σ^w as the collection of infinite geodesic rays originated from $w \in T$. For any $v \in T$, Σ_v^w is defined as the collection of elements of Σ^w passing through v, namely

$$\Sigma_v^w = \{(w, w_1, \ldots)|(w, w_1, \ldots) \in \Sigma^w, w_n = v \text{ for some } n \geq 1\}$$

Remark. Strictly, the notations like π and $|\cdot|$ should be written as $\pi^{(T,\mathcal{A},\phi)}$ and $|\cdot|_{(T,\mathcal{A},\phi)}$ respectively. In fact, if we will need to specify the tree in question, we are going to use such explicit notations.

One of the typical exapmple of a tree is the infinite binary tree. In the next example, we present a class of trees where $\#(S(w))$ is independent of $w \in T$.

Example 3.3. Let $N \geq 2$ be an integer. Let $T_m^{(N)} = \{1, \ldots, N\}^m$ for $m \geq 0$. (We let $T_0^{(N)} = \{\phi\}$, where ϕ represents an empty sequence.) We customarily write $(i_1, \ldots, i_m) \in T_m^{(N)}$ as $i_1 \ldots i_m$. Define $T^{(N)} = \cup_{m \geq 0} T_m^{(N)}$. Define $\pi : T^{(N)} \to T^{(N)}$ by $\pi(i_1 \ldots i_{m+1}) = i_1 \ldots i_m$ for $m \geq 0$ and $\pi(\phi) = \phi$. Furthermore, define

$$\mathcal{A}_{wv}^{(N)} = \begin{cases} 1 & \text{if } w \neq v, \text{ and either } \pi(w) = v \text{ or } \pi(v) = w, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(T^{(N)}, \mathcal{A}^{(N)}, \phi)$ is a locally finite tree with a reference point ϕ. In particular, $(T^{(2)}, \mathcal{A}^{(2)}, \phi)$ is called the infinite binary tree. In this case, $S(i_1 \ldots i_m) = \{i_1 \ldots i_m i_{m+1} | i_{m+1} \in \{1, \ldots, N\}\}$.

It is easy to see that for any infinite geodesic ray $(w_0, w_1, \ldots)$, there exists a geodesic ray originated from ϕ that is equivalent to $(w_0, w_1, \ldots)$. In fact, adding a geodesic $\overline{\phi w_0}$ to $(w_0, w_1, \ldots)$ and removing a loop, one can obtain the infinite geodesic ray having required property. This fact shows the following proposition.

Proposition 3.4. *There exists a natural bijective map from Σ to Σ^ϕ.*

Through this map, we always identify the collection of ends Σ and the collection of infinite geodesic rays originated from ϕ, Σ^ϕ.

Hereafter in this paper, we always assume that $(T, \mathcal{A})$ is a locally finite tree and fix a reference point $\phi \in T$. Accordingly, we omit ϕ in the notations and use Σ, and Σ_v in place of Σ^ϕ and Σ_v^ϕ respectively.

Example 3.5. Let $N \geq 2$ be an integer. In the case of $(T^{(N)}, \mathcal{A}^{(N)}, \phi)$ defined in Example 3.3, the collection of the ends Σ is $\Sigma^{(N)} = \{1, \ldots, N\}^{\mathbb{N}} = \{i_1 i_2 i_3 \ldots, | i_j \in \{1, \ldots, N\} \text{ for any } m \in \mathbb{N}\}$. With the natural product topology, $\Sigma^{(N)}$ is a Cantor set, i.e. perfect and totally disconnected.

Definition 3.6. Let $(T, \mathcal{A}, \phi)$ be a locally finite tree with a reference point ϕ.
(1) For $\omega = (w_0, w_1, \ldots) \in \Sigma$, we define $[\omega]_m$ by $[\omega]_m = w_m$ for any $m \geq 0$. Moreover, let $w \in T$. If $\overline{\phi w} = (w_0, w_1, \ldots, w_{|w|})$, then for any $0 \leq m \leq |w|$, we define $[w]_m = w_m$. For $w \in T$, we define

$$T_w = \{v | v \in T, |v| \geq |w|, \text{ and } [v]_{|w|} = w\}$$

(2) For $w, v \in T$, we define the confluence of w and v, $w \wedge v$, by

$$w \wedge v = w_{\max\{i | i=0,\ldots,|w|, [w]_i = [v]_i\}}$$

(3) For $\omega, \tau \in \Sigma$, if $\omega \neq \tau$, we define the confluence of ω and τ, $\omega \wedge \tau$, by

$$\omega \wedge \tau = [\omega]_{\max\{m | [\omega]_m = [\tau]_m\}}$$

(4) For $\omega, \tau \in \Sigma$, we define $\rho_*(\omega, \tau) \geq 0$ by

$$\rho_*(\omega, \tau) = \begin{cases} 2^{-|\omega \wedge \tau|} & \text{if } \omega \neq \tau, \\ 0 & \text{if } \omega = \tau. \end{cases}$$

It is easy to see that ρ is a metric on Σ and $\{\Sigma_{[\omega]_m}\}_{m \geq 0}$ is a fundamental system of neighborhood of $\omega \in \Sigma$. Moreover, $\{\Sigma_v\}_{v \in T}$ is a countable base of open sets. About this base of open sets, we have the following property.

Lemma 3.7. *Let $(T, \mathcal{A}, \phi)$ be a locally finite tree with a reference point ϕ. Then for any $w, v \in T$, $\Sigma_w \cap \Sigma_v = \emptyset$ if and only if $|w \wedge v| < |w|$ and $|w \wedge v| < |v|$. Furthermore, $\Sigma_w \cap \Sigma_v \neq \emptyset$ if and only if $\Sigma_v \subseteq \Sigma_w$ or $\Sigma_w \subseteq \Sigma_v$.*

Proof. If $|w \wedge v| = |w|$, then $w = w \wedge v$ and hence $w \in \overline{\phi v}$. Therefore $\Sigma_v \subseteq \Sigma_w$. So, $\Sigma_w \cap \Sigma_v \neq \emptyset$. Conversely, if $\omega \in \Sigma_w \cap \Sigma_v$, then there exist $m, n \geq 0$ such that $w = [\omega]_m$ and $v = [\omega]_m$. It follows that

$$w \wedge v = \begin{cases} w & \text{if } m \leq n, \\ v & \text{if } m \leq n. \end{cases}$$

Hence we see that $|w \wedge v| = |w|$ or $|w \wedge v| = |v|$. $\square$

With the help to the above proposition, we may easily verify the following well-known fact. The proof is standard and left to the readers.

Proposition 3.8. *If $(T, \mathcal{A}, \phi)$ is a locally finite tree with a reference point ϕ. Then $\rho_*(\cdot, \cdot)$ is a metric on Σ and (Σ, ρ) is compact and totally disconnected. Moreover, if $\#(S(w)) \geq 2$ for any $w \in T$, then (Σ, ρ) is perfect.*

By the above proposition, if $\#(S(w)) \geq 2$ for any $w \in T$, then Σ is (homeomorphic to) the Cantor set.

4 Partition

In this section, we exactly formulate the notion of a partition introduced in Section 1. A partition is a map from a tree to the collection of nonempty compact subsets of a compact metrizable space and required to preserve natural hierarchical structure of the tree. Consequently, a partition induces a surjective map from the Cantor set, i.e. the collection of ends of the tree, to the compact metrizable space.

Throughout this section, $\mathcal{T} = (T, A, \phi)$ is a locally finite tree with a reference point ϕ.

Definition 4.1 (Partition)**.** Let $(X, \mathcal{O})$ be a compact metrizable space, where $\mathcal{O}$ is the collection of open sets, and let $\mathcal{C}(X, \mathcal{O})$ be the collection of nonempty compact subsets of X. If no confusion can occur, we write $\mathcal{C}(X)$ in place of $\mathcal{C}(X, \mathcal{O})$.
(1) A map $K : T \to \mathcal{C}(X, \mathcal{O})$, where we customarily denote $K(w)$ by K_w for simplicity, is called a partition of X parametrized by $(T, \mathcal{A}, \phi)$ if and only if it satisfies the following conditions (P1) and (P2), which correspond to (1.3) and (1.4) respectively.
(P1) $K_\phi = X$ and for any $w \in T$,

$$K_w = \bigcup_{v \in S(w)} K_v.$$

and $K_w \neq K_v$ for any $v \in S(w)$.
(P2) For any $\omega \in \Sigma$, $\cap_{m \geq 0} K_{[\omega]_m}$ is a single point.
(2) Let $K : T \to \mathcal{C}(X, \mathcal{O})$ be a partition of X parametrized by $(T, \mathcal{A}, \phi)$. Define O_w and B_w for $w \in T$ by

$$O_w = K_w \backslash \left(\bigcup_{v \in (T)_{|w|} \backslash \{w\}} K_v \right),$$

$$B_w = K_w \cap \left(\bigcup_{v \in (T)_{|w|} \backslash \{w\}} K_v \right).$$

If $O_w \neq \emptyset$ for any $w \in T$, then the partition K is called minimal.
(3) Let $K : T \to \mathcal{C}(X, \mathcal{O})$ be a partition of X. Then $(w(1), \ldots, w(m)) \in \cup_{k \geq 0} T^k$ is called a K-chain (or chain for short if no confusion can occur) if and only if $K_{w(i)} \cap K_{w(i+1)} \neq \emptyset$ for any $i = 1, \ldots, m-1$. A K-chain $(w(1), \ldots, w(m))$ is called a K-chain in $\Lambda \subseteq T$ if $w(i) \in \Lambda$ for any $i = 1, \ldots, m$. For subsets $A, B \subseteq X$, A K-chain $(w(1), \ldots, w(m))$ is called a K-chain between A and B if and only if $A \cap K_{w(1)} \neq \emptyset$ and $B \cap K_{w(m)} \neq \emptyset$. We use $\mathcal{CH}_K(A, B)$ to denote the collection of K-chains between x and y. Moreover, we denote the collections of K-chains in Λ between A and B by $\mathcal{CH}_K^\Lambda(A, B)$.

Remark. Since $K_w \neq K_v$ for any $v \in S(w)$ by the condition (P1), it follows that $\#(S(w)) \geq 2$ for any $w \in T$ and $(X, \mathcal{O})$ has no isolated point.

As is shown in Theorem 4.6, a partition can be modified so as to be minimal by restricting it to a suitable subtree.

The next lemma is an assortment of direct consequences from the definition of the partition.

Lemma 4.2. *Let $K : T \to \mathcal{C}(X, \mathcal{O})$ be a partition of X parametrized by (T, A, ϕ).*
(1) *For any $w \in T$, O_w is an open set. $O_v \subseteq O_w$ for any $v \in S(w)$.*
(2) *$O_w \cap O_v = \emptyset$ if $w, v \in T$ and $\Sigma_w \cap \Sigma_v = \emptyset$.*
(3) *If $\Sigma_w \cap \Sigma_v = \emptyset$, then $K_w \cap K_v = B_w \cap B_v$.*

Proof. (1) Note that by (P1), $X = \cup_{w \in (T)_m} K_w$. Hence

$$O_w = K_w \backslash (\cup_{v \in (T)_{|w|} \backslash \{w\}} K_v) = X \backslash (\cup_{v \in (T)_{|w|} \backslash \{w\}} K_v).$$

The rest of the statement is immediate from the property (P2).
(2) By Lemma 3.7, if $u = w \wedge v$, then $|u| < |w|$ and $|u| < |v|$. Let $w' = [w]_{|u|+1}$ and let $v' = [v]_{|u|+1}$. Then $w', v' \in S(u)$ and $w' \neq v'$. Since $O_{w'} \subseteq K_{w'} \backslash K_{v'}$, it follows that $O_{w'} \cap O_{v'} = \emptyset$. Using (1), we see $O_w \cap O_v = \emptyset$.
(3) This follows immediately by (1). □

The condition (P2) provides a natural map from the ends of the tree Σ to the space X.

Proposition 4.3. *Let $K : T \to \mathcal{C}(X, \mathcal{O})$ be a partition of X parametrized by $(T, \mathcal{A}, \phi)$.*
(1) *For $\omega \in \Sigma$, define $\sigma(\omega)$ as the single point $\cap_{m \geq 0} K_{[\omega]_m}$. Then $\sigma : \Sigma \to X$ is continuous and surjective. Moreover. $\sigma(\Sigma_w) = K_w$ for any $w \in T$.*
(2) *The partition $K : T \to \mathcal{C}(X, \mathcal{O})$ is minimal if and only if K_w is the closure of O_w for any $w \in T$. Moreover, if $K : T \to \mathcal{C}(X, \mathcal{O})$ is minimal then O_w coincides with the interior of K_w.*

Proof. (1) Note that $K_w = \cup_{v \in S(w)} K_v$. Hence if $x \in K_w$, there exists $v \in S(w)$ such that $x \in K_v$. Using this fact inductively, we see that, for any $x \in X$, there exists $\omega \in \Sigma$ such that $x \in K_{[\omega]_m}$ for any $m \geq 0$. Since $x \in \cap_{m \geq 0} K_{[\omega]_m}$, (P2) shows that $\sigma(\omega) = x$. Hence ω is surjective. At the same time, it follows that $\sigma(\Sigma_w) = K_w$. Let U be an open set in X. For any $\omega \in \sigma^{-1}(U)$, $K_{[\omega]_m} \subseteq U$ for sufficiently large m. Then $\Sigma_{[\omega]_m} \subseteq \sigma^{-1}(U)$. This shows that $\sigma^{-1}(U)$ is an open set and hence σ is continuous.
(2) Let $\overline{O}_w$ be the closure of O_w. If $K_w = \overline{O}_w$ for any $w \in T$, then $O_w \neq \emptyset$ for any $w \in T$ and hence $K : T \to \mathcal{C}(X, \mathcal{O})$ is minimal. Conversely, assume that $K : T \to \mathcal{C}(X, \mathcal{O})$ is minimal. By Lemma 4.2, $\overline{O}_{[\omega]_m} \supseteq \overline{O}_{[\omega]_{m+1}}$ for any $\omega \in \Sigma$ and any $m \geq 0$. Hence $\{\sigma(\omega)\} = \cap_{m \geq 0} K_{[\omega]_m} = \cap_{m \geq 0} \overline{O}_{[\omega]_m} \subseteq \overline{O}_{[\omega]_n}$ for any $n \geq 0$. This yields that $\sigma(\Sigma_w) \subseteq \overline{O}_w$. Since $\sigma(\Sigma_w) = K_w$, this implies $\overline{O}_w = K_w$.
Now if K is minimal, since O_w is open by Lemma 4.2-(1), it follows that O_w is the interior of K_w. □

Definition 4.4. A partition $K : T \to \mathcal{C}(X, \mathcal{O})$ parametrized by a tree $(T, \mathcal{A}, \phi)$ is called strongly finite if and only if

$$\sup_{x \in X} \#(\sigma^{-1}(x)) < +\infty,$$

where $\sigma : \Sigma \to X$ is the map defined in Proposition 4.3-(1).

Example 4.5. Let (Y, d) be a complete metric space and let $\{F_1, \ldots, F_N\}$ be collection of contractions from (Y, d) to itself, i.e. $F_i : Y \to Y$ and

$$\sup_{x \neq y \in Y} \frac{d(F_i(x), F_i(y))}{d(x, y)} < 1$$

for any $i = 1, \ldots, N$. Then it is well-known that there exists a unique nonempty compact set X such that

$$X = \bigcup_{i=1,\ldots,N} F_i(X).$$

See [8, Section 1.1] for a proof of this fact for example. X is called the self-similar set associated with $\{F_1, \ldots, F_N\}$. Let $(T^{(N)}, \mathcal{A}^{(N)}, \phi)$ be the tree defined in Example 3.3. For any $i_1 \ldots i_m \in T$, set $F_{i_1 \ldots i_m} = F_{i_1} \circ \ldots \circ F_{i_m}$ and define $K_w = F_w(X)$. Then $K : T^{(N)} \to \mathcal{C}(X)$ is a partition of K parametrized by $(T^{(N)}, \mathcal{A}^{(N)}, \phi)$. See [8, Section 1.2]. The associated map from $\Sigma = \{1, \ldots, N\}^{\mathbb{N}}$ to K is sometimes called the coding map. To determine if K is minimal or not is known to be rather delicate issue. See [8, Theorem 1.3.8] for example.

Removing unnecessary vertices of the tree, we can always modify the original partition and obtain a minimal one.

Theorem 4.6. *Let $K : T \to \mathcal{C}(X, \mathcal{O})$ be a partition of X parametrized by $(T, \mathcal{A}, \phi)$. There exist $T' \subseteq T$ and $K' : T' \to \mathcal{C}(X, \mathcal{O})$ such that $(T', \mathcal{A}|_{T' \times T'})$ is a tree, $\phi \in T'$, $K'_w \subseteq K_w$ for any $w \in T'$ and $K|_{T'}$ is a minimal partition of X parametrized by $(T', \mathcal{A}', \phi)$.*

Proof. We define a sequence $\{T^{(m)}\}_{m \geq 0}$ of subsets of T and $\{K_w^{(m)}\}_{w \in T^{(m)}}$ inductively as follows. First let $T^{(0)} = T$ and $K_w^{(0)} = K_w$ for any $w \in T^{(0)}$. Suppose we have defined $T^{(m)}$. Define

$$Q^{(m)} = \left\{ w \,\middle|\, w \in T^{(m)}, K_w \subseteq \bigcup_{v \in (T)_{|w|} \cap T^{(m)}, v \neq w} K_v \right\}.$$

If $Q^{(m)} = \emptyset$, then set $T^{(m+1)} = T^{(m)}$ and $K_w^{(m+1)} = K_w^{(m)}$ for any $w \in T^{(m+1)}$. Otherwise choose $w^{(m)} \in Q^{(m)}$ so that $|w^{(m)}|$ attains the minimum of $\{|v| : v \in Q^{(m)}\}$. Then define

$$T^{(m+1)} = T^{(m)} \backslash T_{w^{(m)}}$$

and

$$K_w^{(m+1)} = \begin{cases} \cup_{v \in T_w \cap (T)_{w^{(m)}} \cap T^{(m+1)}} K_v^{(m)} & \text{if } w^{(m)} \in T_w, \\ K_w^{(m)} & \text{otherwise.} \end{cases}$$

In this way, for any $m \geq 0$ and $w \in T^{(m)}$,

$$K_w^{(m)} = \bigcup_{v \in S(w) \cap T^{(m)}} K_v^{(m)}. \tag{4.4}$$

Note that $Q^{(m+1)} \subset Q^{(m)} \backslash \{w^{(m)}\}$. Since $(T)_n$ is a finite set for all $n \geq 0$, it follow that $(T)_n \cap Q^{(m)} = \emptyset$ and $(T^{(m)})_n$ stays the same for sufficiently large m. Hence $|w^{(m)}| \to \infty$ as $m \to \infty$ and $(T)_n \cap T^{(m)}$ does not depend on m for sufficiently large m. Therefore, letting $T' = \cap_{m \geq 1} T^{(m)}$, we see that $(T', \mathcal{A}|_{T' \times T'})$ is a locally finite tree, $\phi \in T'$. Moreover, note that $K_w^{(m+1)} \subseteq K_w^{(m)}$ for any $w \in T'$. Hence if

$$K'_w = \bigcap_{m \geq 0} K_w^{(m)}$$

for any $w \in T'$, then K'_w is nonempty. By (4.4), it follows that

$$K'_w = \bigcup_{v \in T' \cap S(w)} K_v$$

for any $w \in T'$. Now, the map $K' : T' \to \mathcal{C}(X, \mathcal{O})$ given by $K'(w) = K'_w$ is a minimal partition of X parametrized by $(T', \mathcal{A}|_{T' \times T'}, \phi)$. □

A partition $K : T \to \mathcal{C}(X, \mathcal{O})$ induces natural graph structure on T. In the rest of this section, we show that T can be regarded as the hyperbolic filling of X if the induced graph structure is hyperbolic. See [3], for example, about the notion of hyperbolic fillings.

Definition 4.7. Let $K : T \to \mathcal{C}(X, \mathcal{O})$ be a partition. Then define

$$B_m^h = \{(w, v) | w, v \in (T)_m, K_w \cap K_v \neq \emptyset\}$$

and

$$B^h = \bigcup_{m \geq 0} B_m^h.$$

The symbol "h" in the notation B_m^h and B^h represents the word "horizontal". Moreover we define

$$\mathcal{B}(w, v) = \begin{cases} 1 & \text{if } \mathcal{A}(w, v) = 1 \text{ or } (w, v) \in B^h, \\ 0 & \text{otherwise.} \end{cases}$$

The graph $(T, \mathcal{B})$ is called the resolution graph of X associated with the partition $K : T \to \mathcal{C}(X, \mathcal{O})$. We use $d_{(T,\mathcal{B})}(\cdot, \cdot)$ to denote the shortest path metric, i.e.

$$\begin{aligned} d_{(T,\mathcal{B})}(w, v) = \min\{n | &\text{there exists } (w(1), \ldots, w(n+1)) \in (\mathcal{B})^n \\ &\mathcal{B}(w(i), w(i+1)) = 1 \text{ for any } i = 1, \ldots, n\} \end{aligned}$$

Note that if $(\phi, w(1), w(2), \ldots)$ is an infinite geodesic ray in $(T, \mathcal{B})$ associated with $d_{(T,\mathcal{B})}$ starting from ϕ, then $(\phi, w(1), w(2), \ldots) = (\phi, [\omega]_1, [\omega]_2, \ldots)$ for some $\omega \in \Sigma$.

Proposition 4.8. *Let $\omega, \tau \in \Sigma$. If $\sup_{n \geq 1} d_{(T,\mathcal{B})}([\omega]_n, [\tau]_n) < +\infty$, then $\sigma(\omega) = \sigma(\tau)$.*

By this proposition, if the resolution graph $(T, \mathcal{B})$ is hyperbolic in the sense of Gromov, then X is the hyperbolic boundary of $(T, \mathcal{B})$. In other words, $(T, \mathcal{B})$ is the hyperbolic filling of X.

In fact, in the case of self-similar sets introduced in Example 4.5, Lau and Wang have shown that the resolution graph $(T, \mathcal{B})$ is hyperbolic if the self-similar set satisfies the open set condition in [12].

5 Weight function and associated "visual pre-metric"

Throughout this section, $(T, \mathcal{A}, \phi)$ is a locally finite tree with a reference point ϕ, $(X, \mathcal{O})$ is a compact metrizable topological space and $K : T \to \mathcal{C}(X, \mathcal{O})$ is a partition of X parametrized by $(T, \mathcal{A}, \phi)$.

In this section, we introduce the notion of a weight function, which assigns each vertex of the Tree T a "size" or "weight". Then, through a partition, we will induce a kind of "balls" and "distances" to a compact metric space associated with the weight function.

Definition 5.1 (Weight function)**.** A function $g : T \to (0, 1]$ is called a weight function if and only if it satisfies the following conditions (G1), (G2) and (G3):
(G1) $g(\phi) = 1$
(G2) For any $w \in T$, $g(\pi(w)) \geq g(w)$
(G3) $\lim_{m \to \infty} \sup_{w \in (T)_m} g(w) = 0$.
We denote the collection of all the weight functions by $\mathcal{G}(T)$. Let g be a weight function. We define

$$\Lambda_s^g = \{w | w \in T, g(\pi(w)) > s \geq g(w)\}$$

for any $s \in (0, 1]$. $\{\Lambda_s^g\}_{s \in (0,1]}$ is called the scale associated with g. For $s > 1$, we define $\Lambda_s^g = \{\phi\}$.

Remark. To be exact, one should use $\mathcal{G}(T, A, \phi)$ rather than $\mathcal{G}(T)$ as the notation for the collection of all the weight functions because the notion of weight function apparently depends not only on the set T but also the structure of T as a tree. We use, however, $\mathcal{G}(T)$ for simplicity as long as no confusion may occur.

Remark. In the case of the partitions associated with a self-similar set appearing in Example 4.5, the counterpart of weight functions was called gauge functions in [9]. Also $\{\Lambda_s^g\}_{0 < s \leq 1}$ was called the scale associated with the gauge function g.

Given a weight function g, we consider $g(w)$ as a virtual "size" or "diameter" of Σ_w for each $w \in T$. The set Λ_s^g is the collection of subsets Σ_w's whose sizes are approximately s.

Proposition 5.2. *Suppose that* $g : T \to (0, 1]$ *satisfies* (G1) *and* (G2). g *is a weight function if and only if*

$$\lim_{m\to\infty} g([\omega]_m) = 0 \tag{5.5}$$

for any $\omega \in \Sigma$.

Proof. If g is a weight function, i.e. (G3) holds, then (5.5) is immediate.

Suppose that (G3) does not hold, i.e. there exists $\epsilon > 0$ such that

$$\sup_{w\in(T)_m} g(w) > \epsilon \tag{5.6}$$

for any $m \geq 0$. Define $Z = \{w|w \in T, g(w) > \epsilon\}$ and $Z_m = (T)_m \cap Z$. By (5.6), $Z_m \neq \emptyset$ for any $m \geq 0$. Since $\pi(w) \in Z$ for any $w \in Z$, if $Z_{m,n} = \pi^{n-m}(Z_n)$, where π^k is the k-th iteration of π for $k \in \mathbb{N}$, for $n \geq m$, then $Z_{m,n} \neq \emptyset$ and $Z_{m,n} \supseteq Z_{m,n+1}$ for any $n \geq m$. Set $Z^*_m = \cap_{n\geq m} Z_{m,n}$. Since $(T)_m$ is a finite set and so is $Z_{m,n}$, we see that $Z^*_m \neq \emptyset$ and $\pi(Z^*_{m+1}) = Z^*_m$ for any $m \geq 0$. Note that $Z^*_0 = \{\phi\}$. Inductively, we may construct a sequence $(\phi, w(1), w(2), \ldots)$ satisfying $\pi(w(m + 1)) = w(m)$ and $w(m) \in Z^*_m$ for any $m \geq 0$. Set $\omega = (\phi, w(1), w(2), \ldots)$. Then $\omega \in \Sigma$ and $g([\omega]_m) \geq \epsilon$ for any $m \geq 0$. This contradicts to (5.5). □

Proposition 5.3. *Let* $g : T \to (0, 1]$ *be a weight function and let* $s \in (0, 1]$. *Then*

$$\bigcup_{w\in\Lambda^g_s} \Sigma_w = \Sigma \tag{5.7}$$

and if $w, v \in \Lambda^g_s$ *and* $w \neq v$, *then*

$$\Sigma_w \cap \Sigma_v = \emptyset.$$

Proof. For any $\omega = (w_0, w_1, \ldots) \in \Sigma$, $\{g(w_i)\}_{i=0,1,\ldots}$ is monotonically non-increasing sequence converging to 0 as $i \to \infty$. Hence there exists a unique $m \geq 0$ such that $g(w_{m-1}) > s \geq g(w_m)$. Therefore, there exists a unique $m \geq 0$ such that $[\omega]_m \in \Lambda^g_s$. Now (5.7) is immediate. Assume $w, v \in \Lambda^g_s$ and $\Sigma_v \cap \Sigma_w \neq \emptyset$. Choose $\omega = (w_0, w_1, \ldots) \in \Sigma_v \cap \Sigma_w$. Then there exist $m, n \geq 0$ such that $[\omega]_m = w_m = w$ and $[\omega]_n = w_n = v$. By the above fact, we have $m = n$ and hence $w = v$. □

Introducing a partition $K : T \to \mathcal{C}(X, \mathcal{O})$, one can define weight functions naturally associated with metrics and measures as follows.

Notation. Let d be a metric on X. We define the diameter of a subset $A \subseteq X$ with respect to d, $\mathrm{diam}(A, d)$ by $\mathrm{diam}(A, d) = \sup\{d(x, y)|x, y \in A\}$. Moreover, for $x \in X$ and $r > 0$, we set $B_d(x, r) = \{y|y \in X, d(x, y) < r\}$.

Definition 5.4. (1) Define

$$\mathcal{D}(X,\mathcal{O}) = \{d | d \text{ is a metric on } X \text{ inducing the topology } \mathcal{O} \text{ and } \operatorname{diam}(X,d) = 1\}$$

For $d \in \mathcal{D}(X,\mathcal{O})$, define $g_d : T \to (0,1]$ by $g_d(w) = \operatorname{diam}(K_w, d)$ for any $w \in T$.
(2) Define

$$\mathcal{M}_P(X,\mathcal{O}) = \{\mu | \mu \text{ is a Radon probability measure on } (X,\mathcal{O}) \text{ satisfying } \mu(\{x\}) = 0 \text{ for any } x \in X \text{ and } \mu(K_w) > 0 \text{ for any } w \in T\}$$

For $\mu \in \mathcal{M}_P(X,\mathcal{O})$, define $g_\mu : T \to (0,1]$ be $g_\mu(w) = \mu(K_w)$ for any $w \in T$.

The condition $\operatorname{diam}(X,d) = 1$ in the definition of $\mathcal{D}(X,\mathcal{O})$ is only for the purpose of normalization. Note that since $(X,\mathcal{O})$ is compact, if a metric d on X induces the topology $\mathcal{O}$, then $\operatorname{diam}(X,d) < +\infty$.

Proposition 5.5. (1) *For any $d \in \mathcal{D}(X,\mathcal{O})$, g_d is a weight function.*
(2) *For any $\mu \in \mathcal{M}_P(X,\mathcal{O})$, g_μ is a weight function.*

Proof. (1) The properties (G1) and (G2) are immediate from the definition of g_d. Suppose there exists $\omega \in \Sigma$ such that

$$\lim_{m\to\infty} g_d([\omega]_m) > 0 \tag{5.8}$$

Let ϵ be the above limit. Since $g_d([\omega]_m) = \operatorname{diam}(K_{[\omega]_m}, d) > \epsilon$, there exist $x_m, y_m \in K_{[\omega]_m}$ such that $d(x_m, y_m) \geq \epsilon$. Note that $K_{[\omega]_m} \supseteq K_{[\omega]_{m+1}}$ and hence $x_n, y_n \in K_{[\omega]_m}$ if $n \geq m$. Since X is compact, there exist subsequences $\{x_{n_i}\}_{i\geq 1}$, $\{y_{n_i}\}_{i\geq 1}$ converging to x and y as $i \to \infty$ respectively. It follows that $x, y \in \cap_{m\geq 0} K_{[\omega]_m}$ and $d(x,y) \geq \epsilon > 0$. This contradicts to (P2). Thus we have shown (5.5). By Proposition 5.2, g_d is a weight function. (2) As in the case of metrics, (G1) and (G2) are immediate. Let $\omega \in \Sigma$. Then $\cap_{m\geq 0} K_{[\omega]_m} = \{\sigma(\omega)\}$. Therefore, $g_\mu([\omega]_m) = \mu(K_{[\omega]_m}) \to 0$ as $m \to \infty$. Hence we verify (5.5). Thus by Proposition 5.2, g_μ is a weight function. □

The weight function g_d and g_μ are called the weight functions associated with d and μ respectively. Although the maps $d \to g_d$ and $\mu \to g_\mu$ are not injective at all, we sometimes abuse notations and use d and μ to denote g_d and g_μ respectively.

Through a partition we introduce the notion of "balls" of a compact metric space associated with a weight function.

Definition 5.6. Let $g : T \to (0,1]$ be a weight function.
(1) For $s \in (0,1], w \in \Lambda_s^g, M \geq 0$ and $x \in X$, we define

$$\Lambda_{s,M}^g(w) = \{v | v \in \Lambda_s^g, \text{there exists a } K\text{-chain } (w(1),\ldots,w(k)) \text{ in } \Lambda_s^g \text{ such that } w(1) = w,\ w(k) = v \text{ and } k \leq M+1\}$$

and

$$\Lambda^g_{s,M}(x) = \bigcup_{w \in \Lambda^g_s \text{ and } x \in K_w} \Lambda^g_{s,M}(w).$$

For $x \in X$, $s \in (0,1]$ and $M \geq 0$, define

$$U^g_M(x,s) = \bigcup_{w \in \Lambda^g_{s,M}(x)} K_w.$$

We let $U^g_M(x,s) = X$ if $s \geq 1$.

The family $\{U^g_M(x,s)\}_{s>0}$ is a fundamental system of neighborhood of $x \in X$ as is shown in Proposition 5.7.

Note that

$$\Lambda^g_{s,0}(w) = \{w\} \quad \text{and} \quad \Lambda^g_{s,1}(w) = \{v | v \in \Lambda^g_s, K_v \cap K_w \neq \emptyset\}$$

for any $w \in \Lambda^g_s$ and

$$\Lambda^g_{s,0}(x) = \{w | w \in \Lambda^g_s, x \in K_w\} \quad \text{and} \quad U^g_0(x,s) = \bigcup_{w : w \in \Lambda^g_s, x \in K_w} K_w$$

for any $x \in T$. Moreover,

$$U^g_M(x,s) = \{y | y \in X, \text{there exists } (w(1), \ldots, w(M+1)) \in \mathcal{CH}^{\Lambda^g_s}_K(x,y).\}$$

Proposition 5.7. *Let K be a partition of X parametrized by $(T, \mathcal{A}, \phi)$ and let $g : T \to (0,1]$ be a weight function. For any $s \in (0,1]$ and any $x \in X$, $U^g_0(x,s)$ is a neighborhood of x. Furthermore, $\{U^g_M(x,s)\}_{s \in (0,1]}$ is a fundamental system of neighborhood of x for any $x \in X$.*

Proof. Let d be a metric on X giving the original topology of $(X, \mathcal{O})$. Assume that for any $r > 0$, there exists $y \in B_d(x,r)$ such that $y \notin U^g_0(x,s)$. Then there exists a sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $x_n \to x$ and $x_n \notin U^g_0(x,s)$ for any $n \geq 1$. Since Λ^g_s is a finite set, there exists $w \in \Lambda_s$ which includes infinite members of $\{x_n\}_{n \geq 1}$. By the closedness of K_w, it follows that $x \in K_w$ and $x_n \in K_w \subseteq U^g_0(x,s)$. This contradiction shows that $U^g_0(x,s)$ contains $B_d(x,r)$ for some $r > 0$.
Next note $\min_{w \in \Lambda^g_s} |w| \to \infty$ as $s \downarrow 0$. This along with that fact that g_d is a weight function implies that $\max_{w \in \Lambda^g_s} \mathrm{diam}(K_w, d) \to 0$ as $s \downarrow 0$. Set $\rho_s = \max_{w \in \Lambda^g_s} \mathrm{diam}(K_w, d)$. Then $\mathrm{diam}(U^g_M(x,s), d) \leq (M+1)\rho_s \to 0$ as $s \downarrow 0$. This implies that $\cap_{s \in (0,1]} U^g_M(x,s) = \{x\}$. Thus we have shown that $\{U^g_M(x,s)\}_{s \in (0,1]}$ is a fundamental system of neighborhoods of x. $\square$

We regard $U^g_M(x,s)$ as a virtual "ball' of radius s and center x. In fact, there exists a kind of "pre-metric" $\delta^g_M : X \times X \to [0, \infty)$ such that $\delta^g_M(x,y) > 0$ if and only if $x \neq y$, $\delta^g_M(x,y) = \delta^g_M(y,x)$ and

$$U^g_M(x,s) = \{y | \delta^g_M(x,y) \leq s\}. \tag{5.9}$$

As is seen in the next section, however, the pre-metric δ^g_M does not satisfy the triangle inequality in general.

Definition 5.8. Let $M \geq 0$. Define $\delta^g_M(x, y)$ for $x, y \in X$ by

$$\delta^g_M(x, y) = \inf\{s|s \in (0, 1], y \in U^g_M(x, s)\}.$$

The pre-metric δ^g_M can be thought of as a counterpart of the "visual metric" studied in [2]. Indeed, if there exists $\lambda \in (0, 1)$ such that $g(w) = \lambda^{|w|}$ for any $w \in T$, then

$$\delta^g_M(x, y) = \lambda^{n_M(x,y)},$$

where

$$n_M(x, y) = \max\{n|\text{there exists a chain } (w(1), \ldots, w(M + 1)) \in \mathcal{CH}^{(T)_n}_K(x, y)\}.$$

Proposition 5.9. *For any $M \geq 0$ and any $x, y \in X$,*

$$\delta^g_M(x, y) = \min\{s|s \in (0, 1], y \in U^g_M(x, s)\}.$$

In particular, (5.9) *holds for any $M \geq 0$ and $s \in (0, 1]$.*

Proof. The property (G3) implies that for any $s_* \in (0, 1]$, there exists $n \geq 0$ such that $\cup_{s \geq s_*} \Lambda^g_s \subseteq \cup^n_{m=0}(T)_m$. Hence $\{(w(1), \ldots, w(M + 1))|w(i) \in \cup_{s \geq s_*} \Lambda^g_s\}$ is finite. Let $s_* = \delta^g_M(x, y)$. Then there exist a sequence $\{s_m\}_{m \geq 1} \subseteq [s_*, 1]$ and $(w_m(1), \ldots, w_m(M + 1)) \in (\Lambda^g_{s_m})^{M+1}$ such that $\lim_{m \to \infty} s_m = s_*$ and $(w_m(1), \ldots, w_m(M + 1))$ is a chain between x and y for any $m \geq 1$. Since $\{(w(1), \ldots, w(M+1))|w(i) \in \cup_{s \geq s_*} \Lambda^g_s\}$ is finite, there exists $(w_*(1), \ldots, w_*(M+1))$ such that $(w_*(1), \ldots, w_*(M + 1)) = (w_m(1), \ldots, w_m(M + 1))$ for infinitely many m. For such m, we have $g(\pi(w_*(i))) > s_m \geq g(w_*(i))$ for any $i = 1, \ldots, M + 1$. This implies that $w_*(i) \in \Lambda^g_{s_*}$ for any $i = 1, \ldots, M + 1$ and hence $y \in U^g_M(x, s_*)$. Thus we have shown (1).

□

6 Metrics adapted to weight function

In this section, we consider the first question mentioned in the introduction, which is when a weight function is naturally associated with a metric. Our answer will be given in Theorem 6.11.

The purpose of the next definition is to clarify when the virtual balls $U^g_M(x, s)$ induced by a weight function g can be though of as real "balls" derived from a metric.

Definition 6.1. Let $M \geq 0$. A metric $d \in (X, \mathcal{O})$ is said to be M-adapted to g if and only if there exist $\alpha_1, \alpha_2 > 0$ such that

$$U^g_M(x, \alpha_1 r) \subseteq B_d(x, r) \subseteq U^g_M(x, \alpha_2 r)$$

for any $x \in X$ and any $r > 0$. d is said to be adapted to g if and only if d is M-adapted to g for some $M \geq 0$.

Now the exact meaning of "when a weight function is naturally associated with a metric" is when there is a metric d which is adapted to given weight function g. The number M really makes a difference in the above definition. Namely, in Example 10.9, we construct an example where a weight function does not have any 1-adapted metric but has 2-adapted metric.

By (5.9), a metric $d \in \mathcal{D}(X, \mathcal{O})$ is M-adapted to a weight function g if and only if there exist $c_1, c_2 > 0$ such that

$$c_1 \delta_M^g(x,y) \le d(x,y) \le c_2 \delta_M^g(x,y) \tag{6.1}$$

for any $x, y \in X$. By this equivalence, we may think of a metric adapted to a weight function as a "visual metric" associated with the weight function.

If a metric d is M-adapted to given weight function g, then we think of the virtual balls $U_M^g(x,s)$ as the real balls associated with the metric d.

There is another "pre-metric" associated with a weight function.

Definition 6.2. Let $M \ge 0$. Define $D_M^g(x,y)$ for $x, y \in X$ by

$$D_M^g(x,y) = \inf\Big\{\sum_{i=1}^k g(w(i))\Big| 1 \le k \le M+1, (w(1), \ldots, w(k)) \in \mathcal{CH}_K(x,y)\Big\}$$

It is easy to see that $D_M^g(x,y) \ge 0$, $D_M^g(x,y) = 0$ if and only if $x = y$ and $D_M^g(x,y) = D_M^g(y,x)$. The pre-metric D_M^g is equivalent to the other one, δ_M^g as follows.

Proposition 6.3. *For any $M \ge 0$ and any $x, y \in X$,*

$$\delta_M^g(x,y) \le D_M^g(x,y) \le (M+1)\delta_M^g(x,y).$$

Proof. Set $s_* = \delta_M^g(x,y)$. Due to Proposition 5.9, it follows that there exists a chain $(w(1), \ldots, w(M+1))$ between x and y such that $w(i) \in \Lambda_{s_*}^g$ for any $i = 1, \ldots, M+1$. Then

$$D_M^g(x,y) \le \sum_{i=1}^{M+1} g(w(i)) \le (M+1)s_*$$

Next set $d_* = D_M^g(x,y)$. For any $\epsilon > 0$, there exists a chain $(w(1), \ldots, w(M+1))$ between x and y such that $\sum_{i=1}^{M+1} g(w(i)) < d_* + \epsilon$. In particular, $g(w(i)) < d_* + \epsilon$ for any $i = 1, \ldots, M+1$. Hence for any $i = 1, \ldots, M+1$, there exists $w_*(i) \in \Lambda_{d_*+\epsilon}^g$ such that $K_{w(i)} \subseteq K_{w_*(i)}$. Since $(w_*(1), \ldots, w_*(M+1))$ is a chain between x and y, it follows that $\delta_M^g(x,y) \le d_* + \epsilon$. Thus we have shown $\delta_M^g(x,y) \le D_M^g(x,y)$. $\square$

Combining the above proposition with (6.1), we see that d is M-adapted to g if and only if there exist $C_1, C_2 > 0$ such that

$$C_1 D_M^g(x,y) \le d(x,y) \le C_2 D_M^g(x,y) \tag{6.2}$$

for any $x, y \in X$.

Next we present another condition which is equivalent to a metric being adapted.

Theorem 6.4. *Let $g : T \to (0,1]$ be a weight function and let $M \geq 0$. If $d \in \mathcal{D}(X, \mathcal{O})$, then d is M-adapted to g if and only if the following conditions* (ADa) *and* $(\mathrm{ADb})_{\mathrm{M}}$ *hold:*
(ADa) *There exists $c > 0$ such that* $\mathrm{diam}(K_w, d) \leq cg(w)$ *for any $w \in T$.*
$(\mathrm{ADb})_{\mathrm{M}}$ *For any $x, y \in X$, there exists $(w(1), \ldots, w(k)) \in \mathcal{CH}_K(x, y)$ such that $1 \leq k \leq M + 1$ and*

$$Cd(x, y) \geq \max_{i=1,\ldots,k} g(w(i)),$$

where $C > 0$ is independent of x and y.

Remark. In [2, Proposition 8.4], one find an analogous result in the case of partitions associated with expanding Thurston maps. The condition (ADa) and $(ADb)_M$ corresponds their conditions (ii) and (i) respectively.

Proof. First assuming that (ADa) and $(\mathrm{ADb})_M$ hold, we are going to show (6.1). Let $x, y \in X$. By $(\mathrm{ADb})_{\mathrm{M}}$, there exists a chain $(w(1), \ldots, w(k))$ between x and y such that $1 \leq k \leq M + 1$ and $Cd(x, y) \geq g(w(i))$ for any $i = 1, \ldots, k$. By (G2), there exists $v(i)$ such that $\Sigma_{v(i)} \supseteq \Sigma_{w(i)}$ and $v(i) \in \Lambda^g_{Cd(x,y)}$. Since $(v(1), \ldots, v(k))$ is a chain in $\Lambda^g_{Cd(x,y)}$ between x and y, it follows that $Cd(x, y) \geq \delta^g_M(x, y)$.

Next set $t = \delta^g_M(x, y)$. Then there exists a chain $(w(1), \ldots, w(M + 1)) \in \mathcal{CH}_K(x, y)$ in Λ^g_t. Choose $x_i \in K_{w(i)} \cap K_{w(i+1)}$ for every $i = 1, \ldots, M$. Then

$$\begin{aligned} d(x, y) \leq d(x, x_1) + \sum_{i=1}^{M-1} d(x_i, x_{i+1}) + d(x_M, y) \\ \leq c \sum_{j=1}^{M+1} g(w(i)) \leq c(M + 1)t = c(M + 1)\delta^g_M(x, y). \end{aligned}$$

Thus we have (6.1).

Conversely, assume that (6.1) holds, namely, there exist $c_1, c_2 > 0$ such that $c_1 d(x, y) \leq \delta^g_M(x, y) \leq c_2 d(x, y)$ for any $x, y \in X$. If $x, y \in K_w$, then $w \in \mathcal{CH}_K(x, y)$. Let $m = \min\{k|g(\pi^k(w)) > g(\pi^{k-1}(w)), k \in \mathbb{N}\}$ and set $s = g(w)$. Then $g(\pi^{k-1}(w)) = s$ and $\pi^{k-1}(w) \in \Lambda^g_s$. SInce $\pi^{k-1}(w) \in \mathcal{CH}_K(x, y)$, we have

$$g(w) = s \geq \delta^g_0(x, y) \geq \delta^g_M(x, y) \geq c_1 d(x, y).$$

This immediately yields (ADa).

Set $s_* = c_2 d(x, y)$ for $x, y \in X$. Since $\delta^g_M(x, y) \leq c_2 d(x, y)$, there exists a chain $(w(1), \ldots, w(M + 1))$ in $\Lambda^g_{s_*}$ between x and y. As $g(w(i)) \leq s_*$ for any $i = 1, \ldots, M + 1$, we have $(\mathrm{ADb})_{\mathrm{M}}$. □

Since $(\mathrm{ADb})_M$ implies $(\mathrm{ADb})_N$ for any $N \geq M$, we have the following corollary.

Corollary 6.5. *Let $g : T \to (0,1]$ be a weight function. If $d \in \mathcal{D}(X, \mathcal{O})$ is M-adapted to g for some $M \geq 0$, then it is N-adapted to g for any $N \geq M$.*

Recall that a metric $d \in \mathcal{D}(X, \mathcal{O})$ defines a weight function g_d. So one may ask if d is adapted to the weight function g_d or not. Indeed, we are going to give an example of a metric $d \in \mathcal{D}(X, \mathcal{O})$ which is not adapted to g_d in Example 10.8.

Definition 6.6. Let $d \in \mathcal{D}(X, \mathcal{O})$. d is said to be adapted if d is adapted to g_d.

Proposition 6.7. *Let $d \in \mathcal{D}(X, \mathcal{O})$. d is adapted if and only if there exists a weight function $g : T \to (0, 1]$ to which d is adapted. Moreover, suppose that d is adapted. If*

$$D^d(x, y) = \inf\{\sum_{i=1}^{k} d(w(k))|k \geq 1, (w(1), \ldots, w(k)) \in \mathcal{CH}_K(x, y)\}$$

for any $x, y \in X$, then there exist $c_* > 0$ such that

$$c_* D^d(x, y) \leq d(x, y) \leq D^d(x, y)$$

for any $x, y \in X$.

Proof. Necessity direction is immediate. Assume that d is M-adapted to a weight function g. By $(\mathrm{ADb})_\mathrm{M}$, for any $x, y \in X$ there exist $k \in \{1, \ldots, M+1\}$ and $(w(1), \ldots, w(k)) \in \mathcal{CH}_K(x, y)$ such that

$$Cd(x, y) \geq \max_{i=1,\ldots,k} g(w(i)) \geq \frac{1}{c} \max_{i=1,\ldots,k} g_d(w(i)).$$

This proves $(\mathrm{ADb})_\mathrm{M}$ in the case where the weight function $g = g_d$. So we verify that d is M-adapted to g_d. Now, assuming that d is adapted to d, we see

$$c_1 D^d_M(x, y) \leq d(x, y)$$

by (6.2). Since $D^d_M(x, y)$ is monotonically decreasing as $M \to \infty$, it follows that

$$c_1 D^d(x, y) \leq d(x, y).$$

On the other hand, if $(w(1), \ldots, w(k)) \in \mathcal{CH}_K(x, y)$, then the triangle inequality yields

$$d(x, y) \leq \sum_{i=1}^{k} g_d(w(i)).$$

Hence $d(x, y) \leq D^d(x, y)$. $\square$

One of the main interest is the existence of a metric adapted to given weight function g. In other word, when the pre-metric δ^g_M is equivalent to a metric? In the rest of this section, we are going to look into this problem. To start with, we present a weak version of "triangle inequality" for the family $\{\delta^g_M\}_{M \geq 1}$.

Proposition 6.8.

$$\delta^g_{M_1+M_2+1}(x, z) \leq \max\{\delta^g_{M_1}(x, y), \delta^g_{M_2}(y, z)\}$$

Proof. Set $s_* = \max\{d^g_{M_1}(x, y), \delta^g_{M_2}(y, z)\}$. Then we see that there exist a chain $(w(1), \ldots, w(M_1{+}1))$ between x and y and a chain $(v(1), \ldots, v(M_2{+}1))$ between y and z such that $w(i), v(j) \in \Lambda^g_{s_*}$ for any i and j. Since $(w(1), \ldots, w(M_1 + 1), v(1), \ldots, v(M_2 + 1))$ is a chain between x and z, we obtain the claim of the proposition. □

By this proposition, if $\delta^g_M(x, y) \le c\delta_{2M+1}(x, y)$ for any $x, y \in X$, then $\delta^g_M(x, y)$ is so-called quasimetric, i.e.

$$\delta^g_M(x, y) \le c\big(\delta^g_M(x, z) + \delta^g_M(z, y)\big) \tag{6.3}$$

for any $x, y, z \in X$. The coming theorem shows that δ^g_M being a quasimetric is equivalent to the existence of an adapted metric.

The following notion is used in one of the equivalent conditions for a weight function to have an adapted metric.

Definition 6.9. For $w, v \in T$, the pair (w, v) is said to be m-separated with respect to Λ^g_s if and only if whenever $(w, w(1), \ldots, w(k), v)$ is a chain and $w(i) \in \Lambda^g_s$ for any $i = 1, \ldots, k$, it follows that $k \ge m$.

Proposition 6.10. *For any $x, y \in X$ and $M \ge 1$,*

$$\delta^g_M(x, y) = \sup\{s \mid (w, v) \text{ is } M\text{-separated if } w, v \in \Lambda^g_s,\ x \in K_w \text{ and } y \in K_v\}.$$

The following theorem gives several equivalent conditions under which a weight function possesses an associated "visual" metric.

Theorem 6.11. *Assume that $K : T \to \mathcal{C}(X, \mathcal{O})$ is minimal. Let $M \ge 1$. The following five conditions are equivalent:*
$(\mathrm{EV})_M$ *There exist $\alpha \in (0, 1]$ and $d \in \mathcal{D}(X, \mathcal{O})$ such that d is M-adapted to g^α.*
$(\mathrm{EV2})_M$ *δ^g_M is a quasimetric, i.e. there exists $c > 0$ such that (6.3) holds for any $x, y, z \in X$.*
$(\mathrm{EV3})_M$ *There exists $\gamma \in (0, 1)$ such that $\gamma^n\delta^g_M(x, y) \le \delta^g_{M+n}(x, y)$ for any $x, y \in X$ and $n \ge 1$.*
$(\mathrm{EV4})_M$ *There exists $\gamma \in (0, 1)$ such that $\gamma\delta^g_M(x, y) \le \delta^g_{M+1}(x, y)$ for any $x, y \in X$.*
$(\mathrm{EV5})_M$ *There exists $\gamma \in (0, 1)$ such that if $(w, v) \in \Lambda^g_s \times \Lambda^g_s$ is M-separated with respect to Λ^g_s, then (w, v) is $(M + 1)$-separated with respect to $\Lambda^g_{\gamma s}$.*

Remark. By (5.9), $(\mathrm{EV3})_M$ is equivalent to $(\mathrm{EV6})_M$ and $(\mathrm{EV4})_M$ is equivalent to $(\mathrm{EV7})_M$ defined below:
$(\mathrm{EV6})_M$ For any $n \ge 1$, there exists $\gamma_n \in (0, 1)$ such that $U^g_{M+n}(x, \gamma_n s) \subseteq U^g_M(x, s)$ for any $x \in X$ and any $s \in (0, 1]$.
$(\mathrm{EV7})_M$ There exists $\gamma \in (0, 1)$ such that $U^g_{M+1}(x, \gamma s) \subseteq U^g_M(x, s)$ for any $x \in X$ and any $s \in (0, 1]$.

We use the following lemma to prove this theorem.

Lemma 6.12. *If there exist $\gamma \in (0,1)$ and $M \geq 1$ such that $\gamma\delta^g_M(x,y) \leq \delta_{M+1}(x,y)$ for any $x, y \in X$, then*

$$\gamma^n \delta^g_M(x,y) \leq \delta^g_{M+n}(x,y)$$

for any $x, y \in X$ and $n \geq 1$.

Proof. We use inductive argument. Assume that

$$\gamma^l \delta^g_M(x,y) \leq \delta^g_{M+l}(x,y)$$

for any $x, y \in X$ and $l = 1, \ldots, n$. Suppose $\delta^g_{M+n+1}(x,y) \leq c^{n+1}s$. Then there exists a chain $(w(1), \ldots, w(M+n+2))$ in $\Lambda^g_{c^{n+1}s}$ between x and y. Choose any $z \in K_{w(M+n+1)} \cap K_{w(M+n+2)}$. Then

$$\gamma^n \delta^g_M(x,z) \leq \delta^g_{M+n}(x,z) \leq \gamma^{n+1}s$$

Thus we obtain $\delta^g_M(x,z) \leq \gamma s$. Note that $\delta^g_0(z,y) \leq \gamma^{n+1}s$. By Proposition 6.8,

$$\gamma\delta^g_M(x,y) \leq \delta^g_{M+1}(x,y) \leq \max\{\delta^g_{M+1}(x,z), \delta^g_0(z,y)\} \leq \gamma s.$$

This implies $\delta^g_M(x,y) \leq s$. □

Proof of Theorem 6.11. $(\mathrm{EV})_M \Rightarrow (\mathrm{EV4})_M$: Since d is M-adapted to g^α, by Corollary 6.5, d is $M+1$-adapted to g^α as well. By (6.1), we obtain $(\mathrm{EV4})_M$.
$(\mathrm{EV3})_M \Leftrightarrow (\mathrm{EV4})_M$: This is immediate by Lemma 6.12.
$(\mathrm{EV3})_M \Rightarrow (\mathrm{EV2})_M$: Let $n = M+1$. By Proposition 6.8, we have

$$c_{2M+1}\delta^g_M(x,y) \leq \delta^g_{2M+1}(x,y) \leq \max\{d^g_M(x,z), d^g_M(z,y)\} \leq \delta^g_M(x,z) + \delta^g_M(z,y).$$

$(\mathrm{EV2})_M \Rightarrow (\mathrm{EV})_M$: By [6, Proposition 14.5], there exist $c_1, c_2 > 0$, $d \in \mathcal{D}(X, \mathcal{O})$ and $\alpha \in (0,1]$ such that $c_1\delta^g_M(x,y)^\alpha \leq d(x,y) \leq c_2\delta^g_M(x,y)^\alpha$ for any $x, y \in X$. Note tht $\delta^g_M(x,y)^\alpha = \delta^{g^\alpha}_M(x,y)$. By (6.1), d is M-adapted to g^α.
$(\mathrm{EV4})_M \Rightarrow (\mathrm{EV5})_M$: Assume that $w, v \in \Lambda^g_s$. If w and v are not $(M+1)$-separated with respect to $\Lambda^g_{\gamma s}$, then there exist $w(1), \ldots, w(M) \in \Lambda^g_{\gamma s}$ such that $(w, w(1), \ldots, w(M), v)$ is a chain. Then we can choose $w' \in T_w \cap \Lambda^g_{\gamma s}$ and $v' \in T_v \cap \Lambda^g_{\gamma s}$ so that $(w', w(1), \ldots, w(M), v')$ is a chain. Let $x \in O_{w'}$ and let $y \in O_{v'}$. Then $\delta^g_{M+1}(x,y) \leq \gamma s$. Hence by $(\mathrm{EV4})_M$, $\delta_M(x,y) \leq s$. There exists a chain $(v(1), v(2), \ldots, v(M+1))$ in Λ^g_s between x and y. Since $x \in O_{w'} \subseteq O_w$ and $y \in O_{v'} \subseteq O_v$, we see that $v(0) = w$ and $v(M+1) = v$. Hence w and v are not M-separated with respect to Λ^g_s.
$(\mathrm{EV5})_M \Rightarrow (\mathrm{EV4})_M$: Assume that $\delta^g_{M+1}(x,y) \leq \gamma s$. Then there exists a chain $(w(1), \ldots, w(M+2))$ in $\Lambda^g_{\gamma s}$ between x and y. Let w (resp. v) be the unique element in Λ^g_s satisfying $w(1) \in T_w$ (resp. $w(M+2) \in T_v$). Then (w,v) is not $(M+1)$-separated. By $(\mathrm{EV5})_M$, (w,v) is not M-separated. Hence there exists a chain $(w, v(1), \ldots, v(M-1), v)$ in Λ^g_s. This implies $\delta^g_M(x,y) \leq s$. □

7 Bi-Lipschitz equivalence

In this section, we define the notion of bi-Lipschitz equivalent of weight functions. Originally the definition, Definition 7.1, only concerns the tree structure $(T, \mathcal{A}, \phi)$ and has nothing to do with a partition of a space. Under proper conditions, however, we will show that the bi-Lipschitz equivalence of weight functions is identified with

- absolutely continuity with uniformly bounded Radon-Nikodym derivative from below and above between measures in 7.1.
- usual bi-Lipschitz equivalence between metrics in 7.2.
- Ahlfors regularity of a measure with respect to a metric in 7.3.

As in the previous sections, $(T, \mathcal{A}, \phi)$ is a locally finite tree with a reference point ϕ, $(X, \mathcal{O})$ is a compact metrizable space and $K : T \to \mathcal{C}(X, \mathcal{O})$ is a partition of X parametrized by $(T, \mathcal{A}, \phi)$.

Definition 7.1. Two weight functions $g, h \in \mathcal{G}(T)$ are said to be bi-Lipschitz equivalent if and only if there exist $c_1, c_2 > 0$ such that

$$c_1 g(w) \leq h(w) \leq c_2 g(w)$$

for any $w \in T$. We write $g \underset{\text{BL}}{\sim} h$ if and only if g and h are bi-Lipschitz equivalent.

By the definition, we immediately have the next fact.

Proposition 7.2. *The relation $\underset{BL}{\sim}$ is an equivalent relation on $\mathcal{G}(T)$.*

7.1 bi-Lipschitz equivalence of measures

As we mentioned above, the bi-Lipschitz equivalence between weight functions can be identified with other properties according to classes of weight functions. First we consider the case of weight functions associated with measures.

Definition 7.3. Let $\mu, \nu \in \mathcal{M}_P(X, \mathcal{O})$. We write $\mu \underset{\text{AC}}{\sim} \nu$ if and only if there exist $c_1, c_2 > 0$ such that

$$c_1 \mu(A) \leq \nu(A) \leq c_2 \mu(A) \tag{7.1}$$

for any Borel set $A \subseteq X$.

It is easy to see that $\underset{\text{AC}}{\sim}$ is an equivalence relation and $\mu \underset{\text{AC}}{\sim} \nu$ if and only if μ and ν are mutually absolutely continuous and the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ is uniformly bounded from below and above.

Theorem 7.4. *Assume that the partition $K : T \to \mathcal{C}(X, \mathcal{O})$ is strongly finite. Let $\mu, \nu \in \mathcal{M}_P(X, \mathcal{O})$. Then $g_\mu \underset{BL}{\sim} g_\nu$ if and only if $\mu \underset{AC}{\sim} \nu$. Moreover, the natural map $\mathcal{M}_P(X, P)/\underset{AC}{\sim} \to \mathcal{G}(X)/\underset{BL}{\sim}$ given by $[g_\mu]_{\underset{BL}{\sim}}$ is injective, where $[\,\cdot\,]_{\underset{BL}{\sim}}$ is the equivalence class under $\underset{BL}{\sim}$.*

Proof. By (7.1), we see that $\alpha_1\nu(K_w) \leq \mu(K_w) \leq \alpha_2\nu(K_w)$ and hence $g_\mu \underset{\mathrm{BL}}{\sim} g_\nu$. Conversely, if

$$c_1\mu(K_w) \leq \nu(K_w) \leq c_2\mu(K_w)$$

for any $w \in T$. Let $U \subset X$ be an open set. Assume that $U \neq X$. For any $x \in X$, there exists $w \in T$ such that $x \in K_w \subseteq U$. Moreover, if $K_w \subseteq U$, then there exists $m \in \{1, \ldots, |w|\}$ such that $[w]_m \in T^U(x)$ but $[w]_{m-1} \notin T^U(x)$. Therefore, if

$$T(U) = \{w|w \in T, K_w \subseteq U, K_{\pi(w)} \backslash U \neq \emptyset\},$$

then $T(U) \neq \emptyset$ and $U = \cup_{w \in T(U)} K_w$. Now, since K is strongly finite, there exists $N \in \mathbb{N}$ such that $\#(\sigma^{-1}(x)) \leq N$ for any $x \in X$. Let $y \in U$. If $w(1), \ldots, w(m) \in T(U)$ are mutually different and $y \in K_{w(m)}$ for any $i = 1, \ldots, m$, then there exists $\omega(i) \in \Sigma_{w(i)}$ such that $\sigma(\omega(i)) = y$ for any $i = 1, \ldots, m$. Hence $\#(\sigma^{-1}(y)) \geq m$ and therefore $m \leq N$. By Proposition 13.1, we see that

$$\nu(U) \leq \sum_{w \in T(U)} \nu(K_w) \sum_{w \in T(U)} c_2\mu(K_w) \leq c_2N\mu(U)$$

$$\mu(U) \leq \sum_{w \in T(U)} \mu(K_w) \sum_{w \in T(U)} \frac{1}{c_1}\nu(K_w) \leq \frac{N}{c_1}\nu(U).$$

Hence letting $\alpha_1 = c_1/N$ and $\alpha_2 = c_2N$, we have

$$\alpha_1\mu(U) \leq \nu(U) \leq \alpha_2\mu(U)$$

for any open set $U \subseteq X$. Since μ and ν are Radon measures, this yields (7.1). □

7.2 bi-Lipschitz equivalence of metrics

Under the tightness of weight functions defined below, we will translate bi-Lipschitz equivalence of weight functions to the relations between "balls" and "distances" associated with weight functions in Theorem 7.8.

Definition 7.5. A weight function g is called tight if and only if for any $M \geq 0$, there exists $c > 0$ such that

$$\sup_{x,y \in K_w} \delta_M^g(x, y) \geq cg(w)$$

for any $w \in T$.

Proposition 7.6. *Let g and h be weight functions. Assume that $g \underset{BL}{\sim} h$. If g is tight and $g \underset{BL}{\sim} h$, then h is tight.*

Proof. Since $g \underset{\mathrm{BL}}{\sim} h$, there exist $\gamma_1, \gamma_2 > 0$ such that $\gamma_1 g(w) \leq h(w) \leq \gamma_2 g(w)$ for any $w \in T$. Therefore,

$$\gamma_1 D_M^g(x, y) \leq D_M^h(x, y) \leq \gamma_2 D_M^g(x, y)$$

for any $x, y \in X$ and $M \geq 0$. By Proposition 6.3, there exist $c_1, c_2 > 0$ such that

$$c_1 \delta^g_M(x, y) \leq \delta^h_M(x, y) \leq c_2 \delta^g_M(x, y)$$

for any $x, y \in X$ and $M \geq 0$. Hence

$$\sup_{x,y \in K_w} \delta^h_M(x, y) \geq c_1 \sup_{x,y \in K_w} \delta^g_M(x, y) \geq cg(w) \geq \gamma(\gamma_2)^{-1} h(w)$$

for any $w \in T$. Thus h is tight. □

Any weight function induced from a metric is tight.

Proposition 7.7. *Let $d \in \mathcal{D}(X, \mathcal{O})$. Then g_d is tight.*

Proof. Let $x, y \in K$ and let $(w(1), \ldots, w(M+1)) \in \mathcal{CH}(x, y)$. Set $x_0 = x$ and $x_{M+1} = y$. For $i = 1, \ldots, M$, choose $x_i \in K_{w(i)} \cap K_{w(i+1)}$. Then

$$\sum_{i=1}^{M+1} g_d(x) \geq \sum_{i=1}^{M+1} d(x_{i-1}, x_i) \geq d(x, y).$$

Hence using this and Proposition 6.3, we obtain

$$(M+1)\delta^g_M(x, y) \geq D^{g_d}_M(x, y) \geq d(x, y)$$

and therefore $(M+1)\sup_{x,y \in K_w} \delta^{g_d}_M(x, y) \geq g_d(w)$ for any $w \in T$. Thus g_d is tight. □

Now we give geometric conditions which are equivalent to bi-Lipschitz equivalence of tight weight functions. The essential point is that bi-Lipschitz condition between weight function g and h are equivalent to that between $\delta^g_M(\cdot, \cdot)$ and $\delta^h_M(\cdot, \cdot)$ in the usual sense as is seen in (BL2) and (BL3).

Theorem 7.8. *Let g and h be weight functions. Assume that both g and h are tight. Then the following conditions are equivalent:*
(BL) $g \underset{BL}{\sim} h$.
(BL1) *There exist M_1, M_2 and $c > 0$ such that*

$$\delta^g_{M_1}(x, y) \leq c\delta^h_0(x, y) \quad \text{and} \quad \delta^h_{M_2}(x, y) \leq c\delta^g_0(x, y)$$

for any $x, y \in X$.
(BL2) *There exist $c_1, c_2 > 0$ and $M \geq 0$ such that*

$$c_1 \delta^g_M(x, y) \leq \delta^h_M(x, y) \leq c_2 \delta^g_M(x, y)$$

for any $x, y \in X$.
(BL3) *For any $M \geq 0$, there exist $c_1, c_2 > 0$ such that*

$$c_1 \delta^g_M(x, y) \leq \delta^h_M(x, y) \leq c_2 \delta^g_M(x, y)$$

for any $x, y \in X$.

The proof of this theorem will be given after stating corollaries of it.

If weight functions are induced from adapted metrics, then bi-Lipschitz equivalence of weight functions exactly corresponds the usual bi-Lipschitz equivalence of metrics.

Definition 7.9. (1) Let $d, \rho \in \mathcal{D}(X, \mathcal{O})$. d and ρ are said to be bi-Lipschitz equivalent, $d \underset{\text{BL}}{\sim} \rho$ for short, if and only if there exist $c_1, c_2 > 0$ such that

$$c_1 d(x, y) \leq \rho(x, y) \leq c_2 d(x, y)$$

for any $x, y \in X$.
(2) Define

$$\mathcal{D}_A(X, \mathcal{O}) = \{d | d \in \mathcal{D}(X, \mathcal{O}), d \text{ is adapted.}\}$$

Corollary 7.10. *Let $d, \rho \in \mathcal{D}_A(X, \mathcal{O})$. Then $g_d \underset{BL}{\sim} g_\rho$ if and only if $d \underset{BL}{\sim} \rho$. In particular, the correspondence of $[d]_{\underset{BL}{\sim}}$ with $[g_d]_{\underset{BL}{\sim}}$ gives an well-defined injective map $\mathcal{D}_A(X, \mathcal{O})/\underset{BL}{\sim} \to \mathcal{G}(X)/\underset{BL}{\sim}$.*

The next corollary shows that an adapted metric is adapted to a weight function if and only if they are bi-Lipschitz equivalent in the sense of weight functions.

Corollary 7.11. *Let $d \in \mathcal{D}(X, \mathcal{O})$ and let g be a weight function. Then d is adapted to g and g is tight if and only if $g_d \underset{BL}{\sim} g$ and $d \in \mathcal{D}_A(X, \mathcal{O})$.*

Now we start to prove Theorem 7.8 and its corollaries.

Lemma 7.12. *Let h be a weight function. If $K_w \backslash U_0^h(x, s) \neq \emptyset$, then $s \leq h(w)$.*

Proof. If $\pi^n(w) \in \Lambda_{s,0}^h(x)$ for some $n \geq 0$, then $U_0^h(x, s) \supseteq K_{\pi^n(w)} \supseteq K_w$. This contradicts to the assumption and hence $\pi^n(w) \notin \Lambda_{s,0}^h(x)$ for any $n \geq 0$. Therefore there exists $v \in T_w \cap \Lambda_{s,0}^h(x)$ such that $|v| > |w|$. Then we have $h(w) \geq h(\pi(v)) > s$. $\square$

Proposition 7.13. *Let g and h are weight functions. Assume that g is tight. Let $M \geq 0$. If there exists $\alpha > 0$ such that*

$$\alpha \delta_M^g(x, y) \leq \delta_0^h(x, y) \tag{7.2}$$

for any $x, y \in X$. Then there exists $c > 0$ such that

$$c g(w) \leq h(w)$$

for any $w \in T$.

Proof. Since g is tight, there exists $\beta > 0$ such that, for any $w \in T$,

$$K_w \backslash U_M^g(x, \beta g(w)) \neq \emptyset$$

for some $x \in K_w$. On the other hand, by (7.2), there exists $\gamma > 0$ such that $U^g_M(x, s) \supseteq U^h_0(x, \gamma s)$ for any $x \in X$ and $s \geq 0$. Therefore,

$$K_w \backslash U^h_0(x, \beta\gamma g(w)) \neq \emptyset.$$

By Lemma 7.12, we have $\beta\gamma g(w) \leq h(w)$. $\square$

Lemma 7.14. *Let g and h be weight functions. Assume that g is tight. Then the following conditions are equivalent:*
(A) *There exists $c > 0$ such that $g(w) \leq ch(w)$ for any $w \in T$.*
(B) *For any $M, N \geq 0$ with $N \geq M$, there exists $c > 0$ such that*

$$\delta^g_N(x, y) \leq c\delta^h_M(x, y)$$

for any $x, y \in X$.
(C) *There exist $M, N \geq 0$ and $c > 0$ such that $N \geq M$ and*

$$\delta^g_N(x, y) \leq c\delta^h_M(x, y)$$

for any $x, y \in X$.

Proof. (A) implies

$$D^g_M(x, y) \leq cD^h_M(x, y) \tag{7.3}$$

for any $x, y \in X$ and $M \geq 0$. By Proposition 6.3, we see

$$\delta^g_M(x, y) \leq c(M + 1)\delta^h_M(x, y)$$

for any $x, y \in X$. Since $\delta^g_N(x, y) \leq \delta^g_M(x, y)$ if $N \geq M$, we have (B). Obviously (B) implies (C). Now assume (C). Then we have $\delta^g_N(x, y) \leq c\delta^h_0(x, y)$. Hence Proposition 7.13 yields (A). $\square$

Proof of Theorem 7.8. Lemma 7.14 immediately implies the desired statement. $\square$

Proof of Corollary 7.10. Since d and ρ are adapted, by (6.1), there exist $M \geq 1$ and $c > 0$ such that

$$c\delta^d_M(x, y) \leq d(x, y) \leq \delta^d_M(x, y), \tag{7.4}$$
$$c\delta^\rho_M(x, y) \leq \rho(x, y) \leq \delta^\rho_M(x, y) \tag{7.5}$$

for any $x, y \in X$. Assume $g_d \underset{\mathrm{BL}}{\sim} g_\rho$. Since g_d and g_ρ are tight, we have (BL3) by Theorem 7.8. Hence by (7.4) and (7.5), $d(\cdot, \cdot)$ and $\rho(\cdot, \cdot)$ are bi-Lipschitz equivalent as metrics. The converse direction is immediate. $\square$

Proof of Corollary 7.11. If d is M-adapted to g for some $M \geq 1$, then by (ADa), there exists $c > 0$ such that $d_w \leq cg(w)$ for any $w \in K_w$. Moreover, (6.1) implies

$$d(x, y) \geq c_2\delta^g_M(x, y)$$

for any $x, y \in X$, where c_2 is independent of x and y. Hence the tightness of g shows that there exists $c' > 0$ such that

$$d_w \geq c_2 \sup_{x,y} \delta_M^g(x,y) \geq c' g(w)$$

Thus we have shown that $g_d \underset{\mathrm{BL}}{\sim} g$. Moreover, by Proposition 6.7, d is adapted. Conversely, assume that d is M-adapted and $g_d \underset{\mathrm{BL}}{\sim} g$. Then Theorem 7.8 implies (BL3) with $h = g_d$. At the same time, since d is M-adapted, we have (6.2) with $g = g_d$. Combining these two, we deduce (6.2). Hence d is M-adapted to g. $\square$

7.3 Bi-Lipschitz equivalence between measures and metrics

Finally in this section, we consider what happens if the weight function associated with a measure is bi-Lipschitz equivalent to the weight function associated with a metric.

To state our theorem, we need the following notions.

Definition 7.15. (1) A weight function $g : T \to (0,1]$ is said to be uniformly finite if $\sup\{\#(\Lambda_{s,1}^g(w))|s \in (0,1], w \in \Lambda_s^g\} < +\infty$.
(2) A function $f : T \to (0,\infty)$ is called sub-exponential if and only if there exists $m \geq 0$ and $c_1 \in (0,1)$ such that $f(v) \leq c_1 f(w)$ for any $w \in T$ and any $v \in T_w$ with $|v| \geq |w| + m$. f is called super-exponential if and only if there exists $c_2 \in (0,1)$ such that $f(v) \geq c_2 f(w)$ for any $w \in T$ and $v \in S(w)$. f is called exponential if it is sub-exponential and super-exponential.

The following proposition and the lemma are immediate consequences of the above definitions.

Proposition 7.16. *Let h be a weight function. Then h is super-exponential if and only if there exists $c \geq 1$ such that $ch(w) \geq s \geq h(w)$ whenever $w \in \Lambda_s^h$.*

Proof. Assume that h is super-exponential. Then there exists $c_2 < 1$ such that $h(w) \geq c_2 h(\pi(w))$ for any $w \in T$. If $w \in \Lambda_s^h$, then $h(\pi(w)) > s \geq h(w)$. This implies $(c_2)^{-1} h(w) \geq s \geq h(w)$.

Conversely, assume that $ch(w) \geq s \geq h(w)$ for any w and s with $w \in \Lambda_s^h$. If $h(\pi(w)) > ch(w)$, then $w \in \Lambda_t^h$ for $t \in (ch(w), h(\pi(w))$. This contradicts to the assumption that $ch(w) \geq t \geq h(w)$. Hence $h(\pi(w)) \leq ch(w)$ for any $w \in T$. Thus h is super-exponential. $\square$

Lemma 7.17. *If a weight function $g : T \to (0,1]$ is uniformly finite, then*

$$\sup\{\#(\Lambda_{s,M}(x))|x \in X, s \in (0,1]\} < +\infty$$

for any $M \geq 0$.

Proof. Let $C = \sup\{\#(\Lambda_{s,1}(w))|s \in (0,1], w \in \Lambda_s\}$. Then $\#(\Lambda_{s,M}(x)) \leq C + C^2 + \ldots + C^{M+1}$. $\square$

Definition 7.18. Let $\alpha > 0$. A radon measure μ on X is said to be Ahlfors α-regular with respect to $d \in \mathcal{D}(X, \mathcal{O})$ if and only if there exist $C_1, C_2 > 0$ such that

$$C_1 r^\alpha \leq \mu(B_d(x, r)) \leq C_2 r^\alpha \tag{7.6}$$

for any $r \in [0, \mathrm{diam}(X, d)]$.

Definition 7.19. Let $g : T \to (0, 1]$ be a weight function. We say that K has thick interior with respect to g, or g is thick for short, if and only if there exists $M \geq 1$ and $\alpha > 0$ such that $K_w \supseteq U_M^g(x, \alpha s)$ for some $x \in K_w$ if $s \in (0, 1]$ and $w \in \Lambda_s^g$.

The value of the integer $M \geq 1$ is not crucial in the above definition. In Proposition 8.1, we will show if the condition of the above definition holds for a particular $M \geq 1$, then it holds for all $M \geq 1$.

The thickness is invariant under the bi-Lipschitz equivalence of weight functions as follows.

Proposition 7.20. *Let g and h be weight functions. If g is thick and $g \underset{BL}{\sim} h$, then h is thick.*

Since we need further results on thickness of weight functions, we postpone a proof of this proposition until the next section.

Now we give the main theorem of this sub-section.

Theorem 7.21. *Let $d \in \mathcal{D}_A(X, \mathcal{O})$ and let $\mu \in \mathcal{M}_P(X, \mathcal{O})$. Assume that K is minimal and d is super-exponential and thick. Then $(g_d)^\alpha \underset{BL}{\sim} g_\mu$ and d is uniformly finite if and only if μ is Ahlfors α-regular with respect to d. Moreover, if either/both of the these two conditions is/are satisfied, then μ and d are exponential.*

By the same reason as Proposition 7.20, a proof of this theorem will be given at the end of Section 9.

8 Thickness of weight functions

In this section, we study conditions for a weight function being thick and relation between the notions "thick" and "tight". For instance in Theorem 8.3 we present topological condition (TH1) ensuring that all super-exponential weight functions are thick. In particular, this is the case for partitions of S^2 discussed in Section 2 because partitions satisfying (2.2) are minimal and the condition (TH) in Section 2 yields the condition (TH1). Moreover in this case, all super-exponential weight functions are tight as well by Corollary 8.5.

Proposition 8.1. *g is thick if and only if for any $M \geq 0$, there exists $\beta > 0$ such that, for any $w \in T$, $K_w \supseteq U_M^g(x, \beta g(\pi(w)))$ for some $x \in K_w$.*

Proof. Assume that g is thick. By induction, we are going to show the following claim $(C)_M$ holds for any $M \geq 1$:
$(C)_M$ There exists $\alpha_M > 0$ such that, for any $s \in (0,1]$ and any $w \in \Lambda_s^g$, one find $x \in K_w$ satisfying $K_w \supseteq U_M(x, \alpha_M s)$.

Since g is thick, $(C)_M$ holds for some $M \geq 1$. Since $U_1^g(x,s) \subseteq U_M^g(x,s)$ if $M \geq 1$, $(C)_1$ holds as well. Now, suppose that $(C)_M$ holds. Let $w \in \Lambda_s^g$ and choose x as in $(C)_M$. Then there exists $v \in \Lambda_{\alpha_M s}^g$ such that $v \in T_w$ and $x \in K_v$. Applying $(C)_M$ again, we find $y \in K_v$ such that $K_v \supseteq U_M^g(y, (\alpha_M)^2 s)$. Since $M \geq 1$, it follows that $U_{M+1}^g(y, (\alpha_M)^2 s) \subseteq U_M^g(x, \alpha_M s) \subseteq K_w$. Therefore, letting $b_{M+1} = (b_M)^2$, we have obtained $(C)_{M+1}$. Thus we have shown $(C)_M$ for any $M \geq 1$.

Next, fix $M \geq 1$ and write $\alpha = \alpha_M$. Note that $w \in \Lambda_s^g$ if and only if $g(w) \leq s < g(\pi(w))$. Fix $\epsilon \in (0,1)$. Assume that $g(\pi(w)) > g(w)$, then there exists s_* such that $g(w) \leq s_* < g(\pi(w))$ and $s_* > \epsilon g(\pi(w))$. Hence obtain

$$K_w \supseteq U_M^g(x, \alpha s_*) \supseteq U_M^g(x, \alpha\epsilon g(\pi(w))).$$

If $g(w) = g(\pi(w))$, then there exists $v \in T_w$ such that $g(v) < g(\pi(v)) = g(w) = g(\pi(w))$. Choosing s_* so that $g(v) \leq s_* < g(\pi(v)) = g(\pi(w))$ and $\epsilon g(\pi(w)) < s_*$, we obtain

$$K_w \supseteq K_v \supseteq U_M^g(x, \alpha s_*) \supseteq U_M^g(x, \alpha\epsilon g(\pi(w))).$$

Letting $\beta = \alpha\epsilon$, we obtain the desired statement.
Conversely, assume for any $M \geq 0$, there exists $\beta > 0$ such that, for any $w \in T$, $K_w \supseteq U_M^g(x, \beta g(\pi(w)))$ for some $x \in K_w$. If $w \in \Lambda_s$, then $g(w) \leq s < g(\pi(w))$. Therefore $K_w \supseteq U_M^g(x, \beta s)$. This implies that g is thick. □

Proposition 8.2. *Assume that K is minimal. Let $g : T \to (0,1]$ be a weight function. Then g is thick if and only if, for any $M \geq 0$, there exists $\gamma > 0$ such that, for any $w \in T$, $O_w \supseteq U_M^g(x, \gamma g(\pi(w)))$ for some $x \in O_w$.*

Proof. Assume that g is thick. By Proposition 8.1, for any $M \geq 0$, we may choose $\alpha > 0$ so that for any $w \in T$, there exists $x \in K_w$ such that $K_w \supseteq U_{M+1}^g(x, \alpha g(\pi(w)))$. Set $s_w = g(\pi(w))$. Let $y \in U_M^g(x, \alpha s_w) \backslash O_w$. There exists $v \in (T)_{|w|}$ such that $y \in K_v$ and $w \neq v$. Then we find $v_* \in T_v \cap \Lambda_{\alpha s_w}^g$ satisfying $y \in K_{v_*}$. Since $K_{v_*} \cap U_M^g(x, \alpha s_w) \neq \emptyset$, we have

$$K_{v_*} \subseteq U_{M+1}^g(x, \alpha s_w) \subseteq K_w.$$

Therefore, $K_{v_*} \subseteq \cup_{w' \in T_w, |w'| = |v_*|} K_{w'}$. This implies that $O_{v_*} = \emptyset$. This contradicts to the fact that K is minimal. So, $U_M^g(x, \alpha s_w) \backslash O_w = \emptyset$ and hence $U_M^g(x, \alpha s_w) \subseteq O_w$.
The converse direction is immediate. □

Using the above proposition, we give a proof of Proposition 7.20.

Proof of Proposition 7.20. By Proposition 8.1, there exists $\beta > 0$ such that for any $w \in T$, $K_w \supseteq U_M^g(x, \beta g(\pi(w)))$ for some $x \in K_w$. On the other hand, since there exist$c_1, c_2 > 0$ such that $c_1 h(w) \leq g(w) \leq c_2 h(w)$ for any $w \in T$. It follows that $D_M^g(x,y) \leq c_2 D_M^h(x,y)$ for any $x, y \in X$. Proposition 6.3 implies that there exists $\alpha > 0$ such that $\alpha\delta_M^g(x,y) \leq \delta_M^h(x,y)$ for any $x, y \in X$. Hence $U_M^h(x, \alpha s) \subseteq U_M^g(x, s)$ for any $x \in X$ and any $s \in (0,1]$. Combining them, we see that

$$K_w \supseteq U_M^g(x, \beta g(\pi(w))) \supseteq U_M^h(x, \alpha\beta g(\pi(w))) \supseteq U^h(x, \alpha\beta c_2 h(\pi(w))).$$

Thus by Proposition 8.1, h is thick. $\square$

Theorem 8.3. *Assume that K is minimal. Define $h_* : T \to (0,1]$ by $h_*(w) = 2^{-|w|}$ for any $w \in T$. Then the following conditions are equivalent:*
(TH1) *There exists $m \geq 1$ such that, for any $w \in T$, there exists $v_* \in T_w$ such that $K_{v_*} \subseteq O_w$ and $|v_*| \leq |w| + m$.*
(TH2) *Every super-exponential weight function is thick.*
(TH3) *There exists a sub-exponential weight function which is thick.*
(TH4) *The weight function h_* is thick.*

Proof. (TH1) $\Rightarrow$ (TH2): Assume (TH1). Let g be a super-exponential weight function. Then there exists $\lambda \in (0,1)$ such that $g(w) \geq g(\pi(w))$ for any $w \in T$. Let $w \in \Lambda_s^g$. For any $v \in T_w \cap (T)_{|w|+m}$, it follows that

$$g(v) \geq \lambda^{m+1} g(\pi(w)) > \lambda^{m+1} s$$

So choose $x \in O_{v_*}$. Then $\Lambda_{\lambda^{m+1}s,1}(x) \subseteq \{w' | w' \in (T)_{|v_*|}, K_{w'} \cap K_{v_*} \neq \emptyset\} \subseteq (T)_{|v_*|} \cap T_w$. Hence

$$U_1^g(x, \lambda^{m+1} s) \subseteq K_w$$

Thus we have shown that g is thick.

(TH2) $\Rightarrow$ (TH4): Apparently h_* is an exponential weight function. Hence by (TH2), it is thick.

(TH4) $\Rightarrow$ (TH4): Since h_* is exponential and thick, we havre (TH3).

(TH3) $\Rightarrow$ (TH1): Assume that g is a sub-exponential weight function which is thick. Proposition 8.2 shows that there exists $\gamma \in (0,1)$ and $M \geq 1$ such that for any $w \in T$, $O_w \supseteq U_M^g(x, \gamma g(\pi(w)))$. Choose $v_* \in \Lambda_{\gamma g(\pi(w)),0}^g(x)$. Then $K_{v_*} \subseteq U_M^g(x, \gamma g(\pi(w))) \subseteq O_w$ and $g(\pi(v_*)) > \gamma g(\pi(w)) \geq \gamma g(w)$. Since g is sub-exponential, there exists $k \geq 1$ and $\eta \in (0,1)$ such that $g(u) \leq \eta g(v)$ if $v \in T_v$ and $|u| \geq |w| + k$. Choose l so that $\eta^l < \gamma$ and set $m = kl + 1$. Since $g(\pi(v_*)) > \eta^l g(w)$, we see that $|\pi(v_*)| \leq |w| + m - 1$. Therefore, $|v_*| \leq |w| + m$ and hence we have (TH1). $\square$

Theorem 8.4. *Assume that $K : T \to \mathcal{C}(X, \mathcal{O})$ is minimal, that there exists $\lambda \in (0,1)$ such that if $B_w = \emptyset$, then $\#(T_w \cap \Lambda_{\lambda g(w)}^g) \geq 2$ and that g is thick. Then g is tight.*

Proof. By Proposition 8.2, there exists γ such that, for any $v \in T$, $O_v \supseteq U_M^g(x, \gamma g(\pi(v)))$ for some $x \in K_v$. First suppose that $B_w \neq \emptyset$. Then there exists $x \in K_w$ such that $O_w \supseteq U^g_M(x, \gamma g(\pi(w)))$. For any $y \in B_w$, it follows that $\delta^g_M(x,y) > \gamma g(\pi(w))$. Thus

$$\sup_{x,y \in K_w} \delta^g_M(x,y) \geq \gamma g(\pi(w)).$$

Next if $B_w = \emptyset$, then there exists $u \neq v \in T_w \cap \Lambda^g_{\lambda g(w)}$. If $B_u \neq \emptyset$, then the above discussion implies

$$\sup_{x,y \in K_w} \delta^g_M(x,y) \geq \sup_{x,y \in K_v} \delta^g_M(x,y) \geq \gamma g(\pi(v)) \geq \gamma \lambda g(w).$$

If $B_u = \emptyset$, then $\delta^g_M(x,y) \geq \lambda g(w)$ for any $(x,y) \in K_u \times K_v$. Thus for any $w \in T$, we conclude that

$$\sup_{x,y \in K_w} \delta^g_M(x,y) \geq \gamma \lambda g(w).$$

□

The above theorem immediately implies the following corollary.

Corollary 8.5. *Assume that $(X, \mathcal{O})$ is connected and K is minimal. If g is thick, then g is tight.*

9 Volume doubling property

In this section, we introduce the notion of a relation called "gentle" and written as $\underset{\mathrm{GE}}{\sim}$ between weight functions. This relation is not an equivalence relation in general. In Section 11, however, it will be show to be an equivalence relation among exponential weight functions. As was the case of the bi-Lipschitz equivalence, the gentleness will be identified with other properties in classes of weight functions. In particular, we are going to show that the volume doubling property of a measure with respect to a metric is equivalent to the gentleness of the associated weight functions.

As in the previous sections, $(T, \mathcal{A}, \phi)$ is a locally finite tree with a reference point ϕ, $(X, \mathcal{O})$ is a compact metrizable space and $K : T \to \mathcal{C}(X, \mathcal{O})$ is a partition of X parametrized by $(T, \mathcal{A}, \phi)$.

The notion of gentleness of a weight function to another weight function is defined as follows.

Remark. In the case of the natural partition of a self-similar set in Example 4.5, the main results of this section, Theorems 9.6, 9.8 have been obtained in [9].

Definition 9.1. Let $g : T \to (0, 1]$ be a weight function. A function $f : T \to (0, \infty)$ is said to be gentle with respect to g if and only if there exists $c_G > 0$ such that $f(v) \leq c_G f(w)$ whenever $w, v \in \Lambda^g_s$ and $K_w \cap K_v \neq \emptyset$ for some $s \in (0, 1]$. We write $f \underset{\mathrm{GE}}{\sim} g$ if and only if f is gentle with respect to g.

Alternatively, we have a simpler version of the definition of gentleness under a mild restriction.

Proposition 9.2. *Let $g : T \to (0, 1]$ be an exponential weight function. Let $f : T \to (0, \infty)$. Assume that $f(w) \leq f(\pi(w))$ for any $w \in T$ and f is super-exponential. Then f is gentle with respect to g if and only if there exists $c > 0$ such that $f(v) \leq cf(w)$ whenever $g(v) \leq g(w)$ and $K_v \cap K_w \neq \emptyset$.*

Proof. By the assumption, there exist $c_1, c_2 > 0$ and $m \geq 1$ such that $f(v) \geq c_2 f(w)$, $g(v) \geq c_2 g(w)$ and $g(u) \leq c_1 g(w)$ for any $w \in T$, $v \in S(w)$ and $u \in T_w$ with $|u| \geq |w| + m$.

First suppose that f is gentle with respect to g. Then there exists $c > 0$ such that $f(v') \leq cf(w')$ whenever $w', v' \in \Lambda^g_s$ and $K_{w'} \cap K_{v'} \neq \emptyset$ for some $s \in (0, 1]$. Assume that $g(v) \leq g(w)$ and $K_v \cap K_w \neq \emptyset$. There exists $u \in T_w$ such that $K_u \cap K_v \neq \emptyset$ and $g(\pi(u)) > g(v) \geq g(u)$. Moreover, $g(\pi([u]_m)) < g([u]_m) = g(u)$ for some $m \in [0, |v|]$. Then $[u]_m, v \in \Lambda^g_{g(v)}$ and hence $f(v) \leq f([u]_m) \leq cf(u) \leq cf(w)$.

Conversely, assume that $f(v') \leq cf(w')$ whenever $g(v') \leq g(w')$ and $K_{v'} \cap K_{w'} \neq \emptyset$. Let $w, v \in \Lambda^g_s$ with $K_w \cap K_v \neq \emptyset$. If $g(v) \leq g(w)$, then $f(v) \leq cf(w)$. Suppose $g(v) > g(w)$. Since g is super-exponential,

$$s \geq g(w) \geq c_2 g(\pi(w)) \geq c_2 s \geq c_2 g(v),$$

Set $N = \min\{n | c_2 \geq c_1^n\}$. Choose $u \in T_v$ so that $K_u \cap K_w \neq \emptyset$ and $|u| = |v| + Nm$. Then $g(w) \geq c_2 g(v) \geq (c_1)^N g(v) \geq g(u)$. This implies $f(u) \leq cf(w)$. Since $f(u) \geq (c_2)^{Nm} f(v)$, we have $f(v) \leq c(c_2)^{-Nm} f(w)$. Therefore, f is gentle with respect to g. □

The following is the standard version of the definition of the volume doubling property.

Definition 9.3. Let μ be a radon measure on $(X, \mathcal{O})$ and let $d \in \mathcal{D}(X, O)$. μ is said to have the volume doubling property with respect to the metric d if and only if there exists $C > 0$ such that

$$\mu(B_d(x, 2r)) \leq C\mu(B_d(x, r))$$

for any $x \in X$ and any $r > 0$.

Note that X has no isolated point by the condition (P1). Due to this fact, if a Radon measure μ has the volume doubling property with respect to some $d \in \mathcal{D}(X, \mathcal{O})$ or M-volume doubling property with respect to a weight function defined below, then the normalized version of μ, $\mu/\mu(X)$, belongs to $\mathcal{M}_P(X, O)$. Taking this fact into account, we are mainly interested in (normalized version of) a Radon measure in $\mathcal{M}_P(X, \mathcal{O})$.

Next we define the notion of volume doubling property of a measure with respect to a weight function g as well by means of balls "$U^g_M(x, s)$".

Definition 9.4. Let $\mu \in \mathcal{M}_K(X, \mathcal{O})$ and let g be a weight function. For $M \geq 1$, we say μ has M-volume doubling property with respect to g if and only if there exist $\gamma > (0, 1)$ and $\beta \geq 1$ such that $\mu(U^g_M(x, s)) \leq \beta\mu(U^g_M(x, \gamma s))$ for any $x \in X$ and any $s \in (0, 1]$.

It is rather annoying that the notion of "volume doubling property" of a measures versus a weight function depends on the value $M \geq 1$ while that of a measure versus a metric does not. Under certain conditions including the exponentially and the thickness, however, we will show that if μ has M-volume doubling property for some $M \geq 1$, then it has M-volume doubling property for all $M \geq 1$ in Theorem 9.8.

Naturally, if the weight function is associated with a metric, the volume doubling with respect to the metric and the volume doubling property with respect to the associated weight function is virtually the same as is seen in the next proposition.

Proposition 9.5. *Let $d \in \mathcal{D}(X, \mathcal{O})$, let $\mu \in M_K(X, \mathcal{O})$ and let g be a weight function. Assume that d is adapted to g. Then μ has the volume doubling property with respect to d if and only if there exists $M_* \geq 1$ such that μ has M-volume doubling property with respect to g for any $M \geq M_*$.*

Proof. Since d is adapted to g, for sufficiently large M, there exist $\alpha_1, \alpha_2 > 0$ such that

$$U^g_M(x, \alpha_1 s) \subseteq B_d(x, s) \subseteq U^g_M(x, \alpha_2 s)$$

for any $x \in X$ and any $s \in (0, 1]$. Suppose that μ has the volume doubling property with respect to d. Then there exists $\lambda > 1$ such that

$$\mu(B_d(x, 2^m r)) \leq \lambda^m \mu(B_d(x, r))$$

for any $x \in X$ and any $r \geq 0$ Hence

$$\mu(U^g_M(x, \alpha_1 2^m r)) \subseteq \lambda^m \mu(U^g_M(x, \alpha_2 r)).$$

Choosing m so that $\alpha_1 2^m > \alpha_2$, we see that μ has M-volume doubling property with respect to g if M is sufficiently large. Converse direction is more or less similar. □

By the above proposition, as far as we confine ourselves to adapted metrics, it is enough to consider the volume doubling property of a measure with respect to a weight function. Thus we are going to investigate relations between a measure μ having the volume doubling property with respect to a weight function g and other conditions like

- g is exponential,
- g is uniformly finite,
- μ is super-exponential

- μ is gentle with respect to g.

To begin with, we show that the last four conditions imply the volume doubling property of μ with respect to g.

Theorem 9.6. *Let $g : T \to (0, 1]$ be a weight function and let $\mu \in \mathcal{M}_P(X, \mathcal{O})$. Assume that g is exponential, that g is uniformly finite, that μ is gentle with respect to g and that μ is super-exponential. Then μ has M-volume doubling property with respect to g for any $M \geq 1$.*

Remark. In the case of the natural partition of a self-similar set in Example 4.5, the above theorem has been obtained in [9].

Hereafter in this section, we are going to omit g in notations if no confusion may occur. For example, we write $\Lambda_s, \Lambda_{s,M}(w), \Lambda_{s,M}(w)$ and $U_M(x, s)$ in place of $\Lambda_s^g, \Lambda_{s,M}^g(w), \Lambda_{s,M}^g(x)$ and $U_M^g(x, s)$ respectively.

The following lemma is a step to prove the above theorem.

Lemma 9.7. *Let $g : T \to (0, 1]$ be a weight function and let $\mu \in \mathcal{M}_P(X, \mathcal{O})$. For $s \in (0, 1]$, $\lambda > 1$ and $c > 0$, define*

$$\Theta(s, \lambda, k, c) = \{v | v \in \Lambda_s, \mu(K_u) \leq c\mu(K_v) \text{ for any } u \in \Lambda_{\lambda s,k}((v)_{\lambda s})\},$$

where $(v)_{\lambda s}$ is the unique element of $\{[v]_n | 0 \leq n \leq |v|\} \cap \Lambda_{\lambda s}$. Assume that g is uniformly finite and that there exists $N \geq 1, \lambda > 1$ and $c > 0$ such that $\Lambda_{s,N}(w) \cap \Theta(s, \lambda, 2N + 1, c) \neq \emptyset$ for any $s \in (0, 1]$ and any $w \in \Lambda_s$. Then μ has the N-volume doubling property with respect to g.

Proof. Let $w \in \Lambda_{s,0}(x)$ and let $v \in \Lambda_{s,N}(w) \cap \Theta(s, \lambda, 2N+1, c)$. If $u \in \Lambda_{\lambda s,N}(x)$, then $u \in \Lambda_{\lambda s,2N+1}((v)_{\lambda s})$. Moreover, since $v \in \Lambda_{s,N}(x)$, we see that

$$\mu(K_u) \leq c\mu(K_v) \leq c\mu(U_N(x, s)).$$

Therefore,

$$\mu(U_N(x, \lambda s)) \leq \sum_{u \in \Lambda_{\lambda s,N}(x)} \mu(K_u) \leq \#(\Lambda_{\lambda s,N}(x))c\mu(U_N(x, s)).$$

Since g is uniformly finite, Lemma 7.17 shows that $\#(\Lambda_{\lambda s,N}(x))$ is uniformly bounded with respect to $x \in X$ and $s \in (0, 1]$. $\square$

Proof of Theorem 9.6. Fix $\lambda > 1$. By Proposition 7.16, there exists $c \geq 1$ such that $cg(w) \geq s \geq g(w)$ if $w \in \Lambda_s$. Since g is sub-exponential, there exist $c_1 \in (0, 1)$ and $m \geq 1$ such that $c_1 g(w) \geq g(v)$ whenever $v \in T_w$ and $|v| \geq |w| + m$. Assume that $w \in \Lambda_s$. Set $w_* = (w)_{\lambda s}$. Then $\lambda s \geq g(w_*)$. If $|w| \geq |w_*| + nm$, then $(c_1)^n g(w_*) \geq g(w)$ and hence $(c_1)^n \lambda s \geq g(w) \geq g(w)/c$. This shows that $(c_1)^n \lambda c \geq 1$. Set $l = \min\{n | n \geq 0, (c_1)^n \lambda c < 1\}$. Then we see that $|w| < |w_*| + lm$.

On the other hand, since μ is super-exponential, there exists $c_2 > 0$, such that $\mu(K_u) \geq c_2 \mu(K_{\pi(u)})$ for any $u \in T$. This implies that $\mu(K_{w_*}) \leq (c_2)^{-ml} \mu(K_w)$.

Since μ is gentle, there exists $c_* > 0$ such that $\mu(K_{w(1)}) \leq c_*\mu(K_{w(2)})$ whenever $w(1), w(2)\Lambda_s$ and $K_{w(1)} \cap K_{w(2)} \neq \emptyset$ for some $s \in (0,1]$. Therefore for any $u \in \Lambda_{\lambda s,M}(w_*)$,

$$\mu(K_u) \leq (c_*)^M \mu(K_{w_*}) \leq (c_*)^M (c_2)^{-ml}\mu(K_w).$$

Thus we have shown that

$$\Lambda_s = \Theta(s, \lambda, M, (c_*)^M (c_2)^{-ml})$$

for any $s \in (0,1]$. Now by Lemma 9.7, μ has M-volume doubling property with respect to g for any $M \geq 1$. □

In order to study the converse direction of Theorem 9.6, we need the thickness of K with respect to the weight function in question.

Theorem 9.8. *Let $g : T \to (0,1]$ be a weight function and let $\mu \in \mathcal{M}_P(X, \mathcal{O})$. Assume that g is thick.*
(1) *Suppose that g is exponential and uniformly finite. Then the following conditions are equivalent:*
(VD1) *μ has M-volume doubling property with respect to g for some $M \geq 1$.*
(VD2) *μ has M-volume doubling property with respect to g for any $M \geq 1$.*
(VD3) *μ is gentle with respect to g and μ is super-exponential.*
(2) *Suppose that K is minimal and g is super-exponential. Then* (VD1), (VD2) *and the following condition* (VD4) *are equivalent:*
(VD4) *g is sub-exponential and uniformly finite, μ is gentle with respect to g and μ is super-exponential.*
Moreover, if any of the above conditions (VD1), (VD2) *and* (VD4) *hold, then μ is exponential and*

$$\sup_{w \in T} \#(S(w)) < +\infty.$$

In general, the statement of Theorem 9.8 is false if g is not thick. In fact, in Example 10.10, we will present an example without thickness where d is adapted to g, g is exponential and uniformly finite, μ has the volume doubling property with respect to g but μ is neither gentle to g nor super-exponential.

As for a proof of Theorem 9.8, it is enough to show the following theorem.

Theorem 9.9. *Let $g : T \to (0,1]$ be a weight function and let $\mu \in \mathcal{M}_P(X, \mathcal{O})$. Assume that μ has M-volume doubling property with respect to g for some $M \geq 1$.*
(1) *If g is thick, then μ is gentle with respect to g.*
(2) *If g is thick and g is super-exponential, then μ is super-exponential.*
(3) *If g is thick and K is minimal, then g is uniformly finite.*
(4) *If g is thick, K is minimal, and μ is super-exponential, then*

$$\sup_{w \in T} \#(S(w)) < +\infty.$$

and μ is sub-exponential.
(5) *If g is uniformly finite, μ is gentle with respect to g, μ is sub-exponential, then g is sub-exponential.*

To prove Theorem 9.9, we need several lemmas.

Lemma 9.10. *Let $g : T \to (0,1]$ be a weight function. Assume that K is minimal and g is thick. Let $\mu \in \mathcal{M}_P(X,\mathcal{O})$. If μ has M-volume doubling property with respect to g for some $M \geq 1$, then there exists $c > 0$ such that $\mu(O_w) \geq c\mu(K_w)$ for any $w \in T$.*

Proof. By Proposition 8.2, there exists $\gamma > 0$ such that $O_v \supseteq U_M^g(x,\gamma s)$ for some $x \in K_v$ if $v \in \Lambda_s$. Let $w \in T$. Choose $u \in T_w$ such that $u \in \Lambda_{g(w)/2}$. Then

$$\mu(O_w) \geq \mu(O_u) \geq \mu(U_M^g(x,\gamma g(w)/2)).$$

Since μ has M-volume doubling property with respect to g, there exists $c > 0$ such that

$$\mu(U_M^g(y,\gamma r/2)) \geq c\mu(U_M^g(y,r))$$

for any $y \in X$ and any $r > 0$. Since $U_M(x,g(w)) \supseteq K_w$, it follows that

$$\mu(O_w) \geq \mu(U_M^g(x,\gamma g(w)/2)) \geq c\mu(U_M(x,g(w))) \geq c\mu(K_w).$$

□

Lemma 9.11. *Let $g : T \to (0,1]$ be a weight function. Assume that $\mu \in \mathcal{M}_P(X,\mathcal{O})$ is gentle with respect to g and that g is uniformly finite. Then there exists $c > 0$ such that*

$$c\mu(K_w) \geq \mu(U_M(x,s))$$

if $w \in \Lambda_{s,0}(x)$.

Proof. Since μ is gentle with respect to g, there exists $c_1 > 0$ such that $\mu(K_v) \leq c_1\mu(K_w)$ if $w \in \Lambda_s$ and $v \in \Lambda_{s,1}(w)$. Hence if $v \in \Lambda_{s,M+1}(w)$, it follows that $\mu(K_v) \leq (c_1)^{M+1}\mu(K_w)$. Since $\Lambda_{s,M}(x) \subseteq \Lambda_{s,M+1}(w)$,

$$\begin{aligned}\mu(U_M(x,s)) &\leq \sum_{v\in\Lambda_{s,M}(x)} \mu(K_v) \\ &\leq \sum_{v\in\Lambda_{s,M}(x)} (c_1)^{M+1}\mu(K_w) = (c_1)^{M+1}\#(\Lambda_{s,M}(x))\mu(K_w).\end{aligned}$$

By Lemma 7.17, we obtain the desired statement. □

Proof of Theorem 9.9. (1) Since g is thick, there exists $\beta \in (0,1)$ such that, for any $s \in (0,1]$ and any $w \in \Lambda_s$, $K_w \supseteq U_M(x,\beta s)$ for some $x \in K_w$. By M-volume doubling property of μ, there exists $c > 0$ such that $\mu(U_M(x,\beta s)) \geq c\mu(U_M(x,s))$ for any $s \in (0,1]$ and any $x \in X$. Hence

$$\mu(K_w) \geq \mu(U_M(x,\beta s)) \geq c\mu(U_M(x,s)). \tag{9.1}$$

If $v \in \Lambda_s$ and $K_v \cap K_w \neq \emptyset$, then $U_M(x,s) \supseteq K_v$. (9.1) shows that $\mu(K_w) \geq c\mu(K_v)$. Hence μ is gentle with respect to g.

(2) Let $v \in T\backslash\{\phi\}$. Choose $u \in T_v$ so that $u \in \Lambda_{g(v)/2}$. Applying (9.1) to u and using the volume doubling property repeatedly, we see that there exists $x \in K_u$ such that

$$\mu(K_v) \geq \mu(K_u) \geq \mu(U_M(x, \beta g(v)/2)) \geq c^n \mu(U_M(x, \beta^{1-n} g(v)/2)) \tag{9.2}$$

for any $n \geq 0$. Since g is super-exponential, there exists $n \geq 0$, which is independent of v, such that $\beta^{1-n} g(v)/2 > g(\pi(v))$. By (9.2), we obtain $\mu(K_v) \geq c^n \mu(K_{\pi(w)})$. Thus μ is super-exponential.

(3) Let $w \in \Lambda_s$. Then $\{O_v\}_{v \in \Lambda_{s,1}(w)}$ is mutually disjoint by Lemma 4.2-(2). By (9.1) and Lemma 9.10,

$$\mu(K_w) \geq c\mu(U_M(x, s)) \geq c \sum_{v \in \Lambda_{s,1}(w)} \mu(O_v) \geq c^2 \sum_{v \in \Lambda_{s,1}(w)} \mu(K_v)$$

(The constants c's in (9.1) and Lemma 9.10 may be different but by choosing the smaller one, we may use the same c.) As μ is gentle with respect to g by (1), there exists $c_* > 0$, which is independent of w and s, such that $\mu(K_v) \geq c_* \mu(K_w)$ for any $v \in \Lambda_{s,1}(w)$. Therefore,

$$\mu(K_w) \geq c^2 \sum_{v \in \Lambda_{s,1}(w)} \mu(K_v) \geq c^2 c_* \#(\Lambda_{s,1}(w)) \mu(K_w)$$

Hence $\#(\Lambda_{s,1}(w)) \leq c^{-2}(c_*)^{-1}$ and g is uniformly finite.

(4) By Lemma 9.10, for any $w \in T$, we have

$$\mu(K_w) \geq \mu(\cup_{v \in S(w)} O_v) = \sum_{v \in S(w)} \mu(O_v) \geq c \sum_{v \in S(w)} \mu(K_v).$$

Since μ is super-exponential, there exists $c' > 0$ such that $\mu(K_v) \geq c' \mu(K_w)$ if $w \in T$ and $v \in S(w)$. Hence

$$\mu(K_w) \geq c \sum_{v \in S(w)} \mu(K_v) \geq cc' \#(S(w)) \mu(K_w).$$

Thus $\#(S(w)) \leq (cc')^{-1}$, which is independent of w. Note that $\#(S(w)) \geq 2$ for any $w \in T$. By the above arguments,

$$\mu(O_v) \geq c\mu(K_v) \geq c_* \mu(K_w) \geq c_* \mu(O_w) \tag{9.3}$$

for any $w \in T$ and any $v \in S(w)$, where $c_* = cc'$. Let $v_* \in S(w)$. If $\mu(O_{v_*}) = (1-a)\mu(O_w)$, then

$$\mu(O_w) \geq \sum_{v \in S(w)} \mu(O_v) = (1-a)\mu(O_w) + \sum_{v \in S(w), v \neq v_*} \mu(O_v).$$

This implies $a\mu(O_w) \geq \mu(O_v)$ for any $v \in S(w)\backslash\{v_*\}$. By (9.3), $a \geq c_*$. Therefore, $\mu(O_v) \leq (1-c_*)\mu(O_w)$ for any $v \in S(w)$. This implies

$$c\mu(K_v) \leq \mu(O_v) \leq (1-c_*)^m \mu(O_w) \leq (1-c_*)^m \mu(K_w)$$

if $v \in T_w$ and $|v| = |w| + m$. Choosing m so that $(1 - c_*)^m < c$, we see that μ is sub-exponential.

(5) As μ is sub-exponential, there exists $\alpha \in (0,1)$ and $m \geq 0$ such that $\mu(K_v) \leq \alpha\mu(K_w)$ if $u \in T_w$ and $|u| \geq |w| + m$. Since μ has M-volume doubling property with respect to g, there exist $\lambda, c \in (0,1)$ such that $\mu(U_M(x, \lambda s)) \geq c\mu(U_M(x,s))$ for any $x \in X$ and any $s > 0$. Let $\beta \in (\lambda, 1)$. Assume that g is not sub-exponential. Then for any $n \geq 0$, there exist $w \in T$ and $u \in T_w$ such that $|u| \geq |w| + nm$ and $g(u) \geq \beta g(w)$. In case $g(w) = g(\pi(w))$, we may replace w by $v = [w]_m$ for some $m \in \{0, 1, \ldots, |w|\}$ satisfying $g(\pi(v)) > g(v) = g(w)$ or $g(v) = g(w) = 1$. Consequently we may assume $w \in \Lambda_{g(w)}$. Set $s = g(w)$. Since $\beta > \lambda$, there exists $u_* \in T_u \cap \Lambda_{s\lambda}$. Let $x \in K_{u_*}$. Then by the volume doubling property,

$$\mu(U_M(x, \lambda s)) \geq c\mu(U_M(x,s)) \geq c\mu(K_w).$$

By Lemma 9.11, there exists $c_* > 0$ which is independent of n, w and u such that

$$c_*\mu(K_{u_*}) \geq \mu(U_M(x, \lambda s)).$$

Since μ is sub-exponential,

$$\alpha^n c_*\mu(K_w) \geq c_*\mu(K_{u_*}) \geq \mu(U_M(x, \lambda s)) \geq c\mu(K_w).$$

This implies $\alpha^n c_* \geq c$ for any $n \geq 0$ which is a contradiction. $\square$

At the end of this section, we give a proof of Theorem 7.21 by using Theorem 9.9.

Proof of Theorem 7.21. If is enough to show the case where $\alpha = 1$. Assume that $g_d \underset{\text{BL}}{\sim} g_\mu$ and d is uniformly finite. Since d is adapted, there exists $M \geq 1$ and $\alpha_1, \alpha_2 > 0$ such that

$$U^d_M(x, \alpha_1 r) \subseteq B_d(x,r) \subseteq U^d_M(x, \alpha_2 r)$$

for any $x \in X$ and any $r > 0$.

Assume that $g_d \underset{\text{BL}}{\sim} g_\mu$. Then there exists $c_1, c_2 > 0$ such that

$$c_1 d_w \leq \mu_w \leq c_2 d_w$$

for any $w \in T$. For any $x \in X$, choose $w \in T$ such that $x \in K_w$ and $w \in \Lambda^d_{\alpha_1 r}$, then since d is super-exponential, there exists λ which is independent of x, r and w such that

$$\mu(B_d(x,r)) \geq \mu(U^d_M(x, \alpha_1 r)) \geq \mu(K_w) \geq c_1 d_w \geq c_1\lambda d_{\pi(w)} \geq c_1\lambda\alpha_1 r.$$

On the other hand, since d is uniformly finite, Lemma 7.17 implies

$$\begin{aligned}\mu(B_d(x,r)) \leq \mu(U^d_M(x, \alpha_2 r)) &\leq C \sum_{w \in \Lambda^d_{\alpha_2 r, M}(x)} \mu(K_w) \\ &\leq Cc_2 \sum_{w \in \Lambda^d_{\alpha_2 r, M}(x)} d_w \leq Cc_2 \#(\Lambda^d_{\alpha_2 r, M})\alpha_2 r \leq C_2 r\end{aligned}$$

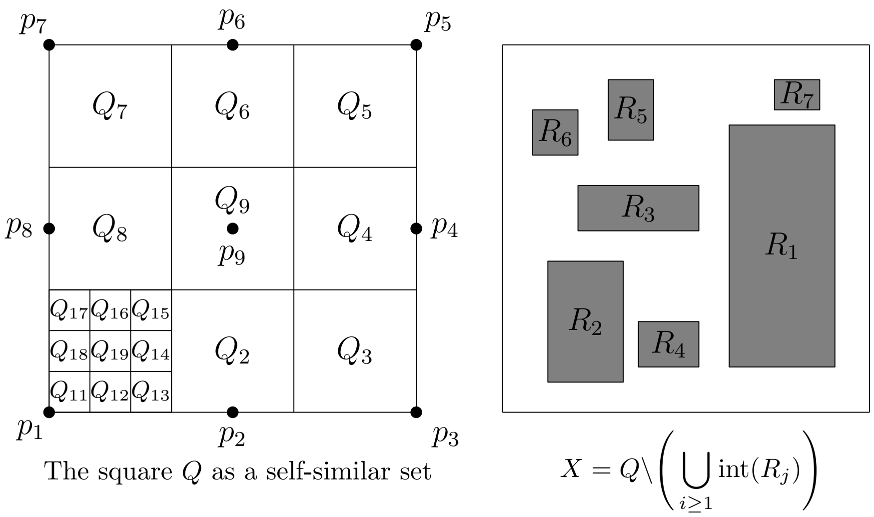


Figure 3: The square Q and its subset X

Conversely, assume (7.6). For any $w \in T$ and any $x \in K_w$,

$$K_w \subseteq U_M^d(x, d_w) \subseteq B_d(x, d_w/c_1).$$

Hence

$$\mu(K_w) \leq \mu(B_d(x, d_w/c_1)) \leq C_2 d_w/c_1.$$

By Proposition 8.1, there exists $z \in K_w$ such that

$$K_w \supseteq U_M^d(z, \beta d_{\pi(w)}) \supseteq B_d(z, \beta d_{\pi(w)}/c_2).$$

By (7.6),

$$\mu(K_w) \geq \mu(B_d(z, \beta d_{\pi(w)}/c_2)) \geq C_1 \beta d_{\pi(w)}/c_2 \geq C_1 \beta d_w/c_2.$$

Thus we have shown that $g_d \underset{\mathrm{BL}}{\sim} g_\mu$. Furthermore, since d is M-adapted for some $M \geq 1$, μ has M-volume doubling property with respect to the weight function g_d. Applying Theorem 9.9-(3), we see that g_d is uniformly finite. In the same way, by Theorem 9.9, both g_d and g_μ are exponential. $\square$

10 Example: subsets of the square

In this section, we give illustrative examples of the results in the previous sections. For simplicity, our examples are the subsets of the square $[0,1]^2$, which is denoted by Q, and trees parametrizing partitions are sub-trees of $(T^{(9)}, \mathcal{A}^{(9)}, \phi)$ defined in Example 3.3. Note that $[0,1]^2$ is divided into 9-squares with the length of the sides $\frac{1}{3}$. As in Example 4.5, the tree $(T^{(9)}, \mathcal{A}^{(9)}, \phi)$ is naturally appears as the tree parametrizing the natural partition associated with this self-similar division. Namely, let $p_1 = (0,0), p_2 = (1/2,0), p_3 = (1,0), p_4 = (1,1/2), p_5 = (1,1), p_6 = (1/2,1), p_7 = (0,1), p_8 = (0,1/2)$ and $p_9 = (1/2,1/2)$. Set $W = \{1, \ldots, 9\}$. Define $F_i : Q \to Q$ by

$$F_i(x) = \frac{1}{3}(x - p_i) + p_i$$

for any $i \in W$. Then F_i is a similitude for any $i \in W$ and

$$Q = \bigcup_{i \in W} F_i(Q).$$

See Figure 3. In this section, we write $(W_*, \mathcal{A}_*, \phi) = (T^{(9)}, \mathcal{A}^{(9)}, \phi)$. Then $(W_*, \mathcal{A}_*, \phi)$ is a locally finite tree with a reference point ϕ. It follows that $|w|_{(W_*, \mathcal{A}_*, \phi)} = m$ if and only if $w \in W_m$ and $\pi^{(W_*, \mathcal{A}_*, \phi)}(w) = w_1 \dots w_{m-1}$ for any $w = w_1 \dots w_m \in W_m$. For simplicity, we use $|w|$ and π in place of $|w|_{(W_*, \mathcal{A}_*, \phi)}$ and $\pi^{(W_*, \mathcal{A}_*, \phi)}$ respectively hereafter. Define $g : W_* \to (0, 1]$ by $g(w) = 3^{-|w|}$ for any $w \in W_*$. Then g is an exponential weight function.

As for the natural associated partition of Q, define $F_w = F_{w_1} \circ \dots \circ F_{w_m}$ and $Q_w = F_w(Q)$. for any $w = w_1 \dots w_m \in W_m$. Set $Q_*(w) = Q_w$ for any $w \in W_*$. (If $w = \phi$, then F_ϕ is the identity map and $Q_\phi = Q$.) Then $Q_* : W_* \to \mathcal{C}(Q, \mathcal{O})$ is a partition of Q parametrized by $(W_*, \mathcal{A}_*, \phi)$, where $\mathcal{O}$ is the natural topology induced by the Euclidean metric. In fact, $\cap_{m \geq 0} Q_{[\omega]_m}$ for any $\omega \in \Sigma$, where $\Sigma = W^{\mathbb{N}}$, is a single point. Define $\sigma : \Sigma \to Q$ by $\{\sigma(\omega)\} = \cap_{m \geq 0} Q_{[\omega]_m}$.

It is easy to see that the partition Q_* is minimal, g is uniformly finite, g is thick with respect to the partition Q_*, and the (restriction of) Euclidean metric d_E on Q is 1-adapted to g.

In order to have more interesting examples, we consider certain class of subsets of Q whose partition is parametrized by a subtree $(T, \mathcal{A}_*|_{T \times T}, \phi)$ of $(W_*, \mathcal{A}_*, \phi)$. Let $\{I_m\}_{m \geq 0}$ be a sequence of subsets of W_* satisfying the following conditions (SQ1), (SQ2) and (SQ3):

(SQ1) For any $m \geq 0$, $I_m \subseteq W_m$ and if $\widehat{I}_{m+1} = \{wi | w \in I_m, i \in W\}$, then $I_{m+1} \supseteq \widehat{I}_{m+1}$.

(SQ2) $Q_w \cap Q_v = \emptyset$ if $w \in \widehat{I}_{m+1}$ and $v \in I_{m+1} \backslash \widehat{I}_{m+1}$.

(SQ3) For any $m \geq 0$, the set $\cup_{w \in I_m} Q_w$ is a disjoint union of rectangles $R^m_j = [a^m_j, b^m_j] \times [c^m_j, d^m_j]$ for $j = 1, \dots, k_m$.

See Figure 3. By (SQ2), we may assume that $k_m \leq k_{m+1}$ and $R^m_j = R^{m+1}_j$ for any m and any $j = 1, \dots, k_m$ without loss of generality. Under this assumption, we may omit m of $R^m_j, a^m_j, b^m_j, c^m_j$ and d^m_j and simply write R_j, a_j, b_j, c_j and d_j respectively.

Notation. As a topology of $Q = [0, 1] \times [0, 1]$, we consider the relative topology induced by the Euclidean metric. We use $\mathrm{int}(A)$ and ∂A to denote the interior and the boundary, respectively, of a subset A of Q with respect to this topology.

Note that $\mathrm{int}(\cup_{w \in I_m} Q_w) = \cup_{j=1,\dots,k_m} \mathrm{int}(R_j)$.

Proposition 10.1. (1) *Define*

$$X^{(m)} = Q \backslash \left(\bigcup_{j=1,\dots,k_m} \mathrm{int}(R_j) \right).$$

then $X^{(m)} \supseteq X^{(m+1)}$ for any $m \geq 0$ and $X = \cap_{m \geq 0} X^{(m)}$ is a non-empty compact set. Moreover, $\partial R_j \subseteq X$ for any $j \geq 1$.

(2) *Define* $(T)_m = \{w|w \in W_m, \mathrm{int}(Q_w) \cap X \neq \emptyset\}$ *for any* $m \geq 0$. *If* $T = \cup_{m\geq 0}(T)_m$ *and* $\mathcal{A} = \mathcal{A}_*|_{T\times T}$, *then* $(T, \mathcal{A}, \phi)$ *is a locally finite tree with the reference point* ϕ *and* $\#(S(w)) \geq 3$ *for any* $w \in T$. *Moreover, let*

$$\Sigma_T = \{\omega|\omega \in \Sigma, [\omega]_m \in (T)_m \text{ for any } m \geq 0\}$$

Then $X = \sigma(\Sigma_T)$.
(3) *Define* $K_w = Q_w \cap X$ *for any* $w \in T$. *Then* $K_w \neq \emptyset$ *and* $K : T \to \mathcal{C}(X)$ *defined by* $K(w) = K_w$ *is a minimal partition of* X *parametrized by* $(T, \mathcal{A}, \phi)$. *Moreover,* $g|_T$ *is exponential and uniformly finite.*

To prove the above proposition, we need the following lemma.

Lemma 10.2. *If* $w \in T$, *then* $\cup_{i\in W, wi\notin T} Q_{wi}$ *is a disjoint union of rectangles and* $\#(\{i|i \in W, wi \in T\}) \in \{3, 5, 7, 8, 9\}$.

Proof. Set $I = \{i|i \in W, wi \notin T\}$. For each $i \in I$, there exists $k_i \geq 1$ such that $Q_{wi} \subseteq R_{k_i}$. Hence $\cup_{i\in I} Q_{wi} = \cup_{i\in I}(Q_w \cap R_{k_i})$. Since $\{R_j\}_{j\geq 1}$ are mutually disjoint, we have the desired conclusion. Assume that $I = W$. Suppose $|i - j| = 1$. Since $Q_{wi} \cap Q_{wj} \neq \emptyset$, we see that$R_{k_i} = R_{k_j}$. Hence $R_{k_1} = \ldots = R_{k_9}$ and $Q_w \subseteq R_{k_1}$. This contradicts with the fact that $\mathrm{int}(Q_w) \cap X \neq \emptyset$. Thus $I \neq W$. Considering all the possible shape of $\cup_{i\in W, wi\notin T} Q_{wi}$, we conclude $\#(\{i|i \in W, wi \in T\}) \in \{3, 5, 7, 8, 9\}$. □

Proof of Proposition 10.1. (1) Since $\{X^{(m)}\}_{m\geq 0}$ is a decreasing sequence of compact sets, it follows that X is a nonempty compact set. By (SQ2), $R_j \cap R_i = \emptyset$ for any $i \neq j$. Therefore, $\partial R_j \subseteq X^{(m)}$ for any $m \geq 0$. Hence $\partial R_j \subseteq X$.
(2) If $w \in (T)_m$, then $\mathrm{int}(Q_{\pi(w)}) \cap X \supseteq \mathrm{int}(Q_w) \cap X \neq \emptyset$. Hence $\pi(w) \in (T)_{m-1}$. Using this inductively, we see that $[w]_k \in (T)_k$ for any $k \in \{0, 1, \ldots, m\}$. This implies that $(T, \mathcal{A}, \phi)$ is a locally finite tree with a reference point ϕ. By Lemma 10.2, we see that $\#(\{i|i \in W, wi \in (T)_{m+1}\}) \geq 3$. Next if $\omega \in \Sigma_T$, then for any $m \geq 0$, there exists $x_m \in \mathrm{int}(Q_{[\omega]_m}) \cap X$. Therefore, $x_m \to \sigma(\omega)$ as $m \to \infty$. Since X is compact, it follows that $\sigma(\omega) \in X$. Conversely,assume that $x \in X$. Set $W_{m,x} = \{w|w \in W_m, x \in Q_w\}$. Note that $\#(\sigma^{-1}(x)) \leq 4$ and $\cup_{w\in W_{m,x}} Q_w$ is a neighborhood of x. Suppose that $(T)_m \cap W_{m,x} \neq \emptyset$ for any $m \geq 0$. Then there exists $w_m \in (T)_m \cap W_{m,x}$ such that $x \in Q_{w_m}$. Since $W_{m,x} = \{[\omega]_m|\omega \in \sigma^{-1}(x)\}$, there exists $\omega \in \sigma^{-1}(x)$ such that $[\omega]_m = w_m$ for infinitely many m. As $\mathrm{int}(Q_{[\omega]_m})$ is monotonically decreasing, it follows that $[\omega]_m \in (T)_m$ for any $m \geq 0$. This implies $x \in \sigma(\Sigma_T)$. Suppose that there exists $m \geq 0$ such that $W_{m,x} \cap (T)_m = \emptyset$. Consequently, for any $w \in W_{m,x}$, $\mathrm{int}(Q_w) \cap X = \emptyset$ and hence there exists $j_w \geq 1$ such that $Q_w \subseteq R_{j_w}$. Note that $Q_w \cap Q_{w'} \neq \emptyset$ for any $w, w' \in W_{m,x}$ and hence $R_{j_w} = R_{j_{w'}}$. Therefore, $\cup_{w\in W_{m,x}} Q_w \subseteq R_j$ for some $j \geq 1$. Since $\cup_{w\in W_{m,x}} Q_w$ is a neighborhood of x, it follows that $x \notin X$. This contradiction concludes the proof.
(3) The fact that K is a partition of X parametrized by $(T, \mathcal{A}|_{T\times T}, \phi)$ is straight forward from (1) and (2). As $K_w \backslash (\cup_{v\in (T)_m, v\neq w} K_v)$ is contained in the sides of the square Q_w, the partition K is minimal. Since $\Lambda_s^{g|_T} = (T)_m$ if

and only if $\frac{1}{3^m} \le s < \frac{1}{3^{m-1}}$, it follows that $g|_T$ is exponential. Furthermore, $\Lambda^{g|_T}_{s,1}(w) \subseteq \{v|v \in W_m, Q_v \cap Q_w \neq \emptyset\}$ for any $w \in (T)_m$. Hence $\#(\Lambda^{g|_T}_{s,1}(w)) \le 8$. This shows that $g|_T$ is uniformly finite. □

Now, we consider when the restriction of the Euclidean metric is adapted.

Definition 10.3. Let $R = [a, b] \times [c, d]$ be a rectangle. The degree of distortion of R, $\kappa(R)$, is defined by

$$\kappa(R) = \max\left\{1, (1-\delta_{c0})(1-\delta_{d1})\frac{|b-a|}{|d-c|}, (1-\delta_{a0})(1-\delta_{b1})\frac{|d-c|}{|b-a|}\right\},$$

where δ_{xy} is the Kronecker delta defined by $\delta_{xy} = 1$ if $x = y$ and $\delta_{xy} = 0$ if $x \neq y$. Moreover, for $\kappa \ge 1$, we define

$$\mathcal{R}^0_\kappa = \{R|R \text{ is a rectangle, } R \subseteq Q \text{ and } \kappa(R) \le \kappa\}$$

and

$$\mathcal{R}^1_\kappa = \{R|R \subseteq Q, R \text{ is a rectangle, there exists } w \in T \text{ such that } Q_w \backslash \mathrm{int}(R) \\ \text{has two connected components and } \kappa(Q_w \cap R) \le \kappa\}$$

The extra factors $(1-\delta_{c0}), (1-\delta_{d1}), (1-\delta_{a0})$ and $(1-\delta_{b1})$ become effective if the rectangle R has an intersection with the boundary of the square Q.

Theorem 10.4. *Let d be the restriction of the Euclidean metric on X. Then d is adapted to $g|_T$ if and only if the following condition* (SQ4) *holds:*
(SQ4) *There exists $\kappa \ge 1$ such that $R_j \in \mathcal{R}^0_\kappa \cup \mathcal{R}^1_\kappa$ for any $j \ge 1$.*

Several lemmas are needed to prove the above theorem.

Lemma 10.5. *Define $N(x, y) = \min\{[-\frac{\log|x_1-y_1|}{\log 3}], [-\frac{\log|x_2-y_2|}{\log 3}]\}$ for any $x = (x_1, x_2), y = (y_1, y_2) \in Q$.*
(1)

$$\frac{1}{3^{N(x,y)+1}} < d(x, y) \le \frac{\sqrt{2}}{3^{N(x,y)}}$$

(2) *If $x, y \in X$, then there exist $w, v, u \in W_{N(x,y)}$ such that $w, v \in T$, $x \in Q_w$, $y \in Q_u$, $Q_w \cap Q_v \neq \emptyset$ and $Q_v \cap Q_v \neq \emptyset$.*

Proof. Set $N = N(x, y)$. Let $n_i = [-\frac{\log|x_i-y_i|}{\log 3}]$ for $i = 1, 2$. Then $N = \min\{n_1, n_2\}$ and

$$\frac{1}{3^{N+1}} < |x_j - y_j| \le \frac{1}{3^N}$$

if $n_j = N$. This yields (1). Since $x, y \in X$, then there exists $w, u \in (T)_m$ such that $x \in K_w$ and $y \in K_u$. Since $|x_1 - y_1| \le 1/3^N$ and $|x_2 - y_2| \le 1/3^N$, we find $v \in W_m$ such that $Q_w \cap Q_v \neq \emptyset$ and $Q_v \cap Q_u \neq \emptyset$. □

Notation. For integers $n, k, l \geq 0$, we set

$$Q(n,k,l) = \left[\frac{k}{3^n}, \frac{(k+1)}{3^n}\right] \times \left[\frac{l}{3^n}, \frac{(l+1)}{3^n}\right]$$

Lemma 10.6. *Assume* (SQ4). *Let* $M = [\log(2\kappa)/\log 3] + 1$ *and* $L = 2[2\kappa] + 9$. *If* $w, v \in (T)_m$ *and* $Q_w \cap Q_v \neq \emptyset$, *then there exists* K*-chain* $(w(1), \ldots, w(L))$ *such that* $w \in T_{w(1)}$, $v \in T_{w(L)}$ *and* $|w(k)| \geq m - M$.

Proof. Case 1: Assume that $Q_w \cap Q_v$ is a line segment. Without loss of generality, we may assume that $Q_w = Q(m, k-1, l)$ and $Q_v = Q(m,k,l)$.
Case 1a: $K_w \cap K_v \neq \emptyset$, then (w, v) is a desired K-chain.
Case 1b: In case $K_w \cap K_v = \emptyset$, then $Q_w \cap Q_v \cap K_w$ and $Q_w \cap Q_v \cap K_v$ are disjoint closed subsets of $Q_w \cap Q_v$. Since $Q_w \cap Q_v$ is connected, there exists $a \in Q_w \cap Q_v$ such that $a \notin K_w \cap K_v$. Since $K_w \cup K_v$ is closed, there exists an open neighborhood of a which has no intersection with $K_w \cap K_v$. This open neighborhood must be contained in R_j for some j. So, we see that $R_j \cap \mathrm{int}(Q_w \cap Q_v) \neq \emptyset$ and $(k-1)/3^m \leq a_j \leq k/3^m \leq b_j \leq (k+1)/3^m$. If $c_i > l/3^m$, then the line segment $[a_j, b_j] \times \{c_j\} \subseteq X$, we see that $K_w \cap K_v \neq \emptyset$. Therefore $c_j \leq l/3^m$. By the same argument we have $d_j \geq (l+1)/3^m$. Now if $R_j \in \mathcal{R}^0_\kappa$, it follows that $|d_j - c_j| \leq 2\kappa/3^m$. Hence the line segment $[a_j, b_j] \times \{c_j\}$ and $[a_j, b_j] \times \{d_j\}$ are covered by at most 4 pieces of K_u's for $u \in (T)_m$ and the line segment $\{a_j\} \times [c_j, d_j]$ and $\{b_j\} \times [c_j, d_j]$ is covered by at most $2\kappa + 2$ pieces of K_u's for $u \in (T)_m$. Since K_w and K_v are pieces of these coverings, we obtain a K-chain $(w(1), \ldots, w(k))$ from these coverings where $w(1) = w, w(k) = v$ and $l \leq 2\kappa + 5$. Next assume $R_j \in \mathcal{R}^1_\kappa$. Note that $2\kappa/3^m \leq 1/3^{m-M}$. By the definition of $\mathcal{R}^1_\kappa$, there exists $u \in (T)_{m-M}$ such that $Q_u \backslash R_j$ has tow connected component. Sifting Q_u up and down, we may find $u' \in (T)_{m-M}$ such that $Q_w \cup Q_v \subseteq Q_{u'}$. Then (u') is a desired K-chain.
Case 2: Assume that $Q_w \cap Q_v$ is a single point. Without loss of generality, we may assume that $Q_w = Q(m, k-1, l-1)$ and $Q_v = Q(m,k,l)$. Choose $u(1), u(2) \in W_m$ so that $Q_{u(1)} = Q(m, k-1, l)$ and $Q_{w(2)} = Q(m, k+1, l-1)$. If neither $u(1)$ nor $u(2)$ does not belong to T. Then there exist $i, j \geq 1$ such that $Q_{u(1)} \subseteq R_i$ and $Q_{u(2)} \subseteq R_j$. Since $Q_{u(1)} \cap Q_{u(2)} \neq \emptyset$, it follows that $R_i = R_j$ and hence $Q_w \cup Q_v \subseteq R_i$. This contradicts to the fact that $w, v \in T$. Hence $u(1) \in T$ or $u(2) \in T$. Let $u(1) \in T$. Then $Q_w \cap Q_{u(1)}$ and $Q_{u(1)} \cap Q_v$ are line segments. By using the method in (1), we find a chain between w and $u(1)$ and a chain between $u(1)$ and v. Connecting these two chains, we obtain the desired chain $(w(1), \ldots, w(L))$. $\square$

Proof of Theorem 10.4. Assume (SQ4). Let $x, y \in X$. Define $N = N(x,y)$ and choose $w, v, u \in W_N$ as in Lemma 10.5. We fix the constants M and L as in Lemma 10.6. There are two cases.
Case 1: Suppose $v \in T$. Applying Lemma 10.6 to two pairs $\{w, v\}$ and $\{v, u\}$ and connecting the two resultant chains, we obtain a K-chain $(w(1), \ldots, w(2L-1)) \in \mathcal{CH}_K(x,y)$ satisfying $w \in T_{w(1)}, u \in\in T_{w(2L-1)}$ and $|w(i)| \geq N - M$ for any i. This concludes Case 1.

Case 2: Suppose $v \notin T$. If $Q_w \cap Q_u \neq \emptyset$. We have K-chain $(w(1), \ldots, w(L)) \in \mathcal{CH}(x, y)$ satisfying $w \in T_{w(1)}$, $u \in T_{w(L)}$ and $|w(i)| \geq N - M$ for any i by Lemma 10.6. Assume $Q_w \cap Q_u = \emptyset$. Without loss of generality, we may assume one of the following tree situations (a), (b) and (c):
(a) $Q_w = Q(N, k-1, l-1)$ and $Q_u = Q(N, k+1, l-1)$.
(b) $Q_w = Q(N, k-1, l-1)$ and $Q_u = Q(N, k+1, l)$.
(c) $Q_w = Q(N, k-1, l-1)$ and $Q_u = Q(N, k+1, l+1)$.

Set $Q_{v(1)} = Q(N, k, l-1)$ and $Q_{v(2)} = Q(N, k, l)$. In each case, $x_1 = k/3^N$ and $y_1 = (k+1)/3^N$.

First consider cases (a) and (b). If either $v(1)$ or $v(2)$ belongs to T, then replacing v by either $v(1)$ or $v(2)$, we end up with Case 1. So we assume that neither $v(1)$ nor $v(2)$ belongs to T. Then there exists $j \geq 1$ such that $Q_{v(1)} \cup Q_{v(2)} \subseteq R_j$. Since $x_1 = k/3^N$ and $y_1 = (k+1)/3^N$, $a_j = k/3^N$ and $b_j = (k+1)/3^N$. Then by the same argument as in the proof of Lemma 10.6, there exists K-chain $(w(1), \ldots, w(L)) \in \mathcal{CH}_K(x, y)$ such that $w \in T_{w(1)}$, $u \in T_{w(L)}$ and $|w(i)| \geq N - M$ for any i.

Next in the situation of (c), $x = (k/3^N, l/3^N)$, $y = ((k+1)/3^N, (l+1)/3^N)$ and $v = v(2)$. Since $v = v(1) \notin T$, there exists $j \geq 1$ such that $Q_v \subseteq R_j$. Note that $x, y \in X \cap Q_v$. Hence $Q_v = R_j$. Choose $v(3), v(4) \in W_N$ so that $Q_{v(3)} = Q(N, k+1, l-1)$ and $Q_{v(4)} = Q(N, k+1, l)$. Then $v(3), v(4) \in T$ and therefore $(w, v(1), v(3), v(4), u)$ is a K-chain between x and y. This concludes Case 2.

As a consequence, we may always find a K-chain $(w(1), \ldots, w(2L-1)) \in \mathcal{CH}_K(x, y)$ satisfying $|w(i)| \geq N(x, y) - M$ for any i. By Lemma 10.5-(1),

$$3^{M+1} d(x, y) \geq 3^M \frac{1}{3^N} \geq \frac{1}{3^{w(i)}} = g(w(i)).$$

Thus we have verified the conditions (ADa) and $(\mathrm{ADb})_{\mathrm{2L-2}}$ in Theorem 6.4. Hence d is $(2L-2)$-adapted to $g|_T$ by Theorem 6.4.

Conversely, assume that d is J-adapted to $g|_T$. By $(\mathrm{ADb})_{\mathrm{J}}$, there exists $C \geq 0$ such that for any $x, y \in X$, there exists a K-chain $(w(1), \ldots, w(J+1)) \in \mathcal{CH}_K(x, y)$ satisfying

$$Cd(x, y) \geq \frac{1}{3^{|w(i)|}} \tag{10.1}$$

for any $i = 1, \ldots, J+1$. Set $M = [\log(\sqrt{2}C)/\log 3] + 1$. Suppose that (SQ4) does not hold; for any $\kappa \geq 1$, there exists $R_j \notin \mathcal{R}^0_\kappa \cup \mathcal{R}^1_\kappa$. In particular, we choose $\kappa \geq 3^{M+2}$. Write $R = R_j$ and set $R = [a, b] \times [c, d]$. Define $\partial R_L = \{a\} \times [c, d]$ and $\partial R_R = \{b\} \times [c, d]/$ (The symbols "L" and "R" correspond to the words "Left" and "Right" respectively.) Without loss of generality, we may assume that $|a - b| \leq |c - d|$. Since $R \notin \mathcal{R}^0_\kappa$, we have $\kappa|b - a| \leq |d - c|$. Let $x = (a, (c+d)/2)$ and let $y = (b, (c+d)/2)$. Set $N = N(x, y)$. There exists $(w(1), \ldots, w(J+1)) \in \mathcal{CH}_K(x, y)$ such that (10.1) holds for any $i = 1, \ldots, J+1$. By Lemma 10.5-(1),

$$|w(i)| \geq N - M \tag{10.2}$$

for any $i = 1, \ldots, J+1$. Define $A = [0,1] \times (c,d)$. If $Q_{w(i)} \subseteq A$, $Q_{w(i)} \cap \partial R_L \neq \emptyset$ and $Q_{w(i)} \cap \partial R_R \neq \emptyset$, then the fact that $R \notin \mathcal{R}^1_\kappa$ along with Lemma 10.5-(1) shows

$$\frac{1}{3^{|w(i)|}} \geq \kappa|b-a| = \kappa d(x,y) \geq \frac{\kappa}{3^{N+1}} \geq \frac{1}{3^{N+M-1}}. \tag{10.3}$$

This contradicts to (10.2) and hence we verify the following claim (I):
(I) If $Q_{w(i)} \subseteq A$, then $Q_{w(i)} \cap \partial R_L = \emptyset$ or $Q_{w(i)} \cap \partial R_R = \emptyset$.

Next we prove that there exists $j \geq 1$ such that $Q_{w(j)} \backslash A \neq \emptyset$. Otherwise, $Q_{w(i)} \subseteq A$ for any $i = 1, \ldots, J+1$. Let $A_L = [0,a] \times (c,d)$ and let $A_R = [b,1] \times (c,d)$. Define $I_L = \{i | i = 1, \ldots, J+1, Q_{w(i)} \cap A_L \neq \emptyset$ and $I_R = \{i | i = 1, \ldots, J+1, Q_{w(i)} \cap A_R \neq \emptyset\}$. Since $K_{(w(i))} \subseteq X \cap A \subseteq A_L \cup A_R$, it follows that $\{1, \ldots, J+1\} = I_L \cup I_R$. Moreover, the claim (I) implies $I_L \cap I_R = \emptyset$. Hence $I_L = \{i | i = 1, \ldots, J+1, K_{w(i)} \subseteq A_L$ and $I_R = \{i | i = 1, \ldots, J+1, K_{w(i)} \subseteq A_R\}$. This contradicts to the fact that $(w(1), \ldots, w(J+1))$ is a K-chain between x and y. Thus we have shown that there exists $j \geq 1$ such that $Q_{w(j)} \backslash A \neq \emptyset$. Define $i_* = \min\{i | i = 1, \ldots, J+1, Q_{w(i)} \backslash A \neq \emptyset\}$. without loss of generality, we may assume that $Q_{w(i_*)} \cap [0,1] \times \{d\} \neq \emptyset$. Set

$$\partial R_L^T = \{a\} \times \left[\frac{c+d}{2}, d - \frac{1}{3^{|w(i_*)|}}\right].$$

Shifting $Q_{w(i)}$'s for $i = 1, \ldots, i_* - 1$ horizontally towards ∂R_L, we obtain a covering of ∂R_L^T. Note that the length of ∂R_L^T is $|d-c|/2 - 1/3^{|w(i_*)|}$ and

$$\frac{|d-c|}{2} - \frac{1}{3^{|w(i_*)|}} \geq \frac{\kappa|b-a|}{2} - \frac{1}{3^{N-M}} = \frac{\kappa}{2} d(x,y) - \frac{1}{3^{N-M}} \geq \frac{\kappa}{2}\frac{1}{3^{N+1}} - \frac{1}{3^{N-M}}.$$

On the other hand, the lengths of the sides of $Q_{w(i)}$'s are no less that $1/3^{N-M}$ by (10.2). Hence

$$i_* - 1 \geq 3^{N-M}\left(\frac{\kappa}{2}\frac{1}{3^{N+1}} - \frac{1}{3^{N-M}}\right) \geq \frac{\kappa}{2}\frac{1}{3^{M+1}} - 1.$$

Since $J + 1 \geq i_*$,

$$2(J+1)3^{M+1} \geq \kappa.$$

This contradicts to the fact that κ can be arbitrarily large. Hence we conclude that (SQ4) holds. □

In the followings, we give four examples. The first one has infinite connected components but still the restriction of the Euclidean metric is adapted.

Example 10.7 (Figure 4)**.** Let X be the self-similar set associated with the contractions $\{F_1, F_3, F_4, F_5, F_7, F_8\}$, i.e. X is the unique nonempty compact set which satisfies

$$X = \bigcup_{i \in S} F_j(X),$$

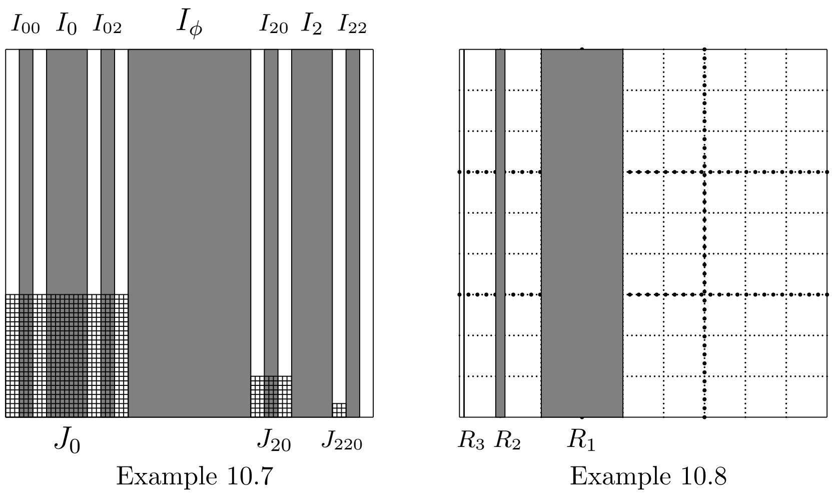


Figure 4: Examples 10.7 and 10.8

where $S = \{1, 3, 4, 5, 7, 8\}$. Then $X = C_3 \times [0, 1]$, where C_3 is the ternary Cantor set. Define $(T)_m = S^m$ and $T = \cup_{m \geq 1}(T)_m$. If $K_w = F_w(X)$ for any $w \in T$, then K is a partition of X parametrized by $(T, \mathcal{A}|_T, \phi)$. Define

$$I_\phi = \left[\frac{1}{3}, \frac{2}{3}\right] \times [0, 1] \quad \text{and} \quad I_{i_1,\ldots,i_n} = \left[\sum_{k=1}^{n} \frac{i_k}{3^k} + \frac{1}{3^{n+1}}, \sum_{k=1}^{n} \frac{i_k}{3^k} + \frac{2}{3^{n+1}}\right] \times [0, 1]$$

for any $1 \geq 0$ and any $i_1, \ldots, i_n \in \{0, 2\}$. Then

$$\{R_j\}_{j \geq 1} = \{I_\phi, I_{i_1,\ldots,i_n} | n \geq 1, i_1, \ldots, i_n \in \{0, 2\}\}.$$

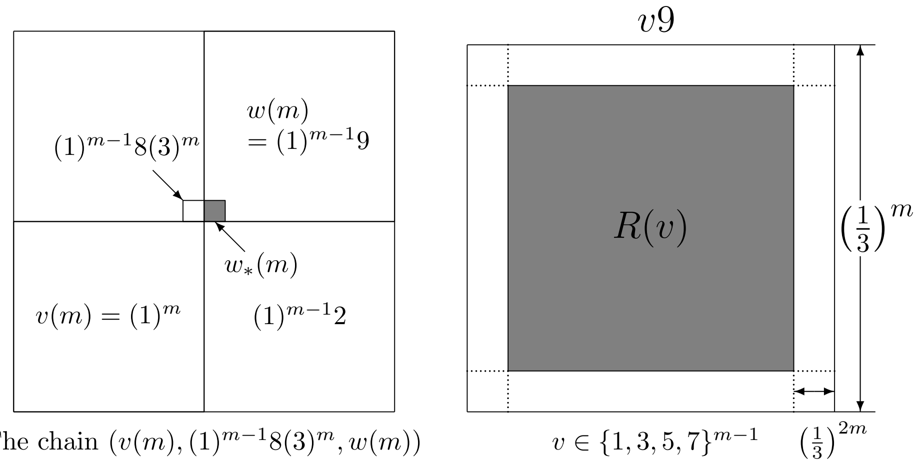
Set $J_{i_1,\ldots,i_n} = [\sum_{k=1}^{n} \frac{i_k}{3^k}, \sum_{k=1}^{n} \frac{i_k}{3^k} + \frac{1}{3^n}] \times [0, \frac{1}{3^n}]$. Then there exists $w \in (T)_n$ such that $J_{i_1,\ldots,i_n} = Q_w$, $Q_w \backslash \mathrm{int}(I_{i_1,\ldots,i_n})$ has two connected component and $\kappa(Q_w \cap I_{i_1,\ldots,i_n}) = 3$. Therefore, $\{R_j\}_{j \geq 1} \subseteq \mathcal{R}_3^1$ and hence d is adapted to $g|_T$.

The second example is the case where the restriction of the Euclidean metric is not adapted.

Example 10.8 (Figure 4)**.** Set $x_j = \frac{1}{3^j} - \frac{1}{3^{2j}}$, $y_j = \frac{1}{3^j} + \frac{1}{3^{2j}}$ and $R_j = [x_j, y_j] \times [0, 1]$ for any $j \geq 1$. Define $X = Q \backslash (\cup_{j \geq 1} \mathrm{int}(R_j))$. Let $T = \{w | w \in W_*, \mathrm{int}(Q_w) \cap X \neq \emptyset\}$ and let $K_w = X \cap Q_w$ for any $w \in T$. Then $K : T \to \mathcal{C}(X)$ is a partition of X parametrized by $(T, \mathcal{A}|_{T \times T}, \phi)$ by Proposition 10.1. In this case, we easily see the following facts:

- $\kappa(R_j) = 3^{2j}/2$ for any $j \geq 1$,
- If $w \in \cup_{m \geq j}(T)_m$, then $Q_w \backslash \mathrm{int}(R_j)$ is a rectangle,
- Set $(1)^n = \underbrace{1 \cdots 1}_{n \text{ times}} \in (T)_n$. Then $Q_{(1)^{j-1}} \backslash \mathrm{int}(R_j)$ has two connected components and $\kappa(Q_{(1)^{j-1}} \cap R_j) = 2 \cdot 3^{j+1}$.

These facts yield that $R_j \notin \mathcal{R}^0_{2 \cdot 3^j} \cup \mathcal{R}^1_{2 \cdot 3^j}$ for sufficiently large j. By Theorem 10.4, d is not adapted to $g|_T$. In fact, $D^g_M((x_j, 0), (y_j, 0)) = 3^{-(j-1)}$ for any



The chain $(v(m), (1)^{m-1}8(3)^m, w(m))$ $\quad$ $v \in \{1,3,5,7\}^{m-1}$

Example 10.9 $\quad$ Example 10.10

Figure 5: Example 10.9 and 10.10

$j \geq 1$ while $d((x_j, 0), (y_i, 0)) = 2{\cdot}3^{-2j}$. Hence the ratio between $D_g^M(\cdot,\cdot)$ and $d(\cdot,\cdot)$ is not bounded for any $M \geq 0$.

Furthermore, let $d_*(x, y) = d(x, y)/\sqrt{2}$ for any $x, y \in X$. Then $g|_T = g_{d_*}$. Since d is not adapted to $g|_T$, it follows that d_* is not adapted to $g|_T$ as well. Hence d_* is not adapted.

The third one is the case when the restriction of the Euclidean metric is not 1-adapted but 2-adapted.

Example 10.9. Define

$$\begin{aligned} w_*(j) &= (1)^{j-1}9(1)^j \\ R_j &= Q_{w_*(j)} \\ k_m &= \left[\frac{m}{2}\right] \end{aligned}$$

for $j \in \mathbb{N}$ and $m \in \mathbb{N}$. Note that $(1)^n = \underbrace{1\ldots1}_{n\text{-times}}$ as is defined in Example 10.8. Then it follows that $T = T^{(9)} \backslash \cup_{j\in\mathbb{N}} T^{(9)}_{w_*(j)}$, where $T^{(9)}_w = \{wi_1i_2\ldots|i_1, i_2, \ldots \in \{1,\ldots,9\}\}$. Let $g(w) = 3^{-|w|}$ for any $w \in T$. Define $w(m) = (1)^{m-1}9$ and $v(m) = (1)^m$. Then $(w(m), (1)^{m-1}8(3)^k, v(m))$ is a chain for $k = 0, 1, \ldots, m$. See Figure 5. Therefore, $w(m)$ and $v(m)$ are 1-separated in $\Lambda^g_{3^{-m}}$ but not 2-separated in $\Lambda^g_{3^{-2m}}$. This means that the condition $(\mathrm{EV5})_M$ for $M = 1$ does not hold. Therefore, there exists no metric which is 1-adapted to g^α for any $\alpha > 0$. On the other hand, since $\kappa(R_j) = 1$ for any $j \in \mathbb{N}$, the restriction of the Euclidean metric to X, which is denoted by d, is adapted to g. In fact, it is easy to see that d is 2-adapted to g. As a consequence, d is not 1-adapted but 2-adapted to g.

In the fourth example, we do not have thickness while the restriction of the Euclidean metric is adapted.

Example 10.10. Define $\Delta Q = (\mathbb{R}^2 \backslash Q) \cap Q$, which is the topological boundary of Q as a subset of $\mathbb{R}^2$. Let $I_0 = \emptyset$ and let $E = \{1, 3, 5, 7\}$. Define $\{I_n\}_{n \geq 0}$ inductively by $I_{2m-1} = \widehat{I}_{2m-1}$ and $I_{2m} = J_m \cup \widehat{I}_{2m}$ for $m \geq 1$, where

$$J_m = \{v9w | v \in E^{m-1}, w \in W_m, Q_w \cap \Delta Q \neq \emptyset\}.$$

$\{I_m\}_{m \geq 0}$ satisfies (SQ1), (SQ2) and (SQ3). In fact, if $J_{m,v} = \{v9w | w \in W_m, Q_w \cap \Delta Q \neq \emptyset\}$ for any $v \in E^{m-1}$, $J_{m,v}$ is a collection of $(3^m - 2)^2$-words in W_{2m}. Set $R(v) = \cup_{u \in J_{m,v}} Q_u$ for any $m \geq 1$ and $v \in E^{m-1}$. See Figure 5. Then $\{R_j\}_{j \geq 1} = \{R(v) | m \geq 1, v \in E^{m-1}\}$. More precisely $R(v) \subseteq Q_{v9}$ and it is a square which has the same center, namely the intersection of two diagonals, as Q_{v9} and the length of the sides is $\frac{1}{3^m}(1 - \frac{2}{3^m})$. Note that the length of the sides of Q_{v9} is $\frac{1}{3^m}$. Hence the relative size of $R(v)$ in comparison with Q_{v9} is monotonically increasing and convergent to 1 as $m \to \infty$. The corresponding tree $(T, \mathcal{A}|_T, \phi)$ and the partition $K : T \to \mathcal{C}(X)$ of $X = Q \backslash \cup_{j \geq 1} \mathrm{int}(R_j)$ has the following properties:
Let d be the restriction of the Euclidean metric to X. Then

(**a**) d is adapted to $g|_T$.

(**b**) $g|_T$ is exponential and uniformly finite.

(**c**) Let μ_* be the restriction of the Lebesgue measure on X. Then μ_* has the volume doubling property with respect to d.

(**d**) μ_* is not gentle with respect to $g|_T$.

(**e**) μ_* is not super-exponential.

(**f**) $g|_T$ is not thick.

In the rest, we present proofs of the above claims.
(**a**) Since $\kappa(R_m) = 1$ for any $m \geq 1$, we see that $\{R_m\}_{m \geq 1} \subseteq \mathcal{R}_1^0$. Hence Theorem 10.4 shows that d is adapted to $g|_T$. In fact, d is 1-adapted to $g|_T$ in this case.
(**b**) This is included in the statement of Proposition 10.1-(3).
(**c**) If $v \in \Lambda_s^{g|_T}$ and $Q_v = K_v$, then $\mu_*(K_v) = 9^{-|v|}$ and hence $\mu_*(K_u) \leq 9^{-|u|} = 9^{-|v|+1} \leq 9\mu_*(K_v)$ for any $u \in \Lambda_{3s}^{g|_T}$. Therefore, $v \in \Theta(s, 3, k, 9)$ for any $k \geq 1$. on the other hand, for any $w \in T$, there exists $v \in \Lambda_{s,1}^{g|_T}(w)$ such that $K_v = Q_v$. Therefore, we see that $\Lambda_{s,1}^{g|_T}(w) \cap \Theta(s, 3, 3, 9) \neq \emptyset$. By Lemma 9.7, we have (**c**).
(**d**) and (**e**) Set $w(m) = (1)^{m-1}9$. Then $K_{w(m)} = Q_{w(m)} \backslash \mathrm{int}(R_m)$, where $R_m = \cup_{w \in J_m} Q_w$. Then $\mu_*(K_{w(m)}) = 4(3^m - 1)3^{-4m}$. On the other hand, if $v(m) = (1)^{m-1}8$, then $\mu_*(K_{v(m)}) = 3^{-2m}$. Since $K_{w(m)} \cap K_{v(m)} \neq \emptyset$, μ_* is not gentle with respect to $g|_T$. Moreover, since $K_{\pi(w(m))}$ contains $Q_{v(m)}$, we have $\mu_*(K_{\pi(w(m))}) \geq 3^{-2m}$. This implies that μ_* is not super-exponential.
(**f**) To clarify the notation, we use $B(x, r) = \{y | y \in Q, |x - y| < r\}$ and $B_*(x, r) = B(x, r) \cap X$. This means that $B_*(x, r)$ is the ball of radius r with respect to the metric d on X. Assume that $g|_T$ is thick. Since K is minimal,

Proposition 8.2 implies that $K_{w(m)} \supseteq B_*(x, c3^{-m})$ for some $x \in K_{w(m)}$, where c is independent of m and x. However, for any $x \in K_{w(m)}$, there exists $y \in X \backslash K_{w(m)}$ such that $|x - y| \leq 2 \cdot 3^{-2m}$. This contradiction shows that $g|_T$ is not thick.

11 Gentleness and exponentially

In this section, we show that the gentleness "$\underset{\mathrm{GE}}{\sim}$" is an equivalence relation among exponential weight functions. Moreover, the thickness of the interior, tightness, the uniformly finiteness and the existence of visual metric will be proven to be invariant under the gentle equivalence.

As in the section 9, $(T, \mathcal{A}, \phi)$ is a locally finite tree with a reference point ϕ, $(X, \mathcal{O})$ is a compact metrizable space and $K : T \to \mathcal{C}(X, \mathcal{O})$ is a partition of X parametrized by $(T, \mathcal{A}, \phi)$.

Definition 11.1. Define $\mathcal{G}_e(T)$ as the collection of exponential weight functions.

Theorem 11.2. *The relation $\underset{\mathrm{GE}}{\sim}$ is an equivalence relation on $\mathcal{G}_e(T)$.*

Several steps of preparation are required to prove the above theorem.

Definition 11.3. (1) Let $A \subseteq T$. For $m \geq 0$, we define $S^m(A) \subseteq T$ as

$$S^m(A) = \bigcup_{w \in A} \{v | v \in (T)_{m+|w|}, [v]_{|w|} = w\}.$$

(2) Let $g : T \to (0, 1]$ be a weight function. For any $w \in T$, define

$$N_g(w) = \min\{n | n \geq 0, \pi^n(w) \in \Lambda^g_{g(w)}\}$$

and $\pi^*_g(w) = \pi^{N_g(w)}(w)$.
(3) $(u, v) \in T \times T$ is called an ordered pair if and only if $u \in T_v$ or $v \in T_u$. Define $|u, v| = ||u| - |v||$ for an ordered pair (u, v).

Note that if $g(w) < 1$, then we have

$$N_g(w) = \min\{n | n \geq 0, g(\pi^{n+1}(w)) > g(w)\}.$$

Therefore, if $g(\pi(w)) > g(w)$ for any $w \in T$, then $N_g(w) = 0$ and $\pi^*_g(w) = w$ for any $w \in T$.

The following lemma is immediate from the definitions.

Lemma 11.4. *Let $g : T \to (0, 1]$ be a super-exponential weight function, i.e. there exists $\gamma \in (0, 1)$ such that $g(w) \geq \gamma g(\pi(w))$ for any $w \in T$. If (u, v) is an ordered pair, then $g(u) \leq \gamma^{-|u,v|} g(v)$.*

Lemma 11.5. *Let $g : T \to (0, 1]$ be a weight function. If g is sub-exponential, then $\sup_{w \in T} N_g(w) < +\infty$.*

Proof. Since g is sub-exponential, there exist $c \in (0,1)$ and $m \geq 0$ such that $cg(w) \geq g(u)$ if $w \in T$, $u \in T_w$ and $|u,v| \geq m$. This immediately imply that $N_g(w) \leq m$. $\square$

Lemma 11.6. *Assume that $g, h \in \mathcal{G}_e(T)$ and h is gentle with respect to g. Then there exist M and N such that $\pi^n(u) \in \Lambda_s^h$ for some $n \in [0,N]$ whenever $s \in (0,1]$, $w \in \Lambda_s^h$ and $u \in S^M(\Lambda_{g(w),1}^g(\pi_g^*(w)))$. Moreover, define $\eta_{s,w}^{g,h} : S^M(\Lambda_{g(w),1}^g(\pi_g^*(w))) \to \Lambda_s^h$ by $\eta_{s,w}^{g,h}(u) = \pi^n(u)$. Then it follows that $\Lambda_{s,1}^h(w) \subseteq \eta_{s,w}^{g,h}(S^M(\Lambda_{g(w),1}^g(\pi_g^*(w))))$. In particular, for any $s \in (0,1]$, any $w \in \Lambda_s^h$ and any $v \in \Lambda_{s,1}^h(w)$, there exists $u \in \Lambda_{g(w),1}^g(\pi_g^*(w))$ such that (u,v) is an ordered pair and $|u,v| \leq \max\{M,N\}$.*

Proof. Since h is sub-exponential, there exist $c_1 \in (0,1)$ and $m \geq 0$ such that $c_1 h(w) \geq h(u)$ for any $w \in T$ and any $u \in S^m(w)$. Let $w \in \Lambda_s^h$ and let $w' = \pi_g^*(w)$. Set $t = g(w)$. Let $v \in \Lambda_{t,1}^g(w')$. As h is gentle with respect to g, there exists $c \geq 1$ such that

$$h(w')/c \leq h(v) \leq ch(w'),$$

where c is independent of s, w and v. By Lemma 11.5 and the fact that h is super-exponential, there exists $c' \geq 1$ such that

$$h(w)/c \leq h(v) \leq c'h(w)$$

for any s, w and v. Using this, the sub-exponentially of h and Proposition 7.16, we see that there exist $c'' > 0$ and M which are independent of s and w such that $c''s \leq h(u) \leq s$ for any $u \in S^M(\Lambda_{t,1}^g(w'))$. Choose k so that $c''(c_1)^{-k} > 1$. Then $h(\pi^{km}(u)) \geq (c_1)^{-k}h(u) \geq c''(c_1)^{-k}s > s$. Set $N = km-1$. Then, for any $u \in S^M(\Lambda_{t,1}^g(w'))$, there exists $n(u)$ such that $n(u) \leq N$ and $\pi^{n(u)}(u) \in \Lambda_s^h$. Now for any $\rho \in \Lambda_{s,1}^h(w)$, there exists $v \in \Lambda_{t,1}^g(w')$ such that (ρ, v) is an ordered pair. Since $\pi^{n(u)}(u) = \rho$ for any $u \in S^M(v)$, it follows that $\eta_{s,w}^{g,h}(S^M(\Lambda_{g(w),1}^g(\pi_g^*(w)))) \supseteq \Lambda_{s,1}^h(w)$. The rest is straight forward. $\square$

Finally we are ready to give a proof of Theorem 11.2.

Proof of Theorem 11.2. Let $g, h, \xi \in \mathcal{G}_e(T)$. Then there exists $\gamma \in (0,1)$ such that $g(w) \geq \gamma g(\pi(w))$, $h(w) \geq \gamma h(\pi(w))$ and $\xi(w) \geq \gamma \xi(\pi(w))$ for any $w \in T$.

First we show $g \underset{\text{GE}}{\sim} g$. By Proposition 7.16, there exists $c \in (0,1)$ such that if $w \in \Lambda_s^g$, then $cg(w) \leq s \leq g(w)$. As a consequence, if $w, v \in \Lambda_s^g$, then $g(w) \leq s/c \leq h(w)/c \leq s/c^2 \leq g(w)/c^2$. Thus $g \underset{\text{GE}}{\sim} g$.

Next assume $g \underset{\text{GE}}{\sim} h$. Suppose that $w, v \in \Lambda_s^h$ and $K_w \cap K_v \neq \emptyset$. Since $v \in \Lambda_{s,1}^h(w)$, Lemma 11.6 implies that there exists $u \in \Lambda_{g(w),1}^g(\pi_g^*(w))$ such that (u,v) is an ordered pair and $|u,v| \leq L$, where $L = \max\{M,N\}$. By Lemma 11.4, $g(v) \geq \gamma^L g(u) \geq \gamma^L g(w)$. Hence $h \underset{\text{GE}}{\sim} g$.

Finally assume that $g \underset{\text{GE}}{\sim} h$ and $h \underset{\text{GE}}{\sim} \xi$. Suppose that $w, v \in \Lambda^\xi_s$ and $K_w \cap K_v \neq \emptyset$. Since $v \in \Lambda^\xi_{s,1}(w)$, Lemma 11.6 implies that there exists $u \in \Lambda^h_{h(w),1}(\pi^*_h(w))$ such that (u, v) is an ordered pair and $|u, v| \leq L$. By Lemma 11.4, it follows that $g(v) \geq \gamma^L g(u)$. Set $s' = h(w)$ and $w' = \pi^*_h(w)$. Note that $w' \in \Lambda^h_s$ and $u \in \Lambda^h_{s',1}(w')$. Again by Lemma 11.6, there exists $a \in \Lambda^g_{g(w'),1}(\pi^*_g(w'))$ such that (u, a) is an ordered pair and $|a, u| \leq L$. By Lemma 11.4, it follows that $g(u) \geq \gamma^L g(a) \geq \gamma^L g(\pi^*_h(w))$. By Lemma 11.5, $N_h(w)$ is uniformly bounded and hence there exists $c_* > 0$ which is independent of s, w and v such that $g(\pi^*_h(w)) \geq c_* g(w)$. Combining these, we obtain $g(v) \geq \gamma^{2L} g(\pi^*_h(w)) \geq \gamma^{2L} c_3 g(w)$. Hence $\xi \underset{\text{GE}}{\sim} g$. Consequently we verify $g \underset{\text{GE}}{\sim} \xi$ by the above arguments. $\square$

Next, we show the invariance of thickness, tightness and uniform finiteness under the equivalence relation $\underset{\text{GE}}{\sim}$.

Theorem 11.7. *Let $g, h \in \mathcal{G}_e(T)$. Suppose $g \underset{\text{GE}}{\sim} h$.*

(1) *Suppose that $\sup_{w \in T} \#(S(w)) < +\infty$. If g is uniformly finite then so is h.*
(2) *If g is thick, then so is h.*
(3) *If g is tight, then so is h.*

We need the next lemma to prove Theorem 11.7.

Lemma 11.8. *Let $g, h \in \mathcal{G}_e(T)$. Assume that g is gentle with respect to h. Then for any $\alpha \in (0, 1]$ and $M \geq 0$, there exists $\gamma \in (0, 1)$ such that*

$$U^g_M(x, \alpha g(w)) \supseteq U^h_M(x, \gamma h(w))$$

for any $w \in T$ and $x \in K_w$.

Proof. Since g and h are exponential, there exist $c_1, c_2 \in (0, 1)$ and $m \geq 1$ such that $h(w) \geq c_2 h(\pi(w)), g(w) \geq c_2 g(\pi(w)), h(v) \leq c_1 h(w)$ and $g(v) \leq c_1 g(w)$ for any $w \in T$ and any $v \in S^m(w)$. Moreover, since g is gentle with respect to g, there exists $c > 1$ such that $g(w) \leq cg(u)$ whenever $w, u \in \Lambda^h_s$ and $K_w \cap K_v \neq \emptyset$. Note that $N_g(w) \leq m$ and $N_h(w) \leq m$ for any $w \in T$.

Let $w \in T$ and let $x \in K_w$. Assume that $\gamma < (c_2)^{lm}$. Let $v \in \Lambda^h_{\gamma h(w),0}(x)$. Then $h(\pi(v)) > \gamma h(w) \geq h(v)$. There exists $k \geq 0$ such that $\pi^k(v) \in \Lambda^h_{h(w)}$. Then $h(\pi^{k+1}(v)) > h(w) \geq h(\pi^k(v))$. Thus we have

$$\gamma h(\pi^{k+1}(v)) \geq h(v)$$

Therefore, it follows that $k + 1 \geq lm$. Let $w_* = \pi^{N_h(w)}(w)$. Then we see that $x \in K_{\pi^{k+1}(v)} \cap K_{w_*}$. Therefore, $c^{-1} g(w_*) \leq g(\pi^{k+1}(v)) \leq cg(w_*)$. Since $k + 1 \geq lm$ and $N_h(w) \leq m$, it follows that

$$g(v) \leq (c_1)^l g(\pi^{k+1}(v)) \leq c(c_1)^l g(w_*) \leq c(c_1)^l (c_2)^{-m} g(w).$$

Now suppose that $(w(1), \ldots, w(M+1))$ is a chain in $\Lambda^h_{\gamma h(w)}$ with $w(1) \in \Lambda^h_{\gamma h(w),0}(x)$. Using the above arguments, we obtain

$$g(w(i)) \leq c^{i-1} g(w(1)) \leq c^i (c_1)^l (c_2)^{-m} g(w) \leq c^{M+1} (c_1)^l (c_2)^{-m} g(w)$$

for any $i = 1, \ldots, M+1$. Choosing l so that $c^{M+1}(c_1)^l(c_2)^{-m} < \alpha$, we see that $U^h_M(x, \gamma h(w)) \subseteq U^g_M(x, \alpha g(w))$. $\square$

Proof of Theorem 11.7. (1) Set $L = \sup_{w \in T} \#(S(w))$. By Lemma 11.6, it follows that $\#(\Lambda^h_{s,1}(w)) \leq L^M \#(\Lambda^g_{g(w),1}(\pi^*_g(w)))$. This suffices to the desired conclusion.
(2) Since g is thick, by Proposition 8.1, for any $M \geq 0$, there exists $\beta > 0$ such that, for any $w \in T$,

$$K_w \supseteq U^g_M(x, \beta g(\pi(w)))$$

for some $x \in K_w$. By Lemma 11.8, there exists $\gamma \in (0,1)$ such that

$$U^g_M(x, \beta g(\pi(w))) \supseteq U^h_M(x, \gamma h(\pi(w)))$$

for any $w \in T$. Thus making use of Proposition 8.1 again, we see that h is thick.
(3) Since g is tight, for any $M \geq 0$, there exists $\alpha > 0$ such that, for any $w \in T$, $K_w \backslash U^g_M(x, \alpha g(w)) \neq \emptyset$ for some $x \in K_w$. By Lemma 11.8, there exists $\gamma \in (0,1)$ such that $U^g_M(x, \alpha g(w)) \supseteq U^h_M(x, \gamma h(w))$ for any $w \in T$ and $x \in K_w$. Hence

$$\sup_{x,y \in K_w} \delta^h_M(x,y) \geq \gamma h(w)$$

for any $w \in T$. Thus we have shown that h is tight. $\square$

Finally, the existence of visual metric is also invariant under $\underset{\mathrm{GE}}{\sim}$ as follows.

Theorem 11.9. *Assume that the partition $K : T \to \mathcal{C}(X, \mathcal{O})$ is minimal. Let $g, h \in \mathcal{G}_e(T)$ and let $M \in \mathbb{N}$. Assume that $g \underset{\mathrm{GE}}{\sim} h$. Then g satisfies the condition $(\mathrm{EV})_M$ if and only if h satisfies the condition $(\mathrm{EV})_M$.*

Proof. Since g and h are exponential, there exists $\lambda \in (0,1)$ and $m \geq 1$ such that

$$g(w') \leq \lambda g(w) \leq g(w'')$$
$$h(w') \leq \lambda h(w) \leq h(w'')$$

if $w \in T$, $w', w'' \in T_w$, $|w'| - |w| \geq m$ and $|w''| - |w| = 1$. Moreover, since $g \underset{\mathrm{GE}}{\sim} h$, there exists $\eta > 1$ such that if $w, v \in \Lambda^g_s$ and $K_w \cap K_v \neq \emptyset$, then $h(w) \leq \eta h(v)$ and if $w, v \in \Lambda^h_s$ and $K_w \cap K_v \neq \emptyset$, then $g(w) \leq \eta g(v)$. Fix $k \in \mathbb{N}$ satisfying $\eta^M \lambda^k < 1$.

Now assume that g satisfies $(\mathrm{EV})_M$. Let $w, v \in \Lambda^h_s$ and assume that (w, v) is M-separated in Λ^h_s. Set $t = g(v)$. Suppose that (w, v) is not M-separated in $\Lambda^g_{\lambda^{km}t}$. Then there exists a chain $(w_*(1), \ldots, w_*(M-1))$ in $\Lambda^g_{\lambda^{km}t}$ such that $(w, w_*(1), \ldots, w_*(M-1), v)$ is a chain. Choose $v_* \in \Lambda^g_{\lambda^{km}t} \cap T_v$ so that $K_{w_*(M-1)} \cap K_{v_*} \neq \emptyset$. Since $g(v_*) \leq \lambda^{km}t = \lambda^{km}g(v)$, it follows that $|v_*| - |v| \geq km$. Then we have

$$h(w_*(i)) \leq \eta^M h(v_*) \leq \eta^M \lambda^k h(v) < h(v).$$

Hence there exists a chain $(w(1), \ldots, w(M-1))$ in Λ^h_s such that $w_*(i) \in T_{w(i)}$ for any $i = 1, \ldots, M-1$. This implies that (w, v) is not M-separated in Λ^h_s. This contradiction implies that (w, v) is M-separated in $\Lambda^g_{\lambda^{km}t}$.

Since $(\mathrm{EV5})_M$ holds for g, we see that (w, v) is $(M+1)$-separated in $\Lambda^g_{\tau\lambda^{km}t}$. Set $t_* = \tau\lambda^{km}t$. Choose $v' \in \Lambda^g_{t_*} \cap T_v$. Then exchanging g and h and using the same argument as above, we see that (w, v) is $(M+1)$-separated in $\Lambda^h_{\lambda^{km}h(v')}$.

Since h is exponential, Proposition 7.16 shows that there exists $c > 0$ such that $cr \leq g(u) \leq r$ for any $r \in (0, 1]$ and any $u \in \Lambda^g_r$. Choose n_* so that $\lambda^{n_*} < c\tau$. Suppose $|v'| - |v| \geq (km + n_*)m$. Then

$$\lambda^{km+n_*} g(v) < c\tau\lambda^{km} g(v) \leq ct_* \leq g(v') \leq \lambda^{km+n_*} g(v).$$

This contradiction yields that $|v'| - |v| < (km + n_*)m$. Therefore, $h(v') \geq \lambda^{(km+n_*)m} h(v) \geq \lambda^{(km+n_*)m} s$. Thus $\lambda^{km} h(v') \geq \lambda^{(km+n_*+k)m} s$. Set $\tau_* = \lambda^{(km+n_*+k)m}$. Then (w, v) is $(M+1)$-separated in $\Lambda^h_{\tau_* s}$. Thus we have shown that $(\mathrm{EV5})_M$ is satisfied for h. $\square$

12 Quasisymmetry

In this section, we are going to identify the gentleness equivalence "$\underset{\mathrm{GE}}{\sim}$" with the quasisymmetry equivalence $\underset{\mathrm{QS}}{\sim}$ among the metrics under certain conditions. As in the last section, $(T, \mathcal{A}, \phi)$ is a locally finite tree with a reference point ϕ, $(X, \mathcal{O})$ is a compact metrizable space and $K : T \to \mathcal{C}(X, \mathcal{O})$ is a partition of X parametrized by $(T, \mathcal{A}, \phi)$ throughout this section.

Definition 12.1 (Quasisymmetry)**.** A metric $\rho \in \mathcal{D}(X, \mathcal{O})$ is said to be quasisymmetric with respect to a metric $d \in \mathcal{D}(X, \mathcal{O})$ if and only if there exists a homeomorphism h from $[0, +\infty)$ to itself such that $h(0) = 0$ and, for any $t > 0$, $\rho(x, z) < h(t)\rho(x, y)$ whenever $d(x, z) < td(x, y)$. We write $\rho \underset{\mathrm{QS}}{\sim} d$ if ρ is quasisymmetric with respect to d.

It is known that $\underset{\mathrm{QS}}{\sim}$ is an equivalence relations on $\mathcal{D}(X, \mathcal{O})$.

Definition 12.2. Let $d \in \mathcal{D}(X, \mathcal{O})$. We say that d is (super-, sub-)exponential if and only if g_d is (super-, sub-)exponential.

Under the uniformly perfectness of a metric space defined below, we can utilize a useful equivalent condition for quasisymmetry obtained in [10]. See the details in the proof of Theorem 12.4.

Definition 12.3. A metric space (X, d) is called uniformly perfect if and only if there exists $\epsilon > 0$ such that $B_d(x, (1+\epsilon)r)\backslash B_d(x, r) \neq \emptyset$ unless $B_d(x, r) = X$.

Lemma 12.4. *Let $d \in \mathcal{D}(X, \mathcal{O})$. If d is super-exponential, then (X, d) is uniformly perfect.*

Proof. Write $d_w = g_d(w)$ for any $w \in T$. Since d is super-exponential, there exists $c_2 \in (0, 1)$ such that $d_w \geq c_2 d_{\pi(w)}$ for any $w \in T$. Therefore, $s \geq d_w > c_2 s$ if $w \in \Lambda^d_s$. For any $x \in X$ and any $r \in (0, 1]$, choose $w \in \Lambda^d_{r/2,0}(x)$. Then $d(x, y) \leq d_w \leq r/2$ for any $y \in K_w$. This shows $K_w \subseteq B_d(x, r)$. Since $\mathrm{diam}(B_d(x, c_2 r/4), d) \leq c_2 r/2 < d_w$, it follows that $K_w \backslash B_d(x, c_2 r/2) \neq \emptyset$. Therefore $B_d(x, r)\backslash B_d(x, c_2 r/2) \neq \emptyset$. This shows that (X, d) is uniformly perfect. □

Definition 12.5. Define

$$\mathcal{D}_{A,e}(X, \mathcal{O}) = \{d | d \in \mathcal{D}(X, \mathcal{O}), d \text{ is adapted and exponential.}\}$$

The next theorem is the main result of this section.

Theorem 12.6. *Let $d \in \mathcal{D}_{A,e}(X, \mathcal{O})$ and let $\rho \in \mathcal{D}(X, \mathcal{O})$. Then $d \underset{\mathrm{QS}}{\sim} \rho$ if and only if $\rho \in \mathcal{D}_{A,e}(X, \mathcal{O})$ and $d \underset{\mathrm{GE}}{\sim} \rho$.*

Remark. In the case of partitions of self-similar sets introduced in Example 4.5, the above theorem has been obtained in [11].

The following corollary is straightforward from the above theorem.

Corollary 12.7. *Let $d, \rho \in \mathcal{D}_{A,e}(X, \mathcal{O})$. Then $d \underset{\mathrm{QS}}{\sim} \rho$ if and only if $d \underset{\mathrm{GE}}{\sim} g$.*

The rest of this section is devoted to a proof of the above theorem.

Proof of Theorem 12.6: Part 1. Assume that d and ρ belongs to $\mathcal{D}_{A,e}(X, \mathcal{O})$. We show that if $d \underset{\mathrm{GE}}{\sim} \rho$, then $d \underset{\mathrm{QS}}{\sim} \rho$. By Lemma 12.4, both (X, d) and (X, ρ) are uniformly perfect. By [10, Theorems 11.5 and 12.3], $d \underset{\mathrm{QS}}{\sim} \rho$ is equivalent to the facts that there exists $\delta \in (0, 1)$ such that

$$\begin{aligned} B_d(x, r) &\supseteq B_\rho(x, \delta\overline{\rho}_d(x, r)) \\ B_\rho(x, r) &\supseteq B_d(x, \delta\overline{d}_\rho(x, r)) \end{aligned} \tag{12.1}$$

and

$$\begin{aligned} \overline{\rho}_d(x, r/2) &\geq \delta\overline{\rho}_d(x, r) \\ \overline{d}_\rho(x, r/2) &\geq \delta\overline{d}_\rho(x, r), \end{aligned} \tag{12.2}$$

where $\overline{\rho}_d(x, r) = \sup_{y \in B_d(x,r)} \rho(x, y)$ and $\overline{d}_d(x, r) = \sup_{y \in B_\rho(x,r)} d(x, y)$.

Now we are going to show (12.1) and (12.2). Since d and ρ are adapted, there exist $\beta \in (0,1)$, $\gamma > 1$ and $M \geq 1$ such that

$$U_M^d(x,\beta r) \subseteq B_d(x,r) \subseteq U_M^d(x,\gamma r)$$
$$U_M^\rho(x,\beta r) \subseteq B_\rho(x,r) \subseteq U_M^\rho(x,\gamma r)$$

for any $x \in X$ and any $r \in (0,1]$. By Lemma 11.8, there exists $\alpha \in (0,1)$ such that $U_M^\rho(x,\rho_w) \supseteq U_M^d(x,\alpha d_w)$ and $U_M^d(x,d_w) \supseteq U_M^\rho(x,\alpha\rho_w)$ for any $w \in T$ and $x \in K_w$. If $w \in \Lambda^d_{\gamma r/\alpha,0}(x)$, then

$$B_d(x,r) \subseteq U_M^d(x,\gamma r) \subseteq U_M^d(x,\alpha d_w) \subseteq U_M^\rho(x,\rho_w), \tag{12.3}$$

where $w \in \Lambda^d_{x,\gamma_1 r/\alpha}$. Hence for any $y \in B_d(x,r)$, there exists $(w(1),\ldots,w(k)) \in \mathcal{CH}_K(x,y)$ such that $k \leq M+1$ and $w(i) \in \Lambda^\rho_{\rho_w}$. Since $\rho(x,y) \leq \sum_{i=1}^k \rho_{w(i)} \leq (M+1)\rho_w$, we have

$$\overline{\rho}_d(x,r) \leq (M+1)\rho_w.$$

Let $w \in \Lambda^d_{\gamma r/\alpha,0}(x)$ as above. Since $\beta/2 < 1 < \gamma/\alpha$, there exists $v \in T_w$ such that $v \in \Lambda^d_{\beta r/2,0}(x)$. Note that $\beta r/2 \geq d_v$. Hence we have

$$B_d\Big(x,\frac{r}{2}\Big) \supseteq U_M^d\Big(x,\frac{\beta r}{2}\Big) \supseteq U_M^d(x,d_v) \supseteq U_M^\rho(x,\alpha\rho_v). \tag{12.4}$$

Since d is sub-exponential, the fact that $w \in \Lambda^d_{\gamma r/\alpha,0}(x)$ and $v \in \Lambda^d_{\beta r/2,0}(x) \cap T_w$ implies that $|v| - |w|$ is uniformly bounded with respect to x, r and w. This and the fact that ρ is super-exponential imply that there exists $c > 0$ which is independent of x, r and w such that $\rho_v \geq c\rho_w$. Now we see that $\alpha\rho_v \geq \eta\overline{\rho}_d(x,r)$, where $\eta = \alpha c/(M+1)$. Hence

$$B_d\Big(x,\frac{r}{2}\Big) \supseteq U_M^\rho(x,\eta\overline{\rho}_d(x,r)) \supseteq B_\rho\Big(x,\frac{\eta}{\gamma}\overline{\rho}_d(x,r)\Big).$$

By the fact that (X,ρ) is uniformly perfect, there exists $c_* \in (0,1)$ such that $B_\rho(y,t) \backslash B_\rho(y,c_*t) \neq \emptyset$ unless $B_\rho(y,c_*t) = X$. Set $\delta = c_*\eta/\gamma$. In case $B_\rho(x,\delta\overline{\rho}_d(x,r)) = X$, then $\overline{\rho}_d(x,r/2) = \overline{\rho}_d(x,r)$. Otherwise, there exists $z \in B_d(x,r/2)$ such that $\rho(x,z) \geq \delta\overline{\rho}_d(x,r)$. In each case, we have $\overline{\rho}_d(x,r/2) \geq \delta\overline{\rho}_d(x,r)$. Furthermore, $B_d(x,r) \supseteq B_\rho(x,\eta\overline{\rho}_d(x,r)/\gamma) \supseteq B_\rho(x,\delta\overline{\rho}_d(x,r))$. Thus we have obtained halves of (12.1) and (12.2). Exchanging d and ρ, we have the other halves of (12.1) and (12.2). □

Lemma 12.8. *Let $d \in \mathcal{D}_A(X,\mathcal{O})$ and let $\rho \in \mathcal{D}(X,\mathcal{O})$. Assume that $d \underset{\mathrm{QS}}{\sim} \rho$. Let $\delta \in (0,1)$ be the constant appearing in* (12.1) *and* (12.2).
(1) *For any $w \in T$ and any $x,y \in K_w$,*

$$\overline{\rho}_d(x,d_w) \leq \delta^{-1}\overline{\rho}_d(y,d_w).$$

(2) *There exists $c > 0$ such that*

$$c\overline{\rho}_d(x,d_w) \leq \rho_w \leq \delta^{-1}\overline{\rho}_d(x,d_w)$$

for any $w \in T$ and any $x \in K_w$.

Proof. Assume $d \underset{\text{QS}}{\sim} \rho$. Lemma 12.4 implies that (X, d) is uniformly perfect. Since $d \underset{\text{QS}}{\sim} \rho$, (X, ρ) is uniformly perfect as well. Hence (12.1) and (12.2) hold.
(1) Since $B_d(x, d_w) \subseteq B_d(y, 2d_w)$, it follows that $\overline{\rho}_d(x, d_w) \le \overline{\rho}_d(y, 2d_w)$. Applying (12.2), we obtain the desired inequality.
(2) For any $x \in K_w$, $K_w \subseteq B_d(x, 2d_w)$. Hence $\rho_w \le \overline{\rho}_d(x, 2d_w)$. By (12.2), we see that

$$\rho_w \le \delta^{-1}\overline{\rho}_d(x, d_w).$$

Set $s = d_w/2$ and choose $v \in T_w$ such that $v \in \Lambda_s^d$. Since d is adapted and tight, there exists $\gamma > 0$ which is independent of w, v and s such that

$$K_v \backslash B_d(z, \gamma d_v) \ne \emptyset$$

for some $z \in K_v$. By (12.1),

$$K_v \backslash B_\rho(z, \delta\overline{\rho}_d(z, \gamma d_v)) \ne \emptyset.$$

Hence $\rho_w \ge \delta\overline{\rho}_d(z, \gamma d_v)$. Since d is super-exponential, there exists $\gamma' > 0$ which is independent of w, v and s such that $\gamma d_v \ge \gamma' d_w$. Choose $n \ge 1$ so that $2^{n-1}\gamma' \ge 1$. Using (12.2) n-times, we have

$$\rho_w \ge \delta\overline{\rho}_d(z, \gamma' d_w) = \delta^{n+1}\overline{\rho}_d(z, d_w).$$

By (1), if $c = \delta^{n+2}$, then $\rho_w \ge d\overline{\rho}_d(x, d_w)$. □

Proof of Theorem 12.6: Part 2. Assume that $d \in \mathcal{D}_{A,e}(X, \mathcal{O})$. We show that if $d \underset{\text{QS}}{\sim} \rho$, then $\rho \in \mathcal{D}_{A,e}(X, \mathcal{O})$ and $d \underset{\text{GE}}{\sim} \rho$. As in the proof of Lemma 12.8, (12.1) and (12.2) hold.
Claim 1 ρ is super-exponential.
Proof of Claim 1: Since d is super-exponential, there exists $c' \in (0, 1)$ such that $d_w \ge c' d_{\pi(w)}$ for any $w \in T$. Choose $l \ge 1$ so that $2^l c' \ge 1$. By Lemma 12.8-(2) and (12.2), if $x \in K_w$, then

$$\rho_w \ge c\overline{\rho}_d(x, d_w) \ge c\delta^l\overline{\rho}_d(z, 2^l d_w) \ge c\delta^l\overline{\rho}_d(x, d_{\pi(w)}) \ge c\delta^{l+1}\rho_{\pi(w)}.$$

Claim 2 ρ is sub-exponential.
Proof of Claim 2: Since d is sub-exponential, there exist $c_1 \in (0, 1)$ and $m \ge 1$ such that

$$d_{v'} \le c_1 d_w$$

for any $w \in T$ and any $v' \in S^m(w)$. Let $w \in T$. If $v \in S^{mj}(w)$ for $j \ge 1$ and $x \in K_v$, then by Lemma 12.8-(1)

$$\rho_v \le \delta^{-1}\overline{\rho}_d(x, d_v) \le \delta^{-1}\overline{\rho}_d(x, (c_1)^j d_w). \tag{12.5}$$

On the other hand, by [10, Proposition 11.7], there exists $\lambda \in (0, 1)$ and $c'' > 0$ such that

$$\overline{\rho}_d(x, c_1 s) \le c''\lambda\overline{\rho}_d(x, s)$$

for any $x \in X$ and any $s \in (0, 1]$. By this, (12.5) and Lemma 12.8-(2),

$$\rho_v \leq \delta^{-1}\overline{\rho}_d(x, (c_1)^j d_w) \leq \delta^{-1}c''\lambda^j\overline{\rho}_d(x, d_w) \leq \delta^{-1}c''\lambda^j c^{-1}\rho_w$$

Choosing j so that $\delta^{-1}c''\lambda^j c^{-1} < 1$, we see that ρ is sub-exponential.
Claim 3 $d \underset{\mathrm{GE}}{\sim} \rho$.
Proof of Claim 3: Since d is super-exponential, there exists $c_2 \in (0, 1)$ such that

$$s \geq d_w > c_2 s \tag{12.6}$$

for any $s \in (0, 1]$ and any $w \in \Lambda_s^d$. Let $w, v \in \Lambda_s^d$ with $K_w \cap K_v \neq \emptyset$. Then $d_w \leq d_v/c_2$. Choose $k \geq 1$ so that $2^k c_2 \geq 1$. If $x \in K_w \cap K_v$, then by Lemma 12.8-(2) and (12.2),

$$\rho_w \leq \delta^{-1}\overline{\rho}_d(x, d_w) \leq \delta^{-1}\overline{\rho}(x, d_v/c_2) \leq \delta^{-(k+1)}\overline{\rho}(x, d_v) \leq c^{-1}\delta^{-(k+1)}\rho_v.$$

Hence $d \underset{\mathrm{GE}}{\sim} \rho$.
Claim 4 ρ is adapted.
Proof of Claim 4: Assume that d is M-adapted. Let $x \in X$ and let $s \in (0, 1]$. Then there exists $\alpha > 0$ which is independent of x and s such that $U_M^d(x, \alpha s) \supseteq B_d(x, s)$. Let $w \in \Lambda_{s,0}^\rho(x)$. Since ρ is super-exponential, there exists $b \in (0, 1)$ which is independent of w and s such that $\rho_w \geq bs$. By Lemma 11.8, there exists $\gamma > 0$ such that $U_M^\rho(x, \rho_w) \supseteq U_M^d(x, \gamma d_w)$ for any $w \in T$ and $x \in K_w$. Choose $p \geq 1$ so that $2^p\gamma/\alpha \geq 1$. Then by Lemma 12.8-(2), (12.1) and (12.2),

$$\begin{aligned} U_M^\rho(x, s) &\supseteq U_M^\rho(x, \rho_w) \supseteq U_M^d(x, \gamma d_w) \\ &\supseteq B_d\Big(x, \frac{\gamma}{\alpha}d_w\Big) \supseteq B_\rho\Big(x, \delta\overline{\rho}_d\big(x, \frac{\gamma}{\alpha}d_w\big)\Big) \supseteq B_\rho(x, \delta^{p+1}\overline{\rho}_d(x, d_w)) \\ &\qquad \supseteq B_\rho(x, \delta^{p+2}\rho_w) \supseteq B_\rho(x, \delta^{p+2}bs). \end{aligned}$$

On the other hand, let $x \in K$ and let $r \in (0, 1]$. Then for any $y \in U_M^\rho(x, r)$, there exists $(w(1), \ldots, w(M+1)) \in \mathcal{CH}_K(x, y)$ such that $w(i) \in \Lambda_r^\rho$ for any i. It follows that

$$\rho(x, y) \leq \sum_{i=1}^{M+1} \rho_{w(i)} \leq (M+1)r.$$

This shows that $U_\rho^M(x, r) \subseteq B_\rho(x, (M+1)r)$. Thus we have shown that ρ is adapted.

Using Theorem 11.7-(2), we see that g_ρ is thick and hence $\rho \in \mathcal{D}_{A,e}(X, \mathcal{O})$. Thus we have shown the desired statement. $\square$

13 Fact from measure theory

Proposition 13.1. *Let $(X, \mathcal{M}, \mu)$ be measurable space and let $N \in \mathbb{N}$. If $U_i \in \mathcal{M}$ for any $i \in \mathbb{N}$ and*

$$\#(\{i | i \in \mathbb{N}, x \in U_i\}) \leq N \tag{13.7}$$

for any $x \in X$, then

$$\sum_{i=1}^{\infty} \mu(U_i) \le N\mu\Big(\bigcup_{i\in\mathbb{N}} U_i\Big).$$

Proof. Set $U = \cup_{i\in\mathbb{N}} U_i$. Define $U_{i_1\dots i_m} = \cap_{j=1,\dots,m} U_{i_j}$. By (13.7), if $m > N$, then $U_{i_1\dots i_m} = \emptyset$. Fix $m \ge 0$ and let rearrange $\{U_{i_1\dots i_m} | i_1 < i_2 < \dots < i_m\}$ so that

$$\{Y_j^m\}_{j\in\mathbb{N}} = \{U_{i_1\dots i_m} | i_1 < i_2 < \dots < i_m\}.$$

Define

$$X_j^m = Y_j^m \backslash \big(\cup_{i\in\mathbb{N}, i\ne j} Y_i^m\big).$$

Then

$$U = \cup_{m=0}^{N} \cup_{j\in\mathbb{N}} X_j^m$$

and $X_j^m \cap X_l^k = \emptyset$ if $(m, j) \ne (k, l)$. This implies

$$\mu(U) = \sum_{m=0}^{N} \sum_{j\in\mathbb{N}} \mu(X_j^m).$$

Set $I_j = \{(k, l) | U_j \supseteq X_l^k \ne \emptyset\}$. Then by (13.7), we have $\#(\{j | (k, l) \in I_j\}) \le N$ for any (k, l). This implies

$$\sum_{j=1}^{\infty} \mu(U_i) \le N \sum_{m=0}^{N} \sum_{j\in\mathbb{N}} \mu(X_j^m) = N\mu(U).$$

□

14 List of definitions, notations and conditions

Definitions

adapted – Definition 6.1, Definition 6.6
Ahlfors regular – Definition 7.18
bi-Lipschitz (metrics) – Definition 7.9
bi-Lipschitz (weight functions) – Definition 7.1
chain – Definition 4.1
degree of distortion – Definition 10.3
end of a tree – Definition 3.2
exponential – Definition 7.15
(super-, sub-)exponential for metrics – Definition 12.2
gentle – Definition 9.1
geodesic – Definition 3.1
infinite binary tree – Example 3.3
infinite geodesic ray – Definition 3.2
locally finite – Definition 3.1
minimal – Definition 4.1

Notations

$\delta^g_M(x,y)$: visual pre-metric – Definition 5.8
$\kappa(\cdot)$ – Definition 10.3
Λ^g_s – Definition 5.1
$\Lambda^g_{s,M}(\cdot)$ – Definition 5.6
π – Definition 3.2
$\pi^{(T,\mathcal{A},\phi)}$ – Remark after Definition 3.2
ρ_* – Definition 3.6
Σ: the collection of ends – Definition 3.2
Σ^w – Definition 3.2
Σ^w_v – Definition 3.2
Σ and Σ_v; abbreviation of Σ^ϕ and Σ^ϕ_v respectively,
$\Sigma^{(N)}$ – Example 3.5
$|w,v|$ – Definition 11.3
$\overline{wv}$: the geodesic between w and v of a tree, – Definition 3.1
$|w|$ – Definition 3.2
$|w|_{(T,\mathcal{A},\phi)}$ – Remark after Definition 3.2
$w \wedge v$ – Definition 3.6
$[\omega]_m$ – Definition 3.6

Equivalence relations
$\underset{\mathrm{AC}}{\sim}$ – Definition 7.3
$\underset{\mathrm{BL}}{\sim}$ relation on weight functions – Definition 7.1
$\underset{\mathrm{BL}}{\sim}$ relation on metrics – Definition 7.9
$\underset{\mathrm{GE}}{\sim}$ – Definition 9.1
$\underset{\mathrm{QS}}{\sim}$ – Definition 12.1

Conditions
(ADa), $(\mathrm{ADb})_{\mathrm{M}}$ – Theorem 6.4
(BL), (BL1), (BL2), (BL3) – Theorem 7.8
$(\mathrm{EV})_M$, $(\mathrm{EV2})_M$, $(\mathrm{EV3})_M$, $(\mathrm{EV4})_M$, $(\mathrm{EV5})_M$ – Theorem 6.11
(G1), (G2), (G3) – Definition 5.1
(P1), (P2) – Definition 4.1
(SQ1), (SQ2), (SQ3) – Section 10
(TH1), (TH2), (TH3), (TH4) – Theorem 8.3
(VD1), (VD2), (VD3), (VD4) – Theorem 9.8

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