Weighted partition of a compact metrizable space, its hyperbolicity and Ahlfors regular conformal dimension

by

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Abstract

Successive divisions of compact metric spaces appear in many different areas of mathematics such as the construction of self-similar sets, Markov partition associated with hyperbolic dynamical systems, dyadic cubes associated with a doubling metric space. The common feature in these is to divide a space into a finite number of subsets, then divide each subset into pieces and repeat this process again and again. In this paper we generalize such successive divisions and call them partitions. Given a partition, we consider the notion of a weight function assigning a "size" to each piece of the partition. Intuitively we believe that a partition and a weight function should provide a "geometry" and an "analysis" on the space of our interest. In this paper, we are going to pursue this idea in three parts. In the first part, the metrizability of a weight function is shown to be equivalent to the Gromov hyperbolicity of the graph associated with the weight function. In the second part, the notions like bi-Lipschitz equivalence, Ahlfors regularity, the volume doubling property and quasisymmetry will be shown to be equivalent to certain properties of weight functions. In particular, we find that quasisymmetry and the volume doubling property are the same notion in the world of weight functions. In the third part, a characterization of the Ahlfors regular conformal dimension of a compact metric space is given as the critical index p of p-energies associated with the partition and the weight function corresponding to the metric.

Contents

1 Introduction

Successive division of a space has played important roles in many areas of mathematics. One of the simplest examples is the binary division of the unit interval $[0,1]$ shown in Figure 1. Let $K_\phi = [0,1]$ and divide K_ϕ in half as $K_0 = [0, \frac{1}{2}]$ and $K_1 = [\frac{1}{2}, 1]$. Next, K_0 and K_1 are divided in half again and yield K_{ij} for each $(i,j) \in \{0,1\}^2$. Repeating this procedure, we obtain $\{K_{i_1\ldots i_m}\}_{i_1,\ldots,i_m \in \{0,1\}}$ satisfying

$$K_{i_1\ldots i_m} = K_{i_1\ldots i_m 0} \cup K_{i_1\ldots i_m 1} \tag{1.1}$$

for any $m \geq 0$ and $i_1 \ldots i_m \in \{0,1\}^m$. In this example, there are two notable properties.

The first one is the role of the (infinite) binary tree

$$T_b = \{\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\} = \bigcup_{m \geq 0} \{0,1\}^m,$$

where $\{0,1\}^0 = \{\phi\}$. The vertex ϕ is called the root or the reference point and T_b is called the tree with the root (or the reference point) ϕ. Note that the correspondence $i_1 \ldots i_m \to K_{i_1\ldots i_m}$ determines a map from the binary tree to the collection of compact subsets of $[0,1]$ with the property (1.1).

Secondly, note that $K_{i_1} \supseteq K_{i_1 i_2} \supseteq K_{i_1 i_2 i_3} \supseteq \ldots$ and

$$\bigcap_{m \geq 1} K_{i_1\ldots i_m} \text{ is a single point} \tag{1.2}$$

for any infinite sequence $i_1 i_2 \ldots$. (Of course, this is the binary expansion and hence the single point is $\sum_{m \geq 1} \frac{i_m}{2^m}$.) In other words, there is a natural map $\sigma : \{0,1\}^{\mathbb{N}} \to [0,1]$ given by

$$\sigma(i_1 i_2 \ldots) = \bigcap_{m \geq 1} K_{i_1\ldots i_m}.$$

Such a successive division of a compact metric space, which may not be as simple as this one, appears various areas in mathematics. One of the typical examples is a self-similar set in fractal geometry. A self-similar set is a union of finite number of contracted copies of itself. Then each contracted copy is again a union of contracted copies and so forth. Another example is the Markov partition associated with a hyperbolic dynamical system. See [1] for details. Also the division of a metric measure space having the volume doubling property by dyadic cubes can be thought of as another example of such a division of a space. See Christ[11] for example.

In general, let X be a compact metrizable topological space with no isolated point. The common properties of the above examples are;

(i) There exists a tree T (i.e. a connected graph without loops) with a root ϕ.

(ii) For any vertex p of T, there is a corresponding nonempty compact subset

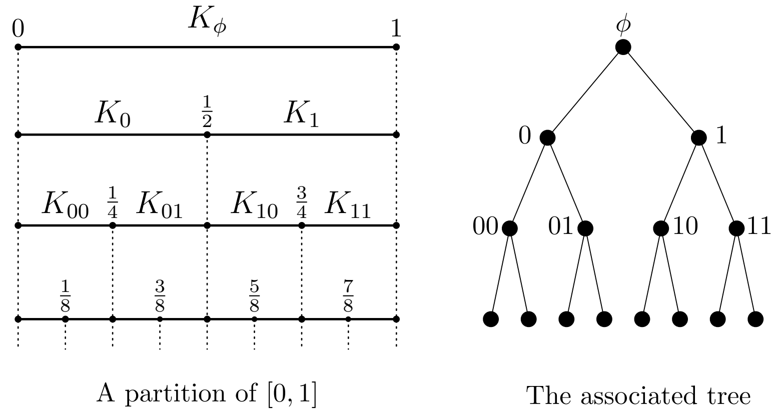


Figure 1: A partition of the unit interval $[0, 1]$ and the associated tree

of X denoted by X_p and $X = X_\phi$.
(iii) Every vertex p of T except ϕ has unique predecessor $\pi(p) \in T$ and

$$X_q = \bigcup_{p \in \{p' | \pi(p') = q\}} X_p \tag{1.3}$$

(iv) The totality of edges of T is $\{(\pi(q), q) | q \in T \backslash \{\phi\}\}$.
(v) For any infinite sequence $(p_0, p_1, p_2, \ldots)$ of vertices of X satisfying $p_0 = \phi$ and $\pi(p_{i+1}) = p_i$ for any $i \geq 1$,

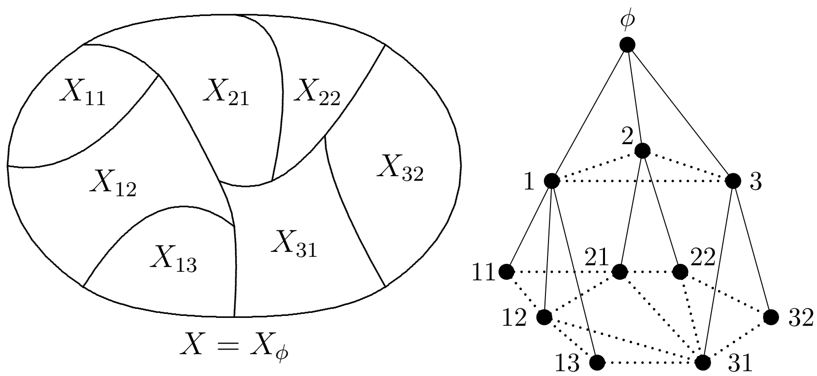
$$\bigcap_{i \geq 1} X_{p_i} \text{ is a single point.} \tag{1.4}$$

See Figure 2 for illustration of the idea. Note that the properties (1.3) and (1.4) correspond to (1.1) and (1.2) respectively. In this paper such $\{X_p\}_{p \in T}$ is called a partition of X parametrized by the tree T. (We will give the precise definition in Section 4.) In addition to the "vertical" edges , which are the edges of the tree, we provide "horizontal" edges to T to describe the combinatorial structure reflecting the topology of X as is seen in Figure 2. More precisely, a horizontal edge is a pair of $(p, q) \in T \times T$ where p and q have the same distance from the root ϕ and $X_p \cap X_q \neq \emptyset$. We call T with horizontal and vertical edges the resolution of X associated with the partition.

Another key notion is a weight function on the tree T. Note that a metric and a measure give weights of the subsets of X. More precisely, let d be a metric on X inducing the original topology of X and let μ be a Radon measure on X where $\mu(X_p) > 0$ for any $p \in T$. Define $\rho_d : T \to (0, 1]$ and $\rho_\mu : T \to (0, 1]$ by

$$\rho_d(p) = \frac{\text{diam}(X_p, d)}{\text{diam}(X, d)} \quad \text{and} \quad \rho_\mu(p) = \frac{\mu(X_p)}{\mu(X)},$$

where $\text{diam}(A, d)$ is the diameter of A with respect to the metric d. Then ρ_d (resp. ρ_μ) is though of as a natural weight of X_p associated with d (resp. μ).



A partition of X

The associated graphs T
Solid lines are vertical edges.
Dotted lines are horizontal edges.

Figure 2: A partition and the associated graphs (up to the 2nd stage)

In both cases where $\# = d$ or $\# = \mu$, the function $\rho_\# : T \to (0, \infty)$ satisfy

$$\rho_\#(\pi(p)) \geq \rho_\#(p) \tag{1.5}$$

for any $p \in T \backslash \{\phi\}$ and

$$\lim_{i \to \infty} \rho_\#(p_i) = 0 \tag{1.6}$$

if $\pi(p_{i+1}) = p_i$ for any $i \geq 1$. (To have the second property (1.6) in case of $\# = \mu$, we must assume that the measure μ is non-atomic, i.e. $\mu(\{x\}) = 0$ for any $x \in X$.)

As we have seen above, given a metric or a measure, we have obtained a weight function $\rho_\#$ satisfying (1.5) and (1.6). In this paper, we are interested in the opposite direction. Namely, given a partition of a compact metrizable topological space parametrized by a tree T, we define the notion of weight functions as the collection of functions from T to $(0, 1]$ satisfying the properties (1.5) and (1.6). Then our main object of interest is the space of weight functions including those derived form metrics and measures. Naively we believe that a partition and a weight function essentially determine a "geometry" and/or an"analysis" of the original set no matter where the weight function comes. It may come from a metric, a measure or else. Keeping this intuition in mind, we are going to develop basic theory of weigh functions in three closely related directions in this paper.

The first direction is to study when a weight function is naturally associated with a metric? In brief, our conclusion will be that a power of a weight function is naturally associated with a metric if and only if the rearrangement of the resolution T associated the weight function is Gromov hyperbolic. To be more precise, given a partition $\{X_w\}_{w \in T}$ of a compact metrizable topological space X with no isolated points and a weight function $\rho : T \to (0, 1]$. In Section 5, we will define $\delta_M^\rho(\cdot, \cdot)$, which is called the visual pre-metric associated with ρ, in the following way: let Λ_s^ρ be the collection of w's in T where the size $\rho(w)$ is almost s. Define a horizontal edge of Λ_s^ρ as $(w, v) \in \Lambda_s^\rho$ with $X_w \cap X_v \neq \emptyset$. For $r \in (0, 1)$, the rearranged resolution $\widetilde{T}^{\rho,r}$ associated with the weight function ρ is defined as the vertices $\cup_{m\geq 0}\Lambda^{\rho}_{r^m}$ with the vertical edges from the tree structure of T and the horizontal edges of $\Lambda^{\rho}_{r^m}$. Then the visual pre-metric $\delta^{\rho}_{M}(x,y)$ for $x, y \in X$ is given by the infimum of s where x and y can be connected by an M-chain of horizontal edges in Λ^{ρ}_{s}. We think a metric d is naturally associated with the weight function ρ if and only if d and Λ^{ρ}_{M} are bi-Lipschitz equivalent on $X \times X$. More precisely, we are going to use a phrase"d is adapted to ρ" instead of "d is naturally associated with ρ". The notion of visual pre-metric is a counterpart of that of visual pre-metric on the boundary of a Gromov hyperbolic metric space, whose detailed account can bee seen in [10], [22] and [14] for example. Now the main conclusion of the first part is Theorem 7.12 saying that the hyperbolicity of the rearranged resolution $\widetilde{T}^{\rho,r}$ is equivalent to the existence of a metric adapted to some power of the weight function. Moreover, if this is the case, the metric adapted to some power of the weight function is shown to be a visual metric in Gromov's sense.

The second direction is to establish relationships of various relations between weight functions, metrics and measures. For examples, Ahlfors regularity and the volume property are relations between measures and metrics. For $\alpha > 0$, a measure μ is α-Ahlfors regular with respect to a metric d if and only if there exist $c_1, c_2 > 0$ such that

$$c_1 r^{\alpha} \leq \mu(B_d(x,r)) \leq c_2 r^{\alpha},$$

where $B_d(x,r) = \{y | y \in X, d(x,y) < r\}$, for any $r \in (0, \mathrm{diam}(X,d)]$ and $x \in X$. See Definition 10.3 for the precise definition of the volume doubling property. On the other hand, bi-Lipschitz and quasisymmetry are equivalence relations between two metrics. (The precise definitions of bi-Lipschitz equivalence and quasisymmetry are given in Definitions 8.9 and 13.1 respectively.) Regarding those relations, we are going to claim the following relationships

$$\text{bi-Lipschitz} = \text{Ahlfors regularity} = \text{being adapted} \tag{1.7}$$

and

$$\text{the volume doubling property} = \text{quasisymmetry}. \tag{1.8}$$

in the framework of weight functions. To illustrate the first claim more explicitly, let us introduce the notion of bi-Lipschitz equivalence of weight functions. Two weight functions ρ_1 and ρ_2 are said to be bi-Lipschitz equivalent if and only if there exist $c_1, c_2 > 0$ such that

$$c_1\rho_1(p) \leq \rho_2(p) \leq c_2\rho_1(p)$$

for any $p \in T$. Now the first claim can be resolved into three parts as follows: let ρ_1 and ρ_2 be two weight functions.

Claim 1: Suppose that $\rho_1 = \rho_{d_1}$ and $\rho_2 = \rho_{d_2}$ for metrics d_1 and d_2 on X. Then ρ_1 and ρ_2 are bi-Lipschitz equivalent if and only if d_1 and d_2 are bi-Lipshitz equivalent as metrics.

Claim 2: Suppose that $\rho_1 = \rho_d$ and $\rho_2 = \rho_\mu$ for a metric d on X and a Radon

measure μ on X. Then $(\rho_1)^\alpha$ and ρ_2 are bi-Lipschitz equivalent if and only if μ is α-Ahlfors regularity of μ with respect to d.
Claim 3: Suppose that $\rho_1 = \rho_d$ for a metric d on X, then ρ_1 and ρ_2 are bi-Lipschitz equivalent if and only if the metric d is adapted to the weight function ρ_2.
One can find the precise statement in Theorem 2.11 in the case of partitions of S^2. The second claim is rationalized in the same manner. See Theorem 2.12 for the exact statement in the case of S^2 for example.

The third direction is a characterization of Ahlfors regular conformal dimension. The Ahlfors regular conformal dimension, AR conformal dimension for short, of a metric space (X, d) is defined as

$$
\begin{aligned}
&\dim_{AR}(X, d) = \\
&\quad \inf\{\alpha | \text{there exist a metric } \rho \text{ on } X \text{ and a Borel regular measure } \mu \text{ on } X \\
&\qquad\qquad \text{such that } \rho \underset{\text{QS}}{\sim} d \text{ and } \mu \text{ is } \alpha\text{-Ahlfors regular with respect to } \rho\},
\end{aligned}
$$

where "$\rho \underset{\text{QS}}{\sim} d$" means that the two metrics ρ and d are quasisymmetric to each other. In [23], Carassco Piaggio has given a characterization of Ahlfors regular conformal dimension in terms of the critical exponent of p-combinatorial modulus of discrete path families. In view of the results from the previous part, we have obtained the ways to express the notions of quasisymmetry and Ahlfors regularity in terms of weight functions. So we are going to translate Carassco Piaggio's work into our framework. However, we are going to use the critical exponent of p-energy instead of p-combinatorial modulus in our work.[1] Furthermore, we are going to define the notion of p-spectral dimension and present a relation between Ahlfors regular conformal dimension and p-spectral dimension. In particular, for $p = 2$, the 2-spectral dimension has been know to appear in the asymptotic behavior of the Brownian motion and the eigenvalue counting function of the Laplacian on certain fractals like the Sierpinski gasket and the Sierpinski carpet. See [6], [4] and [19] for example. For the Sierpinski carpet, we will show that the 2-spectral dimension gives an upper bound of Ahlfors regular conformal dimension.

One of the ideas behind this study is to approximate a space by a series of graphs. Such an idea has already been explored in association with hyperbolic geometry. For example, in [12] and [9], they have constructed an infinite graph whose hyperbolic boundary is homeomorphic to given compact metric space. Their method is first construct a series of coverings of the space, which is a counterpart of our partition, and construct a graph from the series. In [23], Carrasco Piaggio has utilized this series of coverings to study Ahlfors regular conformal dimension of the space. His notion of "relative radius" essentially corresponds to our weight function. In our framework, the original space is homeomorphic to the analogue of hyperbolic boundary of the resolution T of X

[1] This idea of characterizing AR conformal dimension by p-energies was brought to the author by B. Kleiner in a personal communication.

even if it is not hyperbolic in the sense of Gromov. See Theorem 7.5 for details. In other words, the resolution T of X is a version of hyperbolic filling of the original space X. (See [8] for the notion of hyperbolic fillings.) In this respect, our study in this paper may be thought of as a theory of weighted hyperbolic fillings.

The organization of this paper is as follows. In Section 2, we give a summary of the main results of this paper in the case of the 2 dimensional sphere as a showcase of the full theory. In Section 3, we give basic definitions and notations on trees. Section 4 is devoted to the introduction of partitions and related notions. In Section 5, we define the notion of weight function and the associated "visual pre-metric". We study our first question mentioned above, namely, when a weight function is naturally associated with a (power of) metric in Section 6. In Section 7, we are going to relate this question to the hyperbolicity of certain graph associated with a weight function. Section 8 is devoted to justifying the statement (1.7). In Sections 9, 10, 12 and 13, we will study the rationalized version of (1.8) as mathematical statement. In particular, in Section 10, we introduce the key notion of being "gentle". In Section 11, we apply our general theory to certain class of subsets of the square and obtain concrete (counter) examples. From Section 14, we will start arguing a characterization of Ahlfors regular conformal dimension. From Section 14 to 18, we discuss how to obtain a pair of a metric d and a measure μ where μ is α-Ahlfors regular with respect to d for a given order α. The main result of these sections is Theorem 18.1. In Section 19, we will give a characterization of the Ahlfors regular conformal dimension as a critical index p of p-energies. Then we will show the relation of the Ahlfors regular conformal dimension and p-spectral dimension in Section 20. Additionally, we will give another characterization of the Ahlfors regular conformal dimension by p-modulus of curve families in Section 21. Finally in Section B, we present the whereabouts of definitions, notations and conditions appearing in this paper for reader's sake.

2 Summary of the main results; the case of 2-dim. sphere

In this section, we summarize our main results in this paper in the case of 2-dimensional sphere S^2 (or the Riemann sphere in other words), which is denoted by X in what follows. We use d_s to denote the standard spherical geodesic metric on X. Set

$$\mathcal{U} = \{A | A \subseteq X, \text{closed}, \text{int}(A) \neq \emptyset, \partial A \text{ is homeomorphic to the circle } S^1.\}$$

First we divide X into finite number of subsets $X_1, \ldots, X_{N_0}$ belonging to $\mathcal{U}$, i.e.

$$X = \bigcup_{i=1}^{N_0} X_i$$

We assume that $X_i \cap X_j = \partial X_i \cap \partial X_j$ if $i \neq j$. Next each X_i is divided into finite number of its subsets $X_{i1}, X_{i2}, \ldots, X_{iN_i} \in \mathcal{U}$ in the same manner as before. Repeating this process, we obtain $X_{i_1\ldots i_k}$ for any $i_1 \ldots i_k$ satisfying

$$X_{i_1\ldots i_k} = \bigcup_{j=1,\ldots,N_{i_1\ldots i_k}} X_{i_1\ldots i_k j} \tag{2.1}$$

and if $i_1 \ldots i_k \neq j_1 \ldots j_k$, then

$$X_{i_1\ldots i_k} \cap X_{j_1\ldots j_k} = \partial X_{i_1\ldots i_k} \cap \partial X_{j_1\ldots j_k}. \tag{2.2}$$

Note that (2.1) is a counterpart of (1.3). Next define

$$T_k = \{i_1 \ldots i_k | i_j \in \{1, \ldots, N_{i_1\ldots i_{j-1}}\} \text{ for any } j = 1, \ldots, k-1\}$$

for any $k = 0, 1, \ldots$, where T_0 is a one point set $\{\phi\}$. Let $T = \cup_{k \geq 0} T_k$. Then T is naturally though of as a (non-directed) tree whose edges are given by the totality of $(i_1 \ldots i_k, i_1 \ldots i_k i_{k+1})$. We regard the correspondence $w \in T$ to $X_w \in \mathcal{U}$ as a map from T to $\mathcal{U}$, which is denoted by $\mathcal{X}$. Namely, $\mathcal{X}(w) = X_w$ for any $w \in T$. Note that $\mathcal{X}(\phi) = X$. Define

$$\Sigma = \{i_1 i_2 \ldots | i_1 \ldots i_k \in T_k \text{ for any } k \geq 0\},$$

which is the "boundary" of the infinite tree T.

Furthermore we assume that for any $i_1 i_2 \ldots \in \Sigma$

$$\bigcap_{k=1,2,\ldots} X_{i_1\ldots i_k}$$

is a single point, which is denoted by $\sigma(i_1 i_2 \ldots)$. Note that σ is a map from Σ to X. This assumption corresponds to (1.4) and hence the map $\mathcal{X}$ is a partition of X parametrized by the tree T. Since $X = \cup_{w \in T_k} X_w$ for any $k \geq 0$, this map σ is surjective.

In [7, Chapter 5], the authors have constructed "cell decomposition" associated with an expanding Thurston map. This "cell decomposition" is, in fact, an example of a partition formulated above.

Throughout this section, for simplicity, we assume the following conditions (SF) and (TH), where (SF) is called strong finiteness in Definition 4.4 and (TH) ensures the thickness of every exponential weight function. See Definition 8.19 for the "thickness" of a weight function.

(SF)

$$\#(\sigma^{-1}(x)) < +\infty, \tag{2.3}$$

where $\#(A)$ is the number of elements in a set A.

(TH) There exists $m \geq 1$ such that for any $w = i_1 \ldots i_n \in T$, there exists $v = i_1 \ldots i_n i_{n+1} \ldots i_{n+m} \in T$ such that $X_v \subseteq \mathrm{int}(X_w)$.

The main purpose of this paper is to describe metrics and measures of X from a given weight assigned to each piece X_w of the partition $\mathcal{X}$.

Definition 2.1. A map $g : T \to (0,1]$ is called a weight function if and only if it satisfies the following conditions (G1), (G2) and (G3).
(G1) $g(\phi) = 1$
(G2) $g(i_1 \ldots i_k) \geq g(i_1 \ldots i_k i_{k+1})$ for any $i_1 \ldots i_k \in T$ and $i_1 \ldots i_k i_{k+1} \in T$.
(G3)

$$\lim_{m \to 0} \sup_{w \in T_k} g(w) = 0.$$

Define

$$\mathcal{G}(T) = \{g | g : T \to (0,1] \text{ is a weight function.}\}$$

Moreover, we define following conditions (SpE) and (SbE), which represent "super-exponential" and "sub-exponential" respectively:
(SpE) There exists $\lambda \in (0,1)$ such that

$$g(i_1 \ldots i_k i_{k+1}) \geq \lambda g(i_1 \ldots i_k)$$

for any $i_1 \ldots i_k \in T$ and $i_1 \ldots i_k i_{k+1} \in T$.
(SbE) There exist $m \in \mathbb{N}$ and $\gamma \in (0,1)$ such that

$$g(i_1 \ldots i_k i_{k+1} \ldots i_{k+m}) \leq \gamma g(i_1 \ldots i_k)$$

for any $i_1 \ldots i_k \in T$ and $i_1 \ldots i_k i_{k+1} \ldots i_{k+m} \in T$.
Set

$$\mathcal{G}_e(T) = \{g | g : T \to (0,1] \text{ is a weight function satisfying (SpE) and (SbE).}\}.$$

Metrics and measures on X naturally have associated weight functions.

Definition 2.2. Set

$$\mathcal{D}(X) = \{d | d \text{ is a metric on } X \text{ which produces the original topology of } X, \\ \text{and } \mathrm{diam}(X, d) = 1\}$$

and

$$\mathcal{M}(X) = \{\mu | \mu \text{ is a Borel regular probability measure on } X,\ \mu(\{x\}) = 0 \\ \text{for any } x \in T \text{ and } \mu(O) > 0 \text{ for any non-empty open set } O \subseteq X\}$$

For any $d \in \mathcal{D}(X)$, define $g_d : T \to (0,1]$ by $g_d(w) = \mathrm{diam}(X_w, d)$ and for any $\mu \in \mathcal{M}(X)$, define $g_\mu : T \to (0,1]$ by $g_\mu(w) = \mu(X_w)$ for any $w \in T$.

From Proposition 5.5, we have the following fact.

Proposition 2.3. *If $d \in \mathcal{D}(X)$ and $\mu \in \mathcal{M}(X)$, then g_d and g_μ are weight functions.*

So a metric $d \in \mathcal{D}(X)$ has associated weight function g_d. How about the converse direction, i.e. for a given weight function g, is there a metric d such that $g = g_d$? To make this question more rigorous and flexible, we define the notion of "visual pre-metric" $\delta_M^g(\cdot, \cdot)$ associated with a weight function g.

Definition 2.4. Let $g \in \mathcal{G}(T)$. Define

$$\Lambda_s^g = \{i_1 \dots i_k | i_1 \dots i_k \in T, g(i_1 \dots i_{k-1}) > s \geq g(i_1 \dots i_k)\}$$

for $s \in (0, 1]$ and

$$\begin{aligned}\delta_M^g(x, y) = \inf\{s | &\text{there exist } w(1), \dots, w(M+1) \in \Lambda_s^g \text{ such that} \\ &x \in X_{w(1)}, y \in X_{w(M+1)} \text{ and } X_{w(j)} \cap X_{w(j+1)} \neq \emptyset \text{ for any } j = 1, \dots, M\}\end{aligned}$$

for $x, y \in X$. A weight function is called uniformly finite if and only if

$$\sup_{s \in (0,1], w \in \Lambda_s^g} \#(\{v | v \in \Lambda_s^g, X_w \cap X_v \neq \emptyset\}) < +\infty.$$

Although $\delta_M^g(x, y) \geq 0$, $\delta_M^g(x, y) = 0$ if and only if $x = y$ and $\delta_M^g(x, y) = \delta_M^g(y, x)$, the quantity δ_M^g may not satisfy the triangle inequality in general. The visual pre-metric $\delta_M^g(x, y)$ is a counterpart of the visual metric defined in [7]. See Section 5 for details.

If the pre-metric $\delta_M^g(\cdot, \cdot)$ is bi-Lipschitz equivalent to a metric d, we consider d as the metric which is naturally associated with the weight function g.

Definition 2.5. Let $M \geq 1$
(1) A metric $d \in \mathcal{D}(X)$ is said to be M-adapted to a weight function $g \in \mathcal{G}(X)$ if and only if there exist $c_1, c_2 > 0$ such that

$$c_1 d(x, y) \leq \delta_M^g(x, y) \leq c_2 d(x, y)$$

for any $x, y \in X$.
(2) A metric d is said to be M-adapted if and only if it is M-adapted to g_d and it is said to be adapted if it is M-adapted for some $M \geq 1$.
(3) Define

$$\mathcal{D}_{A,e}(X) = \{d | d \in \mathcal{D}(X),\ g_d \in \mathcal{G}_e(T) \text{ and } d \text{ is adapted.}\}$$
$$\mathcal{M}_e(X) = \{\mu | \mu \in \mathcal{M}(X), g_\mu \in \mathcal{G}_e(T)\}$$

The value M really matters. See Example 11.9 for an example.

The following definition is used to describe an equivalent condition for the existence of an adapted metric in Theorem 2.7.

Definition 2.6. Let $g \in \mathcal{G}(T)$. For $r \in (0, 1)$, define $\widetilde{T}^{g,r} = \cup_{m \geq 0} \Lambda_{r^m}^g$. Define the horizontal edges $E_{g,r}^h$ and the vertical edges $E_{g,r}^v$ of $\widetilde{T}^{g,r}$ as

$$E_{g,r}^h = \{(w, v) | w, v \in \Lambda_{r^m}^g \text{ for some } m \geq 0, w \neq v, X_w \cap X_v \neq \emptyset\}$$

and

$$E_{g,r}^v = \{(w, v) | w \in \Lambda_{r^m}^g, v \in \Lambda_{r^{m+1}}^g \text{ for some } m \geq 0, X_w \supseteq X_v\}$$

respectively.

The following theorem is a special case of Theorem 7.12.

Theorem 2.7. *Let $g \in \mathcal{G}(X)$. There exist $M \geq 1$, $\alpha > 0$ and a metric $d \in \mathcal{D}(X)$ such that d is M-adapted to g^α if and only if the graph $(\widetilde{T}^{g,r}, E^h_{g,r} \cup E^v_{g,r})$ is Gromov hyperbolic for some $r \in (0,1)$. Moreover, if this is the case, then the adapted metric d is a visual metric in the Gromov sense.*

Next, we define two equivalent relations $\underset{\mathrm{BL}}{\sim}$ and $\underset{\mathrm{GE}}{\sim}$ on the collection of exponential weight functions. Later, we are going to identify these with known relations according to the types of weight functions.

Definition 2.8. Let $g, h \in \mathcal{G}_e(T)$.
(1) g and h are said to be bi-Lipschitz equivalent if and only if there exist $c_1, c_2 > 0$ such that

$$c_1 g(w) \leq h(w) \leq c_2 g(w)$$

for any $w \in T$. We write $g \underset{\mathrm{BL}}{\sim} h$ if g and h are bi-Lipschitz equivalent.
(2) h is said to be gentle to g if and only if there exists $\gamma > 0$ such that if $w, v \in \Lambda^g_s$ and $X_w \cap X_v \neq \emptyset$, then $h(w) \leq \gamma h(v)$. We write $g \underset{\mathrm{GE}}{\sim} h$ if h is gentle to g.

Clearly, $\underset{\mathrm{BL}}{\sim}$ is an equivalence relation. On the other hand, the fact that $\underset{\mathrm{GE}}{\sim}$ is an equivalence relation is not quite obvious and going to be shown in Theorem 12.2.

Proposition 2.9. *The relations $\underset{BL}{\sim}$ and $\underset{\mathrm{GE}}{\sim}$ are equivalent relations in $\mathcal{G}_e(T)$. Moreover, if $g \underset{BL}{\sim} h$, then $g \underset{\mathrm{GE}}{\sim} h$.*

Some of the properties of a weight function is invariant under the equivalence relation $\underset{\mathrm{GE}}{\sim}$ as follows.

Proposition 2.10. (1) *Being uniformly finite is invariant under the equivalence relation $\underset{\mathrm{GE}}{\sim}$, i.e. if $g \in \mathcal{G}_e(T)$ is uniformly finite, $h \in \mathcal{G}_e(T)$ and $g \underset{\mathrm{GE}}{\sim} h$, then h is uniformly finite.*
(2) *The hyperbolicity of $\widetilde{T}^{g,r}$ is invariant under the equivalence relation $\underset{\mathrm{GE}}{\sim}$.*

The statements (1) and (2) of the above theorem are the special cases of Theorem 12.7 and Theorem 12.9 respectively.

The next theorem shows that bi-Lipschitz equivalence of weight functions can be identified with other properties according to types of involved weight functions.

Theorem 2.11. (1) *For $d, \rho \in \mathcal{D}_{A,e}(X)$, $g_d \underset{BL}{\sim} g_\rho$ if and only if d and g are bi-Lipschitz equivalent as metrics.*
(2) *For $\mu, \nu \in \mathcal{M}(X)$, $g_\mu \underset{BL}{\sim} g_\nu$ if and only if there exist $c_1, c_2 > 0$ such that*

$$c_1 \mu(A) \leq \nu(A) \leq c_2 \mu(A)$$

for any Borel set $A \subseteq X$.
(3) *For $g \in \mathcal{G}_e(X)$ and $d \in \mathcal{D}_{A,e}(X)$, $g \underset{BL}{\sim} g_d$ if and only if d is M-adapted to g for some $M \geq 1$.*
(4) *For $d \in \mathcal{D}_{A,e}(X)$, $\mu \in \mathcal{M}(X)$ and $\alpha > 0$, $(g_d)^\alpha \underset{BL}{\sim} g_\mu$ and g_d is uniformly finite if and only if μ is α-Ahlfors regular with respect to d, i.e. there exist $c_1, c_2 > 0$ such that*

$$c_1 r^\alpha \leq \mu(B_d(x,r)) \leq c_2 r^\alpha$$

for any $r > 0$ and $x \in X$.

The statements (1), (2), (3) and (4) of the above theorem follow from Corollary 8.10, Theorem 8.4, Corollary 8.11 and Theorem 8.21 respectively.

The gentle equivalence relation is identified with "quasisymmetry" between metrics and "volume doubling property" between a metric and a measure.

Theorem 2.12. (1) *Let $d \in \mathcal{D}_{A,e}(X)$ and $\mu \in \mathcal{M}(X)$. Then $g_\mu \in \mathcal{G}_e(T)$, $g_d \underset{\text{GE}}{\sim} g_\mu$ and g_d is uniformly finite if and only if μ has the volume doubling property with respect to d, i.e. there exists $C > 0$ such that*

$$\mu(B_d(x,2r)) \leq C\mu(B_d(x,r))$$

for any $r > 0$ and $x \in X$.
(2) *For $d \in \mathcal{D}_{A,e}(X)$ and $\rho \in \mathcal{D}(X)$, d is quasisymmetric to ρ if and only if $\rho \in \mathcal{D}_{A,e}(X)$ and $g_d \underset{\text{GE}}{\sim} g_\rho$.*

The statement (1) of the above theorem follows from Proposition 10.6 and Theorem 10.9-(2). Note that the condition (TH) implies (TH1) appearing in Theorem 9.3. Consequently every exponential weight function is thick by Theorem 9.3. The statement (2) is immediate from Corollary 13.7.

In [7, Section 17], the authors have shown that the visual metric is quasisymmetric to the chordal metric which is bi-Lipschitz equivalent to the standard geodesic metric d_S on S^2 for certain class of expanding Thurston maps. In view of their proof, they have essentially shown a counterpart of the condition given in Theorem 2.12-(2).

Next we present a characterization of the Ahlfors regular conformal dimension using the critical index p of p-energies.

Definition 2.13. Let $g \in \mathcal{G}_e(T)$ and let $r \in (0,1)$. For $A \subseteq \Lambda^g_{r^m}$, $w \in \Lambda^g_{r^m}$, $M \geq 1$ and $n \geq 0$, define

$$S^n(A) = \{v | v \in \Lambda^g_{r^{m+n}}, X_v \subseteq \cup_{w \in A} X_w\}$$

and

$$\begin{aligned}\Gamma^g_M(w) = \{v | v \in \Lambda^g_{r^m}, &\text{there exists } (v(0), v(1), \ldots, v(M)) \text{ such that} \\ &v(0) = w \text{ and } (v(i), v(i+1)) \in E^h_{g,r} \text{ for any } i = 0, \ldots, M-1\}\end{aligned}$$

The set $S^n(A)$ corresponds to the refinement of A in $\Lambda^g_{r^{m+n}}$ and the set $\Gamma^g_M(w)$ is the M-neighborhood of w in $\Lambda^g_{r^m}$.

Definition 2.14. Let $g \in \mathcal{G}_e(T)$ and let $r \in (0,1)$. For $p > 0$, $w \in \widetilde{T}^{g,r}$, $M \geq 1$ and $n \geq 0$, define

$$\mathcal{E}^g_{M,p,w,n} = \inf\Big\{ \sum_{(u,v)\in E^h_{g,r}, u,v\in\Lambda^g_{r^{m+n}}} |f(u) - f(v)|^p \Big| \\ f : \Lambda^g_{r^{m+n}} \to \mathbb{R}, f|_{S^n(w)} = 1, u|_{\Lambda^g_{r^{m+n}}\setminus S^n(\Gamma^g_M(w))} = 0 \Big\}$$

and

$$\mathcal{E}^g_{M,p} = \liminf_{m\to\infty} \sup_{w\in\widetilde{T}^{g,r}} \mathcal{E}^g_{M,p,w,m}.$$

By Theorem 19.4, we have the following characterization of the Ahlfors regular conformal dimension of (X,d) in terms of $\mathcal{E}^g_{M,p}$.

Theorem 2.15. *Let $d \in \mathcal{D}_{A,e}(X)$ and set $g = g_d$. Assume that d is uniformly finite and M-adapted. If $\mathcal{E}^g_{M,p} = 0$, then there exist $\rho \in \mathcal{D}_{A,e}(X)$ and $\mu \in \mathcal{M}_e(X)$ such that μ is p-Ahlfors regular with respect to ρ and ρ is quasisymmetric to d. Moreover, the Ahlfors regular conformal dimension of (X,d) is (finite and) given by* $\inf\{p|\mathcal{E}^g_{M,p} = 0\}$.

Part I

Partitions, weight functions and their hyperbolicity

3 Tree with a reference point

In this section, we review basic notions and notations on a tree with a reference point.

Definition 3.1. Let T be a countably infinite set and let $\mathcal{A} : T \times T \to \{0,1\}$ which satisfies $\mathcal{A}(w,v) = \mathcal{A}(v,w)$ and $\mathcal{A}(w,w) = 0$ for any $w, v \in T$. We call the pair $(T,\mathcal{A})$ a (non-directed) graph with the vertices T and the adjacent matrix $\mathcal{A}$. An element $(u,v) \in T \times T$ is called a edge of $(T,\mathcal{A})$ if and only if $\mathcal{A}(u,v) = 1$. We will identify the adjacent matrix $\mathcal{A}$ with the collection of edges $\{(u,v)|u,v \in T, \mathcal{A}(u,v) = 1\}$.
(1) The set $\{v|\mathcal{A}(w,v) = 1\}$ is called the neighborhood of w in $(T,\mathcal{A})$. $(T,\mathcal{A})$ is said to be locally finite if the neighborhood of w is a finite set for any $w \in T$.
(2) For $w_0,\ldots,w_n \in T$, $(w_0,w_1,\ldots,w_n)$ is called a path between w_0 and w_n if $\mathcal{A}(w_i,w_{i+1}) = 1$ for any $i = 0,1,\ldots n-1$. A path $(w_0,w_1,\ldots,w_n)$ is called simple if and only if $w_i \neq w_j$ for any i,j with $0 \leq i < j \leq n$ and $|i-j| < n$.

(3) $(T, \mathcal{A})$ is called a (non-directed) tree if and only if there exists a unique simple path between w and v for any $w, v \in T$ with $w \neq v$. For a tree $(T, \mathcal{A})$, the unique simple path between two vertices w and v is called the geodesic between w and v and denoted by $\overline{wv}$. We write $u \in \overline{wv}$ if $\overline{wv} = (w_0, w_1, \ldots, w_n)$ and $u = w_i$ for some i.

In this paper, we always fix a point in a tree as the root of the tree and call the point the reference point.

Definition 3.2. Let $(T, \mathcal{A})$ be a tree and let $\phi \in T$. The triple $(T, \mathcal{A}, \phi)$ is called a tree with a reference point ϕ.
(1) Define $\pi : T \to T$ by

$$\pi(w) = \begin{cases} w_{n-1} & \text{if } w \neq \phi \text{ and } \overline{\phi w} = (w_0, w_1, \ldots, w_{n-1}, w_n), \\ \phi & \text{if } w = \phi \end{cases}$$

and set $S(w) = \{v | \mathcal{A}(w, v) = 1\} \backslash \{\pi(w)\}$.
(2) For $w \in T$, we define $|w| = n$ if and only if $\overline{\phi w} = (w_0, w_1, \ldots, w_n)$. Moreover, we set $(T)_m = \{w | w \in T, |w| = m\}$.
(4) An infinite sequence of vertices $(w_0, w_1, \ldots)$ is called an infinite geodesic ray originated from w_0 if and only if $(w_0, \ldots, w_n) = \overline{w_0 w_n}$ for any $n \geq 0$. Two infinite geodesic rays $(w_0, w_1, \ldots)$ and $(v_0, v_1, \ldots)$ are equivalent if and only if there exists $k \in \mathbb{Z}$ such that $w_{n+k} = v_n$ for sufficiently large n. An equivalent class of infinite geodesic rays is called an end of T. We use Σ to denote the collection of ends of T.
(5) Define Σ^w as the collection of infinite geodesic rays originated from $w \in T$. For any $v \in T$, Σ^w_v is defined as the collection of elements of Σ^w passing through v, namely

$$\Sigma^w_v = \{(w, w_1, \ldots) | (w, w_1, \ldots) \in \Sigma^w, w_n = v \text{ for some } n \geq 1\}$$

Remark. Strictly, the notations like π and $|\cdot|$ should be written as $\pi^{(T,\mathcal{A},\phi)}$ and $|\cdot|_{(T,\mathcal{A},\phi)}$ respectively. In fact, if we will need to specify the tree in question, we are going to use such explicit notations.

One of the typical examples of a tree is the infinite binary tree. In the next example, we present a class of trees where $\#(S(w))$ is independent of $w \in T$.

Example 3.3. Let $N \geq 2$ be an integer. Let $T^{(N)}_m = \{1, \ldots, N\}^m$ for $m \geq 0$. (We let $T^{(N)}_0 = \{\phi\}$, where ϕ represents an empty sequence.) We customarily write $(i_1, \ldots, i_m) \in T^{(N)}_m$ as $i_1 \ldots i_m$. Define $T^{(N)} = \cup_{m \geq 0} T^{(N)}_m$. Define $\pi : T^{(N)} \to T^{(N)}$ by $\pi(i_1 \ldots i_m i_{m+1}) = i_1 \ldots i_m$ for $m \geq 0$ and $\pi(\phi) = \phi$. Furthermore, define

$$\mathcal{A}^{(N)}_{wv} = \begin{cases} 1 & \text{if } w \neq v, \text{ and either } \pi(w) = v \text{ or } \pi(v) = w, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(T^{(N)}, \mathcal{A}^{(N)}, \phi)$ is a locally finite tree with a reference point ϕ. In particular, $(T^{(2)}, \mathcal{A}^{(2)}, \phi)$ is called the infinite binary tree.

It is easy to see that for any infinite geodesic ray $(w_0, w_1, \ldots)$, there exists a geodesic ray originated from ϕ that is equivalent to $(w_0, w_1, \ldots)$. In fact, adding the geodesic $\overline{\phi w_0}$ to $(w_0, w_1, \ldots)$ and removing a loop, one can obtain the infinite geodesic ray having required property. This fact shows the following proposition.

Proposition 3.4. *There exists a natural bijective map from Σ to Σ^ϕ.*

Through this map, we always identify the collection of ends Σ and the collection of infinite geodesic rays originated from ϕ, Σ^ϕ.

Hereafter in this paper, we always assume that $(T, \mathcal{A})$ is a locally finite with a fixed reference point $\phi \in T$. If no confusion can occur, we omit ϕ in the notations. For example, we use Σ, and Σ_v in place of Σ^ϕ and Σ_v^ϕ respectively.

Example 3.5. Let $N \geq 2$ be an integer. In the case of $(T^{(N)}, \mathcal{A}^{(N)}, \phi)$ defined in Example 3.3, the collection of the ends Σ is $\Sigma^{(N)} = \{1, \ldots, N\}^{\mathbb{N}} = \{i_1 i_2 i_3 \ldots, | i_j \in \{1, \ldots, N\}$ for any $m \in \mathbb{N}\}$. With the natural product topology, $\Sigma^{(N)}$ is a Cantor set, i.e. perfect and totally disconnected.

Definition 3.6. Let $(T, \mathcal{A}, \phi)$ be a locally finite tree with a reference point ϕ.
(1) For $\omega = (w_0, w_1, \ldots) \in \Sigma$, we define $[\omega]_m$ by $[\omega]_m = w_m$ for any $m \geq 0$. Moreover, let $w \in T$. If $\overline{\phi w} = (w_0, w_1, \ldots, w_{|w|})$, then for any $0 \leq m \leq |w|$, we define $[w]_m = w_m$. For $w \in T$, we define

$$T_w = \{v | v \in T, w \in \overline{\phi v}\}$$

(2) For $w, v \in T$, we define the confluence of w and v, $w \wedge v$, by

$$w \wedge v = w_{\max\{i | i=0,\ldots,|w|, [w]_i = [v]_i\}}$$

(3) For $\omega, \tau \in \Sigma$, if $\omega \neq \tau$, we define the confluence of ω and τ, $\omega \wedge \tau$, by

$$\omega \wedge \tau = [\omega]_{\max\{m | [\omega]_m = [\tau]_m\}}$$

(4) For $\omega, \tau \in \Sigma$, we define $\rho_*(\omega, \tau) \geq 0$ by

$$\rho_*(\omega, \tau) = \begin{cases} 2^{-|\omega \wedge \tau|} & \text{if } \omega \neq \tau, \\ 0 & \text{if } \omega = \tau. \end{cases}$$

It is easy to see that ρ_* is a metric on Σ and $\{\Sigma_{[\omega]_m}\}_{m \geq 0}$ is a fundamental system of neighborhood of $\omega \in \Sigma$. Moreover, $\{\Sigma_v\}_{v \in T}$ is a countable base of open sets. This base of open sets has the following property.

Lemma 3.7. *Let $(T, \mathcal{A}, \phi)$ be a locally finite tree with a reference point ϕ. Then for any $w, v \in T$, $\Sigma_w \cap \Sigma_v = \emptyset$ if and only if $|w \wedge v| < |w|$ and $|w \wedge v| < |v|$. Furthermore, $\Sigma_w \cap \Sigma_v \neq \emptyset$ if and only if $\Sigma_v \subseteq \Sigma_w$ or $\Sigma_w \subseteq \Sigma_v$.*

Proof. If $|w \wedge v| = |w|$, then $w = w \wedge v$ and hence $w \in \overline{\phi v}$. Therefore $\Sigma_v \subseteq \Sigma_w$. So, $\Sigma_w \cap \Sigma_v \neq \emptyset$. Conversely, if $\omega \in \Sigma_w \cap \Sigma_v$, then there exist $m, n \geq 0$ such that $w = [\omega]_m$ and $v = [\omega]_m$. It follows that

$$w \wedge v = \begin{cases} w & \text{if } m \leq n, \\ v & \text{if } m \leq n. \end{cases}$$

Hence we see that $|w \wedge v| = |w|$ or $|w \wedge v| = |v|$. $\square$

With the help to the above proposition, we may easily verify the following well-known fact. The proof is standard and left to the readers.

Proposition 3.8. *If $(T, \mathcal{A}, \phi)$ is a locally finite tree with a reference point ϕ. Then $\rho_*(\cdot, \cdot)$ is a metric on Σ and (Σ, ρ) is compact and totally disconnected. Moreover, if $\#(S(w)) \geq 2$ for any $w \in T$, then (Σ, ρ) is perfect.*

By the above proposition, if $\#(S(w)) \geq 2$ for any $w \in T$, then Σ is (homeomorphic to) the Cantor set.

4 Partition

In this section, we formulate the notion of a partition introduced in Section 1 exactly. A partition is a map from a tree to the collection of nonempty compact subsets of a compact metrizable topological space with no isolated point and it is required to preserve natural hierarchical structure of the tree. Consequently, a partition induces a surjective map from the Cantor set, i.e. the collection of ends of the tree, to the compact metrizable space.

Throughout this section, $\mathcal{T} = (T, A, \phi)$ is a locally finite tree with a reference point ϕ.

Definition 4.1 (Partition)**.** Let $(X, \mathcal{O})$ be a compact metrizable topological space having no isolated point, where $\mathcal{O}$ is the collection of open sets, and let $\mathcal{C}(X, \mathcal{O})$ be the collection of nonempty compact subsets of X. If no confusion can occur, we write $\mathcal{C}(X)$ in place of $\mathcal{C}(X, \mathcal{O})$.
(1) A map $K : T \to \mathcal{C}(X, \mathcal{O})$, where we customarily denote $K(w)$ by K_w for simplicity, is called a partition of X parametrized by $(T, \mathcal{A}, \phi)$ if and only if it satisfies the following conditions (P1) and (P2), which correspond to (1.3) and (1.4) respectively.
(P1) $K_\phi = X$ and for any $w \in T$,

$$K_w = \bigcup_{v \in S(w)} K_v.$$

(P2) For any $\omega \in \Sigma$, $\cap_{m \geq 0} K_{[\omega]_m}$ is a single point.
(2) Let $K : T \to \mathcal{C}(X, \mathcal{O})$ be a partition of X parametrized by $(T, \mathcal{A}, \phi)$. Define

O_w and B_w for $w \in T$ by

$$O_w = K_w \backslash \left(\bigcup_{v \in (T)_{|w|} \backslash \{w\}} K_v \right),$$

$$B_w = K_w \cap \left(\bigcup_{v \in (T)_{|w|} \backslash \{w\}} K_v \right).$$

If $O_w \neq \emptyset$ for any $w \in T$, then the partition K is called minimal.
(3) Let $K : T \to \mathcal{C}(X, \mathcal{O})$ be a partition of X. Then $(w(1), \ldots, w(m)) \in \cup_{k \geq 0} T^k$ is called a chain of K (or a chain for short if no confusion can occur) if and only if $K_{w(i)} \cap K_{w(i+1)} \neq \emptyset$ for any $i = 1, \ldots, m-1$. A chain $(w(1), \ldots, w(m))$ of K is called a chain of K in $\Lambda \subseteq T$ if $w(i) \in \Lambda$ for any $i = 1, \ldots, m$. For subsets $A, B \subseteq X$, A chain $(w(1), \ldots, w(m))$ of K is called a chain of K between A and B if and only if $A \cap K_{w(1)} \neq \emptyset$ and $B \cap K_{w(m)} \neq \emptyset$. We use $\mathcal{CH}_K(A, B)$ to denote the collection of chains of K between A and B. Moreover, we denote the collections of chains of K in Λ between A and B by $\mathcal{CH}_K^\Lambda(A, B)$.

As is shown in Theorem 4.7, a partition can be modified so as to be minimal by restricting it to a suitable subtree.

The next lemma is an assortment of direct consequences from the definition of the partition.

Lemma 4.2. *Let $K : T \to \mathcal{C}(X, \mathcal{O})$ be a partition of X parametrized by (T, A, ϕ).*
(1) For any $w \in T$, O_w is an open set. $O_v \subseteq O_w$ for any $v \in S(w)$.
(2) $O_w \cap O_v = \emptyset$ if $w, v \in T$ and $\Sigma_w \cap \Sigma_v = \emptyset$.
(3) If $\Sigma_w \cap \Sigma_v = \emptyset$, then $K_w \cap K_v = B_w \cap B_v$.

Proof. (1) Note that by (P1), $X = \cup_{w \in (T)_m} K_w$. Hence

$$O_w = K_w \backslash (\cup_{v \in (T)_{|w|} \backslash \{w\}} K_v) = X \backslash (\cup_{v \in (T)_{|w|} \backslash \{w\}} K_v).$$

The rest of the statement is immediate from the property (P2).
(2) By Lemma 3.7, if $u = w \wedge v$, then $|u| < |w|$ and $|u| < |v|$. Let $w' = [w]_{|u|+1}$ and let $v' = [v]_{|u|+1}$. Then $w', v' \in S(u)$ and $w' \neq v'$. Since $O_{w'} \subseteq K_{w'} \backslash K_{v'}$, it follows that $O_{w'} \cap O_{v'} = \emptyset$. Using (1), we see $O_w \cap O_v = \emptyset$.
(3) This follows immediately by (1). □

The condition (P2) provides a natural map from the ends of the tree Σ to the space X.

Proposition 4.3. *Let $K : T \to \mathcal{C}(X, \mathcal{O})$ be a partition of X parametrized by $(T, \mathcal{A}, \phi)$.*
(1) For $\omega \in \Sigma$, define $\sigma(\omega)$ as the single point $\cap_{m \geq 0} K_{[\omega]_m}$. Then $\sigma : \Sigma \to X$ is continuous and surjective. Moreover. $\sigma(\Sigma_w) = K_w$ for any $w \in T$.
(2) The partition $K : T \to \mathcal{C}(X, \mathcal{O})$ is minimal if and only if K_w is the closure of O_w for any $w \in T$. Moreover, if $K : T \to \mathcal{C}(X, \mathcal{O})$ is minimal then O_w coincides with the interior of K_w.

Proof. (1) Note that $K_w = \cup_{v \in S(w)} K_v$. Hence if $x \in K_w$, then there exists $v \in S(w)$ such that $x \in K_v$. Using this fact inductively, we see that, for any $x \in X$, there exists $\omega \in \Sigma$ such that $x \in K_{[\omega]_m}$ for any $m \geq 0$. Since $x \in \cap_{m \geq 0} K_{[\omega]_m}$, (P2) shows that $\sigma(\omega) = x$. Hence ω is surjective. At the same time, it follows that $\sigma(\Sigma_w) = K_w$. Let U be an open set in X. For any $\omega \in \sigma^{-1}(U)$, $K_{[\omega]_m} \subseteq U$ for sufficiently large m. Then $\Sigma_{[\omega]_m} \subseteq \sigma^{-1}(U)$. This shows that $\sigma^{-1}(U)$ is an open set and hence σ is continuous.
(2) Let $\overline{O}_w$ be the closure of O_w. If $K_w = \overline{O}_w$ for any $w \in T$, then $O_w \neq \emptyset$ for any $w \in T$ and hence $K : T \to \mathcal{C}(X, \mathcal{O})$ is minimal. Conversely, assume that $K : T \to \mathcal{C}(X, \mathcal{O})$ is minimal. By Lemma 4.2, $\overline{O}_{[\omega]_m} \supseteq \overline{O}_{[\omega]_{m+1}}$ for any $\omega \in \Sigma$ and any $m \geq 0$. Hence $\{\sigma(\omega)\} = \cap_{m \geq 0} K_{[\omega]_m} = \cap_{m \geq 0} \overline{O}_{[\omega]_m} \subseteq \overline{O}_{[\omega]_n}$ for any $n \geq 0$. This yields that $\sigma(\Sigma_w) \subseteq \overline{O}_w$. Since $\sigma(\Sigma_w) = K_w$, this implies $\overline{O}_w = K_w$.
Now if K is minimal, since O_w is open by Lemma 4.2-(1), it follows that O_w is the interior of K_w. $\square$

Definition 4.4. A partition $K : T \to \mathcal{C}(X, \mathcal{O})$ parametrized by a tree $(T, \mathcal{A}, \phi)$ is called strongly finite if and only if

$$\sup_{x \in X} \#(\sigma^{-1}(x)) < +\infty,$$

where $\sigma : \Sigma \to X$ is the map defined in Proposition 4.3-(1).

Example 4.5. Let (Y, d) be a complete metric space and let $\{F_1, \ldots, F_N\}$ be collection of contractions from (Y, d) to itself, i.e. $F_i : Y \to Y$ and

$$\sup_{x \neq y \in Y} \frac{d(F_i(x), F_i(y))}{d(x, y)} < 1$$

for any $i = 1, \ldots, N$. Then it is well-known that there exists a unique nonempty compact set X such that

$$X = \bigcup_{i=1,\ldots,N} F_i(X).$$

See [15, Section 1.1] for a proof of this fact for example. X is called the self-similar set associated with $\{F_1, \ldots, F_N\}$. Let $(T^{(N)}, \mathcal{A}^{(N)}, \phi)$ be the tree defined in Example 3.3. For any $i_1 \ldots i_m \in T$, set $F_{i_1 \ldots i_m} = F_{i_1} \circ \ldots \circ F_{i_m}$ and define $K_w = F_w(X)$. Then $K : T^{(N)} \to \mathcal{C}(X)$ is a partition of K parametrized by $(T^{(N)}, \mathcal{A}^{(N)}, \phi)$. See [15, Section 1.2]. The associated map from $\Sigma = \{1, \ldots, N\}^{\mathbb{N}}$ to K is sometimes called the coding map. To determine if K is minimal or not is known to be rather delicate issue. See [15, Theorem 1.3.8] for example.

Example 4.6 (the Sierpinski carpet, Figure 3)**.** This is the special case of self-similar sets presented in the last example. Let $p_1 = (0, 0)$, $p_2 = (\frac{1}{2}, 0)$,

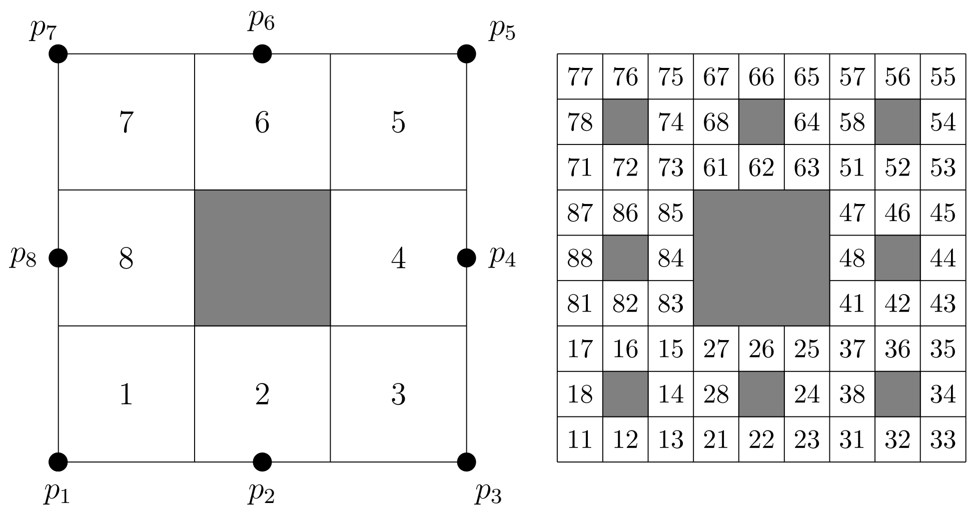


Figure 3: Partition: the Sierpinski carpet

$p_3 = (1, 0)$, $p_4 = (1, \frac{1}{2})$, $p_5 = (1, 1)$, $p_6 = (\frac{1}{2}, 1)$, $p_7 = (0, 1)$ and $p_8 = (0, \frac{1}{2})$. Set $F_i : [0, 1]^2 \to [0, 1]^2$ for $i = 1, \ldots, 8$ by

$$F_i(x) = \frac{1}{3}(x - p_i) + p_i$$

for any $x \in [0, 1]^2$. The unique nonempty compact set X satisfying

$$X = \bigcup_{i=1}^{8} F_i(X)$$

is called the Sierpinski carpet. In this case, the associated tree is $(T^{(8)}, \mathcal{A}^{(8)}, \phi)$. Define $K : T^{(8)} \to \mathcal{C}(X, \mathcal{O})$ by

$$K_{i_1 \ldots i_m} = F_{i_1 \ldots i_m}(X)$$

as in Example 4.5. Then K is a partition of X parametrized by the tree $(T^{(8)}, \mathcal{A}^{(8)}, \phi)$. In Figure 3, K_{ij} is represented by ij for simplicity.

Removing unnecessary vertices of the tree, we can always modify the original partition and obtain a minimal one.

Theorem 4.7. *Let $K : T \to \mathcal{C}(X, \mathcal{O})$ be a partition of X parametrized by $(T, \mathcal{A}, \phi)$. There exist $T' \subseteq T$ and $K' : T' \to \mathcal{C}(X, \mathcal{O})$ such that $(T', \mathcal{A}|_{T' \times T'})$ is a tree, $\phi \in T'$, $K'_w \subseteq K_w$ for any $w \in T'$ and K' is a minimal partition of X parametrized by $(T', \mathcal{A}', \phi)$.*

Proof. We define a sequence $\{T^{(m)}\}_{m \geq 0}$ of subsets of T and $\{K_w^{(m)}\}_{w \in T^{(m)}}$ inductively as follows. First let $T^{(0)} = T$ and $K_w^{(0)} = K_w$ for any $w \in T^{(0)}$.

Suppose we have defined $T^{(m)}$. Define

$$Q^{(m)} = \left\{ w \middle| w \in T^{(m)}, K_w \subseteq \bigcup_{v \in (T)_{|w|} \cap T^{(m)}, v \neq w} K_v \right\}.$$

If $Q^{(m)} = \emptyset$, then set $T^{(m+1)} = T^{(m)}$ and $K_w^{(m+1)} = K_w^{(m)}$ for any $w \in T^{(m+1)}$. Otherwise choose $w^{(m)} \in Q^{(m)}$ so that $|w^{(m)}|$ attains the minimum of $\{|v| : v \in Q^{(m)}\}$. Then define

$$T^{(m+1)} = T^{(m)} \backslash T_{w^{(m)}}$$

and

$$K_w^{(m+1)} = \begin{cases} \cup_{v \in T_w \cap (T)_{|w^{(m)}|} \cap T^{(m+1)}} K_v^{(m)} & \text{if } w^{(m)} \in T_w, \\ K_w^{(m)} & \text{otherwise.} \end{cases}$$

In this way, for any $m \geq 0$ and $w \in T^{(m)}$,

$$K_w^{(m)} = \bigcup_{v \in S(w) \cap T^{(m)}} K_v^{(m)}. \tag{4.4}$$

Note that $Q^{(m+1)} \subset Q^{(m)} \backslash \{w^{(m)}\}$. Since $(T)_n$ is a finite set for any $n \geq 0$, it follows that $(T)_n \cap Q^{(m)} = \emptyset$ and $(T^{(m)})_n$ stays the same for sufficiently large m. Hence $|w^{(m)}| \to \infty$ as $m \to \infty$ and $(T)_n \cap T^{(m)}$ does not depend on m for sufficiently large m. Therefore, letting $T' = \cap_{m \geq 1} T^{(m)}$, we see that $(T', \mathcal{A}|_{T' \times T'})$ is a locally finite tree and $\phi \in T'$. Moreover, note that $K_w^{(m+1)} \subseteq K_w^{(m)}$ for any $w \in T'$. Hence if we set

$$K'_w = \bigcap_{m \geq 0} K_w^{(m)}$$

for any $w \in T'$, then K'_w is nonempty. By (4.4), it follows that

$$K'_w = \bigcup_{v \in T' \cap S(w)} K'_v$$

for any $w \in T'$. Thus the map $K' : T' \to \mathcal{C}(X, \mathcal{O})$ given by $K'(w) = K'_w$ is a minimal partition of X parametrized by $(T', \mathcal{A}|_{T' \times T'}, \phi)$. □

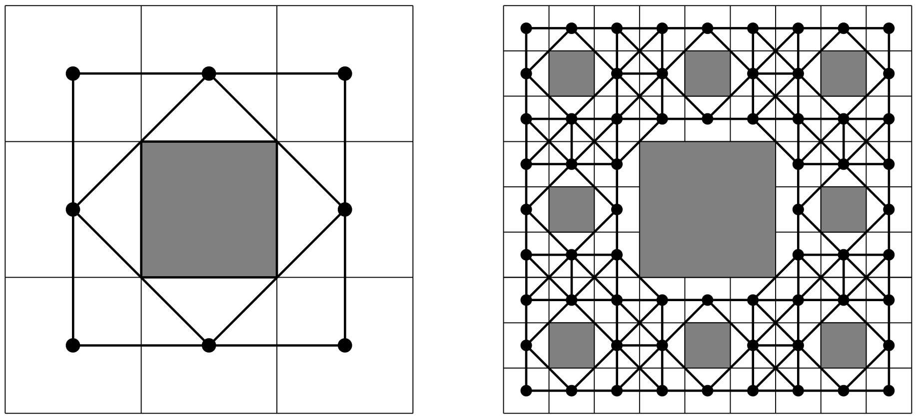
A partition $K : T \to \mathcal{C}(X, \mathcal{O})$ induces natural graph structure on T. In the rest of this section, we show that T can be regarded as the hyperbolic filling of X if the induced graph structure is hyperbolic. See [8], for example, about the notion of hyperbolic fillings.

Definition 4.8. Let $K : T \to \mathcal{C}(X, \mathcal{O})$ be a partition. Then define

$$E_m^h = \{(w, v) | w, v \in (T)_m, K_w \cap K_v \neq \emptyset\}$$

and

$$E^h = \bigcup_{m \geq 0} E_m^h.$$



$((T)_1, E_1^h)$ $\qquad\qquad$ $((T)_2, E_2^h)$

●'s are vertices and thick lines are edges.

Figure 4: Horizontal edges: the Sierpinski carpet

An element $(u, v) \in E^h$ is called a horizontal edge associated with $(T, \mathcal{A}, \phi)$ and $K : T \to \mathcal{C}(X, \mathcal{O})$. The symbol "h" in the notation E_m^h and E^h represents the word "horizontal". On the contrary, an element $(w, v) \in \mathcal{A}$ is called a vertical edge. Moreover we define

$$\mathcal{B}(w, v) = \begin{cases} 1 & \text{if } \mathcal{A}(w, v) = 1 \text{ or } (w, v) \in E^h, \\ 0 & \text{otherwise.} \end{cases}$$

The graph $(T, \mathcal{B})$ is called the resolution of X associated with the partition $K : T \to \mathcal{C}(X, \mathcal{O})$. We use $d_{(T,\mathcal{B})}(\cdot, \cdot)$ to denote the shortest path metric, i.e.

$$\begin{aligned} d_{(T,\mathcal{B})}(w, v) = \min\{n | &\text{there exists } (w(1), \ldots, w(n+1)) \in (\mathcal{B})^n \\ &\mathcal{B}(w(i), w(i+1)) = 1 \text{ for any } i = 1, \ldots, n\} \end{aligned}$$

Remark. The horizontal graph $((T)_m, E_m^h)$ is not necessarily connected. More precisely, $((T)_m, E_m^h)$ is connected for any $m \geq 0$ if and only if X is connected. Note that if X is homeomorphic to the Cantor set, then $E_m^h = \emptyset$ for any $m \geq 0$.

In Figure 4, we present $((T)_1, E_1^h)$ and $((T)_2, E_2^h)$ for the Sierpinski carpet introduced in Example 4.6.

We will show in Lemma 7.1 that if $(\phi, w(1), w(2), \ldots)$ is an infinite geodesic ray in $(T, \mathcal{B})$ with respect to the metric $d_{(T,\mathcal{B})}$ starting from ϕ, then it coincides with $(\phi, [\omega]_1, [\omega]_2, \ldots)$ for some $\omega \in \Sigma$. In other words, the collection of geodesic rays of $(T, \mathcal{B})$ starting from ϕ can be identified with Σ. The following proposition will be proven in Section 7.

Proposition 4.9. *Let* $\omega, \tau \in \Sigma$. *If* $\sup_{n \geq 1} d_{(T,\mathcal{B})}([\omega]_n, [\tau]_n) < +\infty$, *then* $\sigma(\omega) = \sigma(\tau)$.

By this proposition, whether the resolution $(T, \mathcal{B})$ is hyperbolic or not, X can be identified with the quotient space of the geodesic rays under the equivalence relation $\sim$ defined as $\omega \sim \tau$ if and only if $\sup_{n \geq 1} d_{(T,\mathcal{B})}([\omega]_n, [\tau]_n) < +\infty$. In case of a hyperbolic graph, such a quotient space has been called the hyperbolic boundary of the graph in the framework of Gromov theory of hyperbolic metric spaces. We will give detailed accounts on these points later in Section 7.

In [12], Elek has constructed a hyperbolic graph whose hyperbolic boundary is homeomorphic to a given compact subset of $\mathbb{R}^N$. From our point of view, what he has done is to construct a partition of the compact metric space using dyadic cubes as is seen in the next example. However, the resolution $(T, \mathcal{B})$ associated with the partition is slightly different from the original graph constructed by Elek. See the details below.

Example 4.10. Let X be a nonempty compact subset of $\mathbb{R}^N$. For simplicity, we assume that $X \subseteq [0, 1]^N$. We are going to construct a partition of X using the dyadic cubes. Let $S_m = \{(m, i_1, \ldots, i_N) | (i_1, \ldots, i_N) \in \{0, 1, \ldots, 2^m - 1\}^N\}$ and define

$$C(w) = \prod_{j=1}^{N} \Big[\frac{i_j}{2^m}, \frac{i_j + 1}{2^m}\Big]$$

for $w = (m, i_1, \ldots, i_N) \in S_m$. The collection $\{C(w) | w \in \cup_{m \geq 0} S_m\}$ is called the dyadic cubes. (See [11] for example.) Define

$$K_w = X \cap C(w)$$

for $w \in \cup_{m \geq 0} S_m$,

$$(T)_m = \{w | w \in S_m, K_w \neq \emptyset\}$$

for $m \geq 0$ and $T = \cup_{m \geq 0} (T)_m$. Moreover, we define $(w, v) \in \mathcal{A}$ for $(w, v) \in T \times T$ if there exist $m \geq 0$ such that $(w, v) \in (S_m \times S_{m+1}) \cup (S_{m+1} \times S_m)$ and $C(w) \supseteq C(v)$ or $C(w) \subseteq C(v)$. Then $(T, \mathcal{A}, \phi)$ is a tree with a reference point ϕ, where $\phi = (0, 0, \ldots, 0) \in (T)_0$ and the map from $w \in T$ to $K_w \in \mathcal{C}(X, \mathcal{O})$ is a partition of X parametrized by $(T, \mathcal{A}, \phi)$. The hyperbolic graph constructed by Elek is a slight modification of the resolution $(T, \mathcal{B})$. In fact, the vertical edges are the same but Elek's graph has more horizontal edges. Precisely set

$$\widetilde{E}^h_m = \{(w, v) | w, v \in (T)_m, C(w) \cap C(v) \neq \emptyset\}.$$

and define $\widetilde{\mathcal{B}} = \mathcal{A} \cup (\cup_{m \geq 1} \widetilde{E}^h_m)$. Then Elek's graph coincides with $(T, \widetilde{\mathcal{B}})$. Note that in $(T, \mathcal{B})$, the horizontal edges are

$$E^h_m = \{(w, v) | w, v \in (T)_m, K_w \cap K_v \neq \emptyset\}.$$

So, $\widetilde{E}^h_m \supseteq E^h_m$ and hence $\widetilde{\mathcal{B}} \supseteq \mathcal{B}$ in general. In Example 7.18, we are going to show that $(T, \mathcal{B})$ is hyperbolic as a corollary of our general framework.

5 Weight function and associated "visual pre-metric"

Throughout this section, $(T, \mathcal{A}, \phi)$ is a locally finite tree with a reference point ϕ, $(X, \mathcal{O})$ is a compact metrizable topological space with no isolated point and $K : T \to \mathcal{C}(X, \mathcal{O})$ is a partition of X parametrized by $(T, \mathcal{A}, \phi)$.

In this section, we introduce the notion of a weight function, which assigns each vertex of the tree T a "size" or "weight". Then, we will introduce a kind of "balls" and a "distance" of the compact metric space X associated with the weight function.

Definition 5.1 (Weight function)**.** A function $g : T \to (0, 1]$ is called a weight function if and only if it satisfies the following conditions (G1), (G2) and (G3):
(G1) $g(\phi) = 1$
(G2) For any $w \in T$, $g(\pi(w)) \geq g(w)$
(G3) $\lim_{m\to\infty} \sup_{w\in(T)_m} g(w) = 0$.
We denote the collection of all the weight functions by $\mathcal{G}(T)$. Let g be a weight function. We define

$$\Lambda^g_s = \{w | w \in T, g(\pi(w)) > s \geq g(w)\}$$

for any $s \in (0, 1]$. $\{\Lambda^g_s\}_{s\in(0,1]}$ is called the scale associated with g. For $s > 1$, we define $\Lambda^g_s = \{\phi\}$.

Remark. To be exact, one should use $\mathcal{G}(T, A, \phi)$ rather than $\mathcal{G}(T)$ as the notation for the collection of all the weight functions because the notion of weight function apparently depends not only on the set T but also the structure of T as a tree. We use, however, $\mathcal{G}(T)$ for simplicity as long as no confusion may occur.

Remark. In the case of the partitions associated with a self-similar set appearing in Example 4.5, the counterpart of weight functions was called gauge functions in [16]. Also $\{\Lambda^g_s\}_{0<s\leq 1}$ was called the scale associated with the gauge function g.

Given a weight function g, we consider $g(w)$ as a virtual "size" or "diameter" of Σ_w for each $w \in T$. The set Λ^g_s is the collection of subsets Σ_w's whose sizes are approximately s.

Proposition 5.2. *Suppose that* $g : T \to (0, 1]$ *satisfies* (G1) *and* (G2)*.* g *is a weight function if and only if*

$$\lim_{m\to\infty} g([\omega]_m) = 0 \tag{5.5}$$

for any $\omega \in \Sigma$*.*

Proof. If g is a weight function, i.e. (G3) holds, then (5.5) is immediate.

Suppose that (G3) does not hold, i.e. there exists $\epsilon > 0$ such that

$$\sup_{w\in(T)_m} g(w) > \epsilon \tag{5.6}$$

for any $m \geq 0$. Define $Z = \{w|w \in T, g(w) > \epsilon\}$ and $Z_m = (T)_m \cap Z$. By (5.6), $Z_m \neq \emptyset$ for any $m \geq 0$. Since $\pi(w) \in Z$ for any $w \in Z$, if $Z_{m,n} = \pi^{n-m}(Z_n)$ for any $n \geq m$, where π^k is the k-th iteration of π, then $Z_{m,n} \neq \emptyset$ and $Z_{m,n} \supseteq Z_{m,n+1}$ for any $n \geq m$. Set $Z_m^* = \cap_{n \geq m} Z_{m,n}$. Since $(T)_m$ is a finite set and so is $Z_{m,n}$, we see that $Z_m^* \neq \emptyset$ and $\pi(Z_{m+1}^*) = Z_m^*$ for any $m \geq 0$. Note that $Z_0^* = \{\phi\}$. Inductively, we may construct a sequence $(\phi, w(1), w(2), \ldots)$ satisfying $\pi(w(m+1)) = w(m)$ and $w(m) \in Z_m^*$ for any $m \geq 0$. Set $\omega = (\phi, w(1), w(2), \ldots)$. Then $\omega \in \Sigma$ and $g([\omega]_m) \geq \epsilon$ for any $m \geq 0$. This contradicts (5.5). $\square$

Proposition 5.3. *Let $g : T \to (0,1]$ be a weight function and let $s \in (0,1]$. Then*

$$\bigcup_{w \in \Lambda_s^g} \Sigma_w = \Sigma \tag{5.7}$$

and if $w, v \in \Lambda_s^g$ and $w \neq v$, then

$$\Sigma_w \cap \Sigma_v = \emptyset.$$

Proof. For any $\omega = (w_0, w_1, \ldots) \in \Sigma$, $\{g(w_i)\}_{i=0,1,\ldots}$ is monotonically non-increasing sequence converging to 0 as $i \to \infty$. Hence there exists a unique $m \geq 0$ such that $g(w_{m-1}) > s \geq g(w_m)$. Therefore, there exists a unique $m \geq 0$ such that $[\omega]_m \in \Lambda_s^g$. Now (5.7) is immediate. Assume $w, v \in \Lambda_s^g$ and $\Sigma_v \cap \Sigma_w \neq \emptyset$. Choose $\omega = (w_0, w_1, \ldots) \in \Sigma_v \cap \Sigma_w$. Then there exist $m, n \geq 0$ such that $[\omega]_m = w_m = w$ and $[\omega]_n = w_n = v$. By the above fact, we have $m = n$ and hence $w = v$. $\square$

By means of the partition $K : T \to \mathcal{C}(X, \mathcal{O})$, one can define weight functions naturally associated with metrics and measures on the compact metric space X as follows.

Notation. Let d be a metric on X. We define the diameter of a subset $A \subseteq X$ with respect to d, $\mathrm{diam}(A,d)$ by $\mathrm{diam}(A,d) = \sup\{d(x,y)|x, y \in A\}$. Moreover, for $x \in X$ and $r > 0$, we set $B_d(x,r) = \{y|y \in X, d(x,y) < r\}$.

Definition 5.4. (1) Define

$$\mathcal{D}(X,\mathcal{O}) = \{d|d \text{ is a metric on } X \text{ inducing the topology } \mathcal{O} \text{ and } \mathrm{diam}(X,d) = 1\}$$

For $d \in \mathcal{D}(X,\mathcal{O})$, define $g_d : T \to (0,1]$ by $g_d(w) = \mathrm{diam}(K_w, d)$ for any $w \in T$.
(2) Define

$$\mathcal{M}_P(X,\mathcal{O}) = \{\mu|\mu \text{ is a Radon probability measure on } (X,\mathcal{O}) \text{ satisfying } \mu(\{x\}) = 0 \text{ for any } x \in X \text{ and } \mu(K_w) > 0 \text{ for any } w \in T\}$$

For $\mu \in \mathcal{M}_P(X,\mathcal{O})$, define $g_\mu : T \to (0,1]$ by $g_\mu(w) = \mu(K_w)$ for any $w \in T$.

The condition $\mathrm{diam}(X, d) = 1$ in the definition of $\mathcal{D}(X, \mathcal{O})$ is only for the purpose of normalization. Note that since $(X, \mathcal{O})$ is compact, if a metric d on X induces the topology $\mathcal{O}$, then $\mathrm{diam}(X, d) < +\infty$.

Proposition 5.5. (1) *For any $d \in \mathcal{D}(X, \mathcal{O})$, g_d is a weight function.*
(2) *For any $\mu \in \mathcal{M}_P(X, \mathcal{O})$, g_μ is a weight function.*

Proof. (1) The properties (G1) and (G2) are immediate from the definition of g_d. Suppose that there exists $\omega \in \Sigma$ such that

$$\lim_{m\to\infty} g_d([\omega]_m) > 0 \tag{5.8}$$

Let ϵ be the above limit. Since $g_d([\omega]_m) = \mathrm{diam}(K_{[\omega]_m}, d) > \epsilon$, there exist $x_m, y_m \in K_{[\omega]_m}$ such that $d(x_m, y_m) \geq \epsilon$. Note that $K_{[\omega]_m} \supseteq K_{[\omega]_{m+1}}$ and hence $x_n, y_n \in K_{[\omega]_m}$ if $n \geq m$. Since X is compact, there exist subsequences $\{x_{n_i}\}_{i\geq 1}$, $\{y_{n_i}\}_{i\geq 1}$ converging to x and y as $i \to \infty$ respectively. It follows that $x, y \in \cap_{m\geq 0} K_{[\omega]_m}$ and $d(x, y) \geq \epsilon > 0$. This contradicts (P2). Thus we have shown (5.5). By Proposition 5.2, g_d is a weight function.
(2) As in the case of metrics, (G1) and (G2) are immediate. Let $\omega \in \Sigma$. Then $\cap_{m\geq 0} K_{[\omega]_m} = \{\sigma(\omega)\}$. Therefore, $g_\mu([\omega]_m) = \mu(K_{[\omega]_m}) \to 0$ as $m \to \infty$. Hence we verify (5.5). Thus by Proposition 5.2, g_μ is a weight function. □

The weight function g_d and g_μ are called the weight functions associated with d and μ respectively. Although the maps $d \to g_d$ and $\mu \to g_\mu$ are not injective at all, we sometimes abuse notations and use d and μ to denote g_d and g_μ respectively.

Through a partition we introduce the notion of "balls" of a compact metric space associated with a weight function.

Definition 5.6. Let $g : T \to (0, 1]$ be a weight function.
(1) For $s \in (0, 1], w \in \Lambda_s^g, M \geq 0$ and $x \in X$, we define

$$\begin{aligned}\Lambda_{s,M}^g(w) = \{v | v \in \Lambda_s^g, &\text{there exists a chain } (w(1), \ldots, w(k)) \text{ of } K \text{ in } \Lambda_s^g \\ &\text{such that } w(1) = w,\ w(k) = v \text{ and } k \leq M + 1\}\end{aligned}$$

and

$$\Lambda_{s,M}^g(x) = \bigcup_{w\in\Lambda_s^g \text{ and } x\in K_w} \Lambda_{s,M}^g(w).$$

For $x \in X$, $s \in (0, 1]$ and $M \geq 0$, define

$$U_M^g(x, s) = \bigcup_{w\in\Lambda_{s,M}^g(x)} K_w.$$

We let $U_M^g(x, s) = X$ if $s \geq 1$.

In Figure 5, we show examples of $U_M^g(x, s)$ for the Sierpinski carpet introduced in Example 4.6.

The family $\{U_M^g(x,s)\}_{s>0}$ is a fundamental system of neighborhood of $x \in X$ as is shown in Proposition 5.7.

Note that

$$\Lambda_{s,0}^g(w) = \{w\} \quad \text{and} \quad \Lambda_{s,1}^g(w) = \{v | v \in \Lambda_s^g, K_v \cap K_w \neq \emptyset\}$$

for any $w \in \Lambda_s^g$ and

$$\Lambda_{s,0}^g(x) = \{w | w \in \Lambda_s^g, x \in K_w\} \quad \text{and} \quad U_0^g(x,s) = \bigcup_{w: w \in \Lambda_s^g, x \in K_w} K_w$$

for any $x \in X$. Moreover,

$$U_M^g(x,s) = \{y | y \in X, \text{there exists } (w(1), \ldots, w(M+1)) \in \mathcal{CH}_K^{\Lambda_s^g}(x,y).\}$$

Proposition 5.7. *Let K be a partition of X parametrized by $(T, \mathcal{A}, \phi)$ and let $g : T \to (0,1]$ be a weight function. For any $s \in (0,1]$ and any $x \in X$, $U_0^g(x,s)$ is a neighborhood of x. Furthermore, $\{U_M^g(x,s)\}_{s \in (0,1]}$ is a fundamental system of neighborhood of x for any $x \in X$.*

Proof. Let d be a metric on X giving the original topology of $(X, \mathcal{O})$. Assume that for any $r > 0$, there exists $y \in B_d(x,r)$ such that $y \notin U_0^g(x,s)$. Then there exists a sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that $x_n \to x$ and $x_n \notin U_0^g(x,s)$ for any $n \geq 1$. Since Λ_s^g is a finite set, there exists $w \in \Lambda_s$ which includes infinite members of $\{x_n\}_{n \geq 1}$. By the closedness of K_w, it follows that $x \in K_w$ and $x_n \in K_w \subseteq U_0^g(x,s)$. This contradiction shows that $U_0^g(x,s)$ contains $B_d(x,r)$ for some $r > 0$.
Next note $\min_{w \in \Lambda_s^g} |w| \to \infty$ as $s \downarrow 0$. This along with that fact that g_d is a weight function implies that $\max_{w \in \Lambda_s^g} \operatorname{diam}(K_w, d) \to 0$ as $s \downarrow 0$. Set $\rho_s = \max_{w \in \Lambda_s^g} \operatorname{diam}(K_w, d)$. Then $\operatorname{diam}(U_M^g(x,s), d) \leq (M+1)\rho_s \to 0$ as $s \downarrow 0$. This implies that $\cap_{s \in (0,1]} U_M^g(x,s) = \{x\}$. Thus $\{U_M^g(x,s)\}_{s \in (0,1]}$ is a fundamental system of neighborhoods of x. $\square$

We regard $U_M^g(x,s)$ as a virtual "ball' of radius s and center x. In fact, there exists a kind of "pre-metric" $\delta_M^g : X \times X \to [0, \infty)$ such that $\delta_M^g(x,y) > 0$ if and only if $x \neq y$, $\delta_M^g(x,y) = \delta_M^g(y,x)$ and

$$U_M^g(x,s) = \{y | \delta_M^g(x,y) \leq s\}. \tag{5.9}$$

As is seen in the next section, however, the pre-metric δ_M^g may not satisfy the triangle inequality in general.

Definition 5.8. Let $M \geq 0$. Define $\delta_M^g(x,y)$ for $x, y \in X$ by

$$\delta_M^g(x,y) = \inf\{s | s \in (0,1], y \in U_M^g(x,s)\}.$$

Remark. For any $g \in \mathcal{G}(T)$, $M \geq 0$ and $x \in X$, it follows that $\Lambda_{s,M}^g(x) = \{\phi\}$ and hence $U_M^g(x,1) = X$. So, $\delta_M^g(x,y) \leq 1$ for any $x, y \in X$.

The pre-metric δ^g_M can be thought of as a counterpart of the "visual metric" in the sense of Bonk-Meyer in [7] and the "visual pre-metric" in the framework of Gromov hyperbolic metric spaces, whose exposition can be found in [10] and [22]. In fact, if certain rearrangement of the resolution $(X, \mathcal{B})$ is hyperbolic associated with the weight function, then δ^g_M is bi-Lipschitz equivalent to a visual pre-metric in the sense of Gromov. See Theorem 7.12 for details.

Proposition 5.9. *For any $M \geq 0$ and $x, y \in X$,*

$$\delta^g_M(x, y) = \min\{s | s \in (0, 1], y \in U^g_M(x, s)\}. \tag{5.10}$$

In particular, (5.9) *holds for any $M \geq 0$ and $s \in (0, 1]$.*

Proof. The property (G3) implies that for any $t \in (0, 1]$, there exists $n \geq 0$ such that $\cup_{s \geq t} \Lambda^g_s \subseteq \cup^n_{m=0} (T)_m$. Hence $\{(w(1), \ldots, w(M+1)) | w(i) \in \cup_{s \geq t} \Lambda^g_s\}$ is finite. Let $s_* = \delta^g_M(x, y)$. Then there exist a sequence $\{s_m\}_{m \geq 1} \subseteq [s_*, 1]$ and $(w_m(1), \ldots, w_m(M+1)) \in (\Lambda^g_{s_m})^{M+1}$ such that $\lim_{m \to \infty} s_m = s_*$ and $(w_m(1), \ldots, w_m(M+1))$ is a chain between x and y for any $m \geq 1$. Since $\{(w(1), \ldots, w(M+1)) | w(i) \in \cup_{s \geq s_*} \Lambda^g_s\}$ is finite, there exists $(w_*(1), \ldots, w_*(M+1))$ such that $(w_*(1), \ldots, w_*(M+1)) = (w_m(1), \ldots, w_m(M+1))$ for infinitely many m. For such m, we have $g(\pi(w_*(i))) > s_m \geq g(w_*(i))$ for any $i = 1, \ldots, M+1$. This implies that $w_*(i) \in \Lambda^g_{s_*}$ for any $i = 1, \ldots, M+1$ and hence $y \in U^g_M(x, s_*)$. Thus we have shown (5.10). $\square$

6 Metrics adapted to weight function

In this section, we consider the first question mentioned in the introduction, which is when a weight function is naturally associated with a metric. Our answer will be given in Theorem 6.12.

As in the last section, $(T, \mathcal{A}, \phi)$ is a locally finite tree with a reference point ϕ, $(X, \mathcal{O})$ is a compact metrizable topological space with no isolated point and $K : T \to \mathcal{C}(X, \mathcal{O})$ is a partition throughout this section.

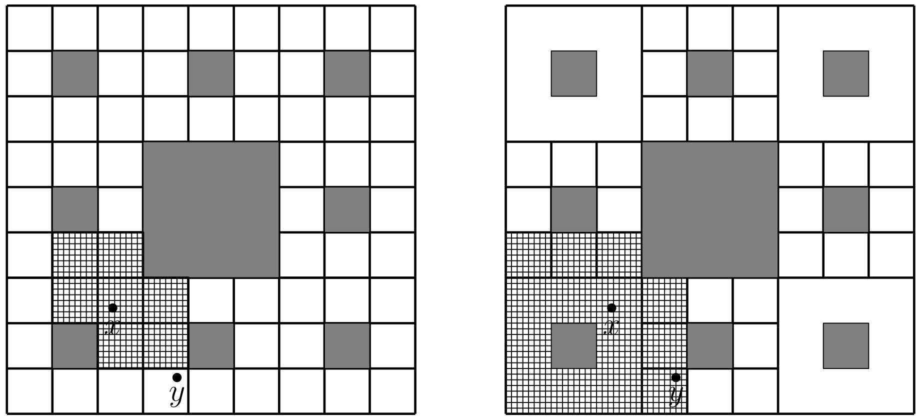
The purpose of the next definition is to clarify when the virtual balls $U^g_M(x, s)$ induced by a weight function g can be though of as real "balls" derived from a metric.

Definition 6.1. Let $M \geq 0$. A metric $d \in \mathcal{D}(X, \mathcal{O})$ is said to be M-adapted to g if and only if there exist $\alpha_1, \alpha_2 > 0$ such that

$$U^g_M(x, \alpha_1 r) \subseteq B_d(x, r) \subseteq U^g_M(x, \alpha_2 r)$$

for any $x \in X$ and any $r > 0$. d is said to be adapted to g if and only if d is M-adapted to g for some $M \geq 0$.

Now our question is the existence of a metric adapted to a given weight function. The number M really makes a difference in the above definition. Namely, in Example 11.9, we construct an example of a weight function to which no metric is 1-adapted but some metric is 2-adapted.



$\Lambda^{\eta}_{\frac{1}{9}}$, The meshed region is $U^{\eta}_1(x, \frac{1}{9})$ $\qquad$ $\Lambda^{g}_{\frac{1}{9}}$, The meshed region is $U^{g}_1(x, \frac{1}{9})$

$\delta^{\eta}_1(x, y) = \frac{1}{3}$ $\qquad$ $\delta^{g}_1(x, y) = \frac{1}{9}$

Figure 5: Visual pre-metrics: the Sierpinski carpet

By (5.9), a metric $d \in \mathcal{D}(X, \mathcal{O})$ is M-adapted to a weight function g if and only if there exist $c_1, c_2 > 0$ such that

$$c_1\delta^g_M(x, y) \leq d(x, y) \leq c_2\delta^g_M(x, y) \tag{6.1}$$

for any $x, y \in X$. By this equivalence, we may think of a metric adapted to a weight function as a "visual metric" associated with the weight function.

If a metric d is M-adapted to a weight function g, then we think of the virtual balls $U^g_M(x, s)$ as the real balls associated with the metric d.

Example 6.2 (Figure 5)**.** Let us consider the case of the Sierpinski carpet introduced in Example 4.6. In this case, the corresponding tree is $(T^{(8)}, \mathcal{A}^{(8)}, \phi)$. Write $T = T^{(8)}$. Define $\eta : T \to (0, 1]$ by $\eta(w) = \frac{1}{3^m}$ for any $w \in (T)_m$. Then η is a weight function and $\Lambda^{\eta}_s = (T)_m$ if and only if $\frac{1}{3^{m-1}} > s \geq \frac{1}{3^m}$. Let d_* be the (restriction of) Euclidean metric. Then d_* is 1-adapted to h. This can be deduced from the following two observations. First, if $w, v \in (T)_m$ and $K_w \cap K_v \neq \emptyset$, then $\sup_{x \in K_w, y \in K_v} d_*(x, y) \leq \frac{2\sqrt{2}}{3^m}$. Second, if $w, v \in (T)_m$ and $K_w \cap K_v = \emptyset$, then $\inf_{x \in K_w, y \in K_v} d_*(x, y) \geq \frac{1}{3^m}$. In fact, these two facts implies that

$$\frac{1}{3}\delta^{\eta}_1(x, y) \leq d_*(x, y) \leq 2\sqrt{2}\delta^{\eta}_1(x, y)$$

for any $x, y \in X$. Next we try another weight function g defined as

$$g(i_1 \dots i_m) = r_{i_1} \cdots r_{i_m}$$

for any $m \geq 0$ and $i_1, \dots, i_m \in \{1, \dots, 8\}$, where

$$r_i = \begin{cases} \frac{1}{9} & \text{if } i \text{ is odd,} \\ \frac{1}{3} & \text{if } i \text{ is even.} \end{cases}$$

Then $\Lambda^g_{\frac{1}{3}} = \{1, \dots, 8\}$ and

$$\Lambda^g_{\frac{1}{3}} = \{1, 3, 5, 7\} \cup \{i_1 i_2 | i_1 \in \{2, 4, 6, 8\}, i_2 \in \{1, \dots, 8\}\}.$$

In this case, the existence of an adapted metric is not immediate. However, by [16, Example 1.7.4], it follows that $\eta \underset{\mathrm{GE}}{\sim} g$. (See Definition 10.1 for the definition of $\underset{\mathrm{GE}}{\sim}$.) By Theorems 12.9 and 7.12, there exists a metric $\rho \in \mathcal{D}(X, \mathcal{O})$ that is adapted to g^α for some $\alpha > 0$. Furthermore, Theorem 13.6 shows that ρ is quasisymmetric to d_*.

There is another "pre-metric" associated with a weight function.

Definition 6.3. Let $M \geq 0$. Define $D^g_M(x, y)$ for $x, y \in X$ by

$$D^g_M(x, y) = \inf\Big\{\sum_{i=1}^k g(w(i))\Big| 1 \leq k \leq M+1, (w(1), \dots, w(k)) \in \mathcal{CH}_K(x, y)\Big\}$$

It is easy to see that $0 \leq D^g_M(x, y) \leq 1$, $D^g_M(x, y) = 0$ if and only if $x = y$ and $D^g_M(x, y) = D^g_M(y, x)$. In fact, the pre-metric D^g_M is equivalent to δ^g_M as follows.

Proposition 6.4. *For any $M \geq 0$ and $x, y \in X$,*

$$\delta^g_M(x, y) \leq D^g_M(x, y) \leq (M+1)\delta^g_M(x, y).$$

Proof. Set $s_* = \delta^g_M(x, y)$. Using Proposition 5.9, we see that there exists a chain $(w(1), \dots, w(M+1))$ between x and y such that $w(i) \in \Lambda^g_{s_*}$ for any $i = 1, \dots, M+1$. Then

$$D^g_M(x, y) \leq \sum_{i=1}^{M+1} g(w(i)) \leq (M+1)s_*$$

Next set $d_* = D^g_M(x, y)$. For any $\epsilon > 0$, there exists a chain $(w(1), \dots, w(M+1))$ between x and y such that $\sum_{i=1}^{M+1} g(w(i)) < d_* + \epsilon$. In particular, $g(w(i)) < d_* + \epsilon$ for any $i = 1, \dots, M+1$. Hence for any $i = 1, \dots, M+1$, there exists $w_*(i) \in \Lambda^g_{d_*+\epsilon}$ such that $K_{w(i)} \subseteq K_{w_*(i)}$. Since $(w_*(1), \dots, w_*(M+1))$ is a chain between x and y, it follows that $\delta^g_M(x, y) \leq d_* + \epsilon$. Thus we have shown $\delta^g_M(x, y) \leq D^g_M(x, y)$. □

Combining the above proposition with (6.1), we see that d is M-adapted to g if and only if there exist $C_1, C_2 > 0$ such that

$$C_1 D^g_M(x, y) \leq d(x, y) \leq C_2 D^g_M(x, y) \tag{6.2}$$

for any $x, y \in X$.

Next we present another condition which is equivalent to a metric being adapted.

Theorem 6.5. *Let $g : T \to (0,1]$ be a weight function and let $M \geq 0$. If $d \in \mathcal{D}(X, \mathcal{O})$, then d is M-adapted to g if and only if the following conditions* (ADa) *and* $(\mathrm{ADb})_{\mathrm{M}}$ *hold:*
(ADa) *There exists $c > 0$ such that $\mathrm{diam}(K_w, d) \leq cg(w)$ for any $w \in T$.*
$(\mathrm{ADb})_{\mathrm{M}}$ *For any $x, y \in X$, there exists $(w(1), \ldots, w(k)) \in \mathcal{CH}_K(x,y)$ such that $1 \leq k \leq M+1$ and*

$$Cd(x,y) \geq \max_{i=1,\ldots,k} g(w(i)),$$

where $C > 0$ is independent of x and y.

Remark. In [7, Proposition 8.4], one find an analogous result in the case of partitions associated with expanding Thurston maps. The condition (ADa) and $(\mathrm{ADb})_{\mathrm{M}}$ corresponds their conditions (ii) and (i) respectively.

Proof. First assuming (ADa) and $(\mathrm{ADb})_M$, we are going to show (6.1). Let $x, y \in X$. By $(\mathrm{ADb})_{\mathrm{M}}$, there exists a chain $(w(1), \ldots, w(k))$ between x and y such that $1 \leq k \leq M+1$ and $Cd(x,y) \geq g(w(i))$ for any $i = 1, \ldots, k$. By (G2), there exists $v(i)$ such that $\Sigma_{v(i)} \supseteq \Sigma_{w(i)}$ and $v(i) \in \Lambda^g_{Cd(x,y)}$. Since $(v(1), \ldots, v(k))$ is a chain in $\Lambda^g_{Cd(x,y)}$ between x and y, it follows that $Cd(x,y) \geq \delta^g_M(x,y)$.

Next set $t = \delta^g_M(x,y)$. Then there exists a chain $(w(1), \ldots, w(M+1)) \in \mathcal{CH}_K(x,y)$ in Λ^g_t. Choose $x_i \in K_{w(i)} \cap K_{w(i+1)}$ for every $i = 1, \ldots, M$. Then

$$\begin{aligned} d(x,y) &\leq d(x,x_1) + \sum_{i=1}^{M-1} d(x_i, x_{i+1}) + d(x_M, y) \\ &\leq c\sum_{j=1}^{M+1} g(w(i)) \leq c(M+1)t = c(M+1)\delta^g_M(x,y). \end{aligned}$$

Thus we have (6.1).

Conversely, assume that (6.1) holds, namely, there exist $c_1, c_2 > 0$ such that $c_1 d(x,y) \leq \delta^g_M(x,y) \leq c_2 d(x,y)$ for any $x, y \in X$. If $x, y \in K_w$, then $w \in \mathcal{CH}_K(x,y)$. Let $m = \min\{k | g(\pi^k(w)) > g(\pi^{k-1}(w)), k \in \mathbb{N}\}$ and set $s = g(w)$. Then $g(\pi^{k-1}(w)) = s$ and $\pi^{k-1}(w) \in \Lambda^g_s$. Since $\pi^{k-1}(w) \in \mathcal{CH}_K(x,y)$, we have

$$g(w) = s \geq \delta^g_0(x,y) \geq \delta^g_M(x,y) \geq c_1 d(x,y).$$

This immediately yields (ADa).

Set $s_* = c_2 d(x,y)$ for $x, y \in X$. Since $\delta^g_M(x,y) \leq c_2 d(x,y)$, there exists a chain $(w(1), \ldots, w(M+1))$ in $\Lambda^g_{s_*}$ between x and y. As $g(w(i)) \leq s_*$ for any $i = 1, \ldots, M+1$, we have $(\mathrm{ADb})_{\mathrm{M}}$. □

Since $(\mathrm{ADb})_M$ implies $(\mathrm{ADb})_N$ for any $N \geq M$, we have the following corollary.

Corollary 6.6. *Let $g : T \to (0,1]$ be a weight function. If $d \in \mathcal{D}(X, \mathcal{O})$ is M-adapted to g for some $M \geq 0$, then it is N-adapted to g for any $N \geq M$.*

Recall that a metric $d \in \mathcal{D}(X, \mathcal{O})$ defines a weight function g_d. So one may ask if d is adapted to the weight function g_d or not. Indeed, we are going to give an example of a metric $d \in \mathcal{D}(X, \mathcal{O})$ which is not adapted to g_d in Example 11.8.

Definition 6.7. Let $d \in \mathcal{D}(X, \mathcal{O})$. d is said to be adapted if d is adapted to g_d.

Proposition 6.8. *Let $d \in \mathcal{D}(X, \mathcal{O})$. d is adapted if and only if there exists a weight function $g : T \to (0, 1]$ to which d is adapted. Moreover, suppose that d is adapted. If*

$$D^d(x, y) = \inf\{\sum_{i=1}^{k} g_d(w(k))|k \geq 1, (w(1), \ldots, w(k)) \in \mathcal{CH}_K(x, y)\}$$

for any $x, y \in X$, then there exist $c_* > 0$ such that

$$c_* D^d(x, y) \leq d(x, y) \leq D^d(x, y)$$

for any $x, y \in X$.

Proof. Necessity direction is immediate. Assume that d is M-adapted to a weight function g. By (ADa) and $(\mathrm{ADb})_{\mathrm{M}}$, for any $x, y \in X$ there exist $k \in \{1, \ldots, M+1\}$ and $(w(1), \ldots, w(k)) \in \mathcal{CH}_K(x, y)$ such that

$$Cd(x, y) \geq \max_{i=1,\ldots,k} g(w(i)) \geq \frac{1}{c} \max_{i=1,\ldots,k} g_d(w(i)).$$

This proves $(\mathrm{ADb})_{\mathrm{M}}$ for the weight function g_d. So we verify that d is M-adapted to g_d. Now, assuming that d is adapted to g_d, we see

$$c_1 D^d_M(x, y) \leq d(x, y)$$

by (6.2). Since $D^d_M(x, y)$ is monotonically decreasing as $M \to \infty$, it follows that

$$c_1 D^d(x, y) \leq d(x, y).$$

On the other hand, if $(w(1), \ldots, w(k)) \in \mathcal{CH}_K(x, y)$, then the triangle inequality yields

$$d(x, y) \leq \sum_{i=1}^{k} g_d(w(i)).$$

Hence $d(x, y) \leq D^d(x, y)$. □

Let us return to the question on the existence of a metric associated with a given weight function g. Strictly speaking, one should try to find a metric adapted to the weight function g itself. In this section, however, we are going to deal with a modified version, i.e. the existence of a metric adapted to g^α for some $\alpha > 0$. Note that if g is a weight function, then so is g^α and $\delta_M^{g^\alpha} = (\delta_M^g)^\alpha$.

To start with, we present a weak version of "triangle inequality" for the family $\{\delta_M^g\}_{M \geq 1}$.

Proposition 6.9.

$$\delta^g_{M_1+M_2+1}(x,z) \le \max\{\delta^g_{M_1}(x,y), \delta^g_{M_2}(y,z)\}$$

Proof. Setting $s_* = \max\{\delta^g_{M_1}(x,y), \delta^g_{M_2}(y,z)\}$, we see that there exist a chain $(w(1), \ldots, w(M_1{+}1))$ between x and y and a chain $(v(1), \ldots, v(M_2{+}1))$ between y and z such that $w(i), v(j) \in \Lambda^g_{s_*}$ for any i and j. Since $(w(1), \ldots, w(M_1 + 1), v(1), \ldots, v(M_2 + 1))$ is a chain between x and z, we obtain the claim of the proposition. □

By this proposition, if $\delta^g_M(x,y) \le c\delta_{2M+1}(x,y)$ for any $x, y \in X$, then $\delta^g_M(x,y)$ is so-called quasimetric, i.e.

$$\delta^g_M(x,y) \le c\big(\delta^g_M(x,z) + \delta^g_M(z,y)\big) \tag{6.3}$$

for any $x, y, z \in X$. The coming theorem shows that δ^g_M being a quasimetric is equivalent to the existence of a metric adapted to g^α for some α.

The following definition and proposition give another characterization of the visual pre-metric δ^g_M.

Definition 6.10. For $w, v \in T$, the pair (w, v) is said to be m-separated with respect to Λ^g_s if and only if whenever $(w, w(1), \ldots, w(k), v)$ is a chain and $w(i) \in \Lambda^g_s$ for any $i = 1, \ldots, k$, it follows that $k \ge m$.

Proposition 6.11. *For any $x, y \in X$ and $M \ge 1$,*

$$\delta^g_M(x,y) = \sup\{s|(w,v) \text{ is } M\text{-separated if } w, v \in \Lambda^g_s,\ x \in K_w \text{ and } y \in K_v\}.$$

The following theorem gives several equivalent conditions on the existence of a metric adapted to a given weight function g. In Theorem 7.12, those condition will be shown to be equivalent to the hyperbolicity of the rearrangement of the resolution $(T, \mathcal{B})$ associated with the weight functiong and the adapted metric is, in fact, a visual metric in the sense of Gromov. See [10] and [22] for details on visual metric in the sense of Gromov.

Theorem 6.12. *Let $M \ge 1$ and let $g \in \mathcal{G}(T)$. The following four conditions are equivalent:*
(EV)$_M$ *There exist $\alpha \in (0,1]$ and $d \in \mathcal{D}(X, \mathcal{O})$ such that d is M-adapted to g^α.*
(EV2)$_M$ *δ^g_M is a quasimetric, i.e. there exists $c > 0$ such that* (6.3) *holds for any $x, y, z \in X$.*
(EV3)$_M$ *There exists $\gamma \in (0,1)$ such that $\gamma^n\delta^g_M(x,y) \le \delta^g_{M+n}(x,y)$ for any $x, y \in X$ and $n \ge 1$.*
(EV4)$_M$ *There exists $\gamma \in (0,1)$ such that $\gamma\delta^g_M(x,y) \le \delta^g_{M+1}(x,y)$ for any $x, y \in X$.*
Moreover, if $K : T \to \mathcal{C}(X, \mathcal{O})$ is minimal, then all the conditions above are equivalent to the following condition (EV5)$_M$*.*
(EV5)$_M$ *There exists $\gamma \in (0,1)$ such that if $(w, v) \in \Lambda^g_s \times \Lambda^g_s$ is M-separated with respect to Λ^g_s, then (w, v) is $(M+1)$-separated with respect to $\Lambda^g_{\gamma s}$.*

The symbol "EV" in the above conditions $(\mathrm{EV})_M$, $(\mathrm{EV1})_M$, …, $(\mathrm{EV5})_M$ represents "Existence of a Visual metric".

We use the following lemma to prove this theorem.

Lemma 6.13. *If there exist $\gamma \in (0,1)$ and $M \geq 1$ such that $\gamma\delta_M^g(x,y) \leq \delta_{M+1}(x,y)$ for any $x, y \in X$, then*

$$\gamma^n \delta_M^g(x,y) \leq \delta_{M+n}^g(x,y)$$

for any $x, y \in X$ and $n \geq 1$.

Proof. We use an inductive argument. Assume that

$$\gamma^l \delta_M^g(x,y) \leq \delta_{M+l}^g(x,y)$$

for any $x, y \in X$ and $l = 1, \ldots, n$. Suppose $\delta_{M+n+1}^g(x,y) \leq \gamma^{n+1}s$. Then there exists a chain $(w(1), \ldots, w(M+n+2))$ in $\Lambda_{\gamma^{n+1}s}^g$ between x and y. Choose any $z \in K_{w(M+n+1)} \cap K_{w(M+n+2)}$. Then

$$\gamma^n \delta_M^g(x,z) \leq \delta_{M+n}^g(x,z) \leq \gamma^{n+1}s.$$

Thus we obtain $\delta_M^g(x,z) \leq \gamma s$. Note that $\delta_0^g(z,y) \leq \gamma^{n+1}s$. By Proposition 6.9,

$$\gamma\delta_M^g(x,y) \leq \delta_{M+1}^g(x,y) \leq \max\{\delta_{M+1}^g(x,z), \delta_0^g(z,y)\} \leq \gamma s.$$

This implies $\delta_M^g(x,y) \leq s$. □

Proof of Theorem 6.12. $(\mathrm{EV})_M \Rightarrow (\mathrm{EV4})_M$: Since d is M-adapted to g^α, by Corollary 6.6, d is $M+1$-adapted to g^α as well. By (6.1), we obtain $(\mathrm{EV4})_M$.
$(\mathrm{EV3})_M \Leftrightarrow (\mathrm{EV4})_M$: This is immediate by Lemma 6.13.
$(\mathrm{EV3})_M \Rightarrow (\mathrm{EV2})_M$: Let $n = M+1$. By Proposition 6.9, we have

$$c_{2M+1}\delta_M^g(x,y) \leq \delta_{2M+1}^g(x,y) \leq \max\{\delta_M^g(x,z), \delta_M^g(z,y)\} \leq \delta_M^g(x,z) + \delta_M^g(z,y).$$

$(\mathrm{EV2})_M \Rightarrow (\mathrm{EV})_M$: By [13, Proposition 14.5], there exist $c_1, c_2 > 0$, $d \in \mathcal{D}(X, \mathcal{O})$ and $\alpha \in (0,1]$ such that $c_1\delta_M^g(x,y)^\alpha \leq d(x,y) \leq c_2\delta_M^g(x,y)^\alpha$ for any $x, y \in X$. Note that $\delta_M^g(x,y)^\alpha = \delta_M^{g^\alpha}(x,y)$. By (6.1), d is M-adapted to g^α.
$(\mathrm{EV4})_M \Rightarrow (\mathrm{EV5})_M$: Assume that $w, v \in \Lambda_s^g$. If w and v are not $(M+1)$-separated with respect to $\Lambda_{\gamma s}^g$, then there exist $w(1), \ldots, w(M) \in \Lambda_{\gamma s}^g$ such that $(w, w(1), \ldots, w(M), v)$ is a chain. Then we can choose $w' \in T_w \cap \Lambda_{\gamma s}^g$ and $v' \in T_v \cap \Lambda_{\gamma s}^g$ so that $(w', w(1), \ldots, w(M), v')$ is a chain. Let $x \in O_{w'}$ and let $y \in O_{v'}$. Then $\delta_{M+1}^g(x,y) \leq \gamma s$. Hence by $(\mathrm{EV4})_M$, $\delta_M^g(x,y) \leq s$. There exists a chain $(v(1), v(2), \ldots, v(M+1))$ in Λ_s^g between x and y. Since $x \in O_{w'} \subseteq O_w$ and $y \in O_{v'} \subseteq O_v$, we see that $v(0) = w$ and $v(M+1) = v$. Hence w and v are not M-separated with respect to Λ_s^g.
$(\mathrm{EV5})_M \Rightarrow (\mathrm{EV4})_M$: Assume that $\delta_{M+1}^g(x,y) \leq \gamma s$. Then there exists a chain $(w(1), \ldots, w(M+2))$ in $\Lambda_{\gamma s}^g$ between x and y. Let w (resp. v) be the unique element in Λ_s^g satisfying $w(1) \in T_w$ (resp. $w(M+2) \in T_v$). Then (w,v) is not $(M+1)$-separated in $\Lambda_{\gamma s}^g$. By $(\mathrm{EV5})_M$, (w,v) is not M-separated in Λ_s^g. Hence there exists a chain $(w, v(1), \ldots, v(M-1), v)$ in Λ_s^g. This implies $\delta_M^g(x,y) \leq s$. □

7 Hyperbolicity of resolutions and the existence of adapted metrics

In this section, we study the hyperbolicity in Gromov's sense of the resolution $(T, \mathcal{B})$ of a compact metric space X. Roughly speaking the hyperbolicity will be shown to be equivalent to the existence of an adapted metric. More precisely, we define the hyperbolicity of a weight function g as that of certain rearranged subgraph of $(T, \mathcal{B})$ associated with g and show that the weight function g is hyperbolic if and only if there exists a metric adapted to g^α for some $\alpha > 0$. Furthermore, in such a case, the adapted metric is shown to be a "visual metric". See Theorem 7.12 for exact statements. Another important point is the "boundary" of the resolution $(T, \mathcal{B})$ is always identified with the original metric space X with or without hyperbolicity of $(T, \mathcal{B})$ as is shown in Theorem 7.5. Furthermore, we are going to obtain counterparts by Elek [12] and Lau-Wang [21] on the constructions of a hyperbolic graph whose hyperbolic boundary is a given compact metric space as by-products of our general framework.

Throughout this section, $(T, \mathcal{A}, \phi)$ is a locally finite tree with a reference point ϕ, $(X, \mathcal{O})$ is a compact metrizable topological space with no isolated point and $K : T \to \mathcal{C}(X, \mathcal{O})$ is a partition of X parametrized by $(T, \mathcal{A}, \phi)$. Moreover, $(T, \mathcal{B})$ is the resolution of X associated with the partition $K : T \to C(X, \mathcal{O})$.

The first lemma claims that the collection of geodesic rays of $(T, \mathcal{B})$ starting from ϕ equals Σ, which is the collection of geodesic rays of the tree $(T, \mathcal{A})$ starting from ϕ.

Lemma 7.1. *If $(w(0), w(1), w(2), \ldots)$ is a geodesic ray from ϕ of $(T, \mathcal{B})$, then $\pi(w(i+1)) = w(i)$ for any $i = 1, 2, \ldots$. In other word, all the edges of a geodesic ray from ϕ are vertical edges and the collection of geodesic rays of $(T, \mathcal{B})$ coincides with Σ.*

Proof. Suppose that $\pi(w(i)) = w(i-1)$ for any $i = 1, \ldots, n$ and $(w(n), w(n+1))$ is a horizontal edge. Then $|w(i)| = i$ for any $i = 0, 1, \ldots, n$ and $|w(n+1)| = n$. Since $d_{(T,\mathcal{B})}(\phi, w(n+1)) = n$, the sequence $(\phi, w(1), \ldots, w(n), w(n+1))$ can not be a geodesic. Hence there exists no horizontal edge in $(w(0), w(1), w(2), \ldots)$. $\square$

The following proposition is the restatement of Proposition 4.9.

Proposition 7.2 (= Proposition 4.9)**.** *Let $\omega, \tau \in \Sigma$. Then*

$$\sup_{n \geq 1} d_{(T,\mathcal{B})}([\omega]_n, [\tau]_n) < +\infty$$

if and only if $\sigma(\omega) = \sigma(\tau)$.

To prove the above proposition, we need to study the structure of geodesics of $(T, d_{(T,\mathcal{B})})$.

Definition 7.3. (1) Let $w, v \in (T)_m$ for some $m \geq 0$. The pair (w, v) is called horizontally minimal if and only if there exists a geodesic of the resolution $(T, \mathcal{B})$

between w and v which consists only of horizontal edges.
(2) Let $w \neq v \in T$. Then a geodesic $\mathbf{b}$ of $(T, \mathcal{B})$ between w and v is called a bridge if and only if there exist $i, j \geq 0$ and a horizontal geodesic $(v(1), \ldots, v(k))$ such that $\pi^i(w) = v(1), \pi^j(v) = v(k)$ and

$$\mathbf{b} = (w, \pi(w), \ldots, \pi^i(w), v(2), \ldots, v(k-1), \pi^j(v), \ldots, \pi(v), v)$$

The number $|v(1)|$ is called the height of the bridge. Also $(w, \pi(w), \ldots, \pi^i(w))$, $(v(1), \ldots, v(k))$ and $(\pi^j(v), \ldots, \pi(v), v)$ are called the ascending part, the horizontal part and the descending part respectively.

Lemma 7.4. *For any $w, v \in T$, there exists a bridge between w and v.*

Proof. Let $(w(1), \ldots, w(m))$ be a geodesic of $(T, \mathcal{B})$ between w and v. Note that there exists no dent, which is a segment $(w(i), \ldots, w(k), w(k+1))$ satisfying $|w(i+1)| = |w(i)| + 1$, $|w(i+1)| = |w(i+2)| = \ldots = |w(k)|$ and $|w(k+1)| = |w(k)| - 1$, because applying π to the dent, we can reduce the length at least by 2. Therefore, if $m_* = \min\{|w(i)| : i = 1, \ldots, m\}$, then there exist $i_* < j_*$ such that $\{i : |w(i)| = m_*\} = \{i : i_* \leq i \leq j_*\}$ and $|w(i)|$ is monotonically nonincreasing on $I_1 = \{i | 1 \leq i \leq i_*\}$ and monotonically nondecreasing on $I_2 = \{i | j_* \leq i \leq m\}$. On I_1, if there exists $(w(i-1), w(i), w(i+1))$ such that $|w(i-1)| = |w(i)|$ and $w(i+1) = \pi(w(i))$, then we modify this part to $(w(i-1), \pi(w(i-1)), w(i+1))$. After the modification, the resulting sequence is also a geodesic. Similarly, on I_2, if there exists $(w(i-1)), w(i), w(i+1))$ such that $w(i-1) = \pi(w(i))$ and $|w(i)| = |w(i+1)|$, then we modify this part to $(w(i-1), \pi(w(i+1)), w(i+1))$. After modification, the resulting sequence is still a geodesic. Iterating those modifications on I_1 and I_2 repeatedly as many times as possible, we obtain a bridge between w and v in the end. $\square$

Proof of Proposition 7.2. Write $\omega_m = [\omega]_m$ and $\tau_m = [\tau]_m$ for any $m \geq 0$. Assume that $\sup_{n \geq 1} d_{(T,\mathcal{B})}(\omega_n, \tau_n) < +\infty$. Let $d \in \mathcal{D}(X, \mathcal{O})$. Then g_d is a weight function by Proposition 5.5. In particular, $\max_{w \in (T)_m} \mathrm{diam}(K_w, d) \to 0$ as $m \to \infty$. Set $x = \sigma(\omega)$ and $y = \sigma(\tau)$. Let $N = \sup_{n \geq 1} d_{(T,\mathcal{B})}(\omega_n, \tau_n) < +\infty$. Then for any m, there exists a bridge $(\omega_m, \ldots, \omega_{m-n}, \ldots, \tau_{m-n}, \ldots, \tau_m)$ between $[\omega]_m$ and $[\tau]_m$, where $(\omega_{m-n}, \ldots, \tau_{m-n})$ is a horizontal geodesic. Since $n \leq d_{(T,\mathcal{B})}(\omega_m, \tau_m) \leq N$ and the length of $(\omega_{m-n}, \ldots, \tau_{m-n})$ is at most N, it follows that $d(x, y) \leq N \max_{w \in (T)_{m-N}} \mathrm{diam}(K_w, d) \to 0$ as $m \to \infty$. Therefore $x = y$.

Conversely, if $x = y$, then $x = y \in K_{[\omega]_m} \cap K_{[\omega]_m}$ for any $m \geq 0$. Therefore $d_{(T,\mathcal{B})}([\omega_m], [\tau]_m) \leq 1$ for any $m \geq 0$. $\square$

Theorem 7.5. *Define an equivalence relation $\sim$ on the collection Σ of the geodesic rays as $\omega \sim \tau$ if and only if $\sup_{n \geq 1} d_{(T,\mathcal{B})}([\omega]_n, [\tau]_n) < +\infty$. Let $\mathcal{O}_*$ be the natural quotient topology of $\Sigma/\!\sim$ induced by the metric ρ_* on Σ. Then*

$$(\Sigma/\!\sim, \mathcal{O}_*) = (X, \mathcal{O}),$$

where we identify $\Sigma/\!\sim$ with X through the map $\sigma : \Sigma \to X$.

For a Gromov hyperbolic graph, the quotient of the collection of geodesic rays by the equivalence relation $\sim$ is called the hyperbolic boundary of the graph. In our framework, however, the above theorem shows that $\Sigma/\sim$ can be always identified with X even if $(T, \mathcal{B})$ is not hyperbolic.

Next we introduce the notion of (Gromov) hyperbolicity of $(T, \mathcal{B})$. Here we give only basic accounts needed in our work. See [10], [22] and [24] for details of the general framework of Gromov hyperbolic metric spaces.

Definition 7.6. Define the Gromov product of $w, v \in T$ in $(T, \mathcal{B})$ with respect to ϕ as

$$(w|v)_{(T,\mathcal{B}),\phi} = \frac{d_{(T,\mathcal{B})}(\phi, w) + d_{(T,\mathcal{B})}(\phi, v) - d_{(T,\mathcal{B})}(w, v)}{2}.$$

The graph $(T, \mathcal{B})$ is called η-hyperbolic (in the sense of Gromov) if and only if

$$(w|v)_{(T,\mathcal{B}),\phi} \geq \min\{(w|u)_{(T,\mathcal{B}),\phi}, (u|v)_{(T,\mathcal{B}),\phi}\} - \eta$$

for any $w, v, u \in T$. $(T, \mathcal{B})$ is called hyperbolic if and only if it is η-hyperbolic for some $\eta \in \mathbb{R}$.

It is known that the hyperbolicity can be defined by the thinness of geodesic triangles.

Definition 7.7. We say that all the geodesic triangles in $(T, \mathcal{B})$ are δ-thin if and only if for any $w, v, u \in T$, if $\mathbf{b}(a, b)$ is geodesic between a and b for each $(a, b) \in \{(w, v), (v, u), (u, w)\}$, then $\mathbf{b}(u, w)$ is contained in δ-neighborhood of $\mathbf{b}(w, v) \cup \mathbf{b}(v, u)$ with respect to $d_{(T,\mathcal{B})}$.

The following theorem is one of the basic facts in the theory of Gromov hyperbolic spaces. A proof can be seen in [24] for example.

Theorem 7.8. *$(T, \mathcal{B})$ is η-hyperbolic for some $\eta > 0$ if and only if all the geodesic triangles in $(T, \mathcal{B})$ are δ-thin for some $\delta > 0$.*

The next theorem gives a criterion of the hyperbolicity of the resolution $(T, \mathcal{B})$. It has explicitly stated and proven by Lau and Wang in [21]. However, Elek had already used essentially the same idea in [12] to construct a hyperbolic graph which is quai-isometric to the hyperbolic cone of a compact metric space. In fact, we are going to recover their works as a part of our general framework later in this section.

Theorem 7.9. *The resolution $(T, \mathcal{B})$ of X is hyperbolic if and only if there exists $L \geq 1$ such that*

$$d_{(T,\mathcal{B})}(w, v) \leq L \tag{7.1}$$

for any horizontally minimal pair $(w, v) \in \cup_{m \geq 1}((T)_m \times (T)_m)$.

Remark. As is shown in the proof, if all the geodesic triangles in $(T, \mathcal{B})$ are δ-thin, then L can be chosen as $4\delta + 1$. Conversely, if (7.1) is satisfied, then $(T, \mathcal{B})$ is $\frac{3}{2}L$-hyperbolic.

Since our terminologies and notations differ much from those in [12] and [21], we are going to present a proof of Theorem 7.9 for reader's sake.

Proof. Assume that all the geodesic triangles in $(T, \mathcal{B})$ are δ-thin. Let $(w, v) \in (T)_m$ be horizontally minimal. Consider the geodesic triangle consists of $p_1 = (w, \pi(w), \ldots, \pi^m(w))$, which is the vertical geodesic between w and ϕ, $p_2 = (v, \pi(v), \ldots, \pi^m(v))$, which is the vertical geodesic between v and ϕ, and $p_3 = (u(1), \ldots, u(k+1))$, which is the horizontal geodesic between w and v. Since all the geodesic triangles in $(T, \mathcal{B})$ are δ-thin, for any i, either there exists $w' \in p_1$ such that $d_{(T,\mathcal{B})}(w', u(i)) \leq \delta$ or there exists $v' \in p_2$ such that $d_{(T,\mathcal{B})}(v', u(i)) \leq \delta$. Suppose that the former is the case. Since $d_{(T,\mathcal{B})}(w, w') = |w| - |w'|$ is the smallest steps from the level $|w| = |u(i)|$ to $|w'|$, it follows that $d_{(T,\mathcal{B})}(w, w') \leq d_{(T,\mathcal{B})}(w', u(i))$. Hence

$$d_{(T,\mathcal{B})}(w, u(i)) \leq d_{(T,\mathcal{B})}(w, w') + d_{(T,\mathcal{B})}(w', u(i)) \leq 2d_{(T,\mathcal{B})}(w', u(i)) \leq 2\delta.$$

Considering the latter case as well, we conclude that either $d_{(T,\mathcal{B})}(w, u(i))) \leq 2\delta$ or $d_{(T,\mathcal{B})}(v, u(i)) \leq 2\delta$ for any i. This shows that $d_{(T,\mathcal{B})}(w, v) \leq 4\delta + 1$.

Conversely, assume (7.1). We are going to show that $(T, \mathcal{B})$ is $\frac{3}{2}L$-hyperbolic, namely,

$$(w(1)|w(2)) \geq \min\{(w(2)|w(3)), (w(3)|w(1))\} - \frac{3}{2}L \tag{7.2}$$

for any $w(1), w(2), w(3) \in T$. For $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$, let $\mathbf{b}_{ij}$ be a bridge between $w(i)$ and $w(j)$ and let m_{ij}, l_{ij} and m_{ji} be the lengths of the ascending part, the horizontal part and the descending part respectively. Also set h_{ij} be the height of the bridge $\mathbf{b}_{ij}$. Then

$$(w(i)|w(j)) = \frac{h_{ij} + m_{ij} + h_{ij} + m_{ji} - (m_{ij} + m_{ji} + l_{ij})}{2} = h_{ij} - \frac{l_{ij}}{2}.$$

Without loss of generality, we may assume that $h_{23} \geq h_{31}$. Then we have three cases;
Case 1: $h_{12} \geq h_{23} \geq h_{31}$,
Case 2: $h_{23} \geq h_{12} \geq h_{31}$,
Case 3: $h_{23} \geq h_{31} \geq h_{12}$.
In Case 1 and Case 2, since $h_{31} - h_{12} \leq 0$ and $l_{12} \leq L$, it follows that

$$(w(3)|w(1)) - (w(1)|w(2)) = h_{31} - h_{12} + \frac{l_{12}}{2} - \frac{l_{31}}{2} \leq \frac{L}{2}$$

Thus (7.2) holds. In Case 3, let $v(1) \in (T)_{h_{31}}$ belong to the ascending part of $\mathbf{b}_{12}$ and let $v(2) \in (T)_{h_{31}}$ belong to the descending part of $\mathbf{b}_{12}$. Moreover, let $\mathbf{b}_{31}^h$ and $\mathbf{b}_{23}^h$ be the horizontal parts of $\mathbf{b}_{31}$ and $\mathbf{b}_{23}$ respectively. Then the combination of $\mathbf{b}_{31}^h$ and $\pi^{h_{23}-h_{31}}(\mathbf{b}_{23}^h)$ gives a chain between $v(1)$ and $v(2)$ whose length is no greater than $l_{31} + l_{23}$. Since the segment of $\mathbf{b}_{12}$ connecting $v(1)$ and $v(2)$ is a geodesic, we have

$$2(h_{31} - h_{12}) + l_{12} \leq l_{31} + l_{23} \leq 2L.$$

Therefore, it follows that $h_{31} - h_{12} \leq L$. This implies

$$(w(3)|w(1)) - (w(1)|w(2)) = h_{31} - h_{12} + \frac{l_{12}}{2} - \frac{l_{31}}{2} \leq L + \frac{L}{2} = \frac{3}{2}L.$$

Thus we have obtained (7.2) in this case as well. $\square$

Note that so far weight functions play no role in the statements of results in this section. In order to take weight functions into account, we are going to introduce an rearranged resolution $(\widetilde{T}^{g,r}, \mathcal{B}_{\widetilde{T}^{g,r}})$ associated with a weight function g and give the definition of hyperbolicity of the weight function g in terms of the rearranged resolution.

Definition 7.10. Let $g \in \mathcal{G}(T)$ and let $r \in (0,1)$. For $m \geq 0$, define $(\widetilde{T}^{g,r})_m = \Lambda^g_{r^m}$ and

$$\widetilde{T}^{g,r} = \bigcup_{m \geq 0} (\widetilde{T}^{g,r})_m.$$

$\widetilde{T}^{g,r}$ is naturally equipped with a tree structure inherited from T. Define $K_{\widetilde{T}^{g,r}} : \widetilde{T}^{g,r} \to \mathcal{C}(X, \mathcal{O})$ by $K_{\widetilde{T}^{g,r}} = K|_{\widetilde{T}^{g,r}}$. The collection of geodesic rays of the tree $\widetilde{T}^{g,r}$ starting from ϕ is denoted by $\Sigma_{\widetilde{T}^{g,r}}$. Define $\sigma_{\widetilde{T}^{g,r}} : \Sigma_{\widetilde{T}^{g,r}} \to X$ by $\sigma_{\widetilde{T}^{g,r}}(\omega) = \cap_{m \geq 0} K_{\omega(m)}$ for any $\omega = (\phi, \omega(1), \ldots) \in \Sigma_{\widetilde{T}^{g,r}}$. For any $w \in \Lambda^g_{r^{m+1}}$, the unique $v \in \Lambda^g_{r^m}$ satisfying $w \in T_v$ is denoted by $\pi^{g,r}(w)$. Also we set $S^{g,r}(w) = \{v | v \in \Lambda^d_{r^{m+1}}, v \in T_w\}$ for $w \in \Lambda^g_{r^m}$. Define the horizontal edges of $\widetilde{T}^{g,r}$ as

$$E^h_{g,r} = \bigcup_{n \geq 1} \{(w,v) | w, v \in (\widetilde{T}^{g,r})_n, K_w \cap K_v \neq \emptyset\}.$$

Moreover, we define the totality of the horizontal and vertical edges $\mathcal{B}_{\widetilde{T}^{g,r}}$ by

$$\mathcal{B}_{\widetilde{T}^{g,r}} = \{(w,v) | (w,v) \in E^h_{g,r} \text{ or } w = \pi^{g,r}(v) \text{ or } v = \pi^{g,r}(w)\}.$$

The graph $(\widetilde{T}^{g,r}, \mathcal{B}_{\widetilde{T}^{g,r}})$ is called the rearranged resolution of X associated with the weight function g.

Remark. Even if $m \neq n$, it may happen that $\Lambda^g_{r^m} \cap \Lambda^g_{r^n} \neq \emptyset$. In such a case, for $w \in \Lambda^g_{r^m} \cap \Lambda^g_{r^n}$, we regard $w \in (\widetilde{T}^{g,r})_m$ and $w \in (\widetilde{T}^{g,r})_n$ as different elements in $\widetilde{T}^{g,r}$. More precisely, the exact definition of $\widetilde{T}^{g,r}$ should be $\widetilde{T}^{g,r} = \cup_{m \geq 0}(\{m\} \times \Lambda^g_{r^m})$ and the associated partition $K_{\widetilde{T}^{g,r}} : \widetilde{T}^{g,r} \to \mathcal{C}(X, \mathcal{O})$ is defined as $K_{\widetilde{T}^{g,r}}((m,w)) = K_w$.

Remark. $\Sigma_{\widetilde{T}^{g,r}}$ and $\sigma_{\widetilde{T}^{g,r}}$ can be naturally identified with Σ and σ respectively.

Definition 7.11. A weight function g is said to be hyperbolic if and only if the rearranged resolution $(\widetilde{T}^{g,r}, \mathcal{B}_{\widetilde{T}^{g,r}})$ is hyperbolic for some $r \in (0,1)$.

The next theorem shows that the hyperbolicity of a weigh function g is equivalent to the existence of a "visual metric" associated with g. It also implies that the quantifier "for some $r \in (0,1)$" in Definition 7.11 can be replaced by "for any $r \in (0,1)$".

Theorem 7.12. *Let g be a weight function. Then the following three conditions are equivalent:*
(1) *There exists $M \geq 1$ such that* $(\mathrm{EV})_M$ *is satisfied, i.e. there exist $d \in \mathcal{D}(X, \mathcal{O})$ and $\alpha > 0$ such that d is M-adapted to g^α.*
(2) *The weight function g is hyperbolic.*
(3) $(\widetilde{T}^{g,r}, \mathcal{B}_{\widetilde{T}^{g,r}})$ *is hyperbolic for any $r \in (0, 1)$.*

Moreover, if any of the above conditions is satisfied, then there exist $c_1, c_2 > 0$ such that

$$c_1 \delta^g_M(x, y) \leq r^{(x|y)_{\widetilde{T}^{g,r}}} \leq c_2 \delta^g_M(x, y). \tag{7.3}$$

for any $x, y \in X$, where

$$(x|y)_{\widetilde{T}^{g,r}} = \sup\Big\{ \lim_{n,m\to\infty} (\omega(n)|\tau(m))_{(\widetilde{T}^{g,r}, \mathcal{B}_{\widetilde{T}^{g,r}}),\phi} \Big| \\ \omega = (\phi, \omega(1), \ldots), \tau = (\phi, \tau(1), \ldots) \in \Sigma_{\widetilde{T}^{g,r}}, \sigma_{\widetilde{T}^{g,r}}(\omega) = x, \sigma_{\widetilde{T}^{g,r}}(\tau) = y \Big\}.$$

Remark. The proof of Theorem 7.12 shows that if every geodesics triangle of $(\widetilde{T}^{g,r}, \mathcal{B}_{\widetilde{T}^{g,r}})$ is η-thin, then $(\mathrm{EV})_{\mathrm{M}}$ is satisfied for $M = \min\{m | m \in \mathbb{N}, 4\eta + 1 \leq m\}$.

By (7.3), if d is M-adapted to g^α, then there exist $c_1, c_2 > 0$ such that

$$c_1 d(x, y) \leq (r^\alpha)^{(x|y)_{\widetilde{T}}} \leq c_2 d(x, y)$$

for any $x, y \in X$. Then the metric d is called a visual metric on the hyperbolic boundary X of $(\widetilde{T}^{g,r}, \mathcal{B}_{\widetilde{T}^{g,r}})$ in the framework of Gromov hyperbolic metric spaces. See [10] and [22] for example.

About the original resolution $(T, \mathcal{B})$, we have the following corollary.

Corollary 7.13. *For $r \in (0, 1)$, define a weight function h_r by $h_r(w) = r^{-|w|}$ for $w \in T$. Then $(T, \mathcal{B})$ is hyperbolic if and only if there exist a metric $d \in \mathcal{D}(X, \mathcal{O})$, $M \geq 1$ and $r \in (0, 1)$ such that d is M-adapted to h_r.*

To show the hyperbolicity of a weigh function g, the existence of adapted metric is (an equivalent condition as we have seen in Theorem 7.12 but) too restrictive in some cases. In fact, the notion of "weakly adapted" metric is often more useful as we will see in Example 7.18 and 7.19.

Definition 7.14. Let $d \in \mathcal{D}(X, \mathcal{O})$. For $r \in (0, 1]$, $s > 0$ and $x \in X$, define

$$\widetilde{B}^r_d(x, s) = \{y | y \in B_d(x, s), \text{there exists a horizontal chain } (w(1), \ldots, w(k)) \\ \text{in } \Lambda^g_r \text{ between } x \text{ and } y \text{ such that } K_{w(i)} \cap B_d(x, s) \neq \emptyset \text{ for any } i = 1, \ldots, k\}.$$

A metric $d \in \mathcal{D}(X, \mathcal{O})$ is said to be weakly M-adapted to a weight function g if and only if there exist $c_1, c_2 > 0$ such that

$$\widetilde{B}^r_d(x, c_1 r) \subseteq U^g_M(x, r) \subseteq B_d(x, c_2 r)$$

for any $x \in X$ and $r \in (0, 1]$.

Since $\widetilde{B}^r_d(x, cr) \subseteq B_d(x, cr)$, we immediately have the following fact.

Proposition 7.15. *If $d \in \mathcal{D}(X, \mathcal{O})$ is M-adapted to a weight function g, then it is weakly M-adapted to g.*

The next proposition gives a sufficient condition for a metric being weakly adapted, which will be applied in Examples 7.18 and 7.19.

Proposition 7.16. *If there exist $c_1, c_2 > 0$ and $M \in \mathbb{N}$ such that*

$$\mathrm{diam}(K_w, d) \leq c_1 r \tag{7.4}$$

for any $w \in \Lambda^g_r$ and

$$\#(\{w | w \in \Lambda^g_r, B_d(x, c_2 r) \cap K_w \neq \emptyset\}) \leq M + 1 \tag{7.5}$$

for any $x \in X$ and $r \in (0, 1]$, then d is weakly M-adapted to g.

Proof. Assume that $y \in U^g_M(x, r)$. Then there exists a chain $(w(1), \ldots, w(M + 1))$ between x and y in Λ^g_r. Choose $x_i \in K_{w(i)} \cap K_{w(i+1)}$ for any i. Set $x_0 = x$ and $x_{M+1} = y$ Then by (7.4), it follows that

$$d(x, y) \leq \sum_{i=0}^{M} d(x_i, x_{i+1}) \leq (M + 1)c_1 r.$$

Hence $y \in B_d(x, (M + 2)c_1 r)$. This implies $U^g_M(x, r) \subseteq B_d(x, (M + 2)c_1 r)$.

Next, let $y \in \widetilde{B}^r_d(x, c_2 r)$. Then there exists a chain $(w(1), \ldots, w(k))$ between x and y in Λ^g_r such that $K_{w(i)} \cap B_d(x, c_2 r) \neq \emptyset$ for any $i = 1, \ldots, k$. We may assume that $w(i) \neq w(j)$ if $i \neq j$. Then by (7.5), we see that $k \leq M + 1$. This yields that $y \in U^g_M(x, r)$. Thus we have shown that d is weakly M-adapted to g. $\square$

Proposition 7.17. *Let g be a weight function. If there exists a metric $d \in \mathcal{D}(X, \mathcal{O})$ that is weakly M-adapted to g^α for some $M \geq 1$ and $\alpha > 0$, then $(\widetilde{T}^{g,r}, \mathcal{B}_{\widetilde{T}^{g,r}})$ is hyperbolic for any $r \in (0, 1]$.*

There have been several works on the construction of a hyperbolic graph whose hyperbolic boundary coincides with a given compact metric space. For example, Elek[12] has studied the case for arbitrary compact subset of $\mathbb{R}^n$ and Lau-Wang[21] has considered self-similar sets satisfying the open set condition. Due to above proposition, we may integrate these works into our framework. See Example 7.18 and 7.19 for details.

For ease of notations, we use $\widetilde{T}$, $\widetilde{\pi}$, $\Sigma_{\widetilde{T}}$, $\sigma_{\widetilde{T}}$ and $\mathcal{B}_{\widetilde{T}}$ to denote $\widetilde{T}^{g,r}$, $\pi^{g,r}$, $\Sigma_{\widetilde{T}^{g,r}}$, $\sigma_{\widetilde{T}^{g,r}}$ and $\mathcal{B}_{\widetilde{T}^{g,r}}$ respectively. Moreover, we write $d_{\widetilde{T}} = d_{(\widetilde{T}, \mathcal{B}_{\widetilde{T}})}$, which is the geodesic metric of $(\widetilde{T}, \mathcal{B}_{\widetilde{T}})$.

Proof. Assume that there exist a metric $d \in \mathcal{D}(X, \mathcal{O})$, $M \geq 1$ and $\alpha > 0$ such that d is weakly M-adapted to g^α. Then there exist $c_1, c_2 > 0$ such that

$$\widetilde{B}^r_d(x, c_1 r) \subseteq U^{g^\alpha}_M(x, r) \subseteq B_d(x, c_2 r)$$

for any $x \in X$ and $r \in (0,1]$. In particular, $d(x,y) \le c_2\delta_M^{g^\alpha}(x,y)$ for any $x,y \in X$. Suppose that $w,v \in (\widetilde{T})_n$, $x_1 \in K_w$, $x_2 \in K_v$ and $(w,v) \in E^h_{g,r^n}$. Since $\delta_1^g(x_1,x_2) \le r^n$, it follows that

$$d(x_1,x_2) \le c_2 r^{\alpha n}. \tag{7.6}$$

Let $m \ge 1$ and fix $r \in (0,1)$. Suppose that there exists $w_*, v_* \in (\widetilde{T})_n$ such that (w_*,v_*) is horizontally minimal and $d_{\widetilde{T}}(w_*,v_*) = 3m+1$. Let $(w(1),\ldots,w(3m+1))$ be the horizontal geodesic of $(\widetilde{T},\mathcal{B}_{\widetilde{T}})$ between w_* and v_*. Let $x \in K_{w_*}$ and let $y \in K_{v_*}$. By (7.6) for any $z \in \cup_{i=1}^{3m+1} K_{w_*}$,

$$d(x,z) \le c_2(3m+1)r^{\alpha n}.$$

If m is sufficiently large, then $c_2 r^{\alpha m}(3m+1) < c_1$. Hence

$$d(x,z) < c_1 r^{\alpha(n-m)}.$$

Set $v(i) = \widetilde{\pi}^m(w(i))$ for $i = 1,\ldots,3m+1$, then $(v(1),\ldots,v(3m+1))$ is a horizontal chain in $(T)_{n-m}$ between x and y and $K_{v(i)} \cap B_d(x, c_1 r^{\alpha(n-m)}) \supseteq K_{w(i)} \cap B_d(x, c_1 r^{\alpha(n-m)}) \ne \emptyset$. Therefore

$$y \in \widetilde{B}_d^r(x, c_1 r^{\alpha(n-m)}) \subseteq U_M^{g^\alpha}(x, r^{\alpha(n-m)}) = U_M^g(x, r^{n-m}).$$

So there exists a horizontal chain $(u(1),\ldots,u(M+1))$ in $(\widetilde{T})_{n-m}$ between x and y. Combining this horizontal chain with vertical geodesics $(w_*,\ldots,\widetilde{\pi}^m(w_*))$ and $(\widetilde{\pi}^m(v_*),\ldots,v_*)$, we have a chain of $(\widetilde{T},\mathcal{B}_{\widetilde{T}})$ between w_* and v_* whose length is $M+2+2m$. Therefore,

$$M + 2 + 2m \ge d_{\widetilde{T}}(w,v) = 3m.$$

Hence $M + 2 \ge m$. Applying Theorem 7.9 to $(\widetilde{T},\mathcal{B}_{\widetilde{T}})$, we verify that $(\widetilde{T},\mathcal{B}_{\widetilde{T}})$ is hyperbolic. $\square$

Proof of Theorem 7.12. (1) $\Rightarrow$ (3) Proposition 7.17 suffices.
(3) $\Rightarrow$ (2) This is immediate.
(2) $\Rightarrow$ (3) Assume that all the geodesic triangles in $(\widetilde{T},\mathcal{B}_{\widetilde{T}})$ are δ-thin. Set $L = \min\{m | m \in \mathbb{N}, 4\delta + 1 \le m\}$. For ease of notation, we use $(w|v)_{\widetilde{T}}$ to denote the Gromov product of w and v in $(\widetilde{T},\mathcal{B}_{\widetilde{T}})$ with respect to ϕ. Let $x \ne y \in X$ and let $\omega = (\phi,\omega(1),\ldots), \tau = (\phi,\tau(1),\ldots) \in \Sigma_{\widetilde{T}}$ satisfy $\sigma_{\widetilde{T}}(\omega) = x$ and $\sigma_{\widetilde{T}}(\tau) = y$. Applying Proposition 7.2 to $\widetilde{T}$, we see that there exists $m_* \in \mathbb{N}$ such that $d_{\widetilde{T}}(\omega(m),\tau(m)) > L$ for any $m \ge m_*$. Let $\mathbf{b}$ be a bridge between $\omega(m_*)$ and $\tau(m_*)$. If k_* is the height of $\mathbf{b}$ and $(\omega(k_*), w(1)\ldots, w(l-1), \tau(k_*))$ is the horizontal part of $\mathbf{b}$, then $\mathbf{b}$ is the concatenation of $(\omega(m_*),\ldots,\omega(k_*))$, $(\omega(k_*), w(1)\ldots, w(l-1), \tau(k_*))$ and $(\tau(k_*),\ldots,\tau(m_*))$. If $m,n \ge m_*$, then $(\omega(m),\ldots,\omega(k_*), w(1),\ldots,w(l-1),\tau(k_*),\ldots,\tau(n))$ is a bridge between $\omega(m)$ and $\tau(n)$. Therefore,

$$(\omega(m)|\tau(n))_{\widetilde{T}} = k_* - \frac{l}{2}.$$

Hence, if we define

$$(\omega, \tau)_{\widetilde{T}} = \lim_{m,n\to\infty} (\omega(m)|\tau(n))_{\widetilde{T}},$$

then $(\omega|\tau)_{\widetilde{T}} = k_* - \frac{l}{2}$.

Applying Theorem 7.9 to $(\widetilde{T}, \mathcal{B}_{\widetilde{T}})$, we see that the length of any horizontal geodesic is no greater than L. Since the length of the horizontal part of $\mathbf{b}$ is l, it follows that $l \le L$. Therefore letting $s_* = \delta_L^g(x, y)$, then we see that

$$s_* \le r^{k_*} \le r^{k_* - \frac{l}{2}} = r^{(\omega|\tau)_{\widetilde{T}}}. \tag{7.7}$$

Choose n_* so that $r^{k_*+n_*} \ge s_* > r^{k_*+n_*+1}$. Then there exists a horizontal chain $(v(1), \dots, v(L+1))$ in $(\widetilde{T})_{k_*+n_*}$ such that $x \in K_{v(1)}$ and $y \in K_{v(L+1)}$. Hence $(\omega(k_*+n_*), v(1), \dots, v(L+1), \tau(k_*+n_*))$ is a chain between $\omega(k_*+n_*)$ and $\tau(k_*+n_*)$. Comparing this chain with $(\omega(k_*+n_*), \dots, \omega(k_*), w(1), \dots, w(l-1), \tau(k_*), \dots, \tau(k_*+n_*))$, we obtain

$$2n_* + l \le L + 2.$$

This implies $k_* + n_* + 1 - \frac{L}{2} - 2 \le k_* - \frac{l}{2}$. Therefore

$$r^{(\omega|\tau)_{\widetilde{T}}} = r^{k_* - \frac{l}{2}} \le r^{k_*+n_*+1} r^{-\frac{L}{2}-2} \le r^{-\frac{L}{2}-2} s_*. \tag{7.8}$$

Set $c_1 = 1$ and $c_2 = r^{-\frac{L}{2}-2}$. Then we have

$$c_1 \delta_L^g(x, y) \le r^{(\omega|\tau)_{\widetilde{T}}} \le c_2 \delta_L^g(x, y).$$

Define $(x|y)_{\widetilde{T}} = \sup\{(\omega|\tau)_{\widetilde{T}} | \omega, \tau \in \Sigma_{\widetilde{T}}, \sigma_{\widetilde{T}}(\omega) = x, \sigma_{\widetilde{T}}(\tau) = y\}$. Then

$$c_1 \delta_L^g(x, y) \le r^{(x|y)_{\widetilde{T}}} \le c_2 \delta_L^g(x, y). \tag{7.9}$$

It is known that if $(\widetilde{T}, \mathcal{B}_{\widetilde{T}})$ is hyperbolic, then $r^{(x|y)_{\widetilde{T}}}$ is a quasimetric. Hence by (7.9), $\delta_L^g(x, y)$ is a quasimetric as well. Thus we have obtained $(\mathrm{EV2})_M$. $\square$

In short, in the above reasonings, we have two steps:
(1) The existence of weakly adapted metric d implies the hyperbolicity of g.
(2) The hyperbolicity of g implies the existence of an adapted metric ρ.
It is notable that the original weakly adapted metric d may essentially differ from the adapted metric d. In fact, in Example 7.18, we are going to present an explicit example where no power of the original weakly adapted metric is bi-Lipschitz equivalent to any adapted metric.

Proof of Corollary 7.13. Note that $(T, \mathcal{B}) = (\widetilde{T}^{h_r, r}, \mathcal{B}_{\widetilde{T}^{h_r, r}})$ for any $r \in (0, 1)$. Assume that $(T, \mathcal{B})$ is hyperbolic. By Theorem 7.12, there exist $d \in \mathcal{D}(X, \mathcal{O})$, $M \ge 1$ and $\alpha > 0$ such that d is adapted to $(h_{1/2})^\alpha$. Since $(h_{1/2})^\alpha = h_{2^{-\alpha}}$, we have the desired statement with $r = 2^{-\alpha}$. Conversely, with the existence of d, M and r, Theorem 7.12 implies that $(\widetilde{T}^{h_r, s}, \mathcal{B}_{\widetilde{T}^{h_r, s}})$ is hyperbolic for any $s \in (0, 1)$. Letting $s = r$, we see that $(T, \mathcal{B})$ is hyperbolic. $\square$

To end this section, we are going to integrate the works by Elek[12] and Lau-Wang[21] into our framework.

Example 7.18. In Example 4.10, we have obtained a partition of a compact metric space in $\mathbb{R}^N$ corresponding to the hyperbolic graph constructed by Elek in [12]. In fact, we have obtained two graphs $(T, \widetilde{\mathcal{B}})$ and $(T, \mathcal{B})$ satisfying $\widetilde{\mathcal{B}} \supseteq \mathcal{B}$. The former coincides with Elek's graph and the latter is the resolution associated with the partition constructed from the Dyadic cubes. In this example, using Propositions 7.16 and 7.17, we are going to show the hyperbolicity of the graph $(T, \mathcal{B})$. The hyperbolicity of the original graph $(T, \widetilde{\mathcal{B}})$ may be shown in a similar fashion.

Let X be a compact subset of $[0,1]^n$ and let $(T, \mathcal{A}, \phi)$ be the tree associated with X constructed from the dyadic cubes in Example 4.10. Also let $K : T \to \mathcal{C}(X, \mathcal{O})$ be the partition of X parametrized by $(T, \mathcal{A}, \phi)$ given in Example 4.10. Set $g(w) = 2^{-m}$ if $w \in (T)_m$. Then $\Lambda_r^g = (T)_m$ if and only if $2^{-m} \le r < 2^{-m+1}$. Let d_* be the Euclidean metric. Then for any $w \in \Lambda_r^g$,

$$\mathrm{diam}(K_w, d_*) \le 2\sqrt{n}r.$$

This shows (7.4). Moreover, if $w \in \Lambda_r^g$ and $K_w \cap B_{d_*}(x, cr) \neq \emptyset$, then $C(w) \subseteq B_{d_*}(x, (c + 2\sqrt{N})r)$. Note that $|C(w)|_n = 2^{-mn}$ for any $w \in (T)_m$, where $|\cdot|_n$ is the n-dimensional Lebesgue measure. Therefore, if $2^{-m} \le r < 2^{-m+1}$, then

$$\begin{aligned}\#(\{w | w \in \Lambda_r^g, B_{d_*}(x, cr) \cap K_w \neq \emptyset\}) &\le \frac{|B_{d_*}(x, (c + 2\sqrt{n})r)|_n}{2^{-mn}} \\ &= |B_{d_*}(0,1)|_N (c + 2\sqrt{n})^n (2^m r)^n \le |B_{d_*}(0,1)|_n (c + \sqrt{n})^n 2^n.\end{aligned}$$

Therefore choosing $M \in \mathbb{N}$ so that $|B_{d_*}(0,1)|_N (c + \sqrt{n})^n 2^n \le M + 1$, we have (7.5). Hence by Proposition 7.16, (the restriction of) d_* is weakly M-adapted to g. Since $(\widetilde{T}^{g,\frac{1}{2}}, \mathcal{B}_{\widetilde{T}^{g,\frac{1}{2}}}) = (T, \mathcal{B})$, Proposition 7.17 yields that $(T, \mathcal{B})$ is hyperbolic and its hyperbolic boundary coincides with X.

As we have mentioned above, in this example, the weakly adapted metric d_* is not necessarily adapted to any power of g. For example, let

$$\begin{aligned}X = [0,1] \bigcup \{(t,t) | t \in [0,1]\} \bigcup \\ \Big(\bigcup_{m \ge 1} \Big\{\Big(\frac{1}{2^m}, s\Big)\Big| s \in \Big[0, \frac{1}{2^m}\Big] \backslash \Big(\frac{1-\epsilon_m}{2^m}, \frac{1+\epsilon_m}{2^m}\Big)\Big\}\Big),\end{aligned}$$

where $\epsilon_m = \dfrac{1}{2^{m^2}}$. Set $x_m = \Big(\dfrac{1}{2^m}, \dfrac{1-\epsilon_m}{2^m}\Big)$ and $y_m = \Big(\dfrac{1}{2^m}, \dfrac{1+\epsilon_m}{2^m}\Big)$. Then $d_*(x_m, y_m) = \dfrac{\epsilon_m}{2^{m-1}}$ and $\delta_1^g(x_m, y_m) = \dfrac{1}{2^{m-1}}$. Then

$$\frac{d_*(x_m, y_m)}{\delta_1^{g^\alpha}(x_m, y_m)} = 2^{\alpha(m-1) - m^2 - m + 1} \to 0$$

as $m \to \infty$ for any $\alpha > 0$. Thus for any $\alpha > 0$, the Euclidean metric is not bi-Lipschitz equivalent to any metric adapted to g^α.

Example 7.19. Let X be the self-similar set associated with the collection of contractions $\{F_1, \ldots, F_N\}$ and let $K : T^{(N)} \to X$ be the partition of K parametrized by $(T^{(N)}, \mathcal{A}^{(N)}, \phi)$ introduced in Example 4.5. We write $T = T^{(N)}$ for simplicity. In this example, we further assume that for any $i = 1, \ldots, N$, $F_i : \mathbb{R}^n \to \mathbb{R}^n$ is a similitude, i.e. there exist an orthogonal matrix A_i, $r_i \in (0, 1)$ and $a_i \in \mathbb{R}^n$ such that $F_i(x) = r_i A_i x + a_i$. Furthermore, we assume that the open set condition holds, i.e. there exists a nonempty open subset O of $\mathbb{R}^n$ such that $F_w(O) \subseteq O$ for any $w \in T$ and $F_w(O) \cap F_v(O) = \emptyset$ if $w, v \in T$ and $T_w \cap T_v = \emptyset$. Define $g(w) = r_{w_1} \cdots r_{w_m}$ for any $w = w_1 \ldots w_m \in T$. In this case, the conditions (7.4) and (7.5) have been known to hold for the Euclidean metric d_*. See [15, Proposition 1.5.8] for example. Hence Proposition 7.16 implies that d_* is weakly M-adapted to g for some $M \in \mathbb{N}$. Using Proposition 7.17, we see that $(\widetilde{T}^{g,r}, \mathcal{B}_{\widetilde{T}^{g,r}})$ is hyperbolic for any $r \in (0, 1)$ and hence the self-similar set X is the hyperbolic boundary of $(\widetilde{T}^{g,r}, \mathcal{B}_{\widetilde{T}^{g,r}})$. This fact has been shown by Lau and Wang in [21]. As in the previous example, the Euclidean metric is not necessarily a visual metric in this case.

Part II

Relations of weight functions

8 Bi-Lipschitz equivalence

In this section, we define the notion of bi-Lipschitz equivalence of weight functions. Originally the definition, Definition 8.1, only concerns the tree structure $(T, \mathcal{A}, \phi)$ and has nothing to do with a partition of a space. Under proper conditions, however, we will show that the bi-Lipschitz equivalence of weight functions is identified with

- absolutely continuity with uniformly bounded Radon-Nikodym derivative from below and above between measures in 8.1.
- usual bi-Lipschitz equivalence between metrics in 8.2.
- Ahlfors regularity of a measure with respect to a metric in 8.3.

As in the previous sections, $(T, \mathcal{A}, \phi)$ is a locally finite tree with a reference point ϕ, $(X, \mathcal{O})$ is a compact metrizable topological space with no isolated point and $K : T \to \mathcal{C}(X, \mathcal{O})$ is a partition of X parametrized by $(T, \mathcal{A}, \phi)$.

Definition 8.1. Two weight functions $g, h \in \mathcal{G}(T)$ are said to be bi-Lipschitz equivalent if and only if there exist $c_1, c_2 > 0$ such that

$$c_1 g(w) \leq h(w) \leq c_2 g(w)$$

for any $w \in T$. We write $g \underset{\mathrm{BL}}{\sim} h$ if and only if g and h are bi-Lipschitz equivalent.

By the definition, we immediately have the next fact.

Proposition 8.2. *The relation* $\underset{BL}{\sim}$ *is an equivalent relation on* $\mathcal{G}(T)$.

8.1 bi-Lipschitz equivalence of measures

As we mentioned above, the bi-Lipschitz equivalence between weight functions can be identified with other properties according to classes of weight functions. First we consider the case of weight functions associated with measures.

Definition 8.3. Let $\mu, \nu \in \mathcal{M}_P(X, \mathcal{O})$. We write $\mu \underset{\text{AC}}{\sim} \nu$ if and only if there exist $c_1, c_2 > 0$ such that

$$c_1\mu(A) \leq \nu(A) \leq c_2\mu(A) \tag{8.1}$$

for any Borel set $A \subseteq X$.

It is easy to see that $\underset{\text{AC}}{\sim}$ is an equivalence relation and $\mu \underset{\text{AC}}{\sim} \nu$ if and only if μ and ν are mutually absolutely continuous and the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ is uniformly bounded from below and above.

Theorem 8.4. *Assume that the partition* $K : T \to \mathcal{C}(X, \mathcal{O})$ *is strongly finite. Let* $\mu, \nu \in \mathcal{M}_P(X, \mathcal{O})$. *Then* $g_\mu \underset{BL}{\sim} g_\nu$ *if and only if* $\mu \underset{AC}{\sim} \nu$. *Moreover, the natural map* $\mathcal{M}_P(X, P)/\underset{AC}{\sim} \to \mathcal{G}(X)/\underset{BL}{\sim}$ *given by* $[g_\mu]_{\underset{BL}{\sim}}$ *is injective, where* $[\,\cdot\,]_{\underset{BL}{\sim}}$ *is the equivalence class under* $\underset{BL}{\sim}$.

Proof. By (8.1), we see that $\alpha_1\nu(K_w) \leq \mu(K_w) \leq \alpha_2\nu(K_w)$ and hence $g_\mu \underset{\text{BL}}{\sim} g_\nu$. Conversely, if

$$c_1\mu(K_w) \leq \nu(K_w) \leq c_2\mu(K_w)$$

for any $w \in T$. Let $U \subset X$ be an open set. Assume that $U \neq X$. For any $x \in X$, there exists $w \in T$ such that $x \in K_w \subseteq U$. Moreover, if $K_w \subseteq U$, then there exists $m \in \{1, \ldots, |w|\}$ such that $K_{[w]_m} \subseteq U$ but $K_{[w]_{m-1}} \backslash U \neq \emptyset$. Therefore, if

$$T(U) = \{w | w \in T, K_w \subseteq U, K_{\pi(w)} \backslash U \neq \emptyset\},$$

then $T(U) \neq \emptyset$ and $U = \cup_{w \in T(U)} K_w$. Now, since K is strongly finite, there exists $N \in \mathbb{N}$ such that $\#(\sigma^{-1}(x)) \leq N$ for any $x \in X$. Let $y \in U$. If $w(1), \ldots, w(m) \in T(U)$ are mutually different and $y \in K_{w(m)}$ for any $i = 1, \ldots, m$, then there exists $\omega(i) \in \Sigma_{w(i)}$ such that $\sigma(\omega(i)) = y$ for any $i = 1, \ldots, m$. Hence $\#(\sigma^{-1}(y)) \geq m$ and therefore $m \leq N$. By Proposition A.1, we see that

$$\nu(U) \leq \sum_{w \in T(U)} \nu(K_w) \leq \sum_{w \in T(U)} c_2\mu(K_w) \leq c_2 N\mu(U)$$

$$\mu(U) \leq \sum_{w \in T(U)} \mu(K_w) \leq \sum_{w \in T(U)} \frac{1}{c_1}\nu(K_w) \leq \frac{N}{c_1}\nu(U).$$

Hence letting $\alpha_1 = c_1/N$ and $\alpha_2 = c_2 N$, we have

$$\alpha_1 \mu(U) \leq \nu(U) \leq \alpha_2 \mu(U)$$

for any open set $U \subseteq X$. Since μ and ν are Radon measures, this yields (8.1). $\square$

8.2 bi-Lipschitz equivalence of metrics

Under the tightness of weight functions defined below, we will translate bi-Lipschitz equivalence of weight functions to the relations between "balls" and "distances" associated with weight functions in Theorem 8.8. The tightness of a weight function ensures that δ_M^g is comparable with g, i.e the diameter with respect to δ_M^g of K_w is bi-Lipschitz equivalent to g.

Definition 8.5. A weight function g is called tight if and only if for any $M \geq 0$, there exists $c > 0$ such that

$$\sup_{x,y \in K_w} \delta_M^g(x,y) \geq cg(w)$$

for any $w \in T$.

Proposition 8.6. *Let g and h be weight functions. Assume that $g \underset{BL}{\sim} h$. If g is tight and $g \underset{BL}{\sim} h$, then h is tight.*

Proof. Since $g \underset{\mathrm{BL}}{\sim} h$, there exist $\gamma_1, \gamma_2 > 0$ such that $\gamma_1 g(w) \leq h(w) \leq \gamma_2 g(w)$ for any $w \in T$. Therefore,

$$\gamma_1 D_M^g(x,y) \leq D_M^h(x,y) \leq \gamma_2 D_M^g(x,y)$$

for any $x, y \in X$ and $M \geq 0$. By Proposition 6.4, for any $M \geq 0$, there exist $c_1, c_2 > 0$ such that

$$c_1 \delta_M^g(x,y) \leq \delta_M^h(x,y) \leq c_2 \delta_M^g(x,y)$$

for any $x, y \in X$. Hence

$$\sup_{x,y \in K_w} \delta_M^h(x,y) \geq c_1 \sup_{x,y \in K_w} \delta_M^g(x,y) \geq c_1 cg(w) \geq c_1 c(\gamma_2)^{-1} h(w)$$

for any $w \in T$. Thus h is tight. $\square$

Any weight function induced from a metric is tight.

Proposition 8.7. *Let $d \in \mathcal{D}(X, \mathcal{O})$. Then g_d is tight.*

Proof. Let $x, y \in X$ and let $(w(1), \ldots, w(M+1)) \in \mathcal{CH}_K(x,y)$. Set $x_0 = x$ and $x_{M+1} = y$. For each $i = 1, \ldots, M$, choose $x_i \in K_{w(i)} \cap K_{w(i+1)}$. Then

$$\sum_{i=1}^{M+1} g_d(x) \geq \sum_{i=1}^{M+1} d(x_{i-1}, x_i) \geq d(x,y).$$

Using this inequality and Proposition 6.4, we obtain

$$(M+1)\delta_M^g(x,y) \geq D_M^{g_d}(x,y) \geq d(x,y)$$

and therefore $(M+1)\sup_{x,y\in K_w} \delta_M^{g_d}(x,y) \geq g_d(w)$ for any $w \in T$. Thus g_d is tight. □

Now we give geometric conditions which are equivalent to bi-Lipschitz equivalence of tight weight functions. The essential point is that bi-Lipschitz condition between weight function g and h are equivalent to that between $\delta_M^g(\cdot,\cdot)$ and $\delta_M^h(\cdot,\cdot)$ in the usual sense as is seen in (BL2) and (BL3).

Theorem 8.8. *Let g and h be weight functions. Assume that both g and h are tight. Then the following conditions are equivalent:*
(BL) $g \underset{BL}{\sim} h$.
(BL1) *There exist M_1, M_2 and $c > 0$ such that*

$$\delta_{M_1}^g(x,y) \leq c\delta_0^h(x,y) \quad \text{and} \quad \delta_{M_2}^h(x,y) \leq c\delta_0^g(x,y)$$

for any $x, y \in X$.
(BL2) *There exist $c_1, c_2 > 0$ and $M \geq 0$ such that*

$$c_1\delta_M^g(x,y) \leq \delta_M^h(x,y) \leq c_2\delta_M^g(x,y)$$

for any $x, y \in X$.
(BL3) *For any $M \geq 0$, there exist $c_1, c_2 > 0$ such that*

$$c_1\delta_M^g(x,y) \leq \delta_M^h(x,y) \leq c_2\delta_M^g(x,y)$$

for any $x, y \in X$.

Before a proof of this theorem, we state two notable corollaries of it. The first one is the case when weight functions are induced from adapted metrics. In such a case bi-Lipschitz equivalence of weight functions exactly corresponds to the usual bi-Lipschitz equivalence of metrics.

Definition 8.9. (1) Let $d, \rho \in \mathcal{D}(X, \mathcal{O})$. d and ρ are said to be bi-Lipschitz equivalent, $d \underset{\text{BL}}{\sim} \rho$ for short, if and only if there exist $c_1, c_2 > 0$ such that

$$c_1 d(x,y) \leq \rho(x,y) \leq c_2 d(x,y)$$

for any $x, y \in X$.
(2) Define

$$\mathcal{D}_A(X,\mathcal{O}) = \{d | d \in \mathcal{D}(X,\mathcal{O}), d \text{ is adapted.}\}$$

Corollary 8.10. *Let $d, \rho \in \mathcal{D}_A(X,\mathcal{O})$. Then $g_d \underset{BL}{\sim} g_\rho$ if and only if $d \underset{BL}{\sim} \rho$. In particular, the correspondence of $[d]_{\underset{BL}{\sim}}$ with $[g_d]_{\underset{BL}{\sim}}$ gives an well-defined injective map $\mathcal{D}_A(X,\mathcal{O})/_{\underset{BL}{\sim}} \to \mathcal{G}(X)/_{\underset{BL}{\sim}}$.*

The next corollary shows that an adapted metric is adapted to a weight function if and only if they are bi-Lipschitz equivalent in the sense of weight functions.

Corollary 8.11. *Let $d \in \mathcal{D}(X, \mathcal{O})$ and let g be a weight function. Then d is adapted to g and g is tight if and only if $g_d \underset{BL}{\sim} g$ and $d \in \mathcal{D}_A(X, \mathcal{O})$.*

Now we start to prove Theorem 8.8 and its corollaries.

Lemma 8.12. *Let h be a weight function. If $x \in K_w$ and $K_w \backslash U_0^h(x, s) \neq \emptyset$, then $s \leq h(w)$.*

Proof. If $\pi^n(w) \in \Lambda_{s,0}^h(x)$ for some $n \geq 0$, then $U_0^h(x, s) \supseteq K_{\pi^n(w)} \supseteq K_w$. This contradicts the assumption and hence $\pi^n(w) \notin \Lambda_{s,0}^h(x)$ for any $n \geq 0$. Therefore there exists $v \in T_w \cap \Lambda_{s,0}^h(x)$ such that $|v| > |w|$. Then we have $h(w) \geq h(\pi(v)) > s$. □

Proposition 8.13. *Let g and h be weight functions. Assume that g is tight. Let $M \geq 0$. If there exists $\alpha > 0$ such that*

$$\alpha \delta_M^g(x, y) \leq \delta_0^h(x, y) \tag{8.2}$$

for any $x, y \in X$. Then there exists $c > 0$ such that

$$cg(w) \leq h(w)$$

for any $w \in T$.

Proof. Since g is tight, there exists $\beta > 0$ such that, for any $w \in T$,

$$K_w \backslash U_M^g(x, \beta g(w)) \neq \emptyset$$

for some $x \in K_w$. On the other hand, by (8.2), there exists $\gamma > 0$ such that $U_M^g(x, s) \supseteq U_0^h(x, \gamma s)$ for any $x \in X$ and $s \geq 0$. Therefore,

$$K_w \backslash U_0^h(x, \beta\gamma g(w)) \neq \emptyset.$$

By Lemma 8.12, we have $\beta\gamma g(w) \leq h(w)$. □

Lemma 8.14. *Let g and h be weight functions. Assume that g is tight. Then the following conditions are equivalent:*
(A) *There exists $c > 0$ such that $g(w) \leq ch(w)$ for any $w \in T$.*
(B) *For any $M, N \geq 0$ with $N \geq M$, there exists $c > 0$ such that*

$$\delta_N^g(x, y) \leq c\delta_M^h(x, y)$$

for any $x, y \in X$.
(C) *There exist $M, N \geq 0$ and $c > 0$ such that $N \geq M$ and*

$$\delta_N^g(x, y) \leq c\delta_M^h(x, y)$$

for any $x, y \in X$.

Proof. (A) implies

$$D^g_M(x,y) \le cD^h_M(x,y) \tag{8.3}$$

for any $x, y \in X$ and $M \ge 0$. By Proposition 6.4, we see

$$\delta^g_M(x,y) \le c(M+1)\delta^h_M(x,y)$$

for any $x, y \in X$. Since $\delta^g_N(x,y) \le \delta^g_M(x,y)$, if $N \ge M$, then we have (B). Obviously (B) implies (C). Now assume (C). Then we have $\delta^g_N(x,y) \le c\delta^h_0(x,y)$. Hence Proposition 8.13 yields (A). □

Proof of Theorem 8.8. Lemma 8.14 immediately implies the desired statement. □

Proof of Corollary 8.10. Since d and ρ are adapted, by (6.1), there exist $M \ge 1$ and $c > 0$ such that

$$c\delta^d_M(x,y) \le d(x,y) \le \delta^d_M(x,y), \tag{8.4}$$

$$c\delta^\rho_M(x,y) \le \rho(x,y) \le \delta^\rho_M(x,y) \tag{8.5}$$

for any $x, y \in X$. Assume $g_d \underset{\mathrm{BL}}{\sim} g_\rho$. Since g_d and g_ρ are tight, we have (BL3) by Theorem 8.8. Hence by (8.4) and (8.5), $d(\cdot,\cdot)$ and $\rho(\cdot,\cdot)$ are bi-Lipschitz equivalent as metrics. The converse direction is immediate. □

Proof of Corollary 8.11. If d is M-adapted to g for some $M \ge 1$, then by (ADa), there exists $c > 0$ such that $d_w \le cg(w)$ for any $w \in K_w$. Moreover, (6.1) implies

$$d(x,y) \ge c_2\delta^g_M(x,y)$$

for any $x, y \in X$, where c_2 is independent of x and y. Hence the tightness of g shows that there exists $c' > 0$ such that

$$d_w \ge c_2 \sup_{x,y} \delta^g_M(x,y) \ge c'g(w).$$

Thus $g_d \underset{\mathrm{BL}}{\sim} g$. Moreover, by Proposition 6.8, d is adapted. Conversely, assume that d is M-adapted and $g_d \underset{\mathrm{BL}}{\sim} g$. Then Theorem 8.8 implies (BL3) with $h = g_d$. At the same time, since d is M-adapted, we have (6.2) with $g = g_d$. Combining these two, we deduce (6.2). Hence d is M-adapted to g. □

8.3 bi-Lipschitz equivalence between measures and metrics

Finally in this section, we consider what happens if the weight function associated with a measure is bi-Lipschitz equivalent to the weight function associated with a metric.

To state our theorem, we need the following notions.

Definition 8.15. (1) A weight function $g : T \to (0, 1]$ is said to be uniformly finite if $\sup\{\#(\Lambda^g_{s,1}(w))|s \in (0, 1], w \in \Lambda^g_s\} < +\infty$.
(2) A function $f : T \to (0, \infty)$ is called sub-exponential if and only if there exist $m \geq 0$ and $c_1 \in (0, 1)$ such that $f(v) \leq c_1 f(w)$ for any $w \in T$ and $v \in T_w$ with $|v| \geq |w| + m$. f is called super-exponential if and only if there exists $c_2 \in (0, 1)$ such that $f(v) \geq c_2 f(w)$ for any $w \in T$ and $v \in S(w)$. f is called exponential if it is both sub-exponential and super-exponential.

The following proposition and the lemma are immediate consequences of the above definitions.

Proposition 8.16. *Let h be a weight function. Then h is super-exponential if and only if there exists $c \geq 1$ such that $ch(w) \geq s \geq h(w)$ whenever $w \in \Lambda^h_s$.*

Proof. Assume that h is super-exponential. Then there exists $c_2 < 1$ such that $h(w) \geq c_2 h(\pi(w))$ for any $w \in T$. If $w \in \Lambda^h_s$, then $h(\pi(w)) > s \geq h(w)$. This implies $(c_2)^{-1}h(w) \geq s \geq h(w)$.

Conversely, assume that $ch(w) \geq s \geq h(w)$ for any w and s with $w \in \Lambda^h_s$. If $h(\pi(w)) > ch(w)$, then $w \in \Lambda^h_t$ for any $t \in (ch(w), h(\pi(w))$. This contradicts the assumption that $ch(w) \geq t \geq h(w)$. Hence $h(\pi(w)) \leq ch(w)$ for any $w \in T$. Thus h is super-exponential. □

Lemma 8.17. *If a weight function $g : T \to (0, 1]$ is uniformly finite, then*

$$\sup\{\#(\Lambda_{s,M}(x))|x \in X, s \in (0, 1]\} < +\infty$$

for any $M \geq 0$.

Proof. Let $C = \sup\{\#(\Lambda_{s,1}(w))|s \in (0, 1], w \in \Lambda_s\}$. Then $\#(\Lambda_{s,M}(x)) \leq C + C^2 + \ldots + C^{M+1}$. □

Definition 8.18. Let $\alpha > 0$. A radon measure μ on X is said to be α-Ahlfors regular with respect to $d \in \mathcal{D}(X, \mathcal{O})$ if and only if there exist $C_1, C_2 > 0$ such that

$$C_1 r^\alpha \leq \mu(B_d(x, r)) \leq C_2 r^\alpha \tag{8.6}$$

for any $r \in [0, \mathrm{diam}(X, d)]$.

Definition 8.19. Let $g : T \to (0, 1]$ be a weight function. We say that K has thick interior with respect to g, or g is thick for short, if and only if there exist $M \geq 1$ and $\alpha > 0$ such that $K_w \supseteq U^g_M(x, \alpha s)$ for some $x \in K_w$ if $s \in (0, 1]$ and $w \in \Lambda^g_s$.

The value of the integer $M \geq 1$ is not crucial in the above definition. In Proposition 9.1, we will show if the condition of the above definition holds for a particular $M \geq 1$, then it holds for all $M \geq 1$.

The thickness is invariant under the bi-Lipschitz equivalence of weight functions as follows.

Proposition 8.20. *Let g and h be weight functions. If g is thick and $g \underset{BL}{\sim} h$, then h is thick.*

Since we need further results on thickness of weight functions, we postpone a proof of this proposition until the next section.

Now we give the main theorem of this sub-section.

Theorem 8.21. *Let $d \in \mathcal{D}_A(X, \mathcal{O})$ and let $\mu \in \mathcal{M}_P(X, \mathcal{O})$. Assume that K is minimal and g_d is super-exponential and thick. Then $(g_d)^\alpha \underset{BL}{\sim} g_\mu$ and g_d is uniformly finite if and only if μ is α-Ahlfors regular with respect to d. Moreover, if either/both of the these two conditions is/are satisfied, then g_μ and g_d are exponential.*

By the same reason as Proposition 8.20, a proof of this theorem will be given at the end of Section 10.

9 Thickness of weight functions

In this section, we study conditions for a weight function being thick and relation between the notions "thick" and "tight". For instance in Theorem 9.3 we present topological condition (TH1) ensuring that all super-exponential weight functions are thick. In particular, this is the case for partitions of S^2 discussed in Section 2 because partitions satisfying (2.2) are minimal and the condition (TH) in Section 2 yields the condition (TH1). Moreover in this case, all super-exponential weight functions are tight as well by Corollary 9.5.

Proposition 9.1. *g is thick if and only if for any $M \geq 0$, there exists $\beta > 0$ such that, for any $w \in T$, $K_w \supseteq U^g_M(x, \beta g(\pi(w)))$ for some $x \in K_w$.*

Proof. Assume that g is thick. By induction, we are going to show the following claim $(C)_M$ holds for any $M \geq 1$:
$(C)_M$ There exists $\alpha_M > 0$ such that, for any $s \in (0, 1]$ and $w \in \Lambda^g_s$, one find $x \in K_w$ satisfying $K_w \supseteq U_M(x, \alpha_M s)$.
Proof of $(C)_M$. Since g is thick, $(C)_M$ holds for some $M \geq 1$. Since $U^g_1(x, s) \subseteq U^g_M(x, s)$ if $M \geq 1$, $(C)_1$ holds as well. Now, suppose that $(C)_M$ holds. Let $w \in \Lambda^g_s$ and choose x as in $(C)_M$. Then there exists $v \in \Lambda^g_{\alpha_M s}$ such that $v \in T_w$ and $x \in K_v$. Applying $(C)_M$ again, we find $y \in K_v$ such that $K_v \supseteq U^g_M(y, (\alpha_M)^2 s)$. Since $M \geq 1$, it follows that $U^g_{M+1}(y, (\alpha_M)^2 s) \subseteq U^g_M(x, \alpha_M s) \subseteq K_w$. Therefore, letting $\alpha_{M+1} = (\alpha_M)^2$, we have obtained $(C)_{M+1}$. Thus we have shown $(C)_M$ for any $M \geq 1$. $\square$

Next, fix $M \geq 1$ and write $\alpha = \alpha_M$. Note that $w \in \Lambda^g_s$ if and only if $g(w) \leq s < g(\pi(w))$. Fix $\epsilon \in (0, 1)$. Assume that $g(\pi(w)) > g(w)$. There exists s_* such that $g(w) \leq s_* < g(\pi(w))$ and $s_* > \epsilon g(\pi(w))$. Hence we obtain

$$K_w \supseteq U^g_M(x, \alpha s_*) \supseteq U^g_M(x, \alpha\epsilon g(\pi(w))).$$

If $g(w) = g(\pi(w))$, then there exists $v \in T_w$ such that $g(v) < g(\pi(v)) = g(w) = g(\pi(w))$. Choosing s_* so that $g(v) \leq s_* < g(\pi(v)) = g(\pi(w))$ and $\epsilon g(\pi(w)) < s_*$, we obtain

$$K_w \supseteq K_v \supseteq U^g_M(x, \alpha s_*) \supseteq U^g_M(x, \alpha\epsilon g(\pi(w))).$$

Letting $\beta = \alpha\epsilon$, we obtain the desired statement.
Conversely, assume that for any $M \geq 0$, there exists $\beta > 0$ such that, for any $w \in T$, $K_w \supseteq U^g_M(x, \beta g(\pi(w)))$ for some $x \in K_w$. If $w \in \Lambda_s$, then $g(w) \leq s < g(\pi(w))$. Therefore $K_w \supseteq U^g_M(x, \beta s)$. This implies that g is thick. □

Proposition 9.2. *Assume that K is minimal. Let $g : T \to (0, 1]$ be a weight function. Then g is thick if and only if, for any $M \geq 0$, there exists $\gamma > 0$ such that, for any $w \in T$, $O_w \supseteq U^g_M(x, \gamma g(\pi(w)))$ for some $x \in O_w$.*

Proof. Assume that g is thick. By Proposition 9.1, for any $M \geq 0$, we may choose $\alpha > 0$ so that for any $w \in T$, there exists $x \in K_w$ such that $K_w \supseteq U^g_{M+1}(x, \alpha g(\pi(w)))$. Set $s_w = g(\pi(w))$. Let $y \in U^g_M(x, \alpha s_w)\backslash O_w$. There exists $v \in (T)_{|w|}$ such that $y \in K_v$ and $w \neq v$. Then we find $v_* \in T_v \cap \Lambda^g_{\alpha s_w}$ satisfying $y \in K_{v_*}$. Since $K_{v_*} \cap U^g_M(x, \alpha s_w) \neq \emptyset$, we have

$$K_{v_*} \subseteq U^g_{M+1}(x, \alpha s_w) \subseteq K_w.$$

Therefore, $K_{v_*} \subseteq \cup_{w' \in T_w, |w'|=|v_*|} K_{w'}$. This implies that $O_{v_*} = \emptyset$, which contradicts the fact that K is minimal. So, $U^g_M(x, \alpha s_w)\backslash O_w = \emptyset$ and hence $U^g_M(x, \alpha s_w) \subseteq O_w$.
The converse direction is immediate. □

Using the above proposition, we give a proof of Proposition 8.20.

Proof of Proposition 8.20. By Proposition 9.1, there exists $\beta > 0$ such that for any $w \in T$, $K_w \supseteq U^g_M(x, \beta g(\pi(w)))$ for some $x \in K_w$. On the other hand, since there exist $c_1, c_2 > 0$ such that $c_1 h(w) \leq g(w) \leq c_2 h(w)$ for any $w \in T$. It follows that $D^g_M(x, y) \leq c_2 D^h_M(x, y)$ for any $x, y \in X$. Proposition 6.4 implies that there exists $\alpha > 0$ such that $\alpha\delta^g_M(x, y) \leq \delta^h_M(x, y)$ for any $x, y \in X$. Hence $U^h_M(x, \alpha s) \subseteq U^g_M(x, s)$ for any $x \in X$ and $s \in (0, 1]$. Combining them, we see that

$$K_w \supseteq U^g_M(x, \beta g(\pi(w))) \supseteq U^h_M(x, \alpha\beta g(\pi(w))) \supseteq U^h(x, \alpha\beta c_2 h(\pi(w))).$$

Thus by Proposition 9.1, h is thick. □

Theorem 9.3. *Assume that K is minimal. Define $h_* : T \to (0, 1]$ by $h_*(w) = 2^{-|w|}$ for any $w \in T$. Then the following conditions are equivalent:*
(TH1)

$$\sup_{w \in T} \min \left\{ |v_*| - |w| \middle| v_* \in T_w, K_{v_*} \subseteq O_w \right\} < \infty.$$

(TH2) *Every super-exponential weight function is thick.*
(TH3) *There exists a sub-exponential weight function which is thick.*
(TH4) *The weight function h_* is thick.*

Proof. (TH1) $\Rightarrow$ (TH2): Assume (TH1). Let m be the supremum in (TH1). Let g be a super-exponential weight function. Then there exists $\lambda \in (0, 1)$ such that $g(w) \geq \lambda g(\pi(w))$ for any $w \in T$. Let $w \in \Lambda_s^g$. By (TH1), there exists $v_* \in T_w \cap (T)_{|w|+m}$ such that $K_{v_*} \subseteq O_w$. For any $v \in T_w \cap (T)_{|w|+m}$,

$$g(v) \geq \lambda^m g(w) \geq \lambda^{m+1} g(\pi(w)) > \lambda^{m+1} s$$

Choose $x \in O_{v_*}$. Let $u \in \Lambda_{\lambda^{m+1}s,1}(x)$. Then there exists $v' \in \Lambda_{\lambda^{m+1}s,0}(x)$ such that $K_{v'} \cap K_u \neq \emptyset$. Since $g(v_*) > \lambda^{m+1}s$ and $x \in O_{v_*}$, it follows that $v' \in T_{v_*}$. Therefore $K_u \cap O_w \supseteq K_u \cap K_{v_*} \neq \emptyset$. This implies that either $u \in T_w$ or $w \in T_u$. Since $g(w) > \lambda^{m+1}s$, it follows that $u \in T_w$. Thus we have shown that $\Lambda_{\lambda^{m+1}s,1}(x) \subseteq T_w$. Hence

$$U_1^g(x, \lambda^{m+1}s) \subseteq K_w.$$

This shows that g is thick.
(TH2) $\Rightarrow$ (TH4): Apparently h_* is an exponential weight function. Hence by (TH2), it is thick.
(TH4) $\Rightarrow$ (TH3): Since h_* is exponential and thick, we have (TH3).
(TH3) $\Rightarrow$ (TH1): Assume that g is a sub-exponential weight function which is thick. Proposition 9.2 shows that there exist $\gamma \in (0,1)$ and $M \geq 1$ such that for any $w \in T$, $O_w \supseteq U_M^g(x, \gamma g(\pi(w)))$. Choose $v_* \in \Lambda^g_{\gamma g(\pi(w)),0}(x)$. Then $K_{v_*} \subseteq U_M^g(x, \gamma g(\pi(w))) \subseteq O_w$ and $g(\pi(v_*)) > \gamma g(\pi(w)) \geq \gamma g(w)$. Since g is sub-exponential, there exists $k \geq 1$ and $\eta \in (0,1)$ such that $g(u) \leq \eta g(v)$ if $v \in T_v$ and $|u| \geq |w| + k$. Choose l so that $\eta^l < \gamma$ and set $m = kl + 1$. Since $g(\pi(v_*)) > \eta^l g(w)$, we see that $|\pi(v_*)| \leq |w| + m - 1$. Therefore, $|v_*| \leq |w| + m$ and hence we have (TH1). $\square$

Theorem 9.4. *Assume that $K : T \to \mathcal{C}(X, \mathcal{O})$ is minimal, that there exists $\lambda \in (0,1)$ such that if $B_w = \emptyset$, then $\#(T_w \cap \Lambda^g_{\lambda g(w)}) \geq 2$ and that g is thick. Then g is tight.*

Proof. By Proposition 9.2, there exists γ such that, for any $v \in T$, $O_v \supseteq U_M^g(x, \gamma g(\pi(v)))$ for some $x \in K_v$. First suppose that $B_w \neq \emptyset$. Then there exists $x \in K_w$ such that $O_w \supseteq U_M^g(x, \gamma g(\pi(w)))$. For any $y \in B_w$, it follows that $\delta_M^g(x,y) > \gamma g(\pi(w))$. Thus

$$\sup_{x,y \in K_w} \delta_M^g(x,y) \geq \gamma g(\pi(w)).$$

Next if $B_w = \emptyset$, then there exists $u \neq v \in T_w \cap \Lambda^g_{\lambda g(w)}$. If $B_u \neq \emptyset$, then the above discussion implies

$$\sup_{x,y \in K_w} \delta_M^g(x,y) \geq \sup_{x,y \in K_v} \delta_M^g(x,y) \geq \gamma g(\pi(v)) \geq \gamma \lambda g(w).$$

If $B_u = \emptyset$, then $\delta_M^g(x,y) \geq \lambda g(w)$ for any $(x,y) \in K_u \times K_v$. Thus for any $w \in T$, we conclude that

$$\sup_{x,y \in K_w} \delta_M^g(x,y) \geq \gamma \lambda g(w).$$

$\square$

The above theorem immediately implies the following corollary.

Corollary 9.5. *Assume that $(X, \mathcal{O})$ is connected and K is minimal. If g is thick, then g is tight.*

10 Volume doubling property

In this section, we introduce the notion of a relation called "gentle" written as $\underset{\text{GE}}{\sim}$ between weight functions. This relation is not an equivalence relation in general. In Section 12, however, it will be shown to be an equivalence relation among exponential weight functions. As was the case of the bi-Lipschitz equivalence, the gentleness will be identified with other properties in classes of weight functions. In particular, we are going to show that the volume doubling property of a measure with respect to a metric is equivalent to the gentleness of the associated weight functions.

As in the previous sections, $(T, \mathcal{A}, \phi)$ is a locally finite tree with a reference point ϕ, $(X, \mathcal{O})$ is a compact metrizable topological space with no isolated point and $K : T \to \mathcal{C}(X, \mathcal{O})$ is a partition of X parametrized by $(T, \mathcal{A}, \phi)$.

The notion of gentleness of a weight function to another weight function is defined as follows.

Remark. In the case of the natural partition of a self-similar set in Example 4.5, the main results of this section, Theorems 10.7 and 10.9 have been obtained in [16].

Definition 10.1. Let $g : T \to (0, 1]$ be a weight function. A function $f : T \to (0, \infty)$ is said to be gentle with respect to g if and only if there exists $c_G > 0$ such that $f(v) \leq c_G f(w)$ whenever $w, v \in \Lambda^g_s$ and $K_w \cap K_v \neq \emptyset$ for some $s \in (0, 1]$. We write $f \underset{\text{GE}}{\sim} g$ if and only if f is gentle with respect to g.

Alternatively, we have a simpler version of the definition of gentleness under a mild restriction.

Proposition 10.2. *Let $g : T \to (0, 1]$ be an exponential weight function. Let $f : T \to (0, \infty)$. Assume that $f(w) \leq f(\pi(w))$ for any $w \in T$ and f is super-exponential. Then f is gentle with respect to g if and only if there exists $c > 0$ such that $f(v) \leq cf(w)$ whenever $g(v) \leq g(w)$ and $K_v \cap K_w \neq \emptyset$.*

Proof. By the assumption, there exist $c_1, c_2 > 0$ and $m \geq 1$ such that $f(v) \geq c_2 f(w)$, $g(v) \geq c_2 g(w)$ and $g(u) \leq c_1 g(w)$ for any $w \in T$, $v \in S(w)$ and $u \in T_w$ with $|u| \geq |w| + m$.

First suppose that f is gentle with respect to g. Then there exists $c > 0$ such that $f(v') \leq cf(w')$ whenever $w', v' \in \Lambda^g_s$ and $K_{w'} \cap K_{v'} \neq \emptyset$ for some $s \in (0, 1]$. Assume that $g(v) \leq g(w)$ and $K_v \cap K_w \neq \emptyset$. There exists $u \in T_w$ such that $K_u \cap K_v \neq \emptyset$ and $g(\pi(u)) > g(v) \geq g(u)$. Moreover, $g(\pi([v]_m)) > g([v]_m) = g(v)$ for some $m \in [0, |v|]$. Then $[v]_m, u \in \Lambda^g_{g(v)}$ and hence $f(v) \leq f([v]_m) \leq cf(u) \leq cf(w)$.

Conversely, assume that $f(v') \leq cf(w')$ whenever $g(v') \leq g(w')$ and $K_{v'} \cap K_{w'} \neq \emptyset$. Let $w, v \in \Lambda_s^g$ with $K_w \cap K_v \neq \emptyset$. If $g(v) \leq g(w)$, then $f(v) \leq cf(w)$. Suppose that $g(v) > g(w)$. Since g is super-exponential, we see that

$$s \geq g(w) \geq c_2 g(\pi(w)) \geq c_2 s \geq c_2 g(v).$$

Set $N = \min\{n | c_2 \geq c_1^n\}$. Choose $u \in T_v$ so that $K_u \cap K_w \neq \emptyset$ and $|u| = |v| + Nm$. Then $g(w) \geq c_2 g(v) \geq (c_1)^N g(v) \geq g(u)$. This implies $f(u) \leq cf(w)$. Since $f(u) \geq (c_2)^{Nm} f(v)$, we have $f(v) \leq c(c_2)^{-Nm} f(w)$. Therefore, f is gentle with respect to g. $\square$

The following is the standard version of the definition of the volume doubling property.

Definition 10.3. Let μ be a radon measure on $(X, \mathcal{O})$ and let $d \in \mathcal{D}(X, O)$. μ is said to have the volume doubling property with respect to the metric d if and only if there exists $C > 0$ such that

$$\mu(B_d(x, 2r)) \leq C\mu(B_d(x, r))$$

for any $x \in X$ and any $r > 0$.

Since $(X, \mathcal{O})$ has no isolated point, if a Radon measure μ has the volume doubling property with respect to some $d \in \mathcal{D}(X, \mathcal{O})$, then the normalized version of μ, $\mu/\mu(X)$, belongs to $\mathcal{M}_P(X, O)$. Taking this fact into account, we are mainly interested in (normalized version of) a Radon measure in $\mathcal{M}_P(X, \mathcal{O})$.

The main theorem of this section is as follows.

Theorem 10.4. *Let $d \in \mathcal{D}(X, \mathcal{O})$ and let $\mu \in \mathcal{M}_P(X, \mathcal{O})$. Assume that d is adapted, that g_d is thick, exponential and uniformly finite and that μ is exponential. Then μ has the volume doubling property with respect to d if and only if $g_d \underset{\mathrm{GE}}{\sim} g_\mu$.*

So, this says that the volume doubling property equals the gentleness in the world of weight functions having certain regularities. This theorem is an immediate corollary of Theorem 10.7 and 10.9.

To describe a refined version of Theorem 10.4, we define the notion of volume doubling property of a measure with respect to a weight function g as well by means of balls "$U_M^g(x, s)$".

Definition 10.5. Let $\mu \in \mathcal{M}_P(X, \mathcal{O})$ and let g be a weight function. For $M \geq 1$, we say μ has M-volume doubling property with respect to g if and only if there exist $\gamma \in (0, 1)$ and $\beta \geq 1$ such that $\mu(U_M^g(x, s)) \leq \beta\mu(U_M^g(x, \gamma s))$ for any $x \in X$ and any $s \in (0, 1]$.

It is rather annoying that the notion of "volume doubling property" of a measures with respect to a weight function depends on the value $M \geq 1$. Under certain conditions including exponentiality and the thickness, however, we will show that if μ has M-volume doubling property for some $M \geq 1$, then it has M-volume doubling property for all $M \geq 1$ in Theorem 10.9.

Naturally, if a metric is adapted to a weight function, the volume doubling with respect to the metric and that with respect to the weight function are virtually the same as is seen in the next proposition.

Proposition 10.6. *Let $d \in \mathcal{D}(X, \mathcal{O})$, let $\mu \in M_P(X, \mathcal{O})$ and let g be a weight function. Assume that d is adapted to g. Then μ has the volume doubling property with respect to d if and only if there exists $M_* \geq 1$ such that μ has M-volume doubling property with respect to g for any $M \geq M_*$.*

Proof. Since d is adapted to g, for sufficiently large M, there exist $\alpha_1, \alpha_2 > 0$ such that

$$U^g_M(x, \alpha_1 s) \subseteq B_d(x, s) \subseteq U^g_M(x, \alpha_2 s)$$

for any $x \in X$ and $s \in (0, 1]$. Suppose that μ has the volume doubling property with respect to d. Then there exists $\lambda > 1$ such that

$$\mu(B_d(x, 2^m r)) \leq \lambda^m \mu(B_d(x, r))$$

for any $x \in X$ and $r \geq 0$. Hence

$$\mu(U^g_M(x, \alpha_1 2^m r)) \subseteq \lambda^m \mu(U^g_M(x, \alpha_2 r)).$$

Choosing m so that $\alpha_1 2^m > \alpha_2$, we see that μ has M-volume doubling property with respect to g if M is sufficiently large. Converse direction is more or less similar. $\square$

By the above proposition, as far as we confine ourselves to adapted metrics, it is enough to consider the volume doubling property of a measure with respect to a weight function. Thus we are going to investigate relations between "the volume doubling property with respect to a weight function" and other conditions like a weight function being exponential, a weight function being uniformly finite, a measure being super-exponential, and a measure being gentle with respect to a weight function. To begin with, we show that the last four conditions imply the volume doubling property of μ with respect to g.

Theorem 10.7. *Let $g : T \to (0, 1]$ be a weight function and let $\mu \in \mathcal{M}_P(X, \mathcal{O})$. Assume that g is exponential, that g is uniformly finite, that μ is gentle with respect to g and that μ is super-exponential. Then μ has M-volume doubling property with respect to g for any $M \geq 1$.*

Hereafter in this section, we are going to omit g in notations if no confusion may occur. For example, we write $\Lambda_s, \Lambda_{s,M}(w), \Lambda_{s,M}(w)$ and $U_M(x, s)$ in place of $\Lambda^g_s, \Lambda^g_{s,M}(w), \Lambda^g_{s,M}(x)$ and $U^g_M(x, s)$ respectively.

The following lemma is a step to prove the above theorem.

Lemma 10.8. *Let $g : T \to (0, 1]$ be a weight function and let $\mu \in \mathcal{M}_P(X, \mathcal{O})$. For $s \in (0, 1]$, $\lambda > 1$ and $c > 0$, define*

$$\Theta(s, \lambda, k, c) = \{v | v \in \Lambda_s, \mu(K_u) \leq c\mu(K_v) \text{ for any } u \in \Lambda_{\lambda s, k}((v)_{\lambda s})\},$$

where $(v)_{\lambda s}$ is the unique element of $\{[v]_n | 0 \leq n \leq |v|\} \cap \Lambda_{\lambda s}$. Assume that g is uniformly finite and that there exist $N \geq 1, \lambda > 1$ and $c > 0$ such that $\Lambda_{s,N}(w) \cap \Theta(s, \lambda, 2N+1, c) \neq \emptyset$ for any $s \in (0, 1]$ and $w \in \Lambda_s$. Then μ has the N-volume doubling property with respect to g.

Proof. Let $w \in \Lambda_{s,0}(x)$ and let $v \in \Lambda_{s,N}(w) \cap \Theta(s, \lambda, 2N+1, c)$. If $u \in \Lambda_{\lambda s,N}(x)$, then $u \in \Lambda_{\lambda s, 2N+1}((v)_{\lambda s})$. Moreover, since $v \in \Lambda_{s,N}(x)$, we see that

$$\mu(K_u) \leq c\mu(K_v) \leq c\mu(U_N(x, s)).$$

Therefore,

$$\mu(U_N(x, \lambda s)) \leq \sum_{u \in \Lambda_{\lambda s,N}(x)} \mu(K_u) \leq \#(\Lambda_{\lambda s,N}(x)) c\mu(U_N(x, s)).$$

Since g is uniformly finite, Lemma 8.17 shows that $\#(\Lambda_{\lambda s,N}(x))$ is uniformly bounded with respect to $x \in X$ and $s \in (0, 1]$. $\square$

Proof of Theorem 10.7. Fix $\lambda > 1$. By Proposition 8.16, there exists $c \geq 1$ such that $cg(w) \geq s \geq g(w)$ if $w \in \Lambda_s$. Since g is sub-exponential, there exist $c_1 \in (0, 1)$ and $m \geq 1$ such that $c_1 g(w) \geq g(v)$ whenever $v \in T_w$ and $|v| \geq |w| + m$. Assume that $w \in \Lambda_s$. Set $w_* = (w)_{\lambda s}$. Then $\lambda s \geq g(w_*)$. If $|w| \geq |w_*| + nm$, then $(c_1)^n g(w_*) \geq g(w)$ and hence $(c_1)^n \lambda s \geq g(w) \geq g(w)/c$. This shows that $(c_1)^n \lambda c \geq 1$. Set $l = \min\{n | n \geq 0, (c_1)^n \lambda c < 1\}$. Then we see that $|w| < |w_*| + lm$.

On the other hand, since μ is super-exponential, there exists $c_2 > 0$ such that $\mu(K_u) \geq c_2 \mu(K_{\pi(u)})$ for any $u \in T$. This implies that $\mu(K_{w_*}) \leq (c_2)^{-ml} \mu(K_w)$. Since μ is gentle, there exists $c_* > 0$ such that $\mu(K_{w(1)}) \leq c_* \mu(K_{w(2)})$ whenever $w(1), w(2) \in \Lambda_s$ and $K_{w(1)} \cap K_{w(2)} \neq \emptyset$ for some $s \in (0, 1]$. Therefore for any $u \in \Lambda_{\lambda s,M}(w_*)$,

$$\mu(K_u) \leq (c_*)^M \mu(K_{w_*}) \leq (c_*)^M (c_2)^{-ml} \mu(K_w).$$

Thus we have shown that

$$\Lambda_s = \Theta(s, \lambda, M, (c_*)^M (c_2)^{-ml})$$

for any $s \in (0, 1]$. Now by Lemma 10.8, μ has M-volume doubling property with respect to g for any $M \geq 1$. $\square$

In order to study the converse direction of Theorem 10.7, we need the thickness of K with respect to the weight function in question.

Theorem 10.9. *Let $g : T \to (0, 1]$ be a weight function and let $\mu \in \mathcal{M}_P(X, \mathcal{O})$. Assume that g is thick.*
(1) *Suppose that g is exponential and uniformly finite. Then the following conditions are equivalent:*
(VD1) *μ has M-volume doubling property with respect to g for some $M \geq 1$.*
(VD2) *μ has M-volume doubling property with respect to g for any $M \geq 1$.*

(VD3) *μ is gentle with respect to g and μ is super-exponential.*
(2) *Suppose that K is minimal and g is super-exponential. Then* (VD1), (VD2) *and the following condition* (VD4) *are equivalent:*
(VD4) *g is sub-exponential and uniformly finite, μ is gentle with respect to g and μ is super-exponential.*
Moreover, if any of the above conditions (VD1), (VD2) *and* (VD4) *hold, then μ is exponential and*

$$\sup_{w \in T} \#(S(w)) < +\infty.$$

In general, the statement of Theorem 10.9 is false if g is not thick. In fact, in Example 11.10, we will present an example without thickness where d is adapted to g, g is exponential and uniformly finite, μ has the volume doubling property with respect to g but μ is neither gentle to g nor super-exponential.

As for a proof of Theorem 10.9, it is enough to show the following theorem.

Theorem 10.10. *Let $g : T \to (0, 1]$ be a weight function and let $\mu \in \mathcal{M}_P(X, \mathcal{O})$. Assume that μ has M-volume doubling property with respect to g for some $M \geq 1$.*
(1) *If g is thick, then μ is gentle with respect to g.*
(2) *If g is thick and g is super-exponential, then μ is super-exponential.*
(3) *If g is thick and K is minimal, then g is uniformly finite.*
(4) *If g is thick, K is minimal, and μ is super-exponential, then*

$$\sup_{w \in T} \#(S(w)) < +\infty.$$

and μ is sub-exponential.
(5) *If g is uniformly finite, μ is gentle with respect to g, μ is sub-exponential, then g is sub-exponential.*

To prove Theorem 10.10, we need several lemmas.

Lemma 10.11. *Let $g : T \to (0, 1]$ be a weight function. Assume that K is minimal and g is thick. Let $\mu \in \mathcal{M}_P(X, \mathcal{O})$. If μ has M-volume doubling property with respect to g for some $M \geq 1$, then there exists $c > 0$ such that $\mu(O_w) \geq c\mu(K_w)$ for any $w \in T$.*

Proof. By Proposition 9.2, there exists $\gamma > 0$ such that $O_v \supseteq U_M^g(x, \gamma s)$ for some $x \in K_v$ if $v \in \Lambda_s$. Let $w \in T$. Choose $u \in T_w$ such that $u \in \Lambda_{g(w)/2}$. Then

$$\mu(O_w) \geq \mu(O_u) \geq \mu(U_M^g(x, \gamma g(w)/2)).$$

Since μ has M-volume doubling property with respect to g, there exists $c > 0$ such that

$$\mu(U_M^g(y, \gamma r/2)) \geq c\mu(U_M^g(y, r))$$

for any $y \in X$ and $r > 0$. Since $U_M(x, g(w)) \supseteq K_w$, it follows that

$$\mu(O_w) \geq \mu(U_M^g(x, \gamma g(w)/2)) \geq c\mu(U_M(x, g(w))) \geq c\mu(K_w).$$

□

Lemma 10.12. *Let $g : T \to (0,1]$ be a weight function. Assume that $\mu \in \mathcal{M}_P(X, \mathcal{O})$ is gentle with respect to g and that g is uniformly finite. Then there exists $c > 0$ such that*

$$c\mu(K_w) \geq \mu(U_M(x,s))$$

if $w \in \Lambda_{s,0}(x)$.

Proof. Since μ is gentle with respect to g, there exists $c_1 > 0$ such that $\mu(K_v) \leq c_1\mu(K_w)$ if $w \in \Lambda_s$ and $v \in \Lambda_{s,1}(w)$. Hence if $v \in \Lambda_{s,M+1}(w)$, then it follows that $\mu(K_v) \leq (c_1)^{M+1}\mu(K_w)$. Since $\Lambda_{s,M}(x) \subseteq \Lambda_{s,M+1}(w)$,

$$\begin{aligned}\mu(U_M(x,s)) \leq &\sum_{v\in\Lambda_{s,M}(x)} \mu(K_v)\\ &\leq \sum_{v\in\Lambda_{s,M}(x)} (c_1)^{M+1}\mu(K_w) = (c_1)^{M+1}\#(\Lambda_{s,M}(x))\mu(K_w).\end{aligned}$$

By Lemma 8.17, we obtain the desired statement. □

Proof of Theorem 10.10. (1) Since g is thick, there exists $\beta \in (0,1)$ such that, for any $s \in (0,1]$ and $w \in \Lambda_s$, $K_w \supseteq U_M(x, \beta s)$ for some $x \in K_w$. By M-volume doubling property of μ, there exists $c > 0$ such that $\mu(U_M(x,\beta s)) \geq c\mu(U_M(x,s))$ for any $s \in (0,1]$ and $x \in X$. Hence

$$\mu(K_w) \geq \mu(U_M(x,\beta s)) \geq c\mu(U_M(x,s)). \tag{10.1}$$

If $v \in \Lambda_s$ and $K_v \cap K_w \neq \emptyset$, then $U_M(x,s) \supseteq K_v$. (10.1) shows that $\mu(K_w) \geq c\mu(K_v)$. Hence μ is gentle with respect to g.
(2) Let $v \in T\backslash\{\phi\}$. Choose $u \in T_v$ so that $u \in \Lambda_{g(v)/2}$. Applying (10.1) to u and using the volume doubling property repeatedly, we see that there exists $x \in K_u$ such that

$$\mu(K_v) \geq \mu(K_u) \geq \mu(U_M(x, \beta g(v)/2)) \geq c^n\mu(U_M(x, \beta^{1-n}g(v)/2)) \tag{10.2}$$

for any $n \geq 0$. Since g is super-exponential, there exists $n \geq 0$, which is independent of v, such that $\beta^{1-n}g(v)/2 > g(\pi(v))$. By (10.2), we obtain $\mu(K_v) \geq c^n\mu(K_{\pi(v)})$. Thus μ is super-exponential.
(3) Let $w \in \Lambda_s$. Then $\{O_v\}_{v\in\Lambda_{s,1}(w)}$ is mutually disjoint by Lemma 4.2-(2). By (10.1) and Lemma 10.11,

$$\mu(K_w) \geq c\mu(U_M(x,s)) \geq c\sum_{v\in\Lambda_{s,1}(w)} \mu(O_v) \geq c^2\sum_{v\in\Lambda_{s,1}(w)} \mu(K_v)$$

As μ is gentle with respect to g by (1), there exists $c_* > 0$, which is independent of w and s, such that $\mu(K_v) \geq c_*\mu(K_w)$ for any $v \in \Lambda_{s,1}(w)$. Therefore,

$$\mu(K_w) \geq c^2\sum_{v\in\Lambda_{s,1}(w)} \mu(K_v) \geq c^2c_*\#(\Lambda_{s,1}(w))\mu(K_w)$$

Hence $\#(\Lambda_{s,1}(w)) \leq c^{-2}(c_*)^{-1}$ and g is uniformly finite.
(4) By Lemma 10.11, for any $w \in T$, we have

$$\mu(K_w) \geq \mu(\cup_{v \in S(w)} O_v) = \sum_{v \in S(w)} \mu(O_v) \geq c \sum_{v \in S(w)} \mu(K_v).$$

Since μ is super-exponential, there exists $c' > 0$ such that $\mu(K_v) \geq c'\mu(K_w)$ if $w \in T$ and $v \in S(w)$. Hence

$$\mu(K_w) \geq c \sum_{v \in S(w)} \mu(K_v) \geq cc'\#(S(w))\mu(K_w).$$

Thus $\#(S(w)) \leq (cc')^{-1}$, which is independent of w. By the above arguments,

$$\mu(O_v) \geq c\mu(K_v) \geq c_*\mu(K_w) \geq c_*\mu(O_w) \tag{10.3}$$

for any $w \in T$ and $v \in S(w)$, where $c_* = cc'$. Let $v_* \in S(w)$. If $\mu(O_{v_*}) = (1-a)\mu(O_w)$, then

$$\mu(O_w) \geq \sum_{v \in S(w)} \mu(O_v) = (1-a)\mu(O_w) + \sum_{v \in S(w), v \neq v_*} \mu(O_v).$$

This implies $a\mu(O_w) \geq \mu(O_v)$ for any $v \in S(w)\backslash\{v_*\}$. By (10.3), $a \geq c_*$. Therefore, $\mu(O_v) \leq (1-c_*)\mu(O_w)$ for any $v \in S(w)$. This implies

$$c\mu(K_v) \leq \mu(O_v) \leq (1-c_*)^m\mu(O_w) \leq (1-c_*)^m\mu(K_w)$$

if $v \in T_w$ and $|v| = |w| + m$. Choosing m so that $(1-c_*)^m < c$, we see that μ is sub-exponential.
(5) As μ is sub-exponential, there exist $\alpha \in (0,1)$ and $m \geq 0$ such that $\mu(K_v) \leq \alpha\mu(K_w)$ if $u \in T_w$ and $|u| \geq |w| + m$. Since μ has M-volume doubling property with respect to g, there exist $\lambda, c \in (0,1)$ such that $\mu(U_M(x, \lambda s)) \geq c\mu(U_M(x,s))$ for any $x \in X$ and $s > 0$. Let $\beta \in (\lambda, 1)$. Assume that g is not sub-exponential. Then for any $n \geq 0$, there exist $w \in T$ and $u \in T_w$ such that $|u| \geq |w| + nm$ and $g(u) \geq \beta g(w)$. In case $g(w) = g(\pi(w))$, we may replace w by $v = [w]_m$ for some $m \in \{0, 1, \ldots, |w|\}$ satisfying $g(\pi(v)) > g(v) = g(w)$ or $g(v) = g(w) = 1$. Consequently we may assume $w \in \Lambda_{g(w)}$. Set $s = g(w)$. Since $\beta > \lambda$, there exists $u_* \in T_u \cap \Lambda_{s\lambda}$. Let $x \in K_{u_*}$. Then by the volume doubling property,

$$\mu(U_M(x, \lambda s)) \geq c\mu(U_M(x,s)) \geq c\mu(K_w).$$

By Lemma 10.12, there exists $c_* > 0$ which is independent of n, w and u such that

$$c_*\mu(K_{u_*}) \geq \mu(U_M(x, \lambda s)).$$

Since μ is sub-exponential,

$$\alpha^n c_*\mu(K_w) \geq c_*\mu(K_{u_*}) \geq \mu(U_M(x, \lambda s)) \geq c\mu(K_w).$$

This implies $\alpha^n c_* \geq c$ for any $n \geq 0$ which is a contradiction. □

At the end of this section, we give a proof of Theorem 8.21 by using Theorem 10.10.

Proof of Theorem 8.21. If is enough to show the case where $\alpha = 1$. Assume that $g_d \underset{\mathrm{BL}}{\sim} g_\mu$ and d is uniformly finite. Since d is adapted, there exist $M \geq 1$ and $\alpha_1, \alpha_2 > 0$ such that

$$U_M^d(x, \alpha_1 r) \subseteq B_d(x, r) \subseteq U_M^d(x, \alpha_2 r)$$

for any $x \in X$ and $r > 0$.

Write $d_w = g_d(w)$ and $\mu_w = g_\mu(w)$. Assume that $g_d \underset{\mathrm{BL}}{\sim} g_\mu$. Then there exist $c_1, c_2 > 0$ such that

$$c_1 d_w \leq \mu_w \leq c_2 d_w$$

for any $w \in T$. For any $x \in X$, choose $w \in T$ so that $x \in K_w$ and $w \in \Lambda^d_{\alpha_1 r}$. Then since d is super-exponential, there exists λ which is independent of x, r and w such that

$$\mu(B_d(x, r)) \geq \mu(U_M^d(x, \alpha_1 r)) \geq \mu(K_w) \geq c_1 d_w \geq c_1 \lambda d_{\pi(w)} \geq c_1 \lambda \alpha_1 r.$$

On the other hand, since d is uniformly finite, Lemma 8.17 implies

$$\begin{aligned} \mu(B_d(x, r)) \leq \mu(U_M^d(x, \alpha_2 r)) &\leq C \sum_{w \in \Lambda^d_{\alpha_2 r, M}(x)} \mu(K_w) \\ &\leq C c_2 \sum_{w \in \Lambda^d_{\alpha_2 r, M}(x)} d_w \leq C c_2 \#(\Lambda^d_{\alpha_2 r, M}) \alpha_2 r \leq C_2 r \end{aligned}$$

Conversely, assume (8.6). For any $w \in T$ and $x \in K_w$,

$$K_w \subseteq U_M^d(x, d_w) \subseteq B_d(x, d_w/\alpha_1).$$

Hence

$$\mu(K_w) \leq \mu(B_d(x, d_w/c_1)) \leq C_2 d_w/\alpha_1.$$

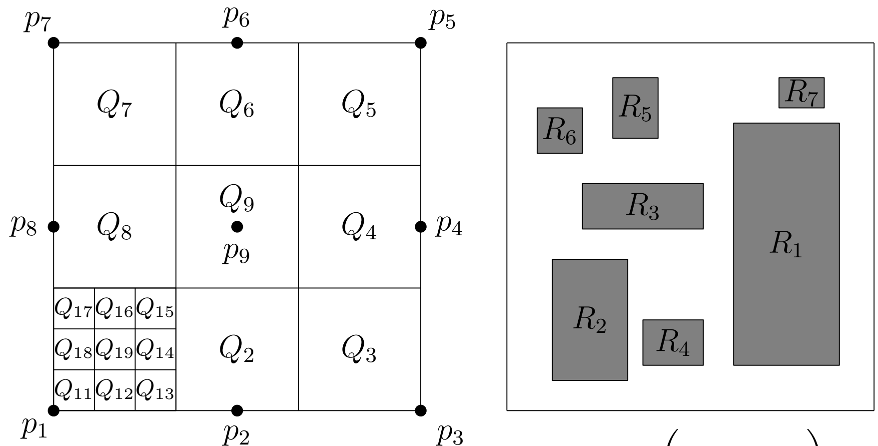
By Proposition 9.1, there exists $z \in K_w$ such that

$$K_w \supseteq U_M^d(z, \beta d_{\pi(w)}) \supseteq B_d(z, \beta d_{\pi(w)}/\alpha_2).$$

By (8.6),

$$\mu(K_w) \geq \mu(B_d(z, \beta d_{\pi(w)}/\alpha_2)) \geq C_1 \beta d_{\pi(w)}/\alpha_2 \geq C_1 \beta d_w/\alpha_2.$$

Thus we have shown that $g_d \underset{\mathrm{BL}}{\sim} g_\mu$. Furthermore, since d is M-adapted for some $M \geq 1$, μ has M-volume doubling property with respect to the weight function g_d. Applying Theorem 10.10-(3), we see that g_d is uniformly finite. In the same way, by Theorem 10.10, both g_d and g_μ are exponential. $\square$



The square Q as a self-similar set

$X = Q \backslash \left(\bigcup_{i \geq 1} \mathrm{int}(R_j) \right)$

Figure 6: The square Q and its subset X

11 Example: subsets of the square

In this section, we give illustrative examples of the results in the previous sections. For simplicity, our examples are subsets of the square $[0,1]^2$ denoted by Q and trees parametrizing partitions are sub-trees of $(T^{(9)}, \mathcal{A}^{(9)}, \phi)$ defined in Example 3.3. Note that $[0,1]^2$ is divided into 9-squares with the length of the sides $\frac{1}{3}$. As in Example 4.5, the tree $(T^{(9)}, \mathcal{A}^{(9)}, \phi)$ is naturally appears as the tree parametrizing the natural partition associated with this self-similar division. Namely, let $p_1 = (0,0), p_2 = (1/2, 0), p_3 = (1,0), p_4 = (1, 1/2), p_5 = (1,1), p_6 = (1/2, 1), p_7 = (0,1), p_8 = (0, 1/2)$ and $p_9 = (1/2, 1/2)$. Set $W = \{1, \ldots, 9\}$. Define $F_i : Q \to Q$ by

$$F_i(x) = \frac{1}{3}(x - p_i) + p_i$$

for any $i \in W$. Then F_i is a similitude for any $i \in W$ and

$$Q = \bigcup_{i \in W} F_i(Q).$$

See Figure 6. In this section, we write $(W_*, \mathcal{A}_*, \phi) = (T^{(9)}, \mathcal{A}^{(9)}, \phi)$, which is a locally finite tree with a reference point ϕ. Set $W_m = \{1, \ldots, 9\}^m$. Then $(W_*)_m = W_m$ and $\pi^{(W_*, \mathcal{A}_*, \phi)}(w) = w_1 \ldots w_{m-1}$ for any $w = w_1 \ldots w_m \in W_m$. For simplicity, we use $|w|$ and π in place of $|w|_{(W_*, \mathcal{A}_*, \phi)}$ and $\pi^{(W_*, \mathcal{A}_*, \phi)}$ respectively hereafter. Define $g : W_* \to (0,1]$ by $g(w) = 3^{-|w|}$ for any $w \in W_*$. Then g is an exponential weight function.

As for the natural associated partition of Q, define $F_w = F_{w_1} \circ \ldots \circ F_{w_m}$ and $Q_w = F_w(Q)$ for any $w = w_1 \ldots w_m \in W_m$. Set $Q_*(w) = Q_w$ for any $w \in W_*$. (If $w = \phi$, then F_ϕ is the identity map and $Q_\phi = Q$.) Then $Q_* : W_* \to \mathcal{C}(Q, \mathcal{O})$ is a partition of Q parametrized by $(W_*, \mathcal{A}_*, \phi)$, where $\mathcal{O}$ is the natural topology induced by the Euclidean metric. In fact, $\cap_{m \geq 0} Q_{[\omega]_m}$ for any $\omega \in \Sigma$, where $\Sigma = W^{\mathbb{N}}$, is a single point. Define $\sigma : \Sigma \to Q$ by $\{\sigma(\omega)\} = \cap_{m \geq 0} Q_{[\omega]_m}$.

It is easy to see that the partition Q_* is minimal, g is uniformly finite, g is thick with respect to the partition Q_*, and the (restriction of) Euclidean metric d_E on Q is 1-adapted to g.

In order to have more interesting examples, we consider certain class of subsets of Q whose partition is parametrized by a subtree $(T, \mathcal{A}_*|_{T\times T}, \phi)$ of $(W_*, \mathcal{A}_*, \phi)$. Let $\{I_m\}_{m\geq 0}$ be a sequence of subsets of W_* satisfying the following conditions (SQ1), (SQ2) and (SQ3):

(SQ1) For any $m \geq 0$, $I_m \subseteq W_m$ and if $\widehat{I}_{m+1} = \{wi | w \in I_m, i \in W\}$, then $I_{m+1} \supseteq \widehat{I}_{m+1}$.

(SQ2) $Q_w \cap Q_v = \emptyset$ if $w \in \widehat{I}_{m+1}$ and $v \in I_{m+1} \backslash \widehat{I}_{m+1}$.

(SQ3) For any $m \geq 0$, the set $\cup_{w\in I_m} Q_w$ is a disjoint union of rectangles $R^m_j = [a^m_j, b^m_j] \times [c^m_j, d^m_j]$ for $j = 1, \ldots, k_m$.

See Figure 6. By (SQ2), we may assume that $k_m \leq k_{m+1}$ and $R^m_j = R^{m+1}_j$ for any m and $j = 1, \ldots, k_m$ without loss of generality. Under this assumption, we may omit m of $R^m_j, a^m_j, b^m_j, c^m_j$ and d^m_j and simply write R_j, a_j, b_j, c_j and d_j respectively.

Notation. As a topology of $Q = [0, 1] \times [0, 1]$, we consider the relative topology induced by the Euclidean metric. We use $\mathrm{int}(A)$ and ∂A to denote the interior and the boundary, respectively, of a subset A of Q with respect to this topology.

Note that $\mathrm{int}(\cup_{w\in I_m} Q_w) = \cup_{j=1,\ldots,k_m} \mathrm{int}(R_j)$.

Proposition 11.1. (1) *Define*

$$X^{(m)} = Q\backslash\Bigg(\bigcup_{j=1,\ldots,k_m} \mathrm{int}(R_j)\Bigg).$$

then $X^{(m)} \supseteq X^{(m+1)}$ for any $m \geq 0$ and $X = \cap_{m\geq 0} X^{(m)}$ is a non-empty compact set. Moreover, $\partial R_j \subseteq X$ for any $j \geq 1$.

(2) *Define $(T)_m = \{w | w \in W_m, \mathrm{int}(Q_w) \cap X \neq \emptyset\}$ for any $m \geq 0$. If $T = \cup_{m\geq 0}(T)_m$ and $\mathcal{A} = \mathcal{A}_*|_{T\times T}$, then $(T, \mathcal{A}, \phi)$ is a locally finite tree with the reference point ϕ and $\#(S(w)) \geq 3$ for any $w \in T$. Moreover, let*

$$\Sigma_T = \{\omega | \omega \in \Sigma, [\omega]_m \in (T)_m \text{ for any } m \geq 0\}$$

Then $X = \sigma(\Sigma_T)$.

(3) *Define $K_w = Q_w \cap X$ for any $w \in T$. Then $K_w \neq \emptyset$ and $K : T \to \mathcal{C}(X)$ defined by $K(w) = K_w$ is a minimal partition of X parametrized by $(T, \mathcal{A}, \phi)$. Moreover, $g|_T$ is exponential and uniformly finite.*

To prove the above proposition, we need the following lemma.

Lemma 11.2. *If $w \in T$, then $\cup_{i\in W, wi\notin T} Q_{wi}$ is a disjoint union of rectangles and $\#(\{i | i \in W, wi \in T\}) \in \{3, 5, 7, 8, 9\}$.*

Proof. Set $I = \{i | i \in W, wi \notin T\}$. For each $i \in I$, there exists $k_i \geq 1$ such that $Q_{wi} \subseteq R_{k_i}$. Hence $\cup_{i\in I} Q_{wi} = \cup_{i\in I}(Q_w \cap R_{k_i})$. Since $\{R_j\}_{j\geq 1}$ are mutually disjoint, we have the desired conclusion. Assume that $I = W$. Suppose $|i-j| = 1$. Since $Q_{wi} \cap Q_{wj} \neq \emptyset$, we see that $R_{k_i} = R_{k_j}$. Hence $R_{k_1} = \ldots = R_{k_9}$ and $Q_w \subseteq R_{k_1}$. This contradicts the fact that $\mathrm{int}(Q_w) \cap X \neq \emptyset$. Thus $I \neq W$. Considering all the possible shapes of $\cup_{i \in W, wi \notin T} Q_{wi}$, we conclude $\#(\{i | i \in W, wi \in T\}) \in \{3, 5, 7, 8, 9\}$. □

Proof of Proposition 11.1. (1) Since $\{X^{(m)}\}_{m \geq 0}$ is a decreasing sequence of compact sets, X is a nonempty compact set. By (SQ2), $R_j \cap R_i = \emptyset$ for any $i \neq j$. Therefore, $\partial R_j \subseteq X^{(m)}$ for any $m \geq 0$. Hence $\partial R_j \subseteq X$.
(2) If $w \in (T)_m$, then $\mathrm{int}(Q_{\pi(w)}) \cap X \supseteq \mathrm{int}(Q_w) \cap X \neq \emptyset$. Hence $\pi(w) \in (T)_{m-1}$. Using this inductively, we see that $[w]_k \in (T)_k$ for any $k \in \{0, 1, \ldots, m\}$. This implies that $(T, \mathcal{A}, \phi)$ is a locally finite tree with a reference point ϕ. By Lemma 11.2, we see that $\#(\{i | i \in W, wi \in (T)_{m+1}\}) \geq 3$. Next if $\omega \in \Sigma_T$, then for any $m \geq 0$, there exists $x_m \in \mathrm{int}(Q_{[\omega]_m}) \cap X$. Therefore, $x_m \to \sigma(\omega)$ as $m \to \infty$. Since X is compact, it follows that $\sigma(\omega) \in X$.

Conversely, assume that $x \in X$. Set $W_{m,x} = \{w | w \in W_m, x \in Q_w\}$. Note that $\#(\sigma^{-1}(x)) \leq 4$ and $\cup_{w \in W_{m,x}} Q_w$ is a neighborhood of x. Suppose that $(T)_m \cap W_{m,x} \neq \emptyset$ for any $m \geq 0$. Then there exists $w_m \in (T)_m \cap W_{m,x}$ such that $x \in Q_{w_m}$. Since $W_{m,x} = \{[\omega]_m | \omega \in \sigma^{-1}(x)\}$, there exists $\omega \in \sigma^{-1}(x)$ such that $[\omega]_m = w_m$ for infinitely many m. As $\mathrm{int}(Q_{[\omega]_m})$ is monotonically decreasing, it follows that $[\omega]_m \in (T)_m$ for any $m \geq 0$. This implies $x \in \sigma(\Sigma_T)$. Suppose that there exists $m \geq 0$ such that $W_{m,x} \cap (T)_m = \emptyset$. By this assumption, $\mathrm{int}(Q_w) \cap X = \emptyset$ for any $w \in W_{m,x}$ and hence there exists $j_w \geq 1$ such that $Q_w \subseteq R_{j_w}$. Note that $Q_w \cap Q_{w'} \neq \emptyset$ for any $w, w' \in W_{m,x}$ and hence $R_{j_w} = R_{j_{w'}}$. Therefore, $\cup_{w \in W_{m,x}} Q_w \subseteq R_j$ for some $j \geq 1$. Since $\cup_{w \in W_{m,x}} Q_w$ is a neighborhood of x, it follows that $x \notin X$. This contradiction concludes the proof.
(3) The fact that K is a partition of X parametrized by $(T, \mathcal{A}|_{T \times T}, \phi)$ is straightforward from (1) and (2). As $K_w \backslash (\cup_{v \in (T)_m, v \neq w} K_v)$ is contained in the sides of the square Q_w, the partition K is minimal. Since $\Lambda_s^{g|_T} = (T)_m$ if and only if $\frac{1}{3^m} \leq s < \frac{1}{3^{m-1}}$, it follows that $g|_T$ is exponential. Furthermore, $\Lambda_{s,1}^{g|_T}(w) \subseteq \{v | v \in W_m, Q_v \cap Q_w \neq \emptyset\}$ for any $w \in (T)_m$. Hence $\#(\Lambda_{s,1}^{g|_T}(w)) \leq 8$. This shows that $g|_T$ is uniformly finite. □

Now, we consider when the restriction of the Euclidean metric is adapted.

Definition 11.3. Let $R = [a, b] \times [c, d]$ be a rectangle. The degree of distortion of R, $\kappa(R)$, is defined by

$$\kappa(R) = \max\left\{1, (1 - \delta_{c0})(1 - \delta_{d1})\frac{|b-a|}{|d-c|}, (1 - \delta_{a0})(1 - \delta_{b1})\frac{|d-c|}{|b-a|}\right\},$$

where δ_{xy} is the Kronecker delta defined by $\delta_{xy} = 1$ if $x = y$ and $\delta_{xy} = 0$ if $x \neq y$. Moreover, for $\kappa \geq 1$, we define

$$\mathcal{R}_\kappa^0 = \{R | R \text{ is a rectangle, } R \subseteq Q \text{ and } \kappa(R) \leq \kappa\}$$

and

$$\mathcal{R}^1_\kappa = \{R | R \subseteq Q, R \text{ is a rectangle, there exists } w \in T \text{ such that } Q_w \backslash \mathrm{int}(R)$$
$$\text{has two connected components and } \kappa(Q_w \cap R) \leq \kappa\}$$

The extra factors $(1-\delta_{c0}), (1-\delta_{d1}), (1-\delta_{a0})$ and $(1-\delta_{b1})$ become effective if the rectangle R has an intersection with the boundary of the square Q.

Theorem 11.4. *Let d be the restriction of the Euclidean metric on X. Then d is adapted to $g|_T$ if and only if the following condition* (SQ4) *holds:*
(SQ4) *There exists $\kappa \geq 1$ such that $R_j \in \mathcal{R}^0_\kappa \cup \mathcal{R}^1_\kappa$ for any $j \geq 1$.*

Several lemmas are needed to prove the above theorem.

Lemma 11.5. *Define $N(x,y) = \min\{[-\frac{\log|x_1-y_1|}{\log 3}], [-\frac{\log|x_2-y_2|}{\log 3}]\}$ for any $x = (x_1, x_2), y = (y_1, y_2) \in Q$, where $[a]$ is the integer part of a real number a.*
(1)
$$\frac{1}{3^{N(x,y)+1}} < d(x,y) \leq \frac{\sqrt{2}}{3^{N(x,y)}}$$
(2) *If $x, y \in X$, then there exist $w, v, u \in W_{N(x,y)}$ such that $w, v \in T$, $x \in Q_w$, $y \in Q_u$, $Q_w \cap Q_v \neq \emptyset$ and $Q_v \cap Q_v \neq \emptyset$.*

Proof. Set $N = N(x,y)$. Let $n_i = [-\frac{\log|x_i-y_i|}{\log 3}]$ for $i = 1, 2$. Then $N = \min\{n_1, n_2\}$ and
$$\frac{1}{3^{N+1}} < |x_j - y_j| \leq \frac{1}{3^N}$$
if $n_j = N$. This yields (1). Since $x, y \in X$, then there exist $w, u \in (T)_m$ such that $x \in K_w$ and $y \in K_u$. Since $|x_1 - y_1| \leq 1/3^N$ and $|x_2 - y_2| \leq 1/3^N$, we find $v \in W_m$ satisfying $Q_w \cap Q_v \neq \emptyset$ and $Q_v \cap Q_u \neq \emptyset$. □

Notation. For integers $n, k, l \geq 0$, we set

$$Q(n,k,l) = \left[\frac{k}{3^n}, \frac{(k+1)}{3^n}\right] \times \left[\frac{l}{3^n}, \frac{(l+1)}{3^n}\right]$$

Lemma 11.6. *Assume* (SQ4)*. Let $M = [\log(2\kappa)/\log 3] + 1$ and $L = 2[2\kappa] + 9$. If $w, v \in (T)_m$ and $Q_w \cap Q_v \neq \emptyset$, then there exists a chain $(w(1), \ldots, w(L))$ of K such that $w \in T_{w(1)}$, $v \in T_{w(L)}$ and $|w(k)| \geq m - M$.*

Proof. Case 1: Assume that $Q_w \cap Q_v$ is a line segment. Without loss of generality, we may assume that $Q_w = Q(m, k-1, l)$ and $Q_v = Q(m, k, l)$.
Case 1a: $K_w \cap K_v \neq \emptyset$, then (w, v) is a desired chain of K.
Case 1b: In case $K_w \cap K_v = \emptyset$, $Q_w \cap Q_v \cap K_w$ and $Q_w \cap Q_v \cap K_v$ are disjoint closed subsets of $Q_w \cap Q_v$. Since $Q_w \cap Q_v$ is connected, there exists $a \in Q_w \cap Q_v$ such that $a \notin K_w \cap K_v$. Since $K_w \cup K_v$ is closed, there exists an open neighborhood of a which has no intersection with $K_w \cap K_v$. This open neighborhood must be contained in R_j for some j. So, we see that $R_j \cap \mathrm{int}(Q_w \cap Q_v) \neq \emptyset$ and

$(k-1)/3^m \leq a_j \leq k/3^m \leq b_j \leq (k+1)/3^m$. Assume $c_i > l/3^m$. Then since the line segment $[a_j, b_j] \times \{c_j\}$ is contained in X, we see that $K_w \cap K_v \neq \emptyset$. Therefore $c_j \leq l/3^m$. By the same argument we have $d_j \geq (l+1)/3^m$. Now if $R_j \in \mathcal{R}^0_\kappa$, it follows that $|d_j - c_j| \leq 2\kappa/3^m$. Hence the line segment $[a_j, b_j] \times \{c_j\}$ and $[a_j, b_j] \times \{d_j\}$ are covered by at most 4 pieces of K_u's for $u \in (T)_m$ and the line segment $\{a_j\} \times [c_j, d_j]$ and $\{b_j\} \times [c_j, d_j]$ is covered by at most $2\kappa + 2$ pieces of K_u's for $u \in (T)_m$. Since K_w and K_v are pieces of these coverings, we obtain a chain $(w(1), \ldots, w(k))$ of K from these coverings where $w(1) = w, w(k) = v$ and $l \leq 2\kappa + 5$. Next assume $R_j \in \mathcal{R}^1_\kappa$. Note that $2\kappa/3^m \leq 1/3^{m-M}$. By the definition of $\mathcal{R}^1_\kappa$, there exists $u \in (T)_{m-M}$ such that $Q_u \backslash R_j$ has two connected component. Sifting Q_u up and down, we may find $u' \in (T)_{m-M}$ such that $Q_w \cup Q_v \subseteq Q_{u'}$. Then (u') is a desired chain of K.
Case 2: Assume that $Q_w \cap Q_v$ is a single point. Without loss of generality, we may assume that $Q_w = Q(m, k-1, l-1)$ and $Q_v = Q(m, k, l)$. Choose $u(1), u(2) \in W_m$ so that $Q_{u(1)} = Q(m, k-1, l)$ and $Q_{w(2)} = Q(m, k+1, l-1)$. If neither $u(1)$ nor $u(2)$ belongs to T, then there exist $i, j \geq 1$ such that $Q_{u(1)} \subseteq R_i$ and $Q_{u(2)} \subseteq R_j$. Since $Q_{u(1)} \cap Q_{u(2)} \neq \emptyset$, it follows that $R_i = R_j$ and hence $Q_w \cup Q_v \subseteq R_i$. This contradicts the fact that $w, v \in T$. Hence $u(1) \in T$ or $u(2) \in T$. Let $u(1) \in T$. Then $Q_w \cap Q_{u(1)}$ and $Q_{u(1)} \cap Q_v$ are line segments. By using the method in (1), we find a chain between w and $u(1)$ and a chain between $u(1)$ and v. Connecting these two chains, we obtain the desired chain $(w(1), \ldots, w(L))$. □

Proof of Theorem 11.4. Assume (SQ4). Let $x, y \in X$. Define $N = N(x, y)$ and choose $w, v, u \in W_N$ as in Lemma 11.5. We fix the constants M and L as in Lemma 11.6. There are two cases.
Case 1: Suppose $v \in T$. Applying Lemma 11.6 to two pairs $\{w, v\}$ and $\{v, u\}$ and connecting the two resultant chains, we obtain a chain $(w(1), \ldots, w(2L-1)) \in \mathcal{CH}_K(x, y)$ satisfying $w \in T_{w(1)}, u \in\in T_{w(2L-1)}$ and $|w(i)| \geq N - M$ for any i. This concludes Case 1.
Case 2: Suppose $v \notin T$. If $Q_w \cap Q_u \neq \emptyset$, then we have a chain $(w(1), \ldots, w(L))$ between x and y satisfying $w \in T_{w(1)}$, $u \in T_{w(L)}$ and $|w(i)| \geq N - M$ for any i by Lemma 11.6. Assume $Q_w \cap Q_u = \emptyset$. Without loss of generality, we may assume one of the following tree situations (a), (b) and (c):
(a) $Q_w = Q(N, k-1, l-1)$ and $Q_u = Q(N, k+1, l-1)$.
(b) $Q_w = Q(N, k-1, l-1)$ and $Q_u = Q(N, k+1, l)$.
(c) $Q_w = Q(N, k-1, l-1)$ and $Q_u = Q(N, k+1, l+1)$.

Set $Q_{v(1)} = Q(N, k, l-1)$ and $Q_{v(2)} = Q(N, k, l)$. In each case, $x_1 = k/3^N$ and $y_1 = (k+1)/3^N$.

First consider cases (a) and (b). If either $v(1)$ or $v(2)$ belongs to T, then replacing v by either $v(1)$ or $v(2)$, we end up with Case 1. So we assume that neither $v(1)$ nor $v(2)$ belongs to T. Then there exists $j \geq 1$ such that $Q_{v(1)} \cup Q_{v(2)} \subseteq R_j$. Since $x_1 = k/3^N$ and $y_1 = (k+1)/3^N$, we have $a_j = k/3^N$ and $b_j = (k+1)/3^N$. Then by the same argument as in the proof of Lemma 11.6, there exists a chain $(w(1), \ldots, w(L)) \in \mathcal{CH}_K(x, y)$ such that $w \in T_{w(1)}$, $u \in T_{w(L)}$ and $|w(i)| \geq N - M$ for any i.

Next in the situation of (c), $x = (k/3^N, l/3^N)$, $y = ((k+1)/3^N, (l+1)/3^N)$ and $v = v(2)$. Since $v = v(1) \notin T$, there exists $j \geq 1$ such that $Q_v \subseteq R_j$. Note that $x, y \in X \cap Q_v$. Hence $Q_v = R_j$. Choose $v(3), v(4) \in W_N$ so that $Q_{v(3)} = Q(N, k+1, l-1)$ and $Q_{v(4)} = Q(N, k+1, l)$. Then $v(3), v(4) \in T$ and therefore $(w, v(1), v(3), v(4), u)$ is a chain of K between x and y. This concludes Case 2.

As a consequence, we may always find a chain $(w(1), \ldots, w(2L-1)) \in \mathcal{CH}_K(x, y)$ satisfying $|w(i)| \geq N(x, y) - M$ for any i. By Lemma 11.5-(1),

$$3^{M+1} d(x, y) \geq 3^M \frac{1}{3^N} \geq \frac{1}{3^{w(i)}} = g(w(i)).$$

Thus we have verified the conditions (ADa) and $(\mathrm{ADb})_{\mathrm{2L-2}}$ in Theorem 6.5. Hence d is $(2L-2)$-adapted to $g|_T$ by Theorem 6.5.

Conversely, assume that d is J-adapted to $g|_T$. By $(\mathrm{ADb})_{\mathrm{J}}$, there exists $C \geq 0$ such that for any $x, y \in X$, there exists a chain $(w(1), \ldots, w(J+1)) \in \mathcal{CH}_K(x, y)$ satisfying

$$Cd(x, y) \geq \frac{1}{3^{|w(i)|}} \tag{11.1}$$

for any $i = 1, \ldots, J+1$. Set $M = [\log(\sqrt{2}C)/\log 3] + 1$. Suppose that (SQ4) does not hold; for any $\kappa \geq 1$, there exists $R_j \notin \mathcal{R}^0_\kappa \cup \mathcal{R}^1_\kappa$. In particular, we choose $\kappa \geq 3^{M+2}$. Write $R = R_j$ and set $R = [a, b] \times [c, d]$. Define $\partial R_L = \{a\} \times [c, d]$ and $\partial R_R = \{b\} \times [c, d]$. (The symbols "L" and "R" correspond to the words "Left" and "Right" respectively.) Without loss of generality, we may assume that $|a - b| \leq |c - d|$. Since $R \notin \mathcal{R}^0_\kappa$, we have $\kappa|b - a| \leq |d - c|$. Let $x = (a, (c+d)/2)$ and let $y = (b, (c+d)/2)$. Set $N = N(x, y)$. There exists $(w(1), \ldots, w(J+1)) \in \mathcal{CH}_K(x, y)$ such that (11.1) holds for any $i = 1, \ldots, J+1$. By Lemma 11.5-(1),

$$|w(i)| \geq N - M \tag{11.2}$$

for any $i = 1, \ldots, J+1$. Define $A = [0, 1] \times (c, d)$. If $Q_{w(i)} \subseteq A$, $Q_{w(i)} \cap \partial R_L \neq \emptyset$ and $Q_{w(i)} \cap \partial R_R \neq \emptyset$, then the fact that $R \notin \mathcal{R}^1_\kappa$ along with Lemma 11.5-(1) shows

$$\frac{1}{3^{|w(i)|}} \geq \kappa|b - a| = \kappa d(x, y) \geq \frac{\kappa}{3^{N+1}} \geq \frac{1}{3^{N+M-1}}. \tag{11.3}$$

This contradicts (11.2) and hence we verify the following claim (I):
(I) If $Q_{w(i)} \subseteq A$, then $Q_{w(i)} \cap \partial R_L = \emptyset$ or $Q_{w(i)} \cap \partial R_R = \emptyset$.

Next we prove that there exists $j \geq 1$ such that $Q_{w(j)} \backslash A \neq \emptyset$. Otherwise, $Q_{w(i)} \subseteq A$ for any $i = 1, \ldots, J+1$. Let $A_L = [0, a] \times (c, d)$ and let $A_R = [b, 1] \times (c, d)$. Define $I_L = \{i | i = 1, \ldots, J+1, Q_{w(i)} \cap A_L \neq \emptyset\}$ and $I_R = \{i | i = 1, \ldots, J+1, Q_{w(i)} \cap A_R \neq \emptyset\}$. Since $K_{(w(i))} \subseteq X \cap A \subseteq A_L \cup A_R$, it follows that $\{1, \ldots, J+1\} = I_L \cup I_R$. Moreover, the claim (I) implies $I_L \cap I_R = \emptyset$. Hence $I_L = \{i | i = 1, \ldots, J+1, K_{w(i)} \subseteq A_L\}$ and $I_R = \{i | i = 1, \ldots, J+1, K_{w(i)} \subseteq A_R\}$. This contradicts the fact that $(w(1), \ldots, w(J+1))$ is a chain of K between x and y. Thus there exists $j \geq 1$ such that $Q_{w(j)} \backslash A \neq \emptyset$. Define $i_* = \min\{i | i = 1, \ldots, J+1, Q_{w(i)} \backslash A \neq \emptyset\}$. Without loss of generality, we may assume that

$Q_{w(i_*)} \cap [0,1] \times \{d\} \neq \emptyset$. Set

$$\partial R_L^T = \{a\} \times \left[\frac{c+d}{2}, d - \frac{1}{3^{|w(i_*)|}}\right].$$

Shifting $Q_{w(i)}$'s for $i = 1, \ldots, i_* - 1$ horizontally towards ∂R_L, we obtain a covering of ∂R_L^T. Note that the length of ∂R_L^T is $|d-c|/2 - 1/3^{|w(i_*)|}$ and

$$\frac{|d-c|}{2} - \frac{1}{3^{|w(i_*)|}} \geq \frac{\kappa|b-a|}{2} - \frac{1}{3^{N-M}} = \frac{\kappa}{2}d(x,y) - \frac{1}{3^{N-M}} \geq \frac{\kappa}{2}\frac{1}{3^{N+1}} - \frac{1}{3^{N-M}}.$$

On the other hand, the lengths of the sides of $Q_{w(i)}$'s are no less that $1/3^{N-M}$ by (11.2). Hence

$$i_* - 1 \geq 3^{N-M}\left(\frac{\kappa}{2}\frac{1}{3^{N+1}} - \frac{1}{3^{N-M}}\right) \geq \frac{\kappa}{2}\frac{1}{3^{M+1}} - 1.$$

Since $J + 1 \geq i_*$, it follows that

$$2(J+1)3^{M+1} \geq \kappa.$$

This contradicts the fact that κ can be arbitrarily large. Hence we conclude that (SQ4) holds. □

In the followings, we give four examples. The first one has infinite connected components but still the restriction of the Euclidean metric is adapted.

Example 11.7 (Figure 7)**.** Let X be the self-similar set associated with the contractions $\{F_1, F_3, F_4, F_5, F_7, F_8\}$, i.e. X is the unique nonempty compact set which satisfies

$$X = \bigcup_{i \in S} F_j(X),$$

where $S = \{1, 3, 4, 5, 7, 8\}$. Then $X = C_3 \times [0,1]$, where C_3 is the ternary Cantor set. Define $(T)_m = S^m$ and $T = \cup_{m \geq 1}(T)_m$. If $K_w = F_w(X)$ for any $w \in T$, then K is a partition of X parametrized by $(T, \mathcal{A}|_T, \phi)$. Define

$$I_\phi = \left[\frac{1}{3}, \frac{2}{3}\right] \times [0,1] \quad \text{and} \quad I_{i_1,\ldots,i_n} = \left[\sum_{k=1}^{n}\frac{i_k}{3^k} + \frac{1}{3^{n+1}}, \sum_{k=1}^{n}\frac{i_k}{3^k} + \frac{2}{3^{n+1}}\right] \times [0,1]$$

for any $1 \geq 0$ and $i_1, \ldots, i_n \in \{0, 2\}$. Then

$$\{R_j\}_{j \geq 1} = \{I_\phi, I_{i_1,\ldots,i_n} | n \geq 1, i_1, \ldots, i_n \in \{0,2\}\}.$$

Set $J_{i_1,\ldots,i_n} = [\sum_{k=1}^{n} \frac{i_k}{3^k}, \sum_{k=1}^{n} \frac{i_k}{3^k} + \frac{1}{3^n}] \times [0, \frac{1}{3^n}]$. Then there exists $w \in (T)_n$ such that $J_{i_1,\ldots,i_n} = Q_w$, $Q_w \backslash \mathrm{int}(I_{i_1,\ldots,i_n})$ has two connected component and $\kappa(Q_w \cap I_{i_1,\ldots,i_n}) = 3$. Therefore, $\{R_j\}_{j \geq 1} \subseteq \mathcal{R}_3^1$ and hence d is adapted to $g|_T$.

The second example is the case where the restriction of the Euclidean metric is not adapted.

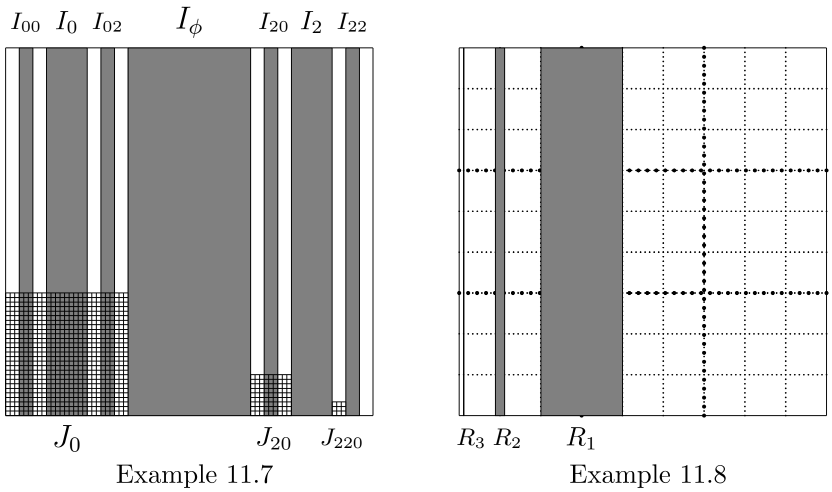


Figure 7: Examples 11.7 and 11.8

Example 11.8 (Figure 7)**.** Set $x_j = \frac{1}{3^j} - \frac{1}{3^{2j}}$, $y_j = \frac{1}{3^j} + \frac{1}{3^{2j}}$ and $R_j = [x_j, y_j] \times [0, 1]$ for any $j \geq 1$. Define $X = Q \backslash (\cup_{j \geq 1} \mathrm{int}(R_j))$. Let $T = \{w | w \in W_*, \mathrm{int}(Q_w) \cap X \neq \emptyset\}$ and let $K_w = X \cap Q_w$ for any $w \in T$. Then $K : T \to \mathcal{C}(X)$ is a partition of X parametrized by $(T, \mathcal{A}|_{T \times T}, \phi)$ by Proposition 11.1. In this case, we easily see the following facts:

- $\kappa(R_j) = 3^{2j}/2$ for any $j \geq 1$,
- If $w \in \cup_{m \geq j} (T)_m$, then $Q_w \backslash \mathrm{int}(R_j)$ is a rectangle,
- Set $(1)^n = \underbrace{1 \cdots 1}_{n \text{ times}} \in (T)_n$. Then $Q_{(1)^{j-1}} \backslash \mathrm{int}(R_j)$ has two connected components and $\kappa(Q_{(1)^{j-1}} \cap R_j) = 2 \cdot 3^{j+1}$.

These facts yield that $R_j \notin \mathcal{R}^0_{2 \cdot 3^j} \cup \mathcal{R}^1_{2 \cdot 3^j}$ for sufficiently large j. By Theorem 11.4, d is not adapted to $g|_T$. In fact, $D^g_M((x_j, 0), (y_j, 0)) = 3^{-(j-1)}$ for any $j \geq 1$ while $d((x_j, 0), (y_i, 0)) = 2 \cdot 3^{-2j}$. Hence the ratio between $D^g_M(\cdot, \cdot)$ and $d(\cdot, \cdot)$ is not bounded for any $M \geq 0$.

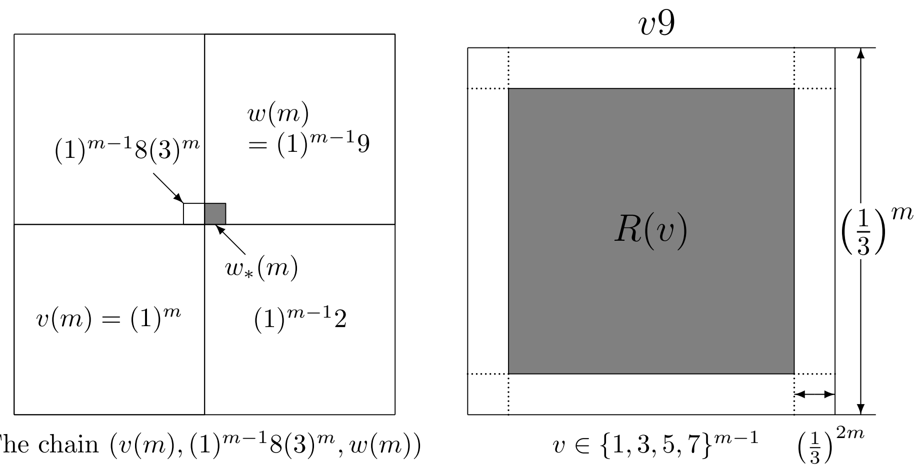
Furthermore, let $d_*(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ for any $x = (x_1, x_2), y = (y_1, y_2) \in X$. Then $g|_T = g_{d_*}$. Note that d and d_* are bi-Lipschitz equivalent. Since d is not adapted to $g|_T$, it follows that d_* is not adapted to $g|_T = g_{d_*}$ as well. Thus d and d_* are not adapted.

The third one is the case when the restriction of the Euclidean metric is not 1-adapted but 2-adapted.

Example 11.9 (Figure 8)**.** Define

$$w_*(j) = (1)^{j-1} 9 (1)^j, \quad R_j = Q_{w_*(j)} \quad \text{and} \quad k_m = \left[\frac{m}{2}\right]$$

for $j \in \mathbb{N}$ and $m \in \mathbb{N}$. Note that $(1)^n = \underbrace{1 \ldots 1}_{n\text{-times}}$ as is defined in Example 11.8. Then it follows that $T = T^{(9)} \backslash \cup_{j \in \mathbb{N}} T^{(9)}_{w_*(j)}$, where $T^{(9)}_w = \{w i_1 i_2 \ldots | i_1, i_2, \ldots \in$



The chain $(v(m), (1)^{m-1}8(3)^m, w(m))$ $\quad$ $v \in \{1,3,5,7\}^{m-1}$

Example 11.9 $\quad$ Example 11.10

Figure 8: Example 11.9 and 11.10

$\{1,\ldots,9\}\}$. Let $g(w) = 3^{-|w|}$ for any $w \in T$. Define $w(m) = (1)^{m-1}9$ and $v(m) = (1)^m$. Then $(w(m), (1)^{m-1}8(3)^k, v(m))$ is a chain for $k = 0, 1, \ldots, m$. See Figure 8. Therefore, $w(m)$ and $v(m)$ are 1-separated in $\Lambda^g_{3^{-m}}$ but not 2-separated in $\Lambda^g_{3^{-2m}}$. This means that the condition $(\mathrm{EV5})_M$ for $M = 1$ does not hold. Therefore, there exists no metric which is 1-adapted to g^α for any $\alpha > 0$. On the other hand, since $\kappa(R_j) = 1$ for any $j \in \mathbb{N}$, the restriction of the Euclidean metric to X, which is denoted by d, is adapted to g. In fact, it is easy to see that d is 2-adapted to g. As a consequence, d is not 1-adapted but 2-adapted to g.

In the fourth example, we do not have thickness while the restriction of the Euclidean metric is adapted.

Example 11.10 (Figure 8)**.** Define $\Delta Q = (\mathbb{R}^2 \backslash \mathrm{int}(Q)) \cap Q$, which is the topological boundary of Q as a subset of $\mathbb{R}^2$. Let $I_0 = \emptyset$ and let $E = \{1, 3, 5, 7\}$. Define $\{I_n\}_{n \geq 0}$ inductively by $I_{2m-1} = \widehat{I}_{2m-1}$ and $I_{2m} = J_m \cup \widehat{I}_{2m}$ for $m \geq 1$, where

$$J_m = \{v9w | v \in E^{m-1}, w \in W_m, Q_w \cap \Delta Q = \emptyset\}.$$

$\{I_m\}_{m \geq 0}$ satisfies (SQ1), (SQ2) and (SQ3). In fact, if $J_{m,v} = \{v9w | w \in W_m, Q_w \cap \Delta Q = \emptyset\}$ for any $v \in E^{m-1}$, $J_{m,v}$ is a collection of $(3^m - 2)^2$-words in W_{2m}. Set $R(v) = \cup_{u \in J_{m,v}} Q_u$ for any $m \geq 1$ and $v \in E^{m-1}$. See Figure 8. Then $\{R_j\}_{j \geq 1} = \{R(v) | m \geq 1, v \in E^{m-1}\}$. More precisely $R(v) \subseteq Q_{v9}$ and $R(v)$ is a square which has the same center, i.e. the intersection of two diagonals, as Q_{v9} and the length of the sides is $\frac{1}{3^m}(1 - \frac{2}{3^m})$. Note that the length of the sides of Q_{v9} is $\frac{1}{3^m}$. Hence the relative size of $R(v)$ in comparison with Q_{v9} is monotonically increasing and convergent to 1 as $m \to \infty$. The corresponding tree $(T, \mathcal{A}|_T, \phi)$ and the partition $K : T \to \mathcal{C}(X)$ of $X = Q \backslash \cup_{j \geq 1} \mathrm{int}(R_j)$ have the following properties:
Let d be the restriction of the Euclidean metric to X. Then

(a) d is adapted to $g|_T$.

(**b**) $g|_T$ is exponential and uniformly finite.

(**c**) Let μ_* be the restriction of the Lebesgue measure on X. Then μ_* has the volume doubling property with respect to d.

(**d**) μ_* is not gentle with respect to $g|_T$.

(**e**) μ_* is not super-exponential.

(**f**) $g|_T$ is not thick.

In the rest, we present proofs of the above claims.
(**a**) Since $\kappa(R_m) = 1$ for any $m \geq 1$, we see that $\{R_m\}_{m\geq 1} \subseteq \mathcal{R}_1^0$. Hence Theorem 11.4 shows that d is adapted to $g|_T$. In fact, d is 1-adapted to $g|_T$ in this case.
(**b**) This is included in the statement of Proposition 11.1-(3).
(**c**) If $v \in \Lambda_s^{g|_T}$ and $Q_v = K_v$, then $\mu_*(K_v) = 9^{-|v|}$ and hence $\mu_*(K_u) \leq 9^{-|u|} = 9^{-|v|+1} \leq 9\mu_*(K_v)$ for any $u \in \Lambda_{3s}^{g|_T}$. Therefore, $v \in \Theta(s, 3, k, 9)$ for any $k \geq 1$. On the other hand, for any $w \in T$, there exists $v \in \Lambda_{s,1}^{g|_T}(w)$ such that $K_v = Q_v$. Therefore, we see that $\Lambda_{s,1}^{g|_T}(w) \cap \Theta(s, 3, 3, 9) \neq \emptyset$. By Lemma 10.8, we have (**c**).
(**d**) and (**e**) Set $w(m) = (1)^{m-1}9$. Then $K_{w(m)} = Q_{w(m)} \backslash \mathrm{int}(R_m)$, where $R_m = \cup_{w\in J_m} Q_w$. Then $\mu_*(K_{w(m)}) = 4(3^m - 1)3^{-4m}$. On the other hand, if $v(m) = (1)^{m-1}8$, then $\mu_*(K_{v(m)}) = 3^{-2m}$. Since $K_{w(m)} \cap K_{v(m)} \neq \emptyset$, μ_* is not gentle with respect to $g|_T$. Moreover, since $K_{\pi(w(m))}$ contains $Q_{v(m)}$, we have $\mu_*(K_{\pi(w(m))}) \geq 3^{-2m}$. This implies that μ_* is not super-exponential.
(**f**) To clarify the notation, we use $B(x, r) = \{y | y \in Q, |x - y| < r\}$ and $B_*(x, r) = B(x, r) \cap X$. This means that $B_*(x, r)$ is the ball of radius r with respect to the metric d on X. Assume that $g|_T$ is thick. Since K is minimal, Proposition 9.2 implies that $K_{w(m)} \supseteq B_*(x, c3^{-m})$ for some $x \in K_{w(m)}$, where c is independent of m and x. However, for any $x \in K_{w(m)}$, there exists $y \in X\backslash K_{w(m)}$ such that $|x - y| \leq 2 \cdot 3^{-2m}$. This contradiction shows that $g|_T$ is not thick.

12 Gentleness and exponentiality

In this section, we show that the gentleness "$\underset{\mathrm{GE}}{\sim}$" is an equivalence relation among exponential weight functions. Moreover, the thickness of the interior, tightness, the uniformly finiteness and the existence of visual metric will be proven to be invariant under the gentle equivalence.

As in the section 10, $(T, \mathcal{A}, \phi)$ is a locally finite tree with a reference point ϕ, $(X, \mathcal{O})$ is a compact metrizable topological space with no isolated point and $K : T \to \mathcal{C}(X, \mathcal{O})$ is a partition of X parametrized by $(T, \mathcal{A}, \phi)$.

Definition 12.1. Define $\mathcal{G}_e(T)$ as the collection of exponential weight functions.

Theorem 12.2. *The relation $\underset{\mathrm{GE}}{\sim}$ is an equivalence relation on $\mathcal{G}_e(T)$.*

Several steps of preparation are required to prove the above theorem.

Definition 12.3. (1) Let $A \subseteq T$. For $m \geq 0$, we define $S^m(A) \subseteq T$ as

$$S^m(A) = \bigcup_{w \in A} \{v | v \in (T)_{m+|w|}, [v]_{|w|} = w\}.$$

(2) Let $g : T \to (0,1]$ be a weight function. For any $w \in T$, define

$$N_g(w) = \min\{n | n \geq 0, \pi^n(w) \in \Lambda^g_{g(w)}\}$$

and $\pi^*_g(w) = \pi^{N_g(w)}(w)$.
(3) $(u, v) \in T \times T$ is called an ordered pair if and only if $u \in T_v$ or $v \in T_u$. Define $|u, v| = ||u| - |v||$ for an ordered pair (u, v).

Note that if $g(w) < 1$, then we have

$$N_g(w) = \min\{n | n \geq 0, g(\pi^{n+1}(w)) > g(w)\}.$$

Therefore, if $g(\pi(w)) > g(w)$ for any $w \in T$, then $N_g(w) = 0$ and $\pi^*_g(w) = w$ for any $w \in T$.

The following lemma is immediate from the definitions.

Lemma 12.4. *Let $g : T \to (0,1]$ be a super-exponential weight function, i.e. there exists $\gamma \in (0,1)$ such that $g(w) \geq \gamma g(\pi(w))$ for any $w \in T$. If (u, v) is an ordered pair, then $g(u) \leq \gamma^{-|u,v|} g(v)$.*

Lemma 12.5. *Let $g : T \to (0,1]$ be a weight function. If g is sub-exponential, then $\sup_{w \in T} N_g(w) < +\infty$.*

Proof. Since g is sub-exponential, there exist $c \in (0,1)$ and $m \geq 0$ such that $cg(w) \geq g(u)$ if $w \in T$, $u \in T_w$ and $|u, v| \geq m$. This immediately implies that $N_g(w) \leq m$. □

Lemma 12.6. *Assume that $g, h \in \mathcal{G}_e(T)$ and h is gentle with respect to g. Then there exist M and N such that if $s \in (0,1]$, $w \in \Lambda^h_s$ and $u \in S^M(\Lambda^g_{g(w),1}(\pi^*_g(w)))$, then one can choose $n(u) \in [0, N]$ so that $\pi^{n(u)}(u) \in \Lambda^h_s$. Moreover, define $\eta^{g,h}_{s,w} : S^M(\Lambda^g_{g(w),1}(\pi^*_g(w))) \to \Lambda^h_s$ by $\eta^{g,h}_{s,w}(u) = \pi^{n(u)}(u)$. Then $\Lambda^h_{s,1}(w) \subseteq \eta^{g,h}_{s,w}(S^M(\Lambda^g_{g(w),1}(\pi^*_g(w))))$. In particular, for any $s \in (0,1]$, $w \in \Lambda^h_s$ and $v \in \Lambda^h_{s,1}(w)$, there exists $u \in \Lambda^g_{g(w),1}(\pi^*_g(w))$ such that (u, v) is an ordered pair and $|u, v| \leq \max\{M, N\}$.*

Proof. Since h is sub-exponential, there exist $c_1 \in (0,1)$ and $m \geq 0$ such that $c_1 h(w) \geq h(u)$ for any $w \in T$ and $u \in S^m(w)$. Let $w \in \Lambda^h_s$ and let $w' = \pi^*_g(w)$. Set $t = g(w)$. Let $v \in \Lambda^g_{t,1}(w')$. As h is gentle with respect to g, there exists $c \geq 1$ such that

$$h(w')/c \leq h(v) \leq ch(w'),$$

where c is independent of s, w and v. By Lemma 12.5 and the fact that h is super-exponential, there exists $c' \geq 1$ such that

$$h(w)/c \leq h(v) \leq c'h(w)$$

for any s, w and v. Using this, h being sub-exponential and Proposition 8.16, we see that there exist $c'' > 0$ and M which are independent of s and w such that $c''s \leq h(u) \leq s$ for any $u \in S^M(\Lambda^g_{t,1}(w'))$. Choose k so that $c''(c_1)^{-k} > 1$. Then $h(\pi^{km}(u)) \geq (c_1)^{-k}h(u) \geq c''(c_1)^{-k}s > s$. Set $N = km-1$. Then, for any $u \in S^M(\Lambda^g_{t,1}(w'))$, there exists $n(u)$ such that $n(u) \leq N$ and $\pi^{n(u)}(u) \in \Lambda^h_s$. Now for any $\rho \in \Lambda^h_{s,1}(w)$, there exists $v \in \Lambda^g_{t,1}(w')$ such that (ρ, v) is an ordered pair. Since $\pi^{n(u)}(u) = \rho$ for any $u \in S^M(v)$, it follows that $\eta^{g,h}_{s,w}(S^M(\Lambda^g_{g(w),1}(\pi^*_g(w)))) \supseteq \Lambda^h_{s,1}(w)$. The rest is straightforward. □

Finally we are ready to give a proof of Theorem 12.2.

Proof of Theorem 12.2. Let $g, h, \xi \in \mathcal{G}_e(T)$. Then there exists $\gamma \in (0,1)$ such that $g(w) \geq \gamma g(\pi(w))$, $h(w) \geq \gamma h(\pi(w))$ and $\xi(w) \geq \gamma\xi(\pi(w))$ for any $w \in T$.

First we show $g \underset{\mathrm{GE}}{\sim} g$. By Proposition 8.16, there exists $c \in (0,1)$ such that if $w \in \Lambda^g_s$, then $cg(w) \leq s \leq g(w)$. As a consequence, if $w, v \in \Lambda^g_s$, then $g(w) \leq s/c \leq g(v)/c$. Thus $g \underset{\mathrm{GE}}{\sim} g$.

Next assume $g \underset{\mathrm{GE}}{\sim} h$. Suppose that $w, v \in \Lambda^h_s$ and $K_w \cap K_v \neq \emptyset$. Since $v \in \Lambda^h_{s,1}(w)$, Lemma 12.6 implies that there exists $u \in \Lambda^g_{g(w),1}(\pi^*_g(w))$ such that (u, v) is an ordered pair and $|u, v| \leq L$, where $L = \max\{M, N\}$. By Lemma 12.4, $g(v) \geq \gamma^L g(u) \geq \gamma^L g(w)$. Hence $h \underset{\mathrm{GE}}{\sim} g$.

Finally assume that $g \underset{\mathrm{GE}}{\sim} h$ and $h \underset{\mathrm{GE}}{\sim} \xi$. Suppose that $w, v \in \Lambda^\xi_s$ and $K_w \cap K_v \neq \emptyset$. Since $v \in \Lambda^\xi_{s,1}(w)$, Lemma 12.6 implies that there exists $u \in \Lambda^h_{h(w),1}(\pi^*_h(w))$ such that (u, v) is an ordered pair and $|u, v| \leq L$. By Lemma 12.4, it follows that $g(v) \geq \gamma^L g(u)$. Set $s' = h(w)$ and $w' = \pi^*_h(w)$. Note that $w' \in \Lambda^h_s$ and $u \in \Lambda^h_{s',1}(w')$. Again by Lemma 12.6, there exists $a \in \Lambda^g_{g(w'),1}(\pi^*_g(w'))$ such that (u, a) is an ordered pair and $|a, u| \leq L$. Lemma 12.4 shows that $g(u) \geq \gamma^L g(a) \geq \gamma^L g(\pi^*_h(w))$. By Lemma 12.5, $N_h(w)$ is uniformly bounded and hence there exists $c_* > 0$ which is independent of s, w and v such that $g(\pi^*_h(w)) \geq c_* g(w)$. Combining these, we obtain $g(v) \geq \gamma^{2L} g(\pi^*_h(w)) \geq \gamma^{2L} c_3 g(w)$. Hence $\xi \underset{\mathrm{GE}}{\sim} g$. Consequently we verify $g \underset{\mathrm{GE}}{\sim} \xi$ by the above arguments. □

Next, we show the invariance of thickness, tightness and uniform finiteness under the equivalence relation $\underset{\mathrm{GE}}{\sim}$.

Theorem 12.7. *Let $g, h \in \mathcal{G}_e(T)$. Suppose $g \underset{\mathrm{GE}}{\sim} h$.*

(1) *Suppose that $\sup_{w \in T} \#(S(w)) < +\infty$. If g is uniformly finite then so is h.*
(2) *If g is thick, then so is h.*
(3) *If g is tight, then so is h.*

We need the next lemma to prove Theorem 12.7.

Lemma 12.8. *Let $g, h \in \mathcal{G}_e(T)$. Assume that g is gentle with respect to h. Then for any $\alpha \in (0, 1]$ and $M \geq 0$, there exists $\gamma \in (0, 1)$ such that*

$$U_M^g(x, \alpha g(w)) \supseteq U_M^h(x, \gamma h(w))$$

for any $w \in T$ and $x \in K_w$.

Proof. Since g and h are exponential, there exist $c_1, c_2 \in (0, 1)$ and $m \geq 1$ such that $h(w) \geq c_2 h(\pi(w)), g(w) \geq c_2 g(\pi(w))$, $h(v) \leq c_1 h(w)$ and $g(v) \leq c_1 g(w)$ for any $w \in T$ and $v \in S^m(w)$. Moreover, since g is gentle with respect to g, there exists $c > 1$ such that $g(w) \leq cg(u)$ whenever $w, u \in \Lambda_s^h$ and $K_w \cap K_v \neq \emptyset$. Note that $N_g(w) \leq m$ and $N_h(w) \leq m$ for any $w \in T$.

Let $w \in T$ and let $x \in K_w$. Assume that $\gamma < (c_2)^{lm}$. Let $v \in \Lambda^h_{\gamma h(w), 0}(x)$. Then $h(\pi(v)) > \gamma h(w) \geq h(v)$. There exists $k \geq 0$ such that $\pi^k(v) \in \Lambda^h_{h(w)}$. Then $h(\pi^{k+1}(v)) > h(w) \geq h(\pi^k(v))$. Thus we have

$$\gamma h(\pi^{k+1}(v)) \geq h(v)$$

Therefore, it follows that $k + 1 \geq lm$. Let $w_* = \pi^{N_h(w)}(w)$. Then we see that $x \in K_{\pi^{k+1}(v)} \cap K_{w_*}$. Therefore, $c^{-1} g(w_*) \leq g(\pi^{k+1}(v)) \leq cg(w_*)$. Since $k + 1 \geq lm$ and $N_h(w) \leq m$, it follows that

$$g(v) \leq (c_1)^l g(\pi^{k+1}(v)) \leq c(c_1)^l g(w_*) \leq c(c_1)^l (c_2)^{-m} g(w).$$

Now suppose that $(w(1), \ldots, w(M + 1))$ is a chain in $\Lambda^h_{\gamma h(w)}$ with $w(1) \in \Lambda^h_{\gamma h(w), 0}(x)$. Using the above arguments, we obtain

$$g(w(i)) \leq c^{i-1} g(w(1)) \leq c^i (c_1)^l (c_2)^{-m} g(w) \leq c^{M+1} (c_1)^l (c_2)^{-m} g(w)$$

for any $i = 1, \ldots, M + 1$. Choosing l so that $c^{M+1}(c_1)^l (c_2)^{-m} < \alpha$, we see that $U_M^h(x, \gamma h(w)) \subseteq U_M^g(x, \alpha g(w))$. $\square$

Proof of Theorem 12.7. (1) Set $L = \sup_{w \in T} \#(S(w))$. By Lemma 12.6, it follows that $\#(\Lambda^h_{s,1}(w)) \leq L^M \#(\Lambda^g_{g(w),1}(\pi^*_g(w)))$. This suffices to the desired conclusion.

(2) By the thickness of g and Proposition 9.1, for any $M \geq 0$, there exists $\beta > 0$ such that, for any $w \in T$,

$$K_w \supseteq U_M^g(x, \beta g(\pi(w)))$$

for some $x \in K_w$. By Lemma 12.8, there exists $\gamma \in (0, 1)$ such that

$$U_M^g(x, \beta g(\pi(w))) \supseteq U_M^h(x, \gamma h(\pi(w)))$$

for any $w \in T$. Thus making use of Proposition 9.1 again, we see that h is thick.

(3) Since g is tight, for any $M \geq 0$, there exists $\alpha > 0$ such that, for any $w \in T$,

$K_w \backslash U^g_M(x, \alpha g(w)) \neq \emptyset$ for some $x \in K_w$. By Lemma 12.8, there exists $\gamma \in (0,1)$ such that $U^g_M(x, \alpha g(w)) \supseteq U^h_M(x, \gamma h(w))$ for any $w \in T$ and $x \in K_w$. Hence

$$\sup_{x,y \in K_w} \delta^h_M(x,y) \geq \gamma h(w)$$

for any $w \in T$. Thus we have shown that h is tight. $\square$

Finally, the existence of visual metric is also invariant under $\underset{\mathrm{GE}}{\sim}$ as follows.

Theorem 12.9. *Assume that the partition $K : T \to \mathcal{C}(X, \mathcal{O})$ is minimal. Let $g, h \in \mathcal{G}_e(T)$ and let $M \in \mathbb{N}$. Assume that $g \underset{\mathrm{GE}}{\sim} h$. Then g is hyperbolic if and only if h is hyperbolic.*

Proof. Since g and h are exponential, there exists $\lambda \in (0,1)$ and $m \geq 1$ such that

$$g(w') \leq \lambda g(w) \leq g(w'')$$
$$h(w') \leq \lambda h(w) \leq h(w'')$$

if $w \in T$, $w', w'' \in T_w$, $|w'| - |w| \geq m$ and $|w''| - |w| = 1$. Moreover, since $g \underset{\mathrm{GE}}{\sim} h$, there exists $\eta > 1$ such that if $w, v \in \Lambda^g_s$ and $K_w \cap K_v \neq \emptyset$, then $h(w) \leq \eta h(v)$ and if $w, v \in \Lambda^h_s$ and $K_w \cap K_v \neq \emptyset$, then $g(w) \leq \eta g(v)$. Fix $k \in \mathbb{N}$ satisfying $\eta^M \lambda^k < 1$.

Now assume that g is hyperbolic. Then by Theorem 6.12 and 7.12, g satisfies $(\mathrm{EV5})_M$. Let $w, v \in \Lambda^h_s$ and assume that (w,v) is M-separated in Λ^h_s. Set $t = g(v)$. Suppose that (w,v) is not M-separated in $\Lambda^g_{\lambda^{km}t}$. Then there exists a chain $(w_*(1), \ldots, w_*(M-1))$ in $\Lambda^g_{\lambda^{km}t}$ such that $(w, w_*(1), \ldots, w_*(M-1), v)$ is a chain. Choose $v_* \in \Lambda^g_{\lambda^{km}t} \cap T_v$ so that $K_{w_*(M-1)} \cap K_{v_*} \neq \emptyset$. Since $g(v_*) \leq \lambda^{km} t = \lambda^{km} g(v)$, it follows that $|v_*| - |v| \geq km$. Then we have

$$h(w_*(i)) \leq \eta^M h(v_*) \leq \eta^M \lambda^k h(v) < h(v).$$

Hence there exists a chain $(w(1), \ldots, w(M-1))$ in Λ^h_s such that $w_*(i) \in T_{w(i)}$ for any $i = 1, \ldots, M-1$. This implies that (w,v) is not M-separated in Λ^h_s. This contradiction implies that (w,v) is M-separated in $\Lambda^g_{\lambda^{km}t}$.

Since $(\mathrm{EV5})_M$ holds for g, we see that (w,v) is $(M+1)$-separated in $\Lambda^g_{\tau\lambda^{km}t}$. Set $t_* = \tau\lambda^{km}t$. Choose $v' \in \Lambda^g_{t_*} \cap T_v$. Then exchanging g and h and using the same argument as above, we see that (w,v) is $(M+1)$-separated in $\Lambda^h_{\lambda^{km}h(v')}$.

Since h is exponential, Proposition 8.16 shows that there exists $c > 0$ such that $cr \leq g(u) \leq r$ for any $r \in (0,1]$ and $u \in \Lambda^g_r$. Choose n_* so that $\lambda^{n_*} < c\tau$. Suppose $|v'| - |v| \geq (km + n_*)m$. Then

$$\lambda^{km+n_*} g(v) < c\tau\lambda^{km} g(v) \leq ct_* \leq g(v') \leq \lambda^{km+n_*} g(v).$$

This contradiction yields that $|v'| - |v| < (km + n_*)m$. Therefore, $h(v') \geq \lambda^{(km+n_*)m} h(v) \geq \lambda^{(km+n_*)m} s$. Thus $\lambda^{km} h(v') \geq \lambda^{(km+n_*+k)m} s$. Set $\tau_* = \lambda^{(km+n_*+k)m}$. Then (w,v) is $(M+1)$-separated in $\Lambda^h_{\tau_* s}$. Thus we have shown that $(\mathrm{EV5})_M$ is satisfied for h. Using Theorem 6.12 and 7.12, we see that h is hyperbolic. $\square$

13 Quasisymmetry

In this section, we are going to identify the equivalence relation, gentleness "$\underset{\text{GE}}{\sim}$" with the quasisymmetry "$\underset{\text{QS}}{\sim}$" among the metrics under certain conditions. As in the last section, $(T, \mathcal{A}, \phi)$ is a locally finite tree with a reference point ϕ, $(X, \mathcal{O})$ is a compact metrizable topological space with no isolated point and $K : T \to \mathcal{C}(X, \mathcal{O})$ is a partition of X parametrized by $(T, \mathcal{A}, \phi)$ throughout this section.

Definition 13.1 (Quasisymmetry)**.** A metric $\rho \in \mathcal{D}(X, \mathcal{O})$ is said to be quasisymmetric to a metric $d \in \mathcal{D}(X, \mathcal{O})$ if and only if there exists a homeomorphism h from $[0, +\infty)$ to itself such that $h(0) = 0$ and, for any $t > 0$, $\rho(x, z) < h(t)\rho(x, y)$ whenever $d(x, z) < td(x, y)$. We write $\rho \underset{\text{QS}}{\sim} d$ if ρ is quasisymmetric to d.

It is known that $\underset{\text{QS}}{\sim}$ is an equivalence relation on $\mathcal{D}(X, \mathcal{O})$.

Definition 13.2. Let $d \in \mathcal{D}(X, \mathcal{O})$. We say that d is (super-, sub-)exponential if and only if g_d is (super-, sub-)exponential.

Under the uniformly perfectness of a metric space defined below, we can utilize a useful equivalent condition for quasisymmetry obtained in [17]. See the details in the proof of Theorem 13.4.

Definition 13.3. A metric space (X, d) is called uniformly perfect if and only if there exists $\epsilon > 0$ such that $B_d(x, (1 + \epsilon)r)\backslash B_d(x, r) \neq \emptyset$ unless $B_d(x, r) = X$.

Lemma 13.4. *Let $d \in \mathcal{D}(X, \mathcal{O})$. If d is super-exponential, then (X, d) is uniformly perfect.*

Proof. Write $d_w = g_d(w)$ for any $w \in T$. Since d is super-exponential, there exists $c_2 \in (0, 1)$ such that $d_w \geq c_2 d_{\pi(w)}$ for any $w \in T$. Therefore, $s \geq d_w > c_2 s$ if $w \in \Lambda_s^d$. For any $x \in X$ and $r \in (0, 1]$, choose $w \in \Lambda_{r/2,0}^d(x)$. Then $d(x, y) \leq d_w \leq r/2$ for any $y \in K_w$. This shows $K_w \subseteq B_d(x, r)$. Since $\mathrm{diam}(B_d(x, c_2 r/4), d) \leq c_2 r/2 < d_w$, it follows that $K_w \backslash B_d(x, c_2 r/2) \neq \emptyset$. Therefore $B_d(x, r)\backslash B_d(x, c_2 r/2) \neq \emptyset$. This shows that (X, d) is uniformly perfect. $\square$

Definition 13.5. Define

$$\mathcal{D}_{A,e}(X, \mathcal{O}) = \{d | d \in \mathcal{D}(X, \mathcal{O}), d \text{ is adapted and exponential.}\}$$

The next theorem is the main result of this section.

Theorem 13.6. *Let $d \in \mathcal{D}_{A,e}(X, \mathcal{O})$ and let $\rho \in \mathcal{D}(X, \mathcal{O})$. Then $d \underset{\text{QS}}{\sim} \rho$ if and only if $\rho \in \mathcal{D}_{A,e}(X, \mathcal{O})$ and $d \underset{\text{GE}}{\sim} \rho$. Moreover, if d is M-adapted and $d \underset{\text{QS}}{\sim} \rho$, then ρ is M-adapted as well.*

Remark. In the case of natural partitions of self-similar sets introduced in Example 4.5, the above theorem has been obtained in [18].

The following corollary is straightforward from the above theorem.

Corollary 13.7. *Let $d, \rho \in \mathcal{D}_{A,e}(X, \mathcal{O})$. Then $d \underset{\mathrm{QS}}{\sim} \rho$ if and only if $d \underset{\mathrm{GE}}{\sim} g$.*

The rest of this section is devoted to a proof of the above theorem.

Proof of Theorem 13.6: Part 1. Assume that d and ρ belong to $\mathcal{D}_{A,e}(X, \mathcal{O})$. We show that if $d \underset{\mathrm{GE}}{\sim} \rho$, then $d \underset{\mathrm{QS}}{\sim} \rho$. By Lemma 13.4, both (X, d) and (X, ρ) are uniformly perfect. By [17, Theorems 11.5 and 12.3], $d \underset{\mathrm{QS}}{\sim} \rho$ is equivalent to the facts that there exists $\delta \in (0, 1)$ such that

$$\begin{aligned} B_d(x, r) &\supseteq B_\rho(x, \delta\overline{\rho}_d(x, r)) \\ B_\rho(x, r) &\supseteq B_d(x, \delta\overline{d}_\rho(x, r)) \end{aligned} \tag{13.1}$$

and

$$\begin{aligned} \overline{\rho}_d(x, r/2) &\geq \delta\overline{\rho}_d(x, r) \\ \overline{d}_\rho(x, r/2) &\geq \delta\overline{d}_\rho(x, r) \end{aligned} \tag{13.2}$$

for any $x \in X$ and $r > 0$, where $\overline{\rho}_d(x, r) = \sup_{y \in B_d(x,r)} \rho(x, y)$ and $\overline{d}_d(x, r) = \sup_{y \in B_\rho(x,r)} d(x, y)$. We are going to show (13.1) and (13.2). Since d and ρ are adapted, there exist $\beta \in (0, 1)$, $\gamma > 1$ and $M \geq 1$ such that

$$\begin{aligned} U^d_M(x, \beta r) \subseteq B_d(x, r) \subseteq U^d_M(x, \gamma r) \\ U^\rho_M(x, \beta r) \subseteq B_\rho(x, r) \subseteq U^\rho_M(x, \gamma r) \end{aligned}$$

for any $x \in X$ and $r \in (0, 1]$. By Lemma 12.8, there exists $\alpha \in (0, 1)$ such that $U^\rho_M(x, \rho_w) \supseteq U^d_M(x, \alpha d_w)$ and $U^d_M(x, d_w) \supseteq U^\rho_M(x, \alpha\rho_w)$ for any $w \in T$ and $x \in K_w$. If $w \in \Lambda^d_{\gamma r/\alpha,0}(x)$, then

$$B_d(x, r) \subseteq U^d_M(x, \gamma r) \subseteq U^d_M(x, \alpha d_w) \subseteq U^\rho_M(x, \rho_w), \tag{13.3}$$

where $w \in \Lambda^d_{x,\gamma_1 r/\alpha}$. Hence for any $y \in B_d(x, r)$, there exists $(w(1), \ldots, w(k)) \in \mathcal{CH}_K(x, y)$ such that $k \leq M + 1$ and $w(i) \in \Lambda^\rho_{\rho_w}$. Since $\rho(x, y) \leq \sum_{i=1}^k \rho_{w(i)} \leq (M + 1)\rho_w$, we have

$$\overline{\rho}_d(x, r) \leq (M + 1)\rho_w.$$

Let $w \in \Lambda^d_{\gamma r/\alpha,0}(x)$ as above. Since $\beta/2 < 1 < \gamma/\alpha$, there exists $v \in T_w$ such that $v \in \Lambda^d_{\beta r/2,0}(x)$. Note that $\beta r/2 \geq d_v$. Hence we have

$$B_d\Big(x, \frac{r}{2}\Big) \supseteq U^d_M\Big(x, \frac{\beta r}{2}\Big) \supseteq U^d_M(x, d_v) \supseteq U^\rho_M(x, \alpha\rho_v). \tag{13.4}$$

Since d is sub-exponential, the fact that $w \in \Lambda^d_{\gamma r/\alpha,0}(x)$ and $v \in \Lambda^d_{\beta r/2,0}(x) \cap T_w$ implies that $|v| - |w|$ is uniformly bounded with respect to x, r and w. This and the fact that ρ is super-exponential imply that there exists $c > 0$ which is independent of x, r and w such that $\rho_v \geq c\rho_w$. Now we see that $\alpha\rho_v \geq \eta\overline{\rho}_d(x, r)$, where $\eta = \alpha c/(M + 1)$. Hence

$$B_d\Big(x, \frac{r}{2}\Big) \supseteq U_M^\rho(x, \eta\overline{\rho}_d(x, r)) \supseteq B_\rho\Big(x, \frac{\eta}{\gamma}\overline{\rho}_d(x, r)\Big).$$

By the fact that (X, ρ) is uniformly perfect, there exists $c_* \in (0, 1)$ such that $B_\rho(y, t)\backslash B_\rho(y, c_*t) \neq \emptyset$ unless $B_\rho(y, c_*t) = X$. Set $\delta = c_*\eta/\gamma$. In case $B_\rho(x, \delta\overline{\rho}_d(x, r)) = X$, we have $\overline{\rho}_d(x, r/2) = \overline{\rho}_d(x, r)$. Otherwise, there exists $z \in B_d(x, r/2)$ such that $\rho(x, z) \geq \delta\overline{\rho}_d(x, r)$. In each case, we have $\overline{\rho}_d(x, r/2) \geq \delta\overline{\rho}_d(x, r)$. Furthermore, $B_d(x, r) \supseteq B_\rho(x, \eta\overline{\rho}_d(x, r)/\gamma) \supseteq B_\rho(x, \delta\overline{\rho}_d(x, r))$. Thus we have obtained halves of (13.1) and (13.2). Exchanging d and ρ, we have the other halves of (13.1) and (13.2). $\square$

Lemma 13.8. *Let $d \in \mathcal{D}_{A,e}(X, \mathcal{O})$ and let $\rho \in \mathcal{D}(X, \mathcal{O})$. Assume that $d \underset{\mathrm{QS}}{\sim} \rho$. Let $\delta \in (0, 1)$ be the constant appearing in* (13.1) *and* (13.2).
(1) *For any $w \in T$ and $x, y \in K_w$,*

$$\overline{\rho}_d(x, d_w) \leq \delta^{-1}\overline{\rho}_d(y, d_w).$$

(2) *There exists $c > 0$ such that*

$$c\overline{\rho}_d(x, d_w) \leq \rho_w \leq \delta^{-1}\overline{\rho}_d(x, d_w)$$

for any $w \in T$ and $x \in K_w$.

Proof. Assume $d \underset{\mathrm{QS}}{\sim} \rho$. Lemma 13.4 implies that (X, d) is uniformly perfect. Since $d \underset{\mathrm{QS}}{\sim} \rho$, (X, ρ) is uniformly perfect as well. Hence (13.1) and (13.2) hold.
(1) Since $B_d(x, d_w) \subseteq B_d(y, 2d_w)$, it follows that $\overline{\rho}_d(x, d_w) \leq \overline{\rho}_d(y, 2d_w)$. Applying (13.2), we obtain the desired inequality.
(2) For any $x \in K_w$, $K_w \subseteq B_d(x, 2d_w)$. Hence $\rho_w \leq \overline{\rho}_d(x, 2d_w)$. By (13.2), we see that

$$\rho_w \leq \delta^{-1}\overline{\rho}_d(x, d_w).$$

Set $s = d_w/2$ and choose $v \in T_w \cap \Lambda_s^d$. Since d is adapted and tight, there exists $\gamma > 0$ which is independent of w, v and s such that

$$K_v\backslash B_d(z, \gamma d_v) \neq \emptyset$$

for some $z \in K_v$. By (13.1),

$$K_v\backslash B_\rho(z, \delta\overline{\rho}_d(z, \gamma d_v)) \neq \emptyset.$$

Hence $\rho_w \geq \delta\overline{\rho}_d(z, \gamma d_v)$. Since d is super-exponential, there exists $\gamma' > 0$ which is independent of w, v and s such that $\gamma d_v \geq \gamma' d_w$. Choose $n \geq 1$ so that $2^{n-1}\gamma' \geq 1$. Using (13.2) n-times, we have

$$\rho_w \geq \delta\overline{\rho}_d(z, \gamma' d_w) = \delta^{n+1}\overline{\rho}_d(z, d_w).$$

By (1), if $c = \delta^{n+2}$, then $\rho_w \geq c\overline{\rho}_d(x, d_w)$. $\square$

Proof of Theorem 13.6: Part 2. Assume that $d \in \mathcal{D}_{A,e}(X, \mathcal{O})$. We show that if $d \underset{\mathrm{QS}}{\sim} \rho$, then $\rho \in \mathcal{D}_{A,e}(X, \mathcal{O})$ and $d \underset{\mathrm{GE}}{\sim} \rho$. As in the proof of Lemma 13.8, (13.1) and (13.2) hold.

Claim 1 ρ is super-exponential.

Proof of Claim 1: Since d is super-exponential, there exists $c' \in (0, 1)$ such that $d_w \geq c' d_{\pi(w)}$ for any $w \in T$. Choose $l \geq 1$ so that $2^l c' \geq 1$. By Lemma 13.8-(2) and (13.2), if $x \in K_w$, then

$$\rho_w \geq c\overline{\rho}_d(x, d_w) \geq c\delta^l \overline{\rho}_d(z, 2^l d_w) \geq c\delta^l \overline{\rho}_d(x, d_{\pi(w)}) \geq c\delta^{l+1}\rho_{\pi(w)}.$$

Claim 2 ρ is sub-exponential.

Proof of Claim 2: Since d is sub-exponential, there exist $c_1 \in (0, 1)$ and $m \geq 1$ such that

$$d_{v'} \leq c_1 d_w$$

for any $w \in T$ and $v' \in S^m(w)$. Let $w \in T$. If $v \in S^{mj}(w)$ for $j \geq 1$ and $x \in K_v$, then by Lemma 13.8-(1)

$$\rho_v \leq \delta^{-1}\overline{\rho}_d(x, d_v) \leq \delta^{-1}\overline{\rho}_d(x, (c_1)^j d_w). \tag{13.5}$$

On the other hand, by [17, Proposition 11.7], there exist $\lambda \in (0, 1)$ and $c'' > 0$ such that

$$\overline{\rho}_d(x, c_1 s) \leq c''\lambda\overline{\rho}_d(x, s)$$

for any $x \in X$ and $s \in (0, 1]$. This together with (13.5) and Lemma 13.8-(2) yields

$$\rho_v \leq \delta^{-1}\overline{\rho}_d(x, (c_1)^j d_w) \leq \delta^{-1}c''\lambda^j\overline{\rho}_d(x, d_w) \leq \delta^{-1}c''\lambda^j c^{-1}\rho_w.$$

Choosing j so that $\delta^{-1}c''\lambda^j c^{-1} < 1$, we see that ρ is sub-exponential.

Claim 3 $d \underset{\mathrm{GE}}{\sim} \rho$.

Proof of Claim 3: Since d is super-exponential, there exists $c_2 \in (0, 1)$ such that

$$s \geq d_w > c_2 s \tag{13.6}$$

for any $s \in (0, 1]$ and $w \in \Lambda^d_s$. Let $w, v \in \Lambda^d_s$ with $K_w \cap K_v \neq \emptyset$. Then $d_w \leq d_v/c_2$. Choose $k \geq 1$ so that $2^k c_2 \geq 1$. If $x \in K_w \cap K_v$, then by Lemma 13.8-(2) and (13.2),

$$\rho_w \leq \delta^{-1}\overline{\rho}_d(x, d_w) \leq \delta^{-1}\overline{\rho}(x, d_v/c_2) \leq \delta^{-(k+1)}\overline{\rho}(x, d_v) \leq c^{-1}\delta^{-(k+1)}\rho_v.$$

Hence $d \underset{\mathrm{GE}}{\sim} \rho$.

Claim 4 ρ is adapted. More precisely, if d is M-adapted, then so is ρ.

Proof of Claim 4: Assume that d is M-adapted. Let $x \in X$ and let $s \in (0, 1]$. Then there exists $\alpha > 0$ which is independent of x and s such that $U^d_M(x, \alpha s) \supseteq B_d(x, s)$. Let $w \in \Lambda^\rho_{s,0}(x)$. Since ρ is super-exponential, there exists $b \in (0, 1)$ which is independent of w and s such that $\rho_w \geq bs$. By Lemma 12.8, there exists $\gamma > 0$ such that $U_M^{\rho}(x, \rho_w) \supseteq U_M^d(x, \gamma d_w)$ for any $w \in T$ and $x \in K_w$. Choose $p \geq 1$ so that $2^p\gamma/\alpha \geq 1$. Then by Lemma 13.8-(2), (13.1) and (13.2),

$$\begin{aligned} U_M^{\rho}(x, s) &\supseteq U_M^{\rho}(x, \rho_w) \supseteq U_M^d(x, \gamma d_w) \\ &\supseteq B_d\Big(x, \frac{\gamma}{\alpha} d_w\Big) \supseteq B_\rho\Big(x, \delta\overline{\rho}_d\big(x, \frac{\gamma}{\alpha} d_w\big)\Big) \supseteq B_\rho(x, \delta^{p+1}\overline{\rho}_d(x, d_w)) \\ &\supseteq B_\rho(x, \delta^{p+2}\rho_w) \supseteq B_\rho(x, \delta^{p+2}bs). \end{aligned}$$

On the other hand, let $x \in K$ and let $r \in (0, 1]$. Then for any $y \in U_M^{\rho}(x, r)$, there exists $(w(1), \ldots, w(M+1)) \in \mathcal{CH}_K(x, y)$ such that $w(i) \in \Lambda_r^{\rho}$ for any i. It follows that

$$\rho(x, y) \leq \sum_{i=1}^{M+1} \rho_{w(i)} \leq (M+1)r.$$

This shows that $U_M^{\rho}(x, r) \subseteq B_\rho(x, (M+1)r)$. Thus we have shown that ρ is M-adapted.

Using Theorem 12.7-(2), we see that g_ρ is thick and hence $\rho \in \mathcal{D}_{A,e}(X, \mathcal{O})$. Thus we have shown the desired statement. $\square$

Part III

Characterization of Ahlfors regular conformal dimension

14 Construction of adapted metric I

In this section, we present a sufficient condition for the existence of an adapted metric to a given weight function. The sufficient condition obtained in this section is useful to construct an Ahlfors regular metric later.

Let $(T, \mathcal{A}, \phi)$ be a locally finite tree with a reference point ϕ and let $(X, \mathcal{O})$ be a compact metrizable topological space with no isolated point. Moreover let $K : T \to \mathcal{C}(X, \mathcal{O})$ be a minimal partition.

Definition 14.1. Let $M \geq 1$. A chain $(w(1), \ldots, w(M+1))$ of K is called a horizontal M-chain of K if and only if $|w(i)| = |w(i+1)|$ and $K_{w(i)} \cap K_{w(i+1)} \neq \emptyset$ for any $i = 1, \ldots, M$. Define

$$\begin{aligned} \Gamma_M(w, K) = \{u | u \in (T)_{|w|},\ &\text{there exists a horizontal } M\text{-chain} \\ &(w(1), \ldots, w(M+1)) \text{ of } K \text{ such that } w(1) = w \text{ and } w(M+1) = u\} \end{aligned}$$

for $w \in T$,

$$J_{M,n}^h(K) = \{(w, u) | w, u \in (T)_n, u \in \Gamma_M(w, T)\}$$

for $n \geq 0$,

$$J_M^h(K) = \bigcup_{n \geq 0} J_{M,n}^h(K),$$

$$J^v_M(K) = \{(w,u)|w,u \in T, w \in \pi(\Gamma_M(u,K)) \text{ or } u \in \pi(\Gamma_M(w,T))\},$$

and

$$J_M(K) = J^v_M(K) \cup J^h_M(K).$$

A sequence $(w(1),\dots,w(m)) \in T$ is called an M-jumping path, or an M-jpath for short, (resp. horizontal M-jumping path, or horizontal M-jpath) of K if and only if $(w(i),w(i+1)) \in J_M(K)$ (resp. $(w(i),w(i+1)) \in J^h_M(K)$) for any $i = 1,\dots,m-1$. Furthermore define

$$U_M(w,K) = \bigcup_{v \in \Gamma_M(w,K)} K_v$$

for any $w \in T$.

Remark. Define a weight function $h_* : T \to (0,1]$ as $h_*(w) = 2^{-|w|}$ for any $w \in T$. Then $(T)_m = \Lambda^{h_*}_{2^{-m}}$ and $\Gamma_M(w,K) = \Lambda^{h_*}_{2^{-|w|},M}(w)$.

Remark. Note that the horizontal vertices E^h_m defined in Definition 4.8 is equal to $J^h_{1,m}$. On the contrary the collection $J^v_1(K)$ of the vertices in the vertical direction in $J_1(K)$ is strictly larger than that of vertical edges in the resolution $(T,\mathcal{B})$ in general.

If no confusion may occur, we are going to omit K in $\Gamma_M(w,K)$, $U_M(w,K)$, $J^h_{M,n}(K)$, $J^v_M(w,K)$, $J^h_M(K)$ and $J_M(K)$ and write $\Gamma_M(w)$, $U_M(w)$, $J^h_{M,n}$, J^v_M, J^h_M and J_M respectively. Moreover, in such a case, we simply say a (horizontal) M-jpath instead of a (horizontal) M-jpath of K.

Definition 14.2. (1) For $w \in T$ and $M \in \mathbb{N}$, define

$$\begin{aligned}\mathcal{C}^M_w = \{&(w(1),\dots,w(m))|\pi(w(i)) \in \Gamma_M(w) \text{ for any } i = 1,\dots,m,\\ &\text{there exist } w(0) \in S(w) \text{ and } w(m+1) \in (T)_{|w|+1}\backslash S(\Gamma_M(w)) \text{ such that}\\ &(w(0),w(1),\dots,w(m),w(m+1)) \text{ is a horizontal } M\text{-jpath.}\}\end{aligned}$$

(2) A function $\varphi : T \to (0,\infty)$ is called M-balanced if and only if

$$\sum_{i=1}^m \varphi(w(i)) \geq \varphi(\pi(w(m)))$$

for any $w \in T$ and $(w(1),\dots,w(m)) \in \mathcal{C}^M_w$.

Remark. If $J^h_M = \emptyset$, then $\mathcal{C}^M_w = \emptyset$ for any $w \in T$ as well. Therefore, in this case, every $\varphi : T \to (0,\infty)$ is M-balanced. This happens if and only if the original set is (homeomorphic to) the Cantor set.

Theorem 14.3. *Define $h_* : T \to (0,1]$ by $h_*(w) = 2^{-|w|}$. Let $g \in \mathcal{G}_e(T)$. Assume that $g \underset{\text{GE}}{\sim} h_*$. If there exists $\varphi : T \to (0,\infty)$ such that φ is M-balanced and $\varphi \underset{BL}{\sim} g$, i.e. there exist $c_1, c_2 > 0$ such that*

$$c_1 g(w) \leq \varphi(w) \leq c_2 g(w)$$

for any $w \in T$, then there exists a metric $\rho \in \mathcal{D}(X,\mathcal{O})$ which is M-adapted to g.

The rest of this section is devoted to a proof of the above theorem. Throughout this section, g and φ are assumed to satisfy the conditions required in Theorem 14.3.

To begin with, we are going to make what the conditions on g mean explicit. $g \underset{\mathrm{GE}}{\sim} h_*$ if and only if there exist $\kappa \in (0,1)$ and $N_0 \in \mathbb{N}$ such that if $|w| = |v|$ and $K_w \cap K_v \neq \emptyset$, then

$$g(w) \geq \kappa g(v). \tag{14.7}$$

and if $w, v \in \Lambda_s^g$ and $K_w \cap K_v \neq \emptyset$, then

$$|w| \leq |v| + N_0. \tag{14.8}$$

Moreover, since g is exponential, there exist $\eta \in (0,1)$ and n_0 such that

$$\eta g(\pi(w)) \leq g(w) \quad \text{and} \quad g(v) \leq \eta g(w) \tag{14.9}$$

if $w \in T$, $v \in T_w$ and $|v| \geq |w| + n_0$. By (14.9),

$$\eta s < g(w) \leq s \tag{14.10}$$

for any $s \in (0,1]$ and $w \in \Lambda_s^g$. In other words, if $g(w) \leq \eta s$, then $w \in \Lambda_t^g$ for some $t < s$.

Definition 14.4. Let $\mathbf{p} = (w(1), \ldots, w(m))$ be an M-jpath and let $\varphi : T \to (0, \infty)$. We define

$$\ell_M^\varphi(\mathbf{p}) = \sum_{i=1}^m \varphi(w(i)).$$

Lemma 14.5. *Assume that $\varphi : T \to (0,\infty)$ is M-balanced. Let $m \geq 2$ and let $\mathbf{p} = (w(1), \ldots, w(m))$ be an M-jpath satisfying $|w(1)| = |w(m)| = |w(i)| - 1$ for any $i = \{2, \ldots, m-1\}$. Then there exists a horizontal M-jpath $\mathbf{p}' = (v(1), \ldots, v(n))$ such that $w(1) = v(1)$, $w(m) = v(n)$ and $\ell_M^\varphi(\mathbf{p}) \geq \ell_M^\varphi(\mathbf{p}')$.*

Proof. If $w(m) \in \Gamma_M(w(1))$, then $p' = (w(1), w(m))$ is the desired horizontal M-jpath. Hence we may assume that $w(m) \notin \Gamma_M(w(1))$ without loss of generality. We use an induction on m. If $m = 2$, then it is trivial. Let $m \geq 3$. Since $w(1) \in \pi(\Gamma_M(w(2)))$, we have $\pi(w(2)) \in \Gamma_M(w(1))$.
Case 1: Suppose that $\pi(w(2)), \ldots, \pi(w(i)) \in \Gamma_M(w(1))$ and $\pi(w(i+1)) \notin \Gamma_M(w(1))$ for some $i \in \{2, \ldots, m-2\}$.
In this case, set $\widetilde{p} = (w(1), \pi(w(i)), w(i+1), \ldots, w(m))$. Since $(w(1), w(2)) \in J_M^v$, there exists $v \in T$ such that $\pi(v) = w(1)$ and $v \in \Gamma_M(w(2))$. Therefore, $(v, w(2), \ldots, w(i))$ is a horizontal M-jpath. Since φ is M-balanced, it follows that

$$\varphi(w(2)) + \ldots, \varphi(w(i)) \geq \varphi(\pi(w(i)).$$

Therefore

$$\ell_M^\varphi(\mathbf{p}) \geq \ell_M^\varphi(\widetilde{p}).$$

Applying induction hypothesis to $(\pi(w(i)), w(i+1), \ldots, w(m-1), w(m))$, we obtain the desired result in this case.

Case 2: Suppose $\pi(w(2)), \ldots, \pi(w(m-1)) \in \Gamma_M(w(1))$.
In this case, since $(w(m-1), w(m)) \in J_M^v$, there exists $u \in \Gamma_M(w(m-1))$ such that $\pi(u) = w(m)$. Then $\pi(u) = w(m) \notin \Gamma_M(w(1))$. Thus as in the previous case, we obtain

$$\varphi(w(2)) + \cdots + \varphi(w(m-1)) \geq \varphi(\pi(w(m-1))).$$

Hence $(w(1), \pi(w(m-1)), w(m))$ is the desired horizontal M-jpath. $\square$

Repeated use of the above lemma yields the following fact.

Lemma 14.6. *Assume that φ is M-balanced. If $\mathbf{p} = (w(1), \ldots, w(m))$ is an M-jpath satisfying $|w(1)| = |w(m)|$. Then there exists an M-jpath $\mathbf{p}' = (v(1), \ldots, v(k))$ such that $v(1) = w(1)$, $v(k) = w(m)$, $|v(i)| \leq |v(1)|$ for any $i = 1, \ldots, k$, $k \leq m$ and*

$$\ell_M^{\varphi}(\mathbf{p}) \geq \ell_M^{\varphi}(\mathbf{p}').$$

Lemma 14.7. *Assume that φ is M-balanced, that there exists $\kappa_0 \in (0,1)$ such that $\varphi(w) \geq \kappa_0 \varphi(v)$ for any $(w, v) \in J_M^h$ and that there exists $r \in (0,1)$ such that*

$$\frac{2r}{1-r} < \kappa_0 \tag{14.11}$$

and $\varphi(w) \leq r\varphi(\pi(w))$ for any $w \in T \backslash \{\phi\}$. Let $\mathbf{p} = (w(1), \ldots, w(m))$ be an M-jpath. If $w, v \in T$, $|w| = |v|$, $w \notin \Gamma_M(v)$, $w(1) \in T_w$ and $w(m) \in T_v$, then

$$\ell_M^{\varphi}(\mathbf{p}) \geq \left(\kappa_0 - \frac{2r}{1-r}\right) \max\{\varphi(w), \varphi(v)\}$$

Proof. Since $w(1) \in T_w$ and $w(m) \in T_v$, there exist $n_1, n_2 \geq 0$ such that $\pi^{n_1}(w(1)) = w$ and $\pi^{n_2}(w(m)) = v$. Set $w_*(i) = \pi^i(w(1))$ for $i = 1, \ldots, n_1$ and $v_*(j) = \pi^j(w(m))$ for $j = 1, \ldots, n_2$. Define

$$\mathbf{p}_* = (w_*(n_1), w_*(n_1 - 1), \ldots, w_*(1), w(1), \ldots, w(m), v_*(1), \ldots, v_*(n_2)).$$

Then $\mathbf{p}_*$ is an M-jpath. Lemma 14.6 implies that there exists an M-jpath $\mathbf{p}' = (u(1), \ldots, u(k))$ such that $u(1) = w, u(k) = v$, $|u(i)| \geq |w|$ for any $i = 1, \ldots, k$ and $\ell_M^{\varphi}(\mathbf{p}_*) \geq \ell_M^{\varphi}(\mathbf{p}')$. Since $w \notin \Gamma_M(v)$, we see that $k \geq 3$. Hence $|u(2)|, |u(k-1)| \in \{|w|, |w| - 1\}$.
Case $|u(2)| = |w| - 1$: In this case, since $(w, u(2)) \in J_M^v$, there exists $w' \in \Gamma_M(w)$ such that $\pi(w') = u(2)$. Hence

$$\varphi(u(2)) = \pi(\pi(w')) \geq \frac{1}{r}\varphi(w') \geq \frac{\kappa_0}{r}\varphi(w).$$

Case $|u(2)| = |w|$: In this case, $u(2) \in \Gamma_M(w)$ and hence $\varphi(u(2)) \geq \kappa_0 \varphi(w)$.
Case $|u(k-1)| = |w| - 1$: As the first case, $\varphi(u(k-1)) \geq \kappa_0 r^{-1} \varphi(v)$.
Case $|u(k-1)| = |w|$: As the second case, $\varphi(u(k-1)) \geq \kappa_0 \varphi(v)$.

Considering all the cases above, we conclude

$$\ell_M^{\varphi}(\mathbf{p}') \geq \varphi(w) + \varphi(v) + \kappa_0 \max\{\varphi(w), \varphi(v)\}. \tag{14.12}$$

If $n_1 \geq 1$ and $n_2 \geq 1$. Then

$$\begin{aligned}\ell_M^{\varphi}(\mathbf{p}_*) &= \ell_M^{\varphi}(\mathbf{p}) + \sum_{i=1}^{n_1} \varphi(w_*(i)) + \sum_{j=1}^{n_2} \varphi(v_*(j)) \\ &\leq \ell_M^{\varphi}(\mathbf{p}) + \frac{1}{1-r}\varphi(w) + \frac{1}{1-r}\varphi(v).\end{aligned}$$

Hence by (14.12),

$$\begin{aligned}\ell_M^{\varphi}(\mathbf{p}) \geq \kappa_0 \max\{\varphi(w), \varphi(v)\} - \frac{r}{1-r}(\varphi(w) + \varphi(v)) \\ \geq \left(\kappa_0 - \frac{2r}{1-r}\right) \max\{\varphi(w), \varphi(v)\}.\end{aligned}$$

If $n_1 \geq 1$ and $n_2 = 0$, then

$$\ell_M^{\varphi}(\mathbf{p}_*) = \ell_M^{\varphi}(\mathbf{p}) + \sum_{i=1}^{n_1} \varphi(w_*(i)) \leq \ell_M^{\varphi}(\mathbf{p}) + \frac{1}{1-r}\varphi(w).$$

By (14.12),

$$\begin{aligned}\ell_M^{\varphi}(\mathbf{p}) \geq \varphi(w) + \varphi(v) + \kappa_0 \max\{\varphi(w), \varphi(v)\} - \frac{1}{1-r}\varphi(w) \\ \geq \left(\kappa_0 - \frac{2r}{1-r}\right) \max\{\varphi(w), \varphi(v)\}\end{aligned}$$

For the rest of the cases, similar arguments as above show the desired estimate. $\square$

Definition 14.8. Let $q \in \mathbb{N}$. Define $T^{(q)} = \cup_{m \geq 0} (T)_{mq}$. Define $\pi_q : T^{(q)} \to T^{(q)}$ by $\pi_q = \pi^q$, which is the q-th iteration of π. We consider $T^{(q)}$ as a tree with the reference point ϕ under the natural tree structure inherited from T. Then $(T^{(q)})_m = (T)_{mq}$. Moreover, set $K^{(q)} = K|_{T^{(q)}}$.

Note that a horizontal M-jpath of $K^{(q)}$ is a horizontal M-jpath of K. Similarly, $\Gamma_M(w, K^{(q)}) = \Gamma_M(w, K)$ and $U_M(w, K^{(q)}) = U_M(w, K)$ for any $w \in T^{(q)}$ and $J_{M,mq}^h(K) = J_{M,m}^h(K^{(q)})$.

Definition 14.9. For a chain $\mathbf{p} = (w(1), \ldots, w(m))$ of K, define $L_g(\mathbf{p})$ by

$$L_g(\mathbf{p}) = \sum_{i=1}^{m} g(w(i)).$$

Lemma 14.10. *Let $g \in \mathcal{G}_e(T)$. For any chain $\mathbf{p} = (w(1), \ldots, w(m))$ of K, there exists a 1-path $\widehat{\mathbf{p}} = (v(1), \ldots, v(k))$ of K such that $w(1) = v(1)$, $w(m) = v(k)$ and*

$$L_g(\mathbf{p}) \geq c\ell_1^g(\widehat{\mathbf{p}}),$$

where $c > 0$ is independent of $\mathbf{p}$ and $\widehat{\mathbf{p}}$.

Proof. By (14.9), there exists $c \geq 1$ and $\lambda \in (0,1)$ such that $g(w) \leq c\lambda^k g(\pi^k(w))$ for any $w \in T$ and any $k \geq 0$.

Now we start to construct a 1-path $\widehat{\mathbf{p}}$ by inserting a sequence between $w(i)$ and $w(i+1)$ for each i with $|w(i)| \neq |w(i+1)|$. If $|w(i)| > |w(i+1)|$, then there exists $v \in T_{w(i+1)}$ such that $|v| = |w(i)|$ and $K_{w(i)} \cap K_v \neq \emptyset$. Let $k_i = |w(i)| - |w(i+1)|$. Then $\pi^{k_i}(v) = w(i+1)$ and $(w(i), v, \pi(v), \ldots, \pi^{k_i}(v))$ is a 1-path. Moreover,

$$\sum_{j=0}^{k_i-1} g(\pi^j(v)) \leq (c\lambda^{k_i} + c\lambda^{k_i-1} + \ldots + c\lambda)g(w(i+1)) \leq \frac{c\lambda}{1-\lambda} g(w(i+1)). \tag{14.13}$$

Next suppose that $|w(i)| < |w(i+1)|$. Then using a similar discussion as above, we find a 1-path $(\pi^{k_i}(v), \pi^{k_i-1}(v), \ldots, v, w(i+1))$ satisfying $\pi^{k_i}(v) = w(i)$ and a counterpart of (14.13). Inserting sequences in this manner, we obtain the desired $\widehat{\mathbf{p}}$. By (14.13), it follows that

$$\Big(1 + \frac{2c\lambda}{1-\lambda}\Big) L_g(\mathbf{p}) \geq \ell_1^g(\widehat{\mathbf{p}}).$$

□

Lemma 14.11. *Assume that $g \in \mathcal{G}_e(T)$. Define $\iota_q : T \to T^{(q)}$ by $\iota_q(w) = \pi^{|w| - I(w)q}(w)$ for any $w \in T$, where $I(w)$ is the integer part of $|w|/q$. If $\mathbf{p} = (w(1), \ldots, w(m))$ is a chain of K, then $\mathbf{p}^{(q)} = (\iota_q(w(1)), \ldots, \iota_q(w(m)))$ is a chain of $K^{(q)}$. Moreover,*

$$L_g(\mathbf{p}) \geq \eta^{q-1} L_g(\mathbf{p}^{(q)}) \tag{14.14}$$

for any chain $\mathbf{p}$ of K, where η is the constant appearing in (14.9).

Proof. Since $\iota_q(w) \in T^{(q)}$, $\mathbf{p}^{(q)}$ is a chain of $K^{(q)}$. By (14.9), it follows that $g(w) \geq \eta^{q-1} g(\iota_q(w))$. This immediately yield (14.14) □

Proof of Theorem 14.3. Fix $M \in \mathbb{N}$. Write $\delta(x,y) = \delta_M^g(x,y)$ for any $x, y \in X$. For $A \subseteq X$, we set

$$\Lambda_s^g(A) = \{w | w \in \Lambda_s^g, K_w \cap A \neq \emptyset\}$$
$$(T)_n(A) = \{w | w \in (T)_n, K_w \cap A \neq \emptyset\}$$

In particular, we write $\Lambda_s^g(x,y) = \Lambda_s^g(\{x,y\})$ and $(T)_n(x,y) = (T)_n(\{x,y\})$.

Since $\varphi \underset{\mathrm{BL}}{\sim} g$, (14.7) implies that there exists $\kappa_0 \in (0,1)$ such that $\varphi(w) \geq \kappa_0 \varphi(v)$ for any $(w,v) \in J_M^h$. Choose $r \in (0,1)$ so that

$$\kappa_0 > \frac{2r}{1-r}. \tag{14.15}$$

Then by the fact that $\varphi \underset{\mathrm{BL}}{\sim} g$ and (14.9), for sufficiently large $q \in \mathbb{N}$, if $v \in T_w$ and $|w| + q \leq |v|$, then

$$\varphi(v) \leq r\varphi(w). \tag{14.16}$$

We fix r and q in (14.15) and (14.16) throughout this proof.
Claim 1: There exists $N_1 \in \mathbb{N}$ such that

$$||w| - |v|| \leq N_1$$

for any $x, y \in X$ and $w, v \in \Lambda^g_{\kappa^M \delta(x,y)}(x, y)$.

Proof of Claim 1. By (14.9), there exists $N' \in \mathbb{N}$ such that if $w \in \Lambda^g_{\kappa^M s}$, $w' \in \Lambda^g_s(x, y)$ and $w \in T_{w'}$, then $|w| \leq |w'| + N'$. Let $w, v \in \Lambda^g_{\kappa^M \delta(x,y)}(x, y)$. Choose $w', v' \in \Lambda^g_{\delta(x,y)}(x, y)$ so that $w \in T_{w'}$ and $v \in T_{v'}$. Since $y \in U^g_M(x, \delta(x, y))$, there exists a chain $(w(1), \ldots, w(M + 3))$ such that $w(1) = w', w(m) = v'$ and $w(j) \in \Lambda^g_{\delta(x,y)}$ for any $j = 1, \ldots, M + 3$. By (14.8), it follows that

$$|w| - N' \leq |w'| \leq |v'| + N_0(M + 2) \leq |v| + N_0(M + 2)$$

Hence letting $N_1 = N_0(M + 2) + N'$, we obtain the desired claim. □

Claim 2: For any $N \in \mathbb{N}$, if $w \in \Lambda^g_s$, $v \in \Lambda^g_{\eta^{N+1} s}$ and $v \in T_w$, then $|v| \geq |w| + N$.

Proof of Claim 2. By (14.9) and (14.10), it follows that

$$\eta^{|v|-|w|} g(w) \leq g(v) \leq \eta^{N+1} s \leq \eta^N g(w).$$

Hence $|v| - |w| \geq N$. □

Claim 3: Set $N_2 = N_1 + q + 1$ and $r_* = \eta^{N_2} \kappa^M$. For any $x, y \in X$, there exists $J = J(x, y) \in \mathbb{N}$ such that $|w| \geq Jq$ for any $w \in \Lambda_{r_* \delta(x,y)}$ and if $v \in (T)_{Jq}(x, y)$, then $v \in \Lambda^g_{s'}$ for some $s' < \kappa^M \delta(x, y)$.

Proof of Claim 3. Set $s = \kappa^M \delta(x, y)$. Let

$$N_3 = \min_{u \in \Lambda^g_s(x,y)} |u|.$$

Then there exists a unique $J = J(x, y)$ such that $N_3 + N_1 < Jq \leq N_3 + N_1 + q$. For any $w \in \Lambda^g_{r_* s}(x, y)$, choose $w_* \in \Lambda^g_s(x, y)$ so that $w \in T_{w_*}$. Then by Claim 1 and 2,

$$|w| \geq |w_*| + N_1 + q \geq N_3 + N_1 + q \geq Jq > N_3 + N_1 \geq |u|.$$

for any $u \in \Lambda^g_s(x, y)$. Let $v \in (T)_{Jq}(x, y)$. There exists $v' \in \Lambda^g_s(x, y)$ such that $v \in T_{v'}$. Since $|v'| < Jq$, there exists $s' < s$ such that $v \in \Lambda^g_{s'}$. □

Let $\mathbf{p} = (w(1), \ldots, w(m))$ be a chain of K. Assume that $x \in K_{w(1)}$ and $y \in K_{w(m)}$. If $g(w(1)) \geq r_* \delta(x, y)$ or $g(w(m)) \geq r_* \delta(x, y)$, then

$$L_g(\mathbf{p}) \geq r_* \delta(x, y). \tag{14.17}$$

Assume that $g(w(1)) < r_* \delta(x, y)$ and $g(w(m)) < r_* \delta(x, y)$. Set $J = J(x, y)$. Then by Claim 3, there exist $w, v \in (T)_{Jq} = (T^{(q)})_J$ such that $\iota_q(w(1)) \in T_w$,

$\iota_q(w(m)) \in T_v$, $w \in \Lambda^g_{s_1}$ for some $s_1 < \kappa^M\delta(x,y)$ and $v \in \Lambda^g_{s_2}$ for some $s_2 < \kappa^M\delta(x,y)$. Note that $x \in K_w$ and $y \in K_v$. Suppose that $v \in \Gamma_M(w)$. Then there exists a horizontal M-chain $(u(1), \ldots, u(M+1))$ such that $u(1) = w$ and $u(M+1) = v$. Since $\max\{s_1, s_2\} < \kappa^M\delta(x,y)$, by (14.7) we see that

$$g(u(i)) \le \kappa^{-\max\{i-1, M-i+1\}}\max\{s_1, s_2\} < \delta(x,y)$$

for any $i = 1, \ldots, M+1$. This implies that $\delta(x,y) \le \max\{g(u(i))|i = 1, \ldots, M+1\} < \delta(x,y)$. Thus it follows that $v \notin \Gamma_M(w)$. Applying Lemma 14.10 to the case $T = T^{(q)}$, we see that there exists a 1-path $\mathbf{p}_1 = (v(1), \ldots, v(k))$ of $T^{(q)}$ such that $v(1) = \iota_q(w(1))$, $v(k) = \iota_q(w(m))$ and

$$L_g(\mathbf{p}^{(q)}) \ge c_0\ell^g_1(\mathbf{p}_1), \tag{14.18}$$

where c_0 is independent of $\mathbf{p}$. Note that a 1-path of $K^{(q)}$ is an M-jpath of $K^{(q)}$ for any $M \ge 1$ and $\ell^g_M(\mathbf{p}_1) = \ell^g_1(\mathbf{p}_1)$. Since $\varphi \underset{\mathrm{BL}}{\sim} g$, we have

$$\ell^g_M(\mathbf{p}_1) \ge c_1\ell^\varphi_M(\mathbf{p}_1), \tag{14.19}$$

where $c_1 > 0$ is independent of $\mathbf{p}$. Applying Lemma 14.7 to the M-jpath $\mathbf{p}_1$ of $K^{(q)}$, we obtain

$$\ell^\varphi_M(\mathbf{p}_1) \ge \Big(\kappa_0 - \frac{2r}{1-r}\Big)\max\{\varphi(w), \varphi(v)\} \ge c_2\Big(\kappa_0 - \frac{2r}{1-r}\Big)\max\{g(w), g(v)\}, \tag{14.20}$$

where c_2 is independent of $\mathbf{p}$. By Claim 3, there exist $w' \in T_w$ and $v' \in T_v$ such that $w', v' \in \Lambda^g_{r_*\delta(x,y)}$. Hence by (14.10), we have $\eta r_*\delta(x,y) < g(w)$ and $\eta r_*\delta(x,y) < g(v)$. So by (14.20),

$$\ell^\varphi_M(\mathbf{p}_1) \ge c_3\delta(x,y), \tag{14.21}$$

where c_3 is independent of $\mathbf{p}$. Finally combining (14.17), (14.18), (14.19) and (14.21) and using Lemma 14.11, we conclude that there exists $c_4 > 0$ such that if $\mathbf{p} = (w(1), \ldots, w(m))$ is a chain of K, $x \in K_{w(1)}$ and $y \in K_{w(m)}$, then

$$L_g(\mathbf{p}) \ge c_4\delta(x,y).$$

This immediately implies

$$c_4\delta^g_M(x,y) \le D^g(x,y) \le D^g_M(x,y) \le (M+1)\delta^g_M(x,y)$$

for any $x, y \in X$. Thus, the metric D^g is M-adapted to g. □

15 Construction of Ahlfors regular metric I

In this section, we discuss the condition for a weight function to induce an Ahlfors regular metric, whose definition is given in Definition 15.1.

As in the last section, $(X, \mathcal{O})$ is a compact metrizable topological space with no isolated point, $(T, \mathcal{A}, \phi)$ is a locally finite tree and $K : T \to \mathcal{C}(X, O)$ is a minimal partition. $K : T \to \mathcal{C}(X, \mathcal{O})$: a partition, minimal. Furthermore, we assume that $\sup_{w \in T} \#(S(w)) < +\infty$ throughout this section.

Definition 15.1. A metric $d \in \mathcal{D}(X, \mathcal{O})$ is called Ahlfors regular if there exists a Borel regular probability measure μ on X which is α-Ahlfors regular with respect to d for some α.

Theorem 15.2. *Let $d \in \mathcal{D}_{A,e}(X, \mathcal{O})$ and assume that d is thick and uniformly finite. Let $\alpha > 0$. There exist a metric $\rho \in \mathcal{D}(X, \mathcal{O})$ and a measure $\mu \in \mathcal{M}_P(X, \mathcal{O})$ such that $\rho \underset{\mathrm{QS}}{\sim} d$ and μ is α-Ahlfors regular with respect to ρ if and only if there exists $g \in \mathcal{G}_e(X)$ such that*

- $g \underset{\mathrm{GE}}{\sim} g_d$,
- *there exist $c > 0$ and $M \geq 1$ such that*

$$cD_M^g(x, y) \leq D^g(x, y) \tag{15.1}$$

  for any $x, y \in X$,

- *there exists $c > 0$ such that*

$$c^{-1} g(w)^\alpha \leq \sum_{v \in T_w \cap (T)_{|w|+n}} g(v)^\alpha \leq c g(w)^\alpha \tag{15.2}$$

  for any $w \in T$ and $n \geq 0$.

Remark. The condition (15.1) is equivalent to the existence of a metric ρ' which is M-adapted to g for some $M \geq 1$.

Proof. Suppose that there exist a metric ρ and a measure μ such that $\rho \underset{\mathrm{QS}}{\sim} d$ and μ is α-Ahlfors regular with respect to μ. By Theorem 13.6, setting $g = g_\rho$, we see that $g \underset{\mathrm{GE}}{\sim} g_d$, g is exponential and ρ is adapted. Hence by Theorem 12.7, g is thick and uniformly finite. Using Proposition 6.8, we verify (15.1). Furthermore, by Theorem 8.21, it follows that $g^\alpha \underset{\mathrm{BL}}{\sim} g_\mu$. This yields (15.2).

Conversely, assume that $g \in \mathcal{G}_e(X)$, $g \underset{\mathrm{GE}}{\sim} g_d$, (15.1) and (15.2). Since $g, g_d \in \mathcal{G}_e(T)$, $g \underset{\mathrm{GE}}{\sim} g_d$ and g_d is tight and thick, Theorem 12.7 shows that g is thick and tight. Define $\rho(x, y) = D^g(x, y)/\sup_{a,b \in X} D^g(a, b)$. By (15.1), it follows that ρ is adapted to g. Moreover, Corollary 8.11 implies that $g_\rho \underset{\mathrm{BL}}{\sim} g$. Therefore $g_\rho \geq g_d$ and hence by Theorem 13.6, we see that $\rho \underset{\mathrm{QS}}{\sim} d$. Choose $x_w \in O_w$ for each $w \in T$. Define

$$\mu_n = \frac{1}{\sum_{w \in (T)_n} g(w)^\alpha} \sum_{w \in (T)_n} g(w)^\alpha \delta_{x_w},$$

where δ_x is Dirac's point mass at x. Note that (15.2) implies that

$$c^{-1} = c^{-1} g(\phi) \leq \sum_{w \in (T)_n} g(w)^\alpha \leq c g(\phi) = c \tag{15.3}$$

Since $(X, \mathcal{O})$ is compact, there exist a sub-sequence $\{\mu_{n_i}\}_{i\geq 1}$ and a Borel regular probability measure μ on X such that $\{\mu_{n_i}\}_{i\geq 1}$ converges weakly to μ as $i \to \infty$. By (15.2), we see that

$$c^{-2}g(w)^{\alpha} \leq \mu(K_w) \leq c^2 g(w)^{\alpha}$$

for any $w \in T$. This implies that $\mu \in \mathcal{M}_P(X, \mathcal{O})$ and $g^{\alpha} \underset{\mathrm{BL}}{\sim} g_{\mu}$. Moreover by Theorem 12.7, g_{ρ} is uniformly finite. Hence by Theorem 8.21, μ is α-Ahlfors regular with respect to d. $\square$

16 Basic framework

From this section, we start proceeding towards the characterization of Ahlfors regular conformal dimension. To begin with, we fix our framework in this section and keep it until the end.

As in the previous sections, $(T, \mathcal{A}, \phi)$ is a locally finite tree with the root ϕ, $(X, \mathcal{O})$ is a compact metrizable topological space with no isolated point, $K : T \to \mathcal{C}(X, \mathcal{O})$ is a minimal partition. We also assume that $\sup_{w\in T} \#(S(w)) < +\infty$.

Our standing assumptions in the following sections are as follows:

Basic Framework Let $d \in \mathcal{D}(X, \mathcal{O})$. For $r \in (0, 1)$, define $h_r : T \to (0, 1]$ by

$$h_r(w) = r^{|w|}$$

for any $w \in T$. We assume the following conditions (BF1) and (BF2) are satisfied:
(BF1) d is M_*-adapted for some $M_* \geq 1$, exponential, thick, and uniformly finite.
(BF2) There exists $r \in (0, 1)$ such that $h_r \underset{\mathrm{BL}}{\sim} d$.

Remark. h_r is an exponential weight function.

Our goal of the rest of this paper is to obtain characterizations of the Ahlfors regular conformal dimension of (X, d) satisfying the above conditions (BF1) and (BF2).

The condition (BF2) may be too restrictive. Replacing the original tree T by its subtree $\widetilde{T}^{g_d,r}$ associated with d defined in Definition 7.10, however, we can realize (BF2) providing (BF1) is satisfied.

To simplify the notation, we write $\widetilde{T}^{d,r}$ in place of $\widetilde{T}^{g_d,r}$ hereafter.

Proposition 16.1. *Assume that d is M_*-adapted, exponential, thick and uniformly finite. For any $r > 0$, if we replace T and $K : T \to \mathcal{C}(X, \mathcal{O})$ by $\widetilde{T}^{d,r}$ and $K_{\widetilde{T}^{d,r}} : \widetilde{T}^{d,r} \to \mathcal{C}(X, \mathcal{O})$ respectively, then (BF1) and (BF2) are satisfied.*

Proof. Since we should handle two different structures (T, K) and $(\widetilde{T}^{d,r}, K_{\widetilde{T}^{d,r}})$ here, we denote Λ^g_s and $U^g_M(x, s)$ by $\Lambda^g_s(T, K)$ and $U^g_M(x, s; T, K)$ respectively to emphasize the dependency on a tree structure T and a partition K. Then it follows that

$$\Lambda^d_{r^n}(\widetilde{T}^{d,r}, K_{\widetilde{T}^{d,r}}) = \Lambda^d_{r^n}(T, K)$$

for any $n \geq 0$ and

$$U^d_M(x, r^n; T, K) = U^{\widetilde{h}_r}_M(x, r^n; \widetilde{T}^{d,r}, K_{\widetilde{T}^{d,r}})$$

Since d is M_*-adapted, there exist $c_1, c_2 > 0$ such that

$$B_d(x, c_1 s) \subseteq U^d_{M_*}(x, s; T, K) \subseteq B_d(x, c_2 s) \tag{16.1}$$

for any $x \in X$ and $s \in (0, 1]$. Let $r^m > s \geq r^{m+1}$. Then

$$\begin{aligned}U^{\widetilde{h}_r}_{M_*}(x, s; \widetilde{T}^{d,r}, K_{\widetilde{T}^{d,r}}) &\subseteq U^{\widetilde{h}_r}_{M_*}(x, r^m; \widetilde{T}^{d,r}, K_{\widetilde{T}^{d,r}}) \\ &= U^d_{M_*}(x, r^m; T, K) \subseteq B_d(x, c_2 r^m) \subseteq B_d(x, c_2 r^{-1} s).\end{aligned}$$

In the same manner, we also obtain

$$B_d(x, c_1 r s) \subseteq U^{\widetilde{h}_r}_{M_*}(x, s; \widetilde{T}^{d,r}, K_{\widetilde{T}^{d,r}}).$$

Thus d is M_* adapted with respect to $(\widetilde{T}^{d,r}, K_{\widetilde{T}^{d,r}})$. It is straightforward to see that d is exponential, tight and uniformly finite with respect to $(\widetilde{T}^{d,r}, K_{\widetilde{T}^{d,r}})$. Since d is exponential, there exists $c > 0$ such that

$$cd(w) \geq s \geq d(w)$$

if $w \in \Lambda^d_s$. This implies that $\widetilde{h}_r \underset{\mathrm{BL}}{\sim} d$ as a weight function of $\widetilde{T}^{d,r}$. Thus (BF1) and (BF2) with respect to $(\widetilde{T}^{d,r}, K_{\widetilde{T}^{d,r}})$ is satisfied. $\square$

Due to this proposition, if d is M_*-adapted, exponential, thick and uniformly finite, then we replace $(T, K), \pi$ and S by $(\widetilde{T}^{d,r}, K_{\widetilde{T}^{d,r}}), \pi^{d,r}$ and $S^{d,r}$ respectively and assume that (BF1) and (BF2) are satisfied hereafter. For ease of notations, we use $(T, K), \pi$ and S to denote $(\widetilde{T}^{d,r}, K_{\widetilde{T}^{d,r}}), \pi^{d,r}$ and $S^{d,r}$.

Note that even after the modification the condition $\sup_{w \in T} \#(S(w)) < +\infty$ still holds as d is exponential. Moreover, since d is uniformly finite, the following notion is well-defined.

Definition 16.2. Define

$$L_* = \sup_{w \in T} \#(\Gamma_1(w))$$

and

$$N_* = \sup_{w \in T} \#(S(w))$$

Notation. We write $U_M(x, s) = U^{h_r}_M(x, s)$ for any $M \geq 0$, $x \in X$ and $s \in (0, 1]$.

By the above definition, it is straightforward to see that the next lemma.

Lemma 16.3. (1) *For any* $w \in T$ *and* $k \geq 1$,

$$\#(S^k(w)) \leq (N_*)^k \tag{16.2}$$

(2) *For any* $w \in T$ *and* $M \geq 1$,

$$\#(\Gamma_M(w)) \leq (L_*)^M \tag{16.3}$$

We present two useful propositions that are natural consequences of the basic framework.

Proposition 16.4. *Assume* (BF1) *and* (BF2). *Let* $M_1 \geq 1$. *Suppose that* $\pi(\Gamma_{M_1+M_*}(v)) \subseteq \Gamma_{M_*}(\pi(v))$ *for any* $v \in T$. *Then*

$$\pi(\Gamma_{M_1+M_*}(v)) \subseteq \Gamma_{M_1+M_*}(w)$$

if $\pi(v) \in \Gamma_{M_1}(w)$.

Proof. There exists a horizontal 1-jpath $(w(1), \ldots, w(M_1+1))$ such that $w(1) = w$ and $w(M_1+1) = \pi(v)$. By (BF3), we see $\pi(u) \in \Gamma_{M_*}(\pi(v))$ for any $u \in \Gamma_{M_1+M_*}(v)$. Hence there exists a horizontal 1-jpath $(v(1), \ldots, v(M_*+1))$ such that $v(1) = \pi(v)$ and $v(M_*+1) = \pi(u)$. As $(w(1), \ldots, w(M_1), v(1), \ldots, v(M_*+1))$ is a horizontal 1-jpath, we see that $\pi(u) \in \Gamma_{M_1+M_*}(w)$. $\square$

Proposition 16.5. *Assume (BF1) and (BF2). There exists* $m_0 \in \mathbb{N}$ *such that, for any* $m \geq m_0$ *and* $w \in T$, $\Gamma_1(v) \subseteq S^m(w)$ *for some* $v \in S^m(w)$.

Proof. Since d is thick, so does h_r. Therefore, there exists $\alpha \in (0,1)$ such that for any $w \in T$,

$$U_{M_*}(x, \alpha r^{|w|}) \subseteq K_w$$

for some $x \in K_w$. Choose m so that $r^m < \alpha$. Then $U_{M_*}(x, r^{|w|+m}) \subseteq K_w$. If $v \in S^m(w)$ and $x \in K_v$, then $U_{M_*}(v) \subseteq U_{M_*}(x, r^{|w|+m}) \subseteq K_w$. Since the partition is minimal, it follows that $\Gamma_{M_*}(v) \subseteq S^m(w)$. $\square$

17 Construction of adapted metric II

In this section, we study the sufficient condition for the existence of an adapted metric to a given weigh function d under the basic framework presented in Section 16. This is the continuation of the study of Section 14.

As in the previous sections, $(T, \mathcal{A}, \phi)$ is a locally finite tree with the root ϕ, $(X, \mathcal{O})$ is a compact metrizable topological space with no isolated point, $K : T \to \mathcal{C}(X, \mathcal{O})$ is a minimal partition. We also assume that $\sup_{w \in T} \#(S(w)) < +\infty$. Moreover, we assume that $d \in \mathcal{D}_{A,e}(X, \mathcal{O})$ and the basic framework given in Section 16, i.e. the conditions (BF1) and (BF2) are satisfied.

In this section, we fix $g \in \mathcal{G}_e(T)$ satisfying $g \underset{\mathrm{GE}}{\sim} d$.

Remark. The modification of T in the previous section does not affect the relation $\underset{\mathrm{GE}}{\sim}$ and the exponentiality of g.

Since g is exponential, there exists $\eta \in (0,1)$ and $n_0 \in \mathbb{N}$ such that for any $w \in T$ and $v \in T_w$ with $|v| \geq |w| + n_0$,

$$\eta g(\pi(w)) \leq g(w) \quad \text{and} \quad g(v) \leq \eta g(w).$$

Since $g \underset{\mathrm{GE}}{\sim} d$, there exists $\kappa = \kappa(d,g) \in (0,1)$ such that
if $w, v \in \Lambda_s^d$ and $K_w \cap K_v \neq \emptyset$, then $g(w) \geq \kappa g(v)$ and
if $w, v \in \Lambda_s^g$ and $K_w \cap K_v \neq \emptyset$, then $d(w) \geq \kappa d(v)$. We fix those constants η and κ through this section.

Lemma 17.1. *Let $g \in \mathcal{G}_e(T)$ satisfy $g \underset{\mathrm{GE}}{\sim} d$. Let $m_1 = \min\{m | r^m \leq \kappa\}$. Then for any $x \in X$ and $s \in (0,1]$,*

$$\max_{w,v \in \Lambda_{s,0}^g(x)} ||w| - |v|| \leq m_1$$

Proof. Assume $w, v \in \Lambda_{s,0}^g(x)$. Then

$$d(w) \geq \kappa d(v) \geq r^{m_1} d(v) \quad \text{and} \quad d(v) \geq \kappa d(w) \geq r^{m_1} d(w).$$

This immediately implies $||w| - |v|| \leq m_1$. □

Proposition 17.2. *Let $g \in \mathcal{G}_e(T)$ satisfy $g \underset{\mathrm{GE}}{\sim} d$. Set $m_* = \min\{m | \kappa^{-M} \eta^m < 1\}$. Then for any $x \in X$ and $s \in (0,1]$, there exists $w_* \in \Lambda_{s,0}^g(x)$ such that*

$$U_M^d(x, r^{|w_*| + m_* n_0}) \subseteq U_M^g(x, s).$$

Furthermore if $v \in \Lambda_{r^{|w_*|+m_* n_0},0}^d(x)$, then $g(v) \geq \kappa \eta^{m_* n_0} s$.

Proof. Let $x \in X$ and $s \in (0,1]$. Choose $w_* \in \Lambda_{s,0}^g(x)$ so that $|w_*|$ attains the maximum of $\{|w'| | w' \in \Lambda_{s,0}^g(x)\}$. Set $r_* = r^{|w_*|+m_* n_0}$. For any $u \in \Lambda_{r_*,M}^d(x)$, there exists $u' \in \Lambda_{r_*,0}^d(x)$ such that $u \in \Gamma_M(u')$. Since $|u'| = |w_*| + m_* n_0 > |w_*|$, it follows that there exist $w \in \Lambda_{s,0}^g(x)$ and $v \in T_{|w|+m_* n_0}$ such that $u' \in T_v$ and $v \in T_w$. If $(v(1), \ldots, v(M+1))$ is a horizontal 1-jpath satisfying $v(1) = v$, then

$$g(v(j)) \leq \kappa^{-(j-1)} g(v) \leq \kappa^{-M} \eta^{m_*} g(w) < g(w).$$

This implies $U_M(v) \subseteq U_g^M(x,s)$. Since $K_u \subseteq U_M(u') \subseteq U_M(v')$, we obtain

$$U_M^d(x, r_*) \subseteq U_M^g(x, s)$$

Let $v_* \in \Lambda_{r_*,0}^d(x)$ satisfy $v_* \in T_{w_*}$. Then

$$g(v_*) \geq \eta^{m_* n_0} g(w) \geq \eta^{m_* n_0} s.$$

Note that if $v \in \Lambda_{r_*,0}^d(x)$, then $K_v \cap K_{v_*} \neq \emptyset$. Hence

$$g(v) \geq \kappa g(v_*) \geq \kappa \eta^{m_* n_0} s.$$

□

Definition 17.3. Let $\varphi : T \to (0, \infty)$. For $M \geq 1$ and $w \in T$, define

$$\langle \varphi \rangle_M(w) = \min_{v \in \Gamma_M(w)} \varphi(v).$$

and

$$\Pi^{\varphi}_M(w) = \min_{v \in \Gamma_M(w)} \frac{\varphi(v)}{\varphi(\pi(v))}$$

Lemma 17.4. *Let $g \in \mathcal{G}_e(T)$ satisfy $g \underset{\mathrm{GE}}{\sim} d$. Suppose that $\pi(\Gamma_{M_1+M_*}(v)) \subseteq \Gamma_{M_*}(\pi(v))$ for any $v \in T$. Then for any $v \in T$,*

$$\langle g \rangle_{M_1+M_*}(v) \geq \Pi^g_{M_1+M_*}(v) \max\{\langle g \rangle_{M_1+M_*}(u) | u \in \Gamma_{M_1}(\pi(v))\}.$$

Proof. For simplicity, set $g_* = \langle g \rangle_{M_1+M_*}$. There exists $v' \in \Gamma_{M_1+M_*}(v)$ such that $g_*(v) = g(v')$. Let $u \in \Gamma_{M_1}(\pi(v))$. Since $\pi(v) \in \Gamma_{M_1}(u)$, Proposition 16.4 implies that $\pi(\Gamma_{M_1+M_*}(v)) \subseteq \Gamma_{M_1+M_*}(u)$. Therefore, $\pi(v') \in \Gamma_{M_1+M_*}(u)$. Hence

$$g_*(v) = g(v') = \frac{g(v')}{g(\pi(v'))} g(\pi(v')) \geq \Pi^g_{M_1+M_*}(v) \max\{g_*(u) | u \in \Gamma_{M_1}(\pi(v))\}$$

□

Theorem 17.5. *Suppose that $\pi(\Gamma_{M_1+M_*}(v)) \subseteq \Gamma_{M_*}(\pi(v))$ for any $v \in T$. Assume that $g \in \mathcal{G}_e(T)$ and $g \underset{\mathrm{GE}}{\sim} d$. If*

$$\sum_{i=1}^{m} \Pi^g_{M_*+M_1}(w(i)) \geq 1 \tag{17.1}$$

for any $w \in T$ and $(w(1), \ldots, w(m)) \in \mathcal{C}^{M_1}_w$, then $\langle g \rangle_{M_1+M_*}$ is M_1-balanced. In particular, there exists a metric $\rho \in \mathcal{D}_{\mathcal{A},e}(X)$ which is M_1-adapted to g and quasisymmetric to d.

Remark. Since $\rho \underset{\mathrm{QS}}{\sim} d$ and d is M_*-adapted, Theorem 13.6 implies that ρ is M_*-adapted. Conversely, as ρ is M_1-adapted, so does d. Thus d and ρ are $\min\{M_1, M_*\}$-adapted. Hence choosing $M_* = \min\{M | \rho \text{ is } M\text{-adapted}\}$, we must have $M_1 \geq M_*$.

Proof. Let $(w(1), \ldots, w(m)) \in C^{M_1}_w$. Then by Lemma 17.4,

$$\sum_{i=1}^{m} \langle g \rangle_{M_1+M_*}(w(i)) \geq \sum_{i=1}^{m} \Pi^g_{M_1+M_*}(w(i)) \max\{\langle g \rangle_{M_1+M_*}(u) | u \in \Gamma_{M_1}(\pi(w(i))\}. \tag{17.2}$$

If $\pi(w(m)) \in \Gamma_{M_1}(\pi(w(i))$ for any $i = 1, \ldots, m$, then (17.2) and (17.1) imply

$$\begin{aligned} \sum_{i=1}^{m} \langle g \rangle_{M_1+M_*}(w(i)) &\geq \sum_{i=1}^{m} \Pi^g_{M_1+M_*}(w(i)) \langle g \rangle_{M_1+M_*}(\pi(w(m))) \\ &\geq \langle g \rangle_{M_1+M_*}(\pi(w(m))). \end{aligned} \tag{17.3}$$

Suppose that there exists $j \in \{1, \ldots, m-1\}$ such that $\pi(w(m)) \notin \Gamma_{M_1}(\pi(w(j)))$ and $\pi(w(m)) \in \Gamma_{M_1}(w(i))$ for any $i \in \{j+1, \ldots, m-1\}$. Note that $j \leq m-2$ because $w(m) \in \Gamma_{M_1}(w(m-1))$. Then $(w(m-1), w(m-2), \ldots, w(j+1)) \in \mathcal{C}^{M_1}_{\pi(w(m))}$. Hence by (17.1)

$$\sum_{k=j+1}^{m-1} \Pi^g_{M_1+M_*}(w(k)) \geq 1.$$

So using (17.2), we see that

$$\begin{aligned}\sum_{i=1}^{m} \langle g \rangle_{M_1+M_*}(w(i)) \geq \sum_{i=j+1}^{m-1} \Pi^g_{M_1+M_*}(w(i)) \max_{u \in \Gamma_{M_1}(\pi(w(i)))} g_{M_1+M_*}(u) \\ \geq \langle g \rangle_{M_1+M_*}(\pi(w(m))).\end{aligned}$$

Hence we obtain (17.3) in this case as well. Thus we have shown that $\langle g \rangle_{M_1+M_*}$ is M_1-balanced. Since $g \underset{\mathrm{GE}}{\sim} d$, we have $g \underset{\mathrm{BL}}{\sim} \langle g \rangle_{M_1+M_*}$. Therefore Theorem 14.3 shows that there exists a metric $\rho \in \mathcal{D}(X)$ such that ρ is M_1-adapted to g. Since $g \underset{\mathrm{BL}}{\sim} \rho$, we see that ρ is exponential. Moreover, the fact that $\rho \underset{\mathrm{GE}}{\sim} d$ implies $\rho \underset{\mathrm{QS}}{\sim} d$ by Theorem 13.6. $\square$

To utilize Theorem 17.5, we need to find M_1 satisfying the property that $\pi(\Gamma_{M_1+M_*}(v)) \subseteq \Gamma_{M_*}(\pi(v))$ for any $v \in T$. On the other hand, the remark after Theorem 17.5 suggests $M_1 \geq M_*$. This requirement make it hard to find M_1 with the desired property. Replacing π by π^k, however, we have the following fact.

Proposition 17.6. *Let $M_1 \in \mathbb{N}$. There exists $k_{M_1} \geq 1$ such that if $v \in T$ and $k \geq k_{M_1}$, then $\pi^k(\Gamma_{M_1+M_*}(v)) \subseteq \Gamma_{M_*}(\pi^k(v))$.*

Proof. Since d is M_*-adapted, it is $M_* + M_1$-adapted as well. Therefore, there exist $c_1, c_2 > 0$ such that

$$U^{h_r}_{M_*+M_1}(x, r^n) \subseteq B_d(x, c_1 r^n)$$

and

$$B_d(x, c_2 r^n) \subseteq U^{h_r}_{M_*}(x, r^n)$$

for any $x \in X$ and $n \geq 0$. Choose $k \in \mathbb{N}$ so that $c_1 r^k < c_2$. For any $v \in T$ and $x \in \mathrm{int}(K_v)$, it follows that

$$U^{h_r}_{M_*+M_1}(x, r^{|v|+k}) \subseteq B_d(x, c_1 r^{|v|+k}) \subseteq B_d(x, c_2 r^{|v|}) \subseteq U^{h_r}_{M_*}(x, r^{|v|+k}).$$

This implies $\cup_{u \in \Gamma_{M_*+M_1}(v)} O_u \subseteq \cup_{w \in \Gamma_{M_*}(\pi^k(v))} K_w$. Since $O_u \subseteq O_{\pi^k(u)}$, we see that $O_u \cap K_w = \emptyset$ if $|w| = |v|+k$ and $w \neq \pi^k(u)$. Therefore, $\pi^k(u) \in \Gamma_{M_*}(\pi^k(v))$ for any $u \in \Gamma_{M_*+M_1}(v)$. $\square$

Definition 17.7. (1) For $w \in T$, define

$$\begin{aligned}\mathcal{C}_{w,k}(N_1, N_2, N) = \{&(w(1), \ldots, w(m)) | (w(1), \ldots, w(m)) \text{ is}\\ &\text{a horizontal } N\text{-jpath, } w(j) \in S^k(\Gamma_{N_2}(w)) \text{ for any } j = 1, \ldots, m,\\ &\Gamma_N(w(1)) \cap S^k(\Gamma_{N_1}(w)) \neq \emptyset \text{ and } \Gamma_N(w(m)) \backslash S^k(\Gamma_{N_2}(w)) \neq \emptyset.\}\end{aligned}$$

(2) For a weight function $g \in \mathcal{G}(T^{(k)})$ on $T^{(k)}$, define

$$\Pi_M^{g,k}(w) = \min_{v \in \Gamma_M(w)} \frac{g(v)}{g(\pi^k(v))}$$

for any $w \in T^{(k)}$.

Note that $\mathcal{C}_w^M = \mathcal{C}_{w,1}(0, M, M)$.
Replacing T by $T^{(k)}$ and applying Theorem 17.5, we obtain the following corollary.

Corollary 17.8. *Let $M_1 \in \mathbb{N}$ and let $k \geq k_{M_1}$, where k_{M_1} is the constant appearing in Proposition 17.6. Assume that $g \in \mathcal{G}_e(T^{(k)})$ and $g \underset{\mathrm{GE}}{\sim} d$ as weight functions on $T^{(k)}$. If*

$$\sum_{i=1}^{m} \Pi_{M_1+M_*}^{g,k}(w(i)) \geq 1$$

for any $w \in T^{(k)}$ and $(w(1), \ldots, w(m)) \in \mathcal{C}_{w,k}(0, M_1, M_1)$, then there exists a metric $\rho \in \mathcal{D}_{\mathcal{A},e}(X)$ such that ρ is M_*-adapted to g and ρ is quasisymmetric to d.

18 Construction of Ahlfors regular metric II

Making use of Theorem 15.2 and Corollary 17.8, we establish an sufficient condition for the existence of an adapted metric d and a measure μ where μ is Ahlfors regular with respect to the metric d.

As in the previous sections, $(T, \mathcal{A}, \phi)$ is a locally finite tree with the root ϕ, $(X, \mathcal{O})$ is a compact metrizable topological space with no isolated point, $K : T \to \mathcal{C}(X, \mathcal{O})$ is a minimal partition. We also assume that $\sup_{w \in T} \#(S(w)) < +\infty$. Furthermore, we continue to adhere to the basic framework in Section 16.

Our main theorem of this section is as follows:

Theorem 18.1. *Let $M_1 \in \mathbb{N}$. Assume that $k \geq \max\{m_0, k_{M_1}, k_{M_*}\}$, where m_0 is the constant appearing in Proposition 16.5 and k_{M_1} and k_{M_*} are the constants appearing in Proposition 17.6. If there exists $\varphi : T^{(k)} \to (0, 1]$ such that*

$$\sum_{i=1}^{m} \varphi(w(i)) \geq 1$$

for any $w \in T^{(k)}$ *and* $(w(1), \ldots, w(m)) \in \mathcal{C}_{w,k}(0, M_1, M_1)$ *and*

$$\sum_{v \in S^k(w)} \varphi(v)^p < \frac{1}{2}(L_*)^{-2(M_1+2M_*)},$$

for any $w \in T^{(k)}$*, then there exist a metric* $\rho \in \mathcal{D}_{\mathcal{A},e}(X)$ *and a Borel regular probability measure* μ *on* X *such that* $\rho \underset{\text{QS}}{\sim} d$ *and* μ *is Ahlfors* p*-regular with respect to the metric* ρ*.*

The rest of this section is devoted to a proof of this theorem. First we present two key lemmas.

Lemma 18.2. *Let* (V, E) *be a non-directed graph. Assume that* $(v, v) \in E$ *for any* $v \in V$*. For* $m \geq 1$ *and* $A \subseteq V$*, define*

$$\begin{aligned} V_m(A) = \{u | &\text{there exists } (x(1), \ldots, x(m+1)) \text{ such that} \\ &x(1) \in A,\ x(k) = u \text{ and } (x(i), x(i+1)) \in E \text{ for any } i = 1, \ldots, m\} \end{aligned}$$

Write $V_m(x) = V_m(\{x\})$ *for* $x \in V$*. For any* $f : V \to [0, \infty)$*, there exists* $\sigma : V \to [0, \infty)$ *such that*

$$f(v) \leq \min\{\sigma(u) | u \in V_m(v)\} \leq \sigma(v) \leq \max_{u \in V_m(v)} f(u)$$

for any $v \in V$ *and*

$$\sum_{v \in U} \sigma(v)^p \leq \left(\max_{v \in V} \#(V_m(v))\right) \sum_{v \in V_m(U)} f(v)^p. \tag{18.4}$$

for any $U \subseteq V$*.*

Proof. Define $\sigma(v) = \max\{f(u) | u \in V_m(v)\}$. Since $v \in V_m(u)$ if and only if $u \in V_m(v)$, it follows that $f(v) \leq \sigma(u)$ for any $u \in V_m(v)$. Hence $f(v) \leq \min\{\sigma(u) | u \in V_m(v)\}$. Moreover

$$\begin{aligned} \sum_{v \in U} \sigma(v)^p &\leq \sum_{v \in U} \sum_{u \in V_m(v)} f(u)^p \\ &= \sum_{u \in V_m(U)} \sum_{v \in V_m(u)} f(u)^p = \sum_{u \in V_m(U)} \#(V_m(u)) f(u)^p. \end{aligned}$$

Hence (18.4) holds. $\square$

Lemma 18.3. *Let* $k \geq k_{M_*}$*. Let* $\kappa_0 \in (0, 1)$ *and let* $f : T^{(k)} \backslash \{\phi\} \to [\kappa_0, 1)$*. Then there exists* $g : T^{(k)} \to (0, 1]$ *such that*

$$g(u) \geq \kappa_0 g(v) \tag{18.5}$$

if $(u,v) \in J^h_{M_*}$,

$$f(u) \le \frac{g(u)}{g(\pi^k(u))} \le \max_{v \in \Gamma_{M_*}(u)} f(v) \tag{18.6}$$

for any $u \in T^{(k)} \backslash \{\phi\}$ *and*

$$\sum_{v \in S^k(w)} \left(\frac{g(v)}{g(\pi^k(v))}\right)^p \le (L_*)^{2M_*} \sup\left\{ \sum_{u \in S^k(w')} f(u)^p \middle| w' \in \Gamma_{M_*}(w)\right\}. \tag{18.7}$$

for any $p > 0$ *and* $w \in T^{(k)}$.

Proof. First we are going to construct $g : \cup_{n \ge 0}(T)_{kn} \to (0,1]$ satisfying (18.5) and (18.6) up to the n-th level inductively. Set $g(\phi) = 1$ and $g(w) = f(w)$ for any $w \in (T)_k$. Then (18.5) and (18.6) are satisfied for any $w \in (T)_k$. Assume that there exists $g : \cup_{n=0}^m (T)_{kn} \to (0,1]$ satisfying (18.5) and (18.6) up to the n-th level. Define $g_1(v) = f(v)g(\pi^k(v))$ for any $v \in (T)_{k(m+1)}$. Set

$$J = \{(u,v) | u, v \in (T)_{k(m+1)}, (u,v) \in J^h_{M_*,k(m+1)}, g_1(u) < \kappa_0 g_1(v)\}.$$

Claim 0: If $(u,v),(v,v') \in J^h_{M_*,k(m+1)}$, then $g_1(u) \ge (\kappa_0)^2 g_1(v')$.
Proof of Claim 0: Note that $\pi^k(\Gamma_{2M_*}(u)) \subseteq \Gamma_{M_*}(\pi^k(u))$ because $k \ge k_{M_*}$. Hence $\pi^k(v') \in \Gamma_{M_*}(\pi^k(u))$ and so we have $(\pi^k(v'), \pi^k(u)) \in J^h_{M_*,km}$. Therefore, by (18.5), if $g_1(u) < (\kappa_0)^2 g_1(v')$, then

$$(\kappa_0)^2 g_1(v') > g_1(u) = f(u)g(\pi^k(u)) \ge \kappa_0 g(\pi^k(u)) \ge (\kappa_0)^2 g(\pi^k(v')).$$

Hence $g_1(v') = f(v')g(\pi^k(v')) > g(\pi^k(v'))$. This contradicts the fact that $f(v') \le 1$. $\square$
Claim 1: If $(u,v) \in J$, then $(v,v') \notin J$ for any $v' \in \Gamma_{M_*}(v)$.
Proof of Claim 1: If $(u,v) \in J$ and $(v,v') \in J$, then we have $g_1(u) < (\kappa_0)^2 g_1(v')$. This is impossible by Claim 0. $\square$
Set $A = \{u | u \in (T)_{k(m+1)}, (u,v) \in J$ for some $v \in (T)_{k(m+1)}\}$. Define $g : T_{k(m+1)} \to (0,1]$ by

$$g(u) = \begin{cases} \kappa_0 \max\{g_1(v) | (u,v) \in J^h_{M_*,k(m+1)}\} & \text{if } u \in A \\ g_1(u) & \text{otherwise.} \end{cases}$$

Let $(u,v) \in J^h_{M_*,k(m+1)}$. Then by Claim 1, there are tree cases, i.e.
Case 1: $u \notin A$ and $v \notin A$.
In this case, $(u,v) \in J$, $g(u) = g_1(u)$ and $g(v) = g_1(v)$. Hence (18.5) holds. Moreover since $g(u) = f(u)g(\pi^k(u))$, we have (18.6).
Case 2: $u \notin A$ and $v \in A$.
Choose $v' \in \Gamma_{M_*}(v)$ satisfying $g_1(v') = \max\{g_1(v'') | v'' \in \Gamma_{M_*}(v)\}$. Then by Claim 0, we have

$$g(u) = g_1(u) \ge (\kappa_0)^2 g_1(v') = \kappa_0 g(v).$$

Hence (18.5) holds. Since $g(u) = f(u)g(\pi^k(u))$, we have (18.6) as well.
Case 3: $u \in A$ and $v \notin A$.
Choose $v' \in \Gamma_{M_*}(u)$ satisfying $g_1(v') = \max\{g_1(v'')|v'' \in \Gamma_{M_*}(u)\}$. Since $v \in \Gamma_{M_*}(u)$, it follows that $g_1(v) \leq g_1(v')$. Since $v \notin A$, we have

$$\kappa_0 g(v) = \kappa_0 g_1(v) \leq \kappa_0 g_1(v') = g(u).$$

Thus (18.5) holds. Since $v'' \in \Gamma_{2M_*}(u)$, it follows that $\pi^k(v') \in \Gamma_{M_*}(\pi^k(u))$. Therefore,

$$\frac{g(u)}{g(\pi^k(u))} = \frac{\kappa_0 g_1(v')}{g(\pi^k(u))} = \frac{\kappa_0 f(v')g(\pi^k(v'))}{g(\pi^k(u))} \leq f(v').$$

On the other hand, since $(u, v') \in J$, we see that

$$\frac{g(u)}{g(\pi^k(u))} = \frac{\kappa_0 g_1(v')}{f(u)g(\pi^k(u))} f(u) = \frac{\kappa_0 g_1(v')}{g_1(u)} f(u) > f(u).$$

Thus we have verified (18.7)

The above arguments on three cases have shown that g satisfies (18.5) and (18.6) up to $(m+1)$-th level. Using this argument inductively, we obtain $g : T \to (0, 1]$ satisfying (18.5) and (18.6) at every level. Next we are going to proof (18.7). Note that $\cup_{v' \in S^k(w)} \Gamma_{M_*}(v') \subseteq \cup_{w' \in \Gamma_{M_*}(w)} S^k(w')$. By (18.6),

$$\begin{aligned}
&\sum_{v \in S^k(w)} \left(\frac{g(v)}{g(\pi^k(v))} \right)^p \leq \sum_{v \in S^k(w)} \sum_{u \in \Gamma_{M_*}(v)} f(u)^p \\
&\leq \sum_{u \in \cup_{v' \in S^k(w)} \Gamma_{M_*}(v')} \sum_{v \in \Gamma_{M_*}(u) \cap S^k(w)} f(u)^p \leq (L_*)^{M_*} \sum_{u \in \cup_{v' \in S^k(w)} \Gamma_{M_*}(v')} f(u)^p \\
&\qquad \leq (L_*)^{M_*} \sum_{w' \in \Gamma_{M_*}(\pi^k(w))} \sum_{u \in S^k(w')} f(u)^p \\
&\qquad\qquad \leq (L_*)^{2M_*} \sup \Big\{ \sum_{u \in S^k(w')} f(u)^p \Big| w' \in \Gamma_{M_*}(w) \Big\}
\end{aligned}$$

□

Proof of Theorem 18.1. Set $\eta = \frac{1}{2}(L_*)^{-2(2M_* + M_1)}$. Assume that $\varphi : T \to [0, 1]$ satisfies

$$\sum_{i=1}^{m} \varphi(w(i)) \geq 1 \tag{18.8}$$

for any $w \in T$ and $(w(1), \ldots, w(m)) \in \mathcal{C}_{w,k}(0, M_1, M_1)$ and

$$\sum_{v \in S^k(w)} \varphi(v)^p < \eta$$

for any $v \in T^{(k)}$. Define

$$\widetilde{\varphi}(v) = \left(\varphi(v)^p + \frac{\eta}{(N_*)^k}\right)^{1/p}$$

for any $v \in T$. Then (18.8) still holds if we replace φ by $\widetilde{\varphi}$. Moreover,

$$\sum_{v \in S^k(w)} \widetilde{\varphi}(v)^p < 2\eta$$

and

$$\left(\frac{\eta}{(N_*)^k}\right)^{1/p} \leq \widetilde{\varphi}(v) \leq \left(\frac{1 + (N_*)^k}{(N_*)^k}\eta\right)^{1/p}.$$

Set $M_3 = M_1 + M_*$. Letting $V = (T)_{km}$ and $f = \widetilde{\varphi}$ and applying Lemma 18.2, we obtain $\psi : T \to [0, \infty)$ satisfying

$$\widetilde{\varphi}(v) \leq \langle\psi\rangle_{M_3}(v) \leq \psi(v) \leq \left(\frac{1 + (N_*)^k}{(N_*)^k}\eta\right)^{1/p}$$

for any $v \in T$ and

$$\begin{aligned}\sum_{v \in S^k(w)} \psi(v)^p \leq (L_*)^{M_3} & \sum_{u \in \cup_{v \in S^k(w)} \Gamma_{M_3}(v)} \widetilde{\varphi}(u)^p \\ & \leq (L_*)^{M_3} \sum_{w' \in \Gamma_{M_3}(w)} \sum_{u \in S^k(w')} \widetilde{\varphi}(u)^p < 2(L_*)^{2M_3}\eta.\end{aligned}$$

Next step is to use Lemma 18.3. Set $\kappa_0 = \left(\frac{\eta}{(N_*)^k}\right)^{1/p}$ and $\kappa_1 = \left(\frac{1+(N_*)^k}{(N_*)^k}\eta\right)^{1/p}$. Note that since $\kappa_1 < 1$ because $\eta < \frac{1}{2}$. Applying Lemma 18.3 with $f = \psi$, we obtain $g : T^{(k)} \to (0, 1]$ satisfying (18.5), (18.6) and (18.7). Define $\tau(w) = g(w)/g(\pi^k(w))$ for any $w \in T^{(k)}\backslash\{\phi\}$. Then by (18.6), for any $w \in T^{(k)}\backslash\{\phi\}$,

$$\kappa_0 \leq \psi(w) \leq \tau(w) \leq \kappa_1$$

and by (18.7)

$$\sum_{v \in S^k(w)} \tau(v)^p < 1$$

for any $w \in T^{(k)}$. To construct desired weight function, we need to modify τ once more. Since $k \geq m_0$, Proposition 16.5 shows that, for any $w \in T^{(k)}$, there exists $v_w \in S^k(w)$ such that $\Gamma_1(v_w) \subseteq S^k(w)$. Define

$$\sigma(v) = \begin{cases} \tau(v) & \text{if } v \neq v_{\pi^k(v)}, \\ \left(1 - \sum_{u \in S^k(w)\backslash\{v\}} \tau(u)^p\right)^{1/p} & \text{if } v = v_{\pi^k(v)}. \end{cases}$$

Then

$$\kappa_0 \leq \tau(v) \leq \sigma(v) \leq \max\{\kappa_1, \big(1-(\kappa_0)^p\big)^{1/p}\} < 1 \tag{18.9}$$

and

$$\sum_{v \in S^k(w)} \sigma(v)^p = 1 \tag{18.10}$$

for any $w \in T^{(k)}$. Since

$$\widetilde{\varphi}(v) \leq \langle \psi \rangle_{M_3}(v) \quad \text{and} \quad \psi(v) \leq \tau(v) \leq \sigma(v),$$

it follows that $\widetilde{\varphi}(v) \leq \langle \sigma \rangle_{M_3}(v)$. Hence

$$\sum_{i=1}^{m} \langle \sigma \rangle_{M_3}(w(i)) \geq 1 \tag{18.11}$$

for any $w \in T^{(k)}$ and $(w(1), \ldots, w(m)) \in \mathcal{C}_{w,k}(0, M_1, M_1)$. Define $\widetilde{g}(w)$ inductively by $\widetilde{g}(\phi) = 1$ and

$$\widetilde{g}(w) = \sigma(w)\widetilde{g}(\pi^k(w)).$$

Suppose that $u, v \in (T)_{kn}$, $u \neq v$ and $u \in \Gamma_1(v)$. Then $\pi^{kl}(u) \neq \pi^{kl}(v)$ and $\pi^{k(l+1)}(u) = \pi^{k(l+1)}$ for some $l \geq 0$. Note that $\pi^{kj}(u) \in T_{\pi^{kl}(u)}$, $\pi^{kj}(v) \in T_{\pi^{kl}(v)}$ and $\pi^{kj}(u) \in \Lambda_1(\pi^{kl}(v))$ for any $j = 0, \ldots, l-1$. Hence we see that

$$\frac{\widetilde{g}(u)}{\widetilde{g}(v)} = \frac{\sigma(u)\sigma(\pi^k(u)) \cdots \sigma(\pi^{kl}(u))}{\sigma(v)\sigma(\pi^k(v)) \cdots \sigma(\pi^{kl}(v))} = \frac{\tau(u) \cdots \tau(\pi^{k(l-1)}(u))\sigma(\pi^{kl}(u))}{\tau(v) \cdots \tau(\pi^{k(l-1)}(v))\sigma(\pi^{kl}(v))}$$

On the other hand,

$$\frac{g(u)}{g(v)} = \frac{\tau(u) \cdots \tau(\pi^{k(l-1)}(u))\tau(\pi^{kl}(u))}{\tau(v) \cdots \tau(\pi^{k(l-1)}(v))\tau(\pi^{kl}(v))}.$$

Thus if $\kappa_2 = \max\{\kappa_1, \big(1-(\kappa_0)^p\big)^{1/p}\}$, then

$$\frac{\widetilde{g}(u)}{\widetilde{g}(v)} = \frac{g(u)}{g(v)} \frac{\sigma(\pi^{kl}(u))}{\sigma(\pi^{kl}(v))} \frac{\tau(\pi^{kl}(v))}{\tau(\pi^{kl}(u))} \geq \kappa_0 \frac{\kappa_0}{\kappa_2} \frac{\kappa_0}{\kappa_1}. \tag{18.12}$$

By (18.9), the weight function $\widetilde{g}$ on $T^{(k)}$ is exponential. By (18.12), $\widetilde{g} \underset{\mathrm{GE}}{\sim} d$ as weight functions on $T^{(k)}$. By (18.11), using Corollary 17.8, we deduce that there exists a metric $\rho \in \mathcal{D}_{\mathcal{A},e}(X)$ such that ρ is M_1-adapted to $\widetilde{g}$, $\rho \underset{\mathrm{BL}}{\sim} \widetilde{g}$ and $\rho \underset{\mathrm{QS}}{\sim} d$. Moreover, by (18.10), $\widetilde{g}$ satisfies the $T^{(k)}$-version of (15.2). Hence applying Theorem 15.2 to $\widetilde{g}$ on $T^{(k)}$, we verify the existence of a Borel regular probability measure μ on X satisfying $\mu(K_w) = \widetilde{g}(w)^p$ for any $w \in T^{(k)}$. This implies $\mu \underset{\mathrm{BL}}{\sim} \widetilde{g}^p \underset{\mathrm{BL}}{\sim} \rho^p$. Hence μ is Ahlfors p-regular with respect to ρ. □

19 Critical index of p-energies and the Ahlfors regular conformal dimension

Finally in this section, we establish the characterization of the Ahlfors regular conformal dimension as the critical index of p-energies.

As in the previous sections, $(T, \mathcal{A}, \phi)$ is a locally finite tree with the root ϕ, $(X, \mathcal{O})$ is a compact metrizable topological space with no isolated point, $K : T \to \mathcal{C}(X, \mathcal{O})$ is a minimal partition. We also assume that $\sup_{w \in T} \#(S(w)) < +\infty$.

Throughout this and the following sections, we fix $d \in \mathcal{D}_{A,\epsilon}(X, \mathcal{O})$ satisfying the basic framework, i.e. (BF1) and (BF2) in Section 16.

First we recall the definition of Ahlfors regular conformal dimension.

Definition 19.1. Let (X, d) be a metric space. The Ahlfors regular conformal dimension, AR conformal dimension for short, of a metric space (X, d) is defined as

$$
\begin{aligned}
&\dim_{AR}(X, d) = \\
&\quad \inf\{\alpha | \text{there exist a metric } \rho \text{ on } X \text{ and a Borel regular measure } \mu \text{ on } X \\
&\qquad\qquad \text{such that } \rho \underset{\text{QS}}{\sim} d \text{ and } \mu \text{ is } \alpha\text{-Ahlfors regular with respect to } \rho\}.
\end{aligned}
$$

The definition of p-energy $\mathcal{E}_p(f|V, E)$ of a function f on a graph (V, E) is as follows.

Definition 19.2. Let $G = (V, E)$ be a (non-directed) graph. For $f : V \to \mathbb{R}$, define

$$
\mathcal{E}_p(f|V, E) = \frac{1}{2} \sum_{(x,y) \in E} |f(x) - f(y)|^p.
$$

If $E = \emptyset$, then we define $\mathcal{E}_p(f, V, E) = 0$ for any $f : V \to \mathbb{R}$. Let $V_1, V_2 \subseteq V$. Assume that $V_1 \cap V_2 = \emptyset$. Define

$$
\mathcal{F}_F(V, E, V_1, V_2) = \{f | f : V \to [0, \infty), f|_{V_1} \geq 1, f|_{V_2} \equiv 0\}
$$

and

$$
\mathcal{E}_p(V, E, V_1, V_2) = \inf\{\mathcal{E}_p(f|V, E) | f \in \mathcal{F}_F(V, E, V_1, V_2)\}
$$

For $p = 2$, on the analogy of electric circuits, the quantity $\mathcal{E}_2(V, E, V_1, V_2)$ is considered as the conductance (and its reciprocal is considered as the resistance) between V_1 and V_2. In the same way, we may regard $\mathcal{E}_p(V, E, V_1, V_2)$ as the p-conductance between V_1 and V_2.

Applying the above definition to the horizontal graphs $((T)_m, J^h_{N,m})$, we define the critical index $I_{\mathcal{E}}(N_1, N_2, N)$ of p-energies.

Definition 19.3. Let N_1, N_2 and N be integers satisfying $N_1 \geq 0$, $N_2 > N_1$ and $N \geq 1$. Define

$$
\mathcal{E}_{p,k}(N_1, N_2, N) = \sup_{w \in T} \mathcal{E}_p((T)_{|w|+k}, J^h_{N,|w|+k}, S^k(\Gamma_{N_1}(w)), S^k(\Gamma_{N_2}(w))^c),
$$

$$\overline{\mathcal{E}}_p(N_1, N_2, N) = \limsup_{k \to \infty} \mathcal{E}_{p,k}(N_1, N_2, N)$$

and

$$\underline{\mathcal{E}}_p(N_1, N_2, N) = \liminf_{k \to \infty} \mathcal{E}_{p,k}(N_1, N_2, N).$$

Furthermore, define

$$I_{\mathcal{E}}(N_1, N_2, N) = \inf\{p | \underline{\mathcal{E}}_p(N_1, N_2, N) = 0\}.$$

The last quantity $I_{\mathcal{E}}(N_1, N_2, N)$ is called the critical index of p-energies. Two values $\overline{\mathcal{E}}_p(N_1, N_2, N)$ and $\underline{\mathcal{E}}_p(N_1, N_2, N)$ represents the asymptotic behavior of the p-conductance between $\Gamma_{N_1}(w)$ and the complement of $\Gamma_{N_2}(w)$ as we refine the graphs between those two sets.

Theorem 19.4. *For any $N \geq 1$,*

$$I_{\mathcal{E}}(N_1, N_2, N) = \dim_{AR}(X, d)$$

if $N_1 + M_* \leq N_2$.

Remark. As is shown in Theorem 19.9, even if we replace $\underline{\mathcal{E}}_p$ by $\overline{\mathcal{E}}_p$, the value of $I_{\mathcal{E}}$ is the same.

Up to now we have considered the critical exponent for p-energies associated with simple graphs $\{((T)_m, J^h_{N,m})\}_{m \geq 0}$. In fact, the critical exponent is robust with respect to certain class of modifications of graphs as will be seen in Theorem 19.9. The admissible class of modified graphs is called a proper system of horizontal networks.

Definition 19.5. A sequence of graphs $\{(\Omega_m, E_m)\}_{m \geq 0}$ is called a proper system of horizontal networks with indices (N, L_0, L_1, L_2) if and only if the following conditions (N1), (N2), (N3), (N4) and (N5) are satisfied:

(N1) For every $m \geq 0$, $\Omega_m = A_m \cup V_m$ where $A_m \subseteq (T)_m$ and $V_m \subseteq X$.

(N2) For any $m \geq 0$ and $w \in (T)_m$, $\Omega_{m,w} \neq \emptyset$, where $\Omega_{m,w}$ is defined as

$$\Omega_{m,w} = (\{w\} \cap A_m) \cup (V_m \cap K_w).$$

(N3) If we define

$$E_m(u, v) = \{(x, y) | (x, y) \in E_m, x \in \Omega_{m,u}, y \in \Omega_{m,v}\},$$

then

$$\#(E_m(u, v)) \leq L_0$$

for any $m \geq 0$ and $u, v \in (T)_m$

(N4) For any $(x, y) \in E_m$, $x \in \Omega_{m,u}$ and $y \in \Omega_{m,v}$ for some $(u, v) \in J^h_{N,m}$.

(N5) For any $u, v \in J^h_{L_1}$, $x \in \Omega_{|u|,u}$ and $y \in \Omega_{|u|,v}$, there exist $(x_1, \ldots, x_n)$ and $(w(1), \ldots, w(n))$ such that $w(i) \in \Gamma_{L_2}(u)$ for any $i \geq 1, \ldots, n$, $(x_i, x_{i+1}) \in E_{|v|}(w(i), w(i+1))$ for any $i = 1, \ldots, n-1$ and $x_1 = x, x_n = y, w(1) = u, w(n) = v$.

Example 19.6. Let $\Omega_*^{(N)} = \{((T)_m, J^h_{N,m})\}_{m\geq 0}$. Then $\Omega_*^{(N)}$ is a proper system of horizontal networks with indices $(N, 1, 1, 1)$.

Example 19.7 (the Sierpinski carpet; Figure 9)**.** Let d be the Euclidean metric (divided by $\sqrt{2}$ so that the diameter of $[0,1]^2$ is one.) Then $h_{1/3} \underset{\text{BL}}{\sim} d$. Obviously, d is 1-adapted to the weight function $h_{1/3}$, exponential and uniformly finite. In this case, the original edges of the horizontal graph $((T)_m, J^h_{1,m})$ contain slanting edges, which are $(w,v) \in (T)_m \times (T_m)$ with $K_w \cap K_v$ being a single point. Even if all the slanting edges are deleted, we still have a proper system of horizontal networks $\{(\Omega^1_m, E^1_m)\}_{m\geq 0}$ given by

$$\Omega^1_m = (T)_m$$

and

$$E^1_m = \{(w,v)|w, v \in (T)_m, \dim_H(K_w \cap K_v, d) = 1\}.$$

$\{(\Omega^1_m, E^1_m)\}_{m\geq 0}$ is a proper system of horizonal networks with indices $(1, 1, 1, 2)$.

There is another natural proper system of horizontal networks. Note that the four points $p_1 = (0,0), p_3 = (1,0), p_5 = (1,1)$ and $p_7 = (0,1)$ are the corners of the square $[0,1]^2$. Define $\Omega^2_0 = \{p_1, p_3, p_5, p_7\}$ and

$$E^2_0 = \{(p_i, p_j)|\text{the line segment } p_ip_j \text{ is one of the four line segments of the boundary of } [0,1]^2.\}$$

For $m \geq 1$, we define

$$\Omega^2_m = \cup_{w\in(T)_m} F_w(\Omega^2_0)$$

and

$$E^2_m = \{(F_w(p_j), F_w(p_j))|w \in (T)_m, (p_i, p_j) \in E^2_0\}$$

Then $\{(\Omega^2_m, E^2_m)\}_{m\geq 0}$ is a proper system of horizontal networks with indices $(1, 5, 1, 1)$. In this case all the vertices are the points in the Sierpinski carpet, and the length between the end points of an edge in E^2_m is 3^{-m}.

Notation. Let $\Omega = \{(\Omega_m, E_m)\}_{m\geq 0}$ be a proper system of horizontal networks. For any $U \subseteq (T)_m$, we define

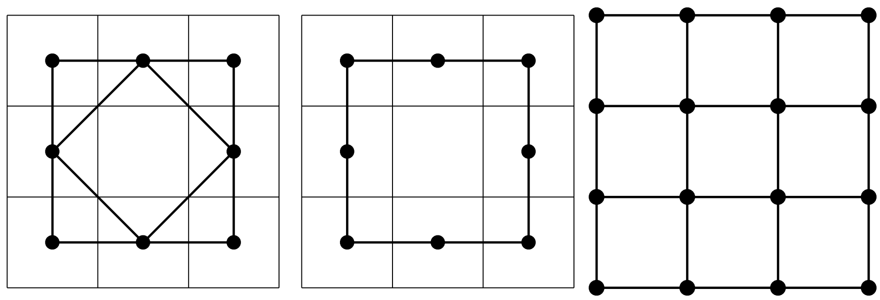
$$\Omega_m(U) = \bigcup_{v\in U} \Omega_{m,v}. \tag{19.13}$$

Furthermore, for $w \in T$, $k \geq 0$ and $n \geq 0$, we define

$$\Omega^k(w,n) = \Omega_{|w|+k}(S^k(\Gamma_n(w))) \tag{19.14}$$

$$\Omega^{k,c}(w,n) = \Omega_{|w|+k}((T)_{|w|+k}\backslash S^k(\Gamma_n(w))) \tag{19.15}$$

In the same manner as the original case, we define the p-conductances and the critical index of p-energies for a proper system of horizontal networks as follows.



$((T)_1, J^h_{1,1})$ (Ω^1_1, E^1_1) (Ω^2_1, E^2_1)

●'s are vertices and thick lines are edges.

Figure 9: Proper systems of horizontal networks: the Sierpinski carpet

Definition 19.8. Let $\Omega = \{(\Omega_m, E_m)\}_{m \geq 0}$ be a proper system of horizontal networks. Define

$$\begin{aligned}
\mathcal{E}_{p,k,w}(N_1, N_2, \Omega) &= \mathcal{E}_p(\Omega_{|w|+k}, E_{|w|+k}, \Omega^k(w, N_1), \Omega^{k,c}(w, N_2)) \\
\mathcal{E}_{p,k}(N_1, N_2, \Omega) &= \sup_{w \in T} \mathcal{E}_{p,k,w}(N_1, N_2, \Omega) \\
\overline{\mathcal{E}}_p(N_1, N_2, \Omega) &= \limsup_{k \to \infty} \mathcal{E}_{p,k}(N_1, N_2, \Omega), \\
\underline{\mathcal{E}}_p(N_1, N_2, \Omega) &= \liminf_{k \to \infty} \mathcal{E}_{p,k}(N_1, N_2, \Omega) \\
\overline{I}_{\mathcal{E}}(N_1, N_2, \Omega) &= \inf\{p | \overline{\mathcal{E}}_p(N_1, N_2, \Omega) = 0\} \\
\underline{I}_{\mathcal{E}}(N_1, N_2, \Omega) &= \inf\{p | \underline{\mathcal{E}}_p(N_1, N_2, \Omega) = 0\}
\end{aligned}$$

Comparing Definitions 19.3 and 19.8, we notice that

$$\mathcal{E}_{p,k}(N_1, N_2, N) = \mathcal{E}_{p,k}(N_1, N_2, \Omega^{(N)}_*).$$

Thus Theorem 19.4 is a corollary of the following theorem.

Theorem 19.9. *Let Ω be a proper system of horizontal networks. If $N_2 \geq N_1 + M_*$, then*

$$\overline{I}_{\mathcal{E}}(N_1, N_2, \Omega) = \underline{I}_{\mathcal{E}}(N_1, N_2, \Omega) = \dim_{AR}(X, d).$$

Before a proof of this theorem, we are going to present a corollary which ensures the finiteness of $\dim_{AR}(X, d)$. To begin with, we need to define growth rates of volumes.

Definition 19.10. Define

$$\overline{N}_* = \limsup_{n\to\infty} \big(\sup_{w\in T} \#(S^n(\Gamma_{N_2}(w)))\big)^{\frac{1}{n}}$$

$$\underline{N}_* = \liminf_{n\to\infty} \big(\sup_{w\in T} \#(S^n(\Gamma_{N_2}(w)))\big)^{\frac{1}{n}}.$$

It is easy to see that

$$\underline{N}_* \leq \overline{N}_* \leq N_*.$$

The quantities $\overline{N}_*$ and $\underline{N}_*$ appear to depend on the value of N_2 but they do not as is shown in the next lemma.

Lemma 19.11.

$$\overline{N}_* = \limsup_{n\to\infty} \big(\sup_{w\in T} \#(S^n(w))\big)^{\frac{1}{n}}$$

$$\underline{N}_* = \liminf_{n\to\infty} \big(\sup_{w\in T} \#(S^n(w))\big)^{\frac{1}{n}}.$$

Proof. Since $S^n(w) \subseteq S^n(\Gamma_{N_2}(w))$, we have

$$\sup_{w\in T} \#(S^n(w)) \leq \sup_{w\in T} \#(S^n(\Gamma_{N_2}(w))).$$

On the other hand, by the fact that $\#(S^n(w)) \leq (N_*)^n$, there exists $w(n) \in T$ such that $\#(S^n(w(n)))$ attains the supremum. Note that

$$S^n(\Gamma_{N_2}(w)) = \bigcup_{v\in\Gamma_{N_2}(w)} S^n(v).$$

Therefore,

$$\#(S^n(\Gamma_{N_2}(w))) \leq \#(\Gamma_{N_2}(w))\#(S^n(w(n)) \leq (L_*)^{N_2} \sup_{w\in T} \#(S^n(w)).$$

$\square$

Corollary 19.12. *Let Ω be a proper system of horizontal networks. Then*

$$\dim_{AR}(X,d) \leq -\frac{\log \underline{N}_*}{\log r} \leq -\frac{\log N_*}{\log r}.$$

Now we start proving the theorem and the corollary.
Using the condition (N5), one can easily obtain the first lemma.

Lemma 19.13. *Let $\Omega = \{(\Omega_m, E_m)\}_{m\geq 0}$ is a proper system of horizontal networks with indices (N, L_0, L_1, L_2). Then Ω is a proper system of horizontal networks with indices $(N, L_0, nL_1, (n-1)L_1 + L_2)$ for any $n \geq 1$.*

Lemma 19.14. *Let $\Omega = \{(\Omega_m, E_m)\}_{m\geq 0}$ be a proper system of horizontal networks. Assume $N_2 \geq N_1 + M_*$. If ρ is a metric on X with* $\mathrm{diam}(X, \rho) = 1$ *and $\rho \underset{\mathrm{QS}}{\sim} d$. Then for any $p > 0$, there exists $c > 0$ such that*

$$\mathcal{E}_{p,k,w}(N_1, N_2, \Omega) \leq c \sum_{u \in S^k(\Gamma_{N_2}(w))} \left(\frac{\rho(u)}{\rho(w)}\right)^p$$

for any $w \in T$ and $k \geq 0$.

Proof. Since $\rho \underset{\mathrm{QS}}{\sim} d$, Theorem 13.6 implies that $\rho \in \mathcal{D}_{A,e}(X)$ and $d \underset{\mathrm{GE}}{\sim} \rho$. Moreover, ρ is M_*-adapted. For $w \in T$, define

$$f_w(x) = \begin{cases} \min\left\{\dfrac{\rho(K_x, U_{N_2}(w)^c)}{\rho(U_{N_1}(w), U_{N_2}(w)^c)}, 1\right\} & \text{if } x \in A_m, \\ \min\left\{\dfrac{\rho(x, U_{N_2}(w)^c)}{\rho(U_{N_1}(w), U_{N_2}(w)^c)}, 1\right\} & \text{if } x \in V_m. \end{cases}$$

for any $x \in \cup_{k\geq 0}\Omega_{|w|+k}$. If is easy to see that $f_w(x) = 1$ for any $x \in \Omega^k(w, N_1)$ and $f_w(x) = 0$ for any $x \in \Omega^{k,c}(w, N_2)$. Since $N_2 \geq N_1 + M_*$, we have

$$U_{N_2}(w) \supseteq U^d_{M_*}(x, r^{|w|}) \supseteq U^\rho_{M_*}(x, \gamma\rho(w)) \supseteq B_\rho(x, \gamma'\rho(w))$$

for any $x \in \mathrm{int}(U_{N_1}(w))$. Thus

$$\rho(U_{N_1}(w), (U_{N_2}(w))^c) \geq \gamma'\rho(w).$$

Let (N, L_0, L_1, L_2) be the indices of Ω. Since $\rho \geq d$, there exists $\kappa \in (0, 1)$ such that

$$\kappa\rho(u) \leq \rho(v)$$

if $|u| = |v|$ and $K_u \cap K_v \neq \emptyset$. For any $(u, v) \in J^h_{N,|w|+k}$ and $(x, y) \in E_{|w|+k}(u, v)$, if $(w(1), \ldots, w(N+1)$ is a horizontal N-chain between u and v, then

$$\begin{aligned} |f_w(x) - f_w(y)| &\leq \frac{\sup_{a\in K_v, b\in K_w} \rho(a, b)}{\gamma'\rho(w)} \\ &\leq \frac{1}{\gamma'\rho(w)} \sum_{i=1}^{N+1} \rho(w(i)) \leq (N+1)\kappa^{-N}(\gamma')^{-1}\frac{\rho(u)}{\rho(w)} \end{aligned}$$

Hence

$$\sum_{(x,y)\in E_{|w|+k}(u,v)} |f_w(x) - f_w(y)|^p \leq L_0((N+1)\kappa^{-N}(\gamma')^{-1})^p \left(\frac{\rho(u)}{\rho(w)}\right)^p.$$

Set $c_1 = L_0((N+1)\kappa^{-N}(\gamma')^{-1})^p$. Then the above inequality and the condition

(N4) imply that

$$\begin{aligned}
&\mathcal{E}_{p,k,w}(N_1, N_2, \Omega) \\
&\qquad \le \frac{1}{2} \sum_{u \in S^k(\Gamma_{N_2}(w))} \sum_{v \in \Gamma_N(u)} \sum_{(x,y) \in E_{|w|+k}(u,v)} |f_w(x) - f_w(y)|^p \\
&\qquad\qquad\qquad\qquad \le c_1 (L_*)^N \sum_{u \in [S^k(\Gamma_{N_2}(w))} \left(\frac{\rho(u)}{\rho(w)}\right)^p.
\end{aligned}$$

□

The next lemma yields the fact that $I_{\mathcal{E}}(N_1, N_2, \Omega) \le \dim_{AR}(X, d)$ if $N_2 \ge N_1 + M_*$.

Lemma 19.15. *Let $\Omega = \{(\Omega_m, E_m)\}_{m \ge 0}$ be a proper system of horizontal networks. Assume $N_2 \ge N_1 + M_*$. If $\dim_{AR}(X, d) < p$, then $\overline{\mathcal{E}}_p(N_1, N_2, \Omega) = 0$.*

Proof. Since $\dim_{AR}(X, d) < p$, there exist $q \in [\dim_{AR}(X, d), p)$, a metric ρ, a Borel regular measure μ and constants $c_1, c_2 > 0$ such that $d \underset{\text{QS}}{\sim} \rho$ and

$$c_1 r^q \le \mu(B_\rho(x, r)) \le c_2 r^q$$

for any $x \in X$ and $r > 0$. By Lemma 19.14,

$$\begin{aligned}
\mathcal{E}_{p,k,w}(N_1, N_2, \Omega) &\le c \sum_{u \in S^k(\Gamma_{N_2}(w))} \left(\frac{\rho(u)}{\rho(w)}\right)^p \\
&\le c \max_{u \in S^k(\Gamma_{N_2}(w))} \left(\frac{\rho(u)}{\rho(w)}\right)^{p-q} \sum_{u \in S^k(\Gamma_{N_2}(w))} \left(\frac{\rho(u)}{\rho(w)}\right)^q. \qquad (19.16)
\end{aligned}$$

Since ρ is exponential, there exist $\lambda \in (0, 1)$ and $c > 0$ such that $\rho(v) \le c\lambda^k \rho(\pi^k(v))$ for any $v \in T$. Choose κ as in the proof of Lemma 19.14. If $u \in S^k(\Gamma_{N_2}(w))$, then

$$\rho(u) \le c\lambda^k \rho(\pi^k(u)) \le c\lambda^k \kappa^{-N_2} \rho(w). \qquad (19.17)$$

On the other hand, by Theorem 8.21(7.21), there exist $\gamma_1, \gamma_2 > 0$ such that

$$\gamma_2 \mu(K_u) \le \rho(u)^q \le \gamma_3 \mu(K_u)$$

for any $u \in T$. This implies

$$\sum_{u \in S^k(v)} \rho(u)^q \le \gamma_3 L_* \mu(K_v) \le (\gamma_2)^{-1} \gamma_3 L_* \rho(u)^q \le (\gamma_2)^{-1} \gamma_3 L_* (\kappa^{-N_2} \rho(w))^q$$

for any $v \in (T)_{|w|}$. Thus

$$\begin{aligned}
\sum_{u \in S^k(\Gamma_{N_2}(w))} \left(\frac{\rho(u)}{\rho(w)}\right)^q &= \sum_{v \in \Gamma_{N_2}(w)} \sum_{u \in S^k(v)} \left(\frac{\rho(u)}{\rho(w)}\right)^q \\
&\le \#(\Gamma_{N_2}(w)) (\gamma_2)^{-1} \gamma_3 L_* \kappa^{-N_2 q} \le (L_*)^{N_2+1} (\gamma_2)^{-1} \gamma_3 \kappa^{-N_2 q}. \qquad (19.18)
\end{aligned}$$

Combining (19.16), (19.17) and (19.18), we obtain

$$\mathcal{E}_{p,k,w}(N_1, N_2, \Omega) \leq c' \lambda^{(p-q)k} \tag{19.19}$$

where c' is independent of w. Therefore, we conclude that $\overline{\mathcal{E}}_p(N_1, N_2, \Omega) = 0$. $\square$

The following lemma enable us to apply Theorem 18.1 and to construct desired pair of a metric and a measure with Ahlfors regularity.

Lemma 19.16. *Let $\Omega = \{(\Omega_m, E_m)\}_{m\geq 0}$ be a proper system of horizontal networks and let $N \in \mathbb{N}$. If $\underline{\mathcal{E}}_p(N_1, N_2, \Omega) = 0$, then for any $\eta > 0$ and $k_0 \in \mathbb{N}$, there exists $k_* \geq k_0$ and $\varphi : T^{(k_*)} \backslash \{\phi\} \to [0, 1]$ such that, for any $w \in T^{(k_*)}$,*

$$\sum_{i=1}^{m} \varphi(w(i)) \geq 1 \tag{19.20}$$

for any $(w(1), \ldots, w(m)) \in \mathcal{C}_{w,k_*}(N_1, N_2, N)$ and

$$\sum_{v \in S^{k_*}(w)} \varphi(v)^p < \eta. \tag{19.21}$$

Proof. As $\underline{\mathcal{E}}_p(N_1, N_2, \Omega) = 0$, for any $\eta_0 > 0$ and $k_0 \in \mathbb{N}$, there exists $k_* \geq k_0$ such that $\mathcal{E}_{p,k_*,w}(N_1, N_2, \Omega) < \eta_0$ for any $w \in T$. Hence there exists $f_w : (T)_{|w|+k_*} \to [0, 1]$ such that $f_w(x) = 1$ for any $x \in \Omega^{k_*}(w, N_1)$, $f_w(x) = 0$ for any $x \in \Omega^{k_*,c}(w, N_2)$ and

$$\sum_{(x,y) \in E_{|w|+k_*}} |f_w(x) - f_w(y)|^p < \eta_0$$

Let (N_0, L_0, L_1, L_2) be the indices of Ω. Set $n_0 = \min\{n | N \leq nL_1\}$ and $\overline{N} = (n_0 - 1)L_1 + L_2$. Note that $N \leq \overline{N}$ because $L_1 \leq L_2$. Define $E_m(U) = \cup_{u_1,u_2 \in U} E_m(u_1, u_2)$ for $U \subseteq (T)_m$. Define $\varphi_w : (T)_{|w|+k_*} \to [0, 1)$ by

$$\varphi_w(v) = \begin{cases} \Big(\displaystyle\sum_{(x,y) \in E_{|w|+k_*}(\Gamma_{\overline{N}}(v))} |f_w(x) - f_w(y)|^p \Big)^{1/p} & \text{if } v \in S^{k_*}(\Gamma_{N_2}(w)), \\ 0 & \text{otherwise.} \end{cases}$$

Let $(w(1), \ldots, w(m)) \in \mathcal{C}_{w,k_*}(N_1, N_2, N)$. By definition, there exist $w(0)$ and $w(m+1) \in (T)_{|w|+k_*}$ such that $w(0) \in S^{k_*}(\Gamma_{N_1}(w))$, $w(m+1) \notin S^{k_*}(\Gamma_{N_2}(w))$ and $(w(0), w(1), \ldots, w(m), w(m+1))$ is a horizontal N-jpath. Choose $x_i \in \Omega_{|w|+k_*,w(i)}$ for $i = 0, \ldots, m+1$. By Lemma 19.13, for any $i = 0, \ldots, m$, there exist $(x^i_1, \ldots, x^i_{l_i})$ and $(w^i(1), \ldots, w^i(l_i))$ such that $x^i_1 = x_i$, $x^i_{l_i} = x_{i+1}$, $(x^i_j, x^i_{j+1}) \in E_{|w|+k_*}(w^i(j), w^i(j+1))$ and $w^i(j) \in \Gamma_{\overline{N}}(w(i))$ for any $j = 1, \ldots, l_i - 1$. Concatenating the paths $(x^i_1, \ldots, x^i_{l_i})$ for $i = 0, \ldots, m$ and removing all loops form it, we obtain a path $(z_1, \ldots, z_n)$. By the nature of the construction, for any $i = 1, \ldots, n-1$, $(z_i, z_{i+1}) \in E_{|w|+k_*}(\Gamma_{\overline{N}}(w(j))$ for some $j = 0, \ldots, m$. Hence

$$\sum_{i=1}^{m} \varphi_w(w(i)) \geq \sum_{i=1}^{n-1} |f_w(z_i) - f_w(z_{i+1})| \geq f_w(z_1) - f_w(z_n) = 1. \tag{19.22}$$

Since $\#(\{u|u \in (T)_m, x \in K_u) \leq L_*$, we see $\#((u_1, u_2)|(x, y) \in E_m(u_1, u_2)\}) \leq (L_*)^2$ for any $(x, y) \in L_m$. Making use of this fact, we see that

$$\begin{aligned}\#(\{v|(x,y) \in E_{|w|+k_*}(\Gamma_{\overline{N}}(v))\}) &\leq \\ &\sum_{(u_1,u_2):(x,y)\in E_{|w|+k_*}(u_1,u_2)} \#(\Gamma_{\overline{N}}(u_1) \cap \Gamma_{\overline{N}}(u_2)) \\ &\leq \sum_{(u_1,u_2):(x,y)\in E_{|w|+k_*}(u_1,u_2)} (L_*)^{\overline{N}} \leq (L_*)^{\overline{N}+2}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{v\in(T)_{|w|+k_*}} \varphi_w(v)^p &= \sum_{v\in(T)_{|w|+k_*}} \sum_{(x,y)\in E_{|w|+k_*}(\Gamma_{\overline{N}}(v))} |f_w(x) - f_w(y)|^p \\ &\leq \sum_{(x,y)\in E_{|w|+K_*}} \#(\{v|(x,y) \in \Gamma_{\overline{N}}(v)\})|f_w(x) - f_w(y)|^p < (L_*)^{\overline{N}+2}\eta_0.\end{aligned}$$

Define $\varphi : T^{(k_*)}\backslash\{\phi\} \to [0, 1]$ by

$$\varphi(v) = \max\{\varphi_w(v)|w \in (T)_{k_*(|v|-1)}\}.$$

By (19.22), we obtain (19.20). Since if $\varphi_w(v) > 0$, then $\pi^{k_*}(v) \in \Gamma_{N_2}(w)$, it follows that

$$\varphi(v) = \max\{\varphi_w(v)|w \in \Gamma_{N_2}(\pi^{k_*}(v))\}.$$

Therefore

$$\begin{aligned}\sum_{v\in S^{k_*}(w)} \varphi(v)^p &\leq \sum_{v\in S^{k_*}(w)} \sum_{w'\in\Gamma_{N_2}(w)} \varphi_{w'}(v)^p \\ &< \#(\Gamma_{N_2}(w))(L_*)^{\overline{N}+2}\eta_0 \leq (L_*)^{N_2+\overline{N}+2}\eta_0.\end{aligned}$$

So, letting $\eta_0 = (L_*)^{-(N_2+\overline{N}+2)}\eta$, we have shown (19.21). $\square$

Finally, we are going to complete the proof of Theorem 19.9.

Proof of Theorem 19.9. Suppose $N_2 \geq N_1 + M_*$. By Lemma 19.15, it follows that $\overline{I}_{\mathcal{E}}(N_1, N_2, \Omega) \leq \dim_{AR}(X, d)$. To prove the opposite inequality, we assume that $\underline{I}_{\mathcal{E}}(0, N_2, \Omega.) < p$. Set $k_0 = \max\{m_0, k_{N_1}, k_{M_*}\}$. Since $\underline{\mathcal{E}}_p(0, N_2, \Omega) = 0$, Lemma 19.16 yields a function φ satisfying the assumptions of Theorem 18.1. Hence by Theorem 18.1, we find a metric ρ which is quasisymmetric to d and a measure μ which is Ahlfors p-regular with respect to ρ. This immediately shows that $\dim_{AR}(X,d) \leq p$. Hence we obtain $\dim_{AR}(X,d) \leq \underline{I}_{\mathcal{E}}(0, N_2, \Omega) \leq \underline{I}_{\mathcal{E}}(N_1, N_2, \Omega)$. Thus we have obtained

$$\dim_{AR}(X,d) \leq \underline{I}_{\mathcal{E}}(N_1, N_2, \Omega) \leq \overline{I}_{\mathcal{E}}(N_1, N_2, \Omega) \leq \dim_{AR}(X,d).$$

□

Proof of Corollary 19.12. Applying Lemma 19.14 in the case where $\rho = d$, we obtain

$$\mathcal{E}_{p,k,w}(N_1, N_2, \Omega) \leq c \sum_{u \in S^k(\Gamma_{M_*}(w)])} \left(\frac{d(u)}{d(w)}\right)^p \leq c' \#(S^k(\Gamma_{M_*}(w))) r^{pk}.$$

Set $\overline{N}_k(M) = \sup_{w \in T} \#(S^k(\Gamma_M(w)))$. Then

$$\mathcal{E}_{p,k}(N_1, N_2, \Omega) \leq c\overline{N}_k(N_2) r^{pk} \tag{19.23}$$

If $q - \epsilon > \liminf_{k \to \infty} -\frac{\log \overline{N}_k(N_2)}{n \log r}$, then there exists $\{k_j\}_{j \geq 1}$ such that

$$r^{\epsilon k_j} \geq \overline{N}_{k_j}(N_2) r^{q k_j}.$$

Hence by (19.23), $I_{\mathcal{E}}(N_1, N_2, \Omega) \leq q$. This implies

$$I_{\mathcal{E}}(N_1, N_2, \Omega) \leq \liminf_{k \to \infty} -\frac{\log \overline{N}_k(N_2)}{n \log r}.$$

The rest of the statement follows from the fact that

$$\#(S^k(\Gamma_M(w))) \leq \#(\Gamma_M(w))(N_*)^k \leq (L_*)^M (N_*)^k.$$

□

20 Relation with p-spectral dimensions

By Theorem 19.9, we see that

$$\lim_{k \to \infty} \mathcal{E}_{p,k}(N_1, N_2, \Omega) = 0$$

if $p > \dim_{AR}(X,d)$ and

$$\liminf_{k \to \infty} \mathcal{E}_{p,k}(N_1, N_2, \Omega) > 0$$

if $p < \dim_{AR}(X,d)$. So, how about the rate of decrease and/or increase of $\mathcal{E}_{p,k}(N_1, N_2, \Omega)$ as $k \to \infty$? In this section, we define and investigate the rates and present another characterization of the Ahlfors regular conformal dimension in terms of them.

As in the previous sections, $(T, \mathcal{A}, \phi)$ is a locally finite tree with the root ϕ, $(X, \mathcal{O})$ is a compact metrizable topological space with no isolated point, $K : T \to \mathcal{C}(X, \mathcal{O})$ is a minimal partition. We also assume that $\sup_{w \in T} \#(S(w)) < +\infty$. Furthermore we fix $d \in \mathcal{D}_{A,\epsilon}(X, \mathcal{O})$ satisfying (BF1) and (BF2) in Section 16.

In this framework, the rates are defined as follows.

Definition 20.1. Define

$$\overline{R}_p(N_1, N_2, \Omega) = \limsup_{n\to\infty} \mathcal{E}_{p,n}(N_1, N_2, \Omega)^{\frac{1}{n}}$$
$$\underline{R}_p(N_1, N_2, \Omega) = \liminf_{n\to\infty} \mathcal{E}_{p,n}(N_1, N_2, \Omega)^{\frac{1}{n}}.$$

Ideally, we expect that $\overline{R}_p(N_1, N_2, \Omega) = \underline{R}_p(N_1, N_2, \Omega)$ and that

$$\frac{1}{(R_p)^m}\mathcal{E}_p(f|\Omega, E_m) \to \mathcal{E}_p(f)$$

as $m \to \infty$ for f belonging to some reasonably large class of functions, where we write $R_p = \overline{R}_p(N_1, N_2, \Omega)$. In particular, for a class of (random) self-similar sets including the Sierpinski gasket and the (generalized) Sierpinski carpets, it is known that

$$\overline{R}_2(N_1, N_2, \Omega) = \underline{R}_2(N_1, N_2, \Omega). \tag{20.1}$$

Moreover, the rate R_2 is called the resistance scaling ratio and $\mathcal{E}_2(f)$ has known to induce the "Brownian motion" $(\{X_t\}_{t>0}, \{P_x\}_{x\in X})$ and the "Laplacian" Δ through the formula

$$\mathcal{E}_2(f) = \int_X f\Delta f d\mu$$
$$E_x(f(X_t)) = (e^{-t\Delta}f)(x),$$

where $E_x(\cdot)$ is the expectation with respect to $P_x(\cdot)$. See [6], [3], [2], [20] and [15] for details

Now we start to study the relation between R_p's for different values of p.

Lemma 20.2. *Let $\Omega = \{(\Omega_m, E_m)\}_{m\geq 0}$ be a proper system of horizontal networks. If $p < q$, then there exists $c > 0$ such that*

$$\mathcal{E}_{p,k}(N_1, N_2, \Omega)^{\frac{1}{p}} \leq c\mathcal{E}_{q,k}(N_1, N_2, \Omega)^{\frac{1}{q}} \sup_{w\in T} \#(S^k(\Gamma_{N_2}(w)))^{\frac{1}{p}-\frac{1}{q}}.$$

Proof. Let $(Y, \mathcal{M}, \mu)$ is a measurable space. Assume $\mu(Y) < \infty$. Then by the Hölder inequality,

$$\int_Y |u|^p d\mu \leq \left(\int_Y |u|^q d\mu\right)^{\frac{p}{q}} \mu(Y)^{\frac{q-p}{q}}.$$

Applying this to $\mathcal{E}_p(f|\Omega_{|w|+k}, E_{|w|+k})$, we obtain

$$\begin{aligned}
&\mathcal{E}_p(f|\Omega_{|w|+k}, E_{|w|+k})^{\frac{1}{p}} \\
&\leq \mathcal{E}_q(f|\Omega_{|w|+k}, E_{|w|+k})^{\frac{1}{q}} \Big(\#(\{(x,y)|(x,y) \in E_{|w|+k}, x \in S^k(\Gamma_{N_2}(w))\})\Big)^{\frac{1}{p}-\frac{1}{q}} \\
&\qquad \leq \mathcal{E}_q(f|\Omega_{|w|+k}, E_{|w|+k})^{\frac{1}{q}} \left(\sum_{u\in S^k(\Gamma_{N_2}(w))} \sum_{v\in\Gamma_N(u)} \#(E_m(u,v))\right)^{\frac{1}{p}-\frac{1}{q}} \\
&\qquad\qquad \leq \mathcal{E}_q(f|\Omega_{|w|+k}, E_{|w|+k})^{\frac{1}{q}} \left(L_0(L_*)^N \#(S^k(\Gamma_{N_2}(w)))\right)^{\frac{1}{p}-\frac{1}{q}}
\end{aligned}$$

for any $f \in \mathcal{F}(\Omega_{|w|+k}, E_{|w|+k}, \Omega^k(w, N_1), \Omega^{k,c}(w, N_2))$. Set $c = \big(L_0(L_*)^N\big)^{\frac{1}{p}-\frac{1}{q}}$. Then the above inequality implies that

$$\mathcal{E}_{p,k,w}(N_1, N_2, \Omega)^{\frac{1}{p}} \leq c\mathcal{E}_{q,k,w}(N_1, N_2, \Omega)^{\frac{1}{q}} \#(S^k(\Gamma_{N_2}(w)))^{\frac{1}{p}-\frac{1}{q}}.$$

This immediately verifies the desired inequality. $\square$

Lemma 20.2 immediately implies the following fact.

Lemma 20.3. *Let $\Omega = \{(\Omega_m, E_m)\}_{m \geq 0}$ be a proper system of horizontal networks. If $p < q$, then*

$$\overline{R}_p(N_1, N_2, \Omega)^{\frac{1}{p}} \leq \overline{R}_q(N_1, N_2, \Omega)^{\frac{1}{q}} (\overline{N}_*)^{\frac{1}{p}-\frac{1}{q}} \tag{20.2}$$

$$\underline{R}_p(N_1, N_2, \Omega)^{\frac{1}{p}} \leq \underline{R}_q(N_1, N_2, \Omega)^{\frac{1}{q}} (\overline{N}_*)^{\frac{1}{p}-\frac{1}{q}} \tag{20.3}$$

Using this lemma, we can show the continuity and the monotonicity of $\underline{R}_p$ and $\overline{R}_p$.

Proposition 20.4. *Let $\Omega = \{(\Omega_m, E_m)\}_{m \geq 0}$ be a proper system of horizontal networks.*
(1) $p > 0$,

$$\underline{R}_p(N_1, N_2, \Omega) \leq \overline{R}_p(N_1, N_2, \Omega) \leq r^p \overline{N}_*$$

(2) $\overline{R}_p(N_1, N_2, \Omega)$ and $\underline{R}_p(N_1, N_2, \Omega)$ are continuous and monotonically non-increasing as a function of p.
(3) If $N_2 \geq N_1 + M_*$, then $\overline{R}_p(N_1, N_2, \Omega) < 1$ for any $p > \dim_{AR}(X, d)$.

Proof. (1) By (19.23),

$$\mathcal{E}_{p,k}(N_1, N_2, \Omega) \leq c' \overline{N}_k(N_2) r^{pk}.$$

This immediately implies the desired inequality.
(2) Since $|f(x) - f(y)| \leq 1$ if $0 \leq f(x) \leq 1$ for any $x \in \Omega_m$, we see that

$$\mathcal{E}_{p,n,w}(N_1, N_2, \Omega) \geq \mathcal{E}_{q,n,w}(N_1, N_2, \Omega)$$

whenever $p < q$. Hence $\underline{R}_p$ and $\overline{R}_p$ are monotonically decreasing. Set $R_p = \overline{R}_p(N_1, N_2, \Omega)$. By (20.2), if $p < q$, then

$$R_q \leq R_p \leq (R_q)^{\frac{p}{q}} (\overline{N}_*)^{1-\frac{p}{q}}$$

This shows that $\lim_{p \uparrow q} R_p = R_q$. Exchanging p and q, we obtain

$$(R_q)^{\frac{p}{q}} (\overline{N}_*)^{1-\frac{p}{q}} \leq R_p \leq R_q$$

if $q < p$. This implies $\lim_{p \downarrow q} R_p = R_q$. Thus R_p is continuous. The same discussion works for $\underline{R}_p(N_1, N_2, \Omega)$ as well.
(3) This follows from (19.19). $\square$

Consequently, we obtain another characterization of the AR conformal dimension.

Theorem 20.5. *Let* $\Omega = \{(\Omega_m, E_m)\}_{m \geq 0}$ *be a proper system of horizontal networks. Assume that* $N_2 \geq N_1 + M_*$. *Then*

$$\begin{aligned}\dim_{AR}(X, d) &= \inf\{p|\overline{R}_p(N_1, N_2, \Omega) < 1\} = \max\{p|\overline{R}_p(N_1, N_2, \Omega) = 1\} \quad (20.4)\\ &= \inf\{p|\underline{R}_p(N_1, N_2, \Omega) < 1\} = \max\{p|\underline{R}_p(N_1, N_2, \Omega) = 1\}. \quad (20.5)\end{aligned}$$

In particular, if $p_* = \dim_{AR}(X, d)$, *then*

$$\lim_{n \to \infty} \mathcal{E}_{p_*, n}(N_1, N_2, \Omega)^{\frac{1}{n}} = 1. \tag{20.6}$$

Proof. Write $\underline{R}_p = \underline{R}_p(N_1, N_2, \Omega)$ and $p_* = \dim_{AR}(X, d)$. Since $\underline{R}_p$ is continuous, $\lim_{p \downarrow d_*} \underline{R}_p \leq 1$. If this limit is less than 1, the continuity of $\underline{R}_p$ implies that $\underline{R}_{p_* + \epsilon} < 1$ for sufficiently small $\epsilon > 0$. Then $\mathcal{E}_{p_* + \epsilon}(N_1, N_2, \Omega) = 0$. Hence $p_* + \epsilon \leq d_*$. This contradiction shows that $\lim_{p \downarrow p_*} \underline{R}_p = 1$. Consequently, $\underline{R}_{p_*} = 1$. Since $\underline{R}_p < 1$ if $p > d_*$, we may verify (20.4) for $\underline{R}_p(N_1, N_2, \Omega)$. The same discussion works for $\overline{R}_p(N_1, N_2, \Omega)$ as well. Consequently, $\overline{R}_{p_*}(N_1, N_2, \Omega) = \underline{R}_{p_*}(N_1, N_2, \Omega) = 1$. Hence we have (20.6) $\square$

Next we define the (upper and lower) p-spectral dimension $\overline{d}_p^S(N_1, N_2, \Omega)$ and $\underline{d}_p^S(N_1, N_2, \Omega)$.

Definition 20.6. Define $\overline{d}_p^S(N_1, N_2, \Omega)$ and $\underline{d}_p^S(N_1, N_2, \Omega)$ by

$$\begin{aligned}\overline{d}_p^S(N_1, N_2, \Omega) &= \frac{p \log \overline{N}_*}{\log \overline{N}_* - \log \overline{R}_p(N_1, N_2, \Omega)}\\ \underline{d}_p^S(N_1, N_2, \Omega) &= \frac{p \log \overline{N}_*}{\log \overline{N}_* - \log \underline{R}_p(N_1, N_2, \Omega)}.\end{aligned}$$

The quantities $\overline{d}_p^S(N_1, N_2, \Omega)$ and $\underline{d}_P^S(N_1, N_2, \Omega)$ are called the upper p-spectral dimension and the lower p-spectral dimension respectively.

Note that $\overline{d}_p^S(N_1, N_2, \Omega)$ and $\underline{d}_p^S(N_1, N_2, \Omega)$ coincide with the unique numbers $\overline{d}, \underline{d} \in \mathbb{R}$ which satisfy

$$\overline{N}_* \Big(\frac{\overline{R}_p(N_1, N_2, \Omega)}{\overline{N}_*}\Big)^{\overline{d}/p} = 1 \quad \text{and} \quad \overline{N}_* \Big(\frac{\underline{R}_p(N_1, N_2, \Omega)}{\overline{N}_*}\Big)^{\underline{d}/p} = 1 \tag{20.7}$$

respectively.

For the Sierpinski gasket and the generalized Sierpinski carpets, the equality (20.1) implies $\overline{d}_2^S(N_1, N_1, \Omega) = \underline{d}_2^S(N_1, N_2, \Omega)$, which is called the spectral dimension and written as d^S. The spectral dimension has been known to represent asymptotic behaviors of the Brownian motion and the Laplacian. See [4],

[6] and [19] for example. For example, if $p(t, x, y)$ is the transition density of the Brownian motion, then

$$c_1 t^{-d^S/2} \le p(t, x, x) \le c_2 t^{-d^S/2}$$

for any $t \in (0, 1]$ and $x \in X$. Moreover, let $N(\cdot)$ is the eigenvalue counting function of Δ, i.e.

$$N(\lambda) = \text{the number of eigenvalues} \le \lambda \text{ taking the multiplicity into account.}$$

Then

$$c_1 \lambda^{d^S/2} \le N(\lambda) \le c_2 \lambda^{d^S/2}$$

for any $\lambda \ge 1$. Immediately by the above definition, we obtain the following lemma.

Lemma 20.7. (a) $\overline{R}_p(N_1, N_2, \Omega) < 1$ *if and only if* $\overline{d}_p^S(N_1, N_2, \Omega) < p$,
(b) $\overline{R}_p(N_1, N_2, \Omega) = 1$ *if and only if* $\overline{d}_p^S(N_1, N_2, \Omega) = p$
(c) $\overline{R}_p(N_1, N_2, \Omega) > 1$ *if and only if* $\overline{d}_p^S(N_1, N_2, \Omega) > p$.

Finally we present the relation between p-spectral dimension and the Ahlfors regular conformal dimension.

Theorem 20.8. *Let* $\Omega = \{(\Omega_m, E_m)\}_{m \ge 0}$ *be a proper system of horizontal networks.*
(1) *If* $\underline{R}_p(N_1, N_2, \Omega) < 1$, *then*

$$\dim_{AR}(X, d) \le \underline{d}_p^S(N_1, N_2, \Omega) \le \overline{d}_p^S(N_1, N_2, \Omega) < p.$$

(2) *If* $\overline{R}_p(N_1, N_2, \Omega) \ge 1$, *then*

$$\dim_{AR}(X, d) \ge \overline{d}_p^S(N_1, N_2, \Omega) \ge \underline{d}_p^S(N_1, N_2, \Omega) \ge p.$$

For the case of the Sierpinski gasket and the (generalized) Sierpinski carpets, the above theorem shows that either

$$\dim_{AR}(X, d_*) \le d^S < 2$$

or

$$\dim_{AR}(X, d_*) \ge d^S \ge 2,$$

where d_* is the restriction of the Euclidean metric. For the standard planar Sierpinski carpet in Example 19.7, it has been shown in [5] that

$$d^S \le 1.805$$

by rigorous numerical estimate. This gives an upper estimate of the Ahlfors regular conformal dimension of the Sierpinski carpet.

Proof of Theorem 20.8. (1) Write $\underline{R}_p = \underline{R}_p(N_1, N_2, \Omega)$ and $\underline{d}_p = \underline{d}_p^S(N_1, N_2, \Omega)$. Suppose that $\underline{d}_p < q < p$. Using (20.7), we see that there exists $\epsilon > 0$ such that

$$(1+\epsilon)\overline{N}_*\Big(\frac{\underline{R}_p}{\overline{N}_*}\Big)^{q/p} < 1.$$

Choose $\{n_j\}_{j\ge 1}$ so that $\underline{R}_p = \lim_{j\to\infty} \mathcal{E}_{p,n_j}(N_1, N_2, \Omega)^{\frac{1}{n_j}}$. Then for sufficiently large j, we have

$$\mathcal{E}_{p,n_j}(N_1, N_2, \Omega) \le ((1+\epsilon)\underline{R}_p)^{n_j} \text{ and } \sup_{w\in T} \#(S^{n_j}(\Gamma_{M_1}(w))) \le ((1+\epsilon)\overline{N}_*)^{n_j}.$$

Hence by Lemma 20.2, as $j \to 0$,

$$\mathcal{E}_{q,n_j}(0, M_1, \Omega) \le c\Bigg((1+\epsilon)\overline{N}_*\Big(\frac{\underline{R}_p}{\overline{N}_*}\Big)^{\frac{q}{p}}\Bigg)^{n_j} \to 0.$$

Therefore $\underline{I}_{\mathcal{E}}(0, M_1, \Omega) \le q$. Using Theorem 19.9, we have $\dim_{AR}(X, d) \le \underline{d}_p$.
(2) Set $\overline{R}_p = \overline{R}(N_1, N_2, \Omega)$. Assume that $\overline{R}_p > 1$. Let $q \in (p, \overline{d}_p)$. By (20.7),

$$\frac{\overline{N}_*}{1-\epsilon}\Bigg((1-\epsilon)^2\frac{\overline{R}_p}{\overline{N}_*}\Bigg)^{\frac{q}{p}} > 1$$

for sufficiently small $\epsilon > 0$. Choose $\{n_j\}_{j\ge 1}$ so that $\mathcal{E}_{p,n_j}(N_1, N_2, \Omega)^{\frac{1}{n_j}} \to \overline{R}_p$ as $j \to \infty$. Using Lemma 20.2, we have

$$((1-\epsilon)\overline{R}_p)^{n_j\frac{q}{p}} \le c\mathcal{E}_{q,n_j}(N_1, N_2, \Omega)\Bigg(\frac{\overline{N}_*}{1-\epsilon}\Bigg)^{n_j\frac{q-p}{p}}$$

for sufficiently large j. This implies

$$1 \le \Bigg(\frac{\overline{N}_*}{1-\epsilon}\Bigg((1-\epsilon)^2\frac{\overline{R}_p}{\overline{N}_*}\Bigg)^{\frac{q}{p}}\Bigg)^{n_j} \le c\mathcal{E}_{q,n_j}(N_1, N_2, \Omega).$$

Thus we have $\overline{\mathcal{E}}_q(N_1, N_2, \Omega) > 0$. Hence $q \le \dim_{AR}(X, d)$. Consequently by Theorem 19.9, we have $\overline{d}_p \le \dim_{AR}(X, d)$.
If $\overline{R}_p = 1$, then $\overline{d}_p = p \le \dim_{AR}(X, d)$ by Theorem 20.5. $\square$

21 Combinatorial modulus of curves

Originally in [23], the characterization of the Ahlfors regular conformal dimension has been given in terms of the critical exponent of p-combinatorial modulus of curve families. In this section, we are going to show a direct correspondence between p-energies and p-combinatorial moduli and reproduce Piaggio's result in [23] within our framework.

As in the previous sections, $(T, \mathcal{A}, \phi)$ is a locally finite tree with the root ϕ, $(X, \mathcal{O})$ is a compact metrizable topological space with no isolated point, $K : T \to \mathcal{C}(X, \mathcal{O})$ is a minimal partition. We also assume that $\sup_{w \in T} \#(S(w)) < +\infty$. Furthermore, we fix $d \in \mathcal{D}_{A,\epsilon}(X, \mathcal{O})$ satisfying the basic framework, i.e. (BF1) and (BF2) in Section 16.

Definition 21.1. Let (V, E) be a non-directed graph. Set

$$\mathcal{P}(V, E) = \{(x(1), \ldots, x(n)) | x(i) \in V \text{ for any } i = 1, \ldots, n \text{ and } (x(i), x(i+1)) \in E \text{ for any } i = 1, \ldots, n-1\}.$$

For $U_1, U_2 \subseteq V$ with $U_1 \cap U_2 = \emptyset$, set

$$\mathcal{C}(V, E, U_1, U_2) = \{(x(1), \ldots, x(m)) | \text{ there exist } x(0), x(m+1) \in V \text{ such that } (x(0), \ldots, x(m+1)) \in \mathcal{P}(V, E), x(0) \in U_1, x(m+1) \in U_2\}.$$

Define

$$\mathcal{F}_M(V, E, U_1, U_2) = \{f | f : V \to [0, \infty), \sum_{i=1}^{m} f(x(i)) \geq 1 \text{ for any } (x(1), \ldots, x(m)) \in \mathcal{C}(V, E, U_1, U_2)\}$$

and

$$\mathrm{Mod}_p(V, E, U_1, U_2) = \inf\{\sum_{x \in V} |f(x)|^p | f \in \mathcal{F}_M(V, E, U_1, U_2)\},$$

which is called the p-modulus of curves connecting U_1 and U_2.

Definition 21.2. Let (V, E) be a non-directed graph. Assume that $U_1, U_2 \subseteq V$ and $U_1 \cap U_2 = \emptyset$.
(1) For $f \in \mathcal{F}_M(V, E, U_1, U_2)$, define

$$F(f)(x) = \min\{\sum_{i=1}^{k} f(x(i)) | (x(1), \ldots, x(m)) \in \mathcal{P}(V, E), x(1) \in U_2, x(k) = x\}.$$

(2) For $g \in \mathcal{F}_F(V, E, U_1, U_2)$, define

$$G(g)(x) = \sum_{(x,y) \in E} |g(x) - g(y)|.$$

The following version of discrete Hölder inequality will be used several times. It is obtained by applying the ordinary Hölder inequality to a sum of Dirac measures.

Lemma 21.3. *Let $C_h(p, n) = \max\{n^{p-1}, 1\}$. For any $a_1, \ldots, a_n \in \mathbb{R}$,*

$$\Big(\sum_{i=1}^{n} |a_i|\Big)^p \leq C_h(p, n) \sum_{i=1}^{n} |a_i|^p.$$

Lemma 21.4. *Let (V, E) be a non-directed graph. Assume that $U_1, U_2 \subseteq V$ and $U_1 \cap U_2 = \emptyset$. Define $L(V, E) = \max\{\#(\{y|(x, y) \in E\})|x \in V\}$.*
(1) For any $f \in \mathcal{F}_M(V, E, U_1, U_2)$, $F(f) \in \mathcal{F}_F(V, E, U_1, U_2)$ and

$$\mathcal{E}_p(F(f)|V, E) \leq C_h(p, 2)L(V, E) \sum_{x \in V} f(x)^p. \tag{21.8}$$

(2) For any $g \in \mathcal{F}_F(V, E, U_1, U_2)$, $G(g) \in \mathcal{F}_M(V, E, U_1, U_2)$ and

$$\sum_{x \in V} G(g)(x)^p \leq 2C_h(p, L(E, V))\mathcal{E}_p(g|V, E). \tag{21.9}$$

Proof. (1) The claim that $F(f) \in \mathcal{F}_F(V, E, U_1, U_2)$ is immediate by the definition. If $(x(1), \dots, x(m)) \in \mathcal{P}(V, E)$, $x(0) = x$ and $x(m) = y$, then

$$F(f)(x) + \sum_{i=1}^{m} f(x(i)) \geq F(f)(y) \quad \text{and}$$

$$F(f)(y) + \sum_{i=0}^{m-1} f(x(i)) \geq F(f)(x).$$

Therefore,

$$|F(f)(x) - F(f)(y)| \leq \\ \min\{\sum_{i=1}^{m} f(x(i))|(x(0), \dots, x(m)) \in \mathcal{P}(V, E), x(0) = x, x(m) = y\}.$$

This implies

$$|F(f)(x) - F(f)(y)| \leq f(x) + f(y)$$

if $(x, y) \in E$. Thus by Lemma 21.3,

$$\mathcal{E}_p(F(f)|V, E) = \frac{1}{2} \sum_{(x,y) \in E} |F(f)(x) - F(f)(y)|^p \leq \frac{1}{2} \sum_{(x,y) \in E} (f(x) + f(y))^p \\ \frac{C_h(p, 2)}{2} \sum_{(x,y) \in E} (f(x)^p + f(y)^p) \leq C_h(p, 2)L(V, E) \sum_{x \in V} f(x)^p.$$

(2) Let $(x(1), \dots, x(m)) \in \mathcal{C}(V, E, U_1, U_2)$. Then $(x(0), x(1)) \in E$ for some $x(0) \in V_1$ and $(x(m), x(m+1)) \in V_2$ for some $x(m+1) \in V_2$. Set $j = \min\{i|i \in \{0, \dots, m+1\}, x(i) \in U_2\} - 1$. Since $G(g)(x(i)) \geq |g(x(i-1) - g(x(i))|$ for any $i = 1, \dots, j-1$ and $G(g)(x(j)) \geq |g(x(j-1)) - g(x(j))| + |g(x(j)) - g(x(j+1))|$, we see that

$$\sum_{i=1}^{m} G(g)(x(i)) \geq \sum_{i=1}^{j} G(g)(x(i)) \geq \sum_{i=1}^{j+1} |g(x(i)) - g(x(i-1))| \\ \geq \sum_{i=1}^{j+1} g(x(i)) - g(x(i-1)) \geq 1.$$

Thus $G(g) \in \mathcal{F}_M(V, E, U_1, U_2)$. Moreover, by Lemma 21.3

$$\sum_{x \in V} G(g)(x)^p \leq C_h(p, L(E, V)) \sum_{x \in V} \sum_{y:(x,y) \in E} |g(x) - g(y)|^p$$
$$\leq 2C_h(p, L(E, V))\mathcal{E}_p(g|V, E).$$

□

Taking infimums in (21.8) and (21.9), we obtain the following proposition giving a direct connection between p-energy and p-modulus.

Proposition 21.5. *Let (V, E) be a non-directed graph. Assume that $U_1, U_2 \subseteq V$ and $U_1 \cap U_2 = \emptyset$. Then*

$$\mathcal{E}_p(V, E, U_1, U_2) \leq C_h(p, 2)L(V, E)\mathrm{Mod}_p(V, E, U_1, U_2)$$

and

$$\mathrm{Mod}_p(V, E, U_1, U_2) \leq 2C_h(p, L(V, E))\mathcal{E}_p(V, E, U_1, U_2).$$

Next we give a definition of the critical index of p-moduli.

Definition 21.6. Let $\Omega = \{(\Omega_m, E_m)\}_{m \geq 0}$ be a proper system of horizontal networks. Define

$$\begin{aligned}
\mathcal{M}_{p,k}(N_1, N_2, \Omega) &= \sup_{w \in T} \mathrm{Mod}_p(\Omega_{|w|+k}, E_{|w|+k}, \Omega^k(w, N_1), \Omega^{k,c}(w, N_2)) \\
\overline{\mathcal{M}}_p(N_1, N_2, \Omega) &= \limsup_{k \to \infty} \mathcal{M}_{p,k}(N_1, N_2, \Omega), \\
\underline{\mathcal{M}}_p(N_1, N_2, \Omega) &= \liminf_{k \to \infty} \mathcal{M}_{p,k}(N_1, N_2, \Omega) \\
\overline{I}_{\mathcal{M}}(N_1, N_2, \Omega) &= \inf\{p | \overline{\mathcal{M}}_p(N_1, N_2, \Omega) = 0\} \\
\underline{I}_{\mathcal{M}}(N_1, N_2, \Omega) &= \inf\{p | \underline{\mathcal{M}}_p(N_1, N_2, \Omega) = 0\}
\end{aligned}$$

Due to Proposition 21.5, $\mathcal{E}_{p,n}(N_1, N_2, \Omega)$ and $\mathcal{M}_{p,n}(N_1, N_2, \Omega)$ can be compared in the following way.

Lemma 21.7. *Let $\Omega = \{(\Omega_m, E_m)\}_{m \geq 0}$ be a proper system of horizontal networks with indices (N, L_0, L_1, L_2). Then*

$$\mathcal{E}_{p,n}(N_1, N_2, \Omega) \leq C_h(p, 2)L_0(L_*)^{N+1}\mathcal{M}_{p,n}(N_1, N_2, \Omega)$$

and

$$\mathcal{M}_{p,n}(N_1, N_2, \Omega) \leq 2C_h(p, L_0(L_*)^{N+1})\mathcal{E}_{p,n}(N_1, N_2, \Omega).$$

Proof. It is enough to show that $L(\Omega_m, E_m) \leq L_0(L_*)^{N+1}$. Let $x \in \Omega_m$. If $x = K_w \cap \Omega_m$, then By (N4),

$$\{y | y \in \Omega_m, (x, y) \in E_m\} \subseteq \bigcup_{w: x \in K_w} \bigcup_{v \in \Gamma_N(w)} \bigcup_{(x,y) \in E_m(w,v)} \{y\}$$

Using (N3), we see that

$$\#(\{y|y \in \Omega_m, (x,y) \in E_m\}) \leq \#(\{w|x \in K_w\})\#(\Gamma_N(w)))L_0$$
$$\leq L_*(L_*)^N L_0.$$

If $x = w \in (T)_m \cap \Omega_m$, similar arguments show that $\#(y|y \in \Omega_m, (x,y) \in E_m\}) \leq (L_*)^N L_0$. Thus we have $L(\Omega, E_m) \leq (L_*)^{N+1} L_0$. □

The above lemma combined with Theorem 19.9 immediately yields the following characterization of the Ahlfors regular conformal dimension by the critical exponents of discrete moduli.

Theorem 21.8. *Let $\Omega = \{(\Omega_m, E_m)\}_{m \geq 0}$ be a proper system of horizontal networks. If $N_2 \geq N_1 + M_*$, then*

$$\overline{I}_{\mathcal{M}}(N_1, N_2, \Omega) = \underline{I}_{\mathcal{M}}(N_1, N_2, \Omega) = \dim_{AR}(X, d).$$

22 Positivity at the critical value

One of the advantages of the use of discrete moduli is to show the positivity of $\underline{\mathcal{M}}_p(N_1, N_2, \Omega)$ and $\underline{\mathcal{E}}_p(N_1, N_2, \Omega)$ at the critical value $p_* = \dim_{AR}(X, d)$.

As in the previous sections, $(T, \mathcal{A}, \phi)$ is a locally finite tree with the root ϕ, $(X, \mathcal{O})$ is a compact metrizable topological space with no isolated point, $K : T \to \mathcal{C}(X, \mathcal{O})$ is a minimal partition. We also assume that $\sup_{w \in T} \#(S(w)) < +\infty$. Furthermore, we fix $d \in \mathcal{D}_{A,\epsilon}(X, \mathcal{O})$ satisfying the basic framework, i.e. (BF1) and (BF2) in Section 16.

Theorem 22.1. *Let $\Omega = \{(\Omega_m, E_m)\}_{m \geq 0}$ be a proper system of horizontal networks. Suppose $N_2 \geq N_1 + M_*$. Let $p_* = \dim_{AR}(X, d)$. Then*

$$\underline{\mathcal{M}}_{p_*}(N_1, N_2, \Omega) > 0 \quad \text{and} \quad \underline{\mathcal{E}}_{p_*}(N_1, N_2, \Omega) > 0.$$

First step of a proof is to modify the original proper system of horizontal networks.

Lemma 22.2. *Let $\Omega = \{(\Omega_m, E_m)\}_{m \geq 0}$ be a proper system of horizontal networks with indices (N, L_0, L_1, L_2).*
(1)

$$\#(\Omega_{m,w}) \leq L_0(L_*)^N$$

for any $m \geq 0$ and $w \in (T)_m$.
(2) *Define*

$$J^h_{M,m}[\Omega] = \{(x,y)|m \geq 0, v, w \in (T)_m, v \in \Gamma_M(w), x \in \Omega_{m,v}, y \in \Omega_{m,w}\}.$$

Set $\overline{\Omega}^M = \{(\Omega_m, J^h_{M,m}[\Omega])\}_{m \geq 0}$. Then $\overline{\Omega}^M$ is a proper system of horizontal networks with indices $(M, L_0^2(L_*)^{2N}, M, M)$. Moreover, there exists $c > 0$ such that

$$\mathcal{E}_p(f|\Omega_m, J^h_{M,m}[\Omega]) \leq c\mathcal{E}_p(f|\Omega_m, E_m) \tag{22.1}$$

for any $m \geq 0$ and $f : \Omega_m \to \mathbb{R}$.

Proof. (1) Note that

$$\{(x,y)|(x,y)\in E_m, x\in \Omega_{m,w}\}\subseteq \bigcup_{v\in\Gamma_N(w)} E_m(u,v).$$

Using (N3), we have

$$\begin{aligned}\#(\Omega_{m,w}) &\le \#(\{(x,y)|(x,y)\in E_m, x\in\Omega_{m,w}\})\\ &\le \sum_{v\in\Gamma_N(w)} \#(E_m(w,v)) \le (L_*)^N L_0.\end{aligned}$$

(2) By (1),

$$\begin{aligned}&\#(\{(x,y)|(x,y)\in J^h_{M,m}(\Omega), x\in\Omega_{m,v}, y\in\Omega_{m,w}\})\\ &\qquad = \#(\Omega_{m,v}\times\Omega_{m,w}) \le (L_0)^2(L_*)^{2M}.\end{aligned}$$

This shows that $\overline{\Omega}^M$ is a proper system of horizontal networks with indices $(M,(L_0)^2(L_*)^{2M},M,M)$. Assume that $M\le L_1$ for the moment. Let $(x,y)\in \Omega_{m,v}\times\Omega_{m,u}$ for some $u,v\in(T)_m$ with $u\in\Gamma_M(v)$. Since $M\le L_1$, the condition (N5) implies that there exist $(x_1,\ldots,x_n)$ and $(w(1),\ldots,w(n))$ such that $w(i)\in\Gamma_{L_2}(u)$ for any $i\ge 1,\ldots,n$, $(x_i,x_{i+1})\in\Omega_m(w(i),w(i+1))$ for any $i=1,\ldots,n-1$ and $x_1=x, x_n=y, w(1)=u, w(n)=v$. Since $n-1$ is no greater than the total number of edges in $\Gamma_{L_2}(u)$, we have

$$n-1\le \#\Big(\bigcup_{v_1,v_2\in\Gamma_{L_2}(u)} E_m(v_1,v_2)\Big)\le (L_*)^{2L_2}L_0.$$

For any $f:\Omega_m\to\mathbb{R}$, by Lemma 21.3

$$|f(x)-f(y)|^p\le C_h(p,n-1)\sum_{i=1}^{n-1}|u(x(i))-u(x(i+1))|^p.$$

Let $(z_1,z_2)\in E_m$. Consider how many $(x,y)\in J^h_{M,m}$ there are for which (z_1,z_2) appears as $(x(i),x(i+1))$ in the above inequality. We start with counting the number of possible u's. Since z_1 and z_2 must belong to $\Gamma_{L_2}(u)$, the possible number of u's is no greater than $\#(\Gamma_{L_2}(z_1)\cap\Gamma_{L_2}(z_2))\le (L_*)^{L_2}$. For each u,

$$\#(\{(x,y)|x\in\Omega_{m,u}, y\in\cup_{v\in\Gamma_{L_1}(u)}\Omega_{m,v}\})\le (L_*)^{L_1}(L_0)^2(L_*)^{2M}.$$

Combining those facts, we see that the possible number of (x,y) for which (z_1,z_2) appears as $(x(i),x(i+1))$ is at most $(L_*)^{L_2+L_1+2M}(L_0)^2$, which is denoted by C_1. Then it follows that

$$\mathcal{E}_p(f|\Omega_m, J^h_{M,m}[\Omega])\le C_1C_h(p,(L_*)^{2L_2}L_0)\mathcal{E}_p(f|\Omega_m,E_m).$$

So, we have finished the proof if $M\le M_1$. For general situation, choosing n_0 so that $M\le n_0L_1$, we see that Ω is a proper system of horizontal networks with indices $(N,L_0,n_0L_1,(n_0-1)L_1+L_2)$. Thus replacing L_1 and L_2 by n_0L_1 and $(n_0-1)L_1+L_2$ respectively, we complete the proof for general cases. □

Lemma 22.3. *Let $\Omega = \{(\Omega_m, E_m)\}_{m\geq 0}$ be a proper system of horizontal networks with indices (N, L_0, L_1, L_2). Then*

$$\mathcal{M}_{p,k+l}(0, M, \Omega_*^{(J)}) \leq C\mathcal{M}_{p,k}(0, M, \overline{\Omega}^{2M+J})\mathcal{M}_{p,l}(0, M, \Omega_*^{(J)}),$$

for any $k, l, M, J \in \mathbb{N}$ and $p > 0$, where $C = L_* C_h(p, (L_*)^{N+1}L_0)$.

Notation.

$$\mathcal{Q}_{w,k}(M_1, M_2, \Omega) = \mathcal{F}_M(\Omega_{|w|+k}, E_{|w|+k}, , \Omega^k(w, M_1), \Omega^{k,c}(w, M_2)) \qquad (22.2)$$

and

$$\mathcal{C}_{w,k}(M_1, M_2, \Omega) = \mathcal{C}(\Omega_{|w|+k}, E_{|w|+k}, \Omega^k(w, M_1), \Omega^{k,c}(w, M_2)) \qquad (22.3)$$

for $M_1, M_2 \geq 1$.

Proof. Let $f \in \mathcal{Q}_{w,k}(0, M, \overline{\Omega}^{2M+N})$ and let $g_v \in \mathcal{Q}_{v,l}(0, M, \Omega_*^{(J)})$ for any $v \in (T)_{|w|+k}$. Define

$$h(u) = \max\{f(x)g_v(u) | x \in \Omega_{|w|+k,v}, v \in \Gamma_M(\pi^l(u))\}\chi_{S^{k+l}(\Gamma_M(w))}(u)$$

Claim 1. $h \in \mathcal{Q}_{w,k+l}(0, M, \Omega_*^{(J)})$.
Proof of Claim 1: Let $(u(1), \ldots, u(m)) \in \mathcal{C}_{w,k+l}(0, M, J)$. Set $v(i) = \pi^l(u(i))$ for $i = 1, \ldots, m$. Let $v_*(1) = v(1)$ and let $i_1 = 1$. Define $v_*(n)$ and i_n inductively as

$$i_{n+1} = \max\{j | v(j) \in \Gamma_{2M}(v_*(n))\} + 1$$

and $v_*(n+1) = v(i_{n+1})$ while $\max\{j | v(j) \in \Gamma_{2M}(v_*(n))\} < m$. In this way, we construct $(v_*(1), \ldots, v_*(n_*))$ satisfying $\max\{j | v(j) \in \Gamma_{2M}(v_*(n_*))\} = m$. Since $v(i_{n+1} - 1) \in \Gamma_{2M}(v_*(n))$, it follows that $v_*(n+1) \in \Gamma_{2M+J}(v_*(n))$. Hence $(v_*(1), \ldots, v_*(n_*)) \in \mathcal{C}_{w,k}(0, M, 2M+J)$. Moreover, $\Gamma_M(v(i)) \cap \Gamma_M(v(j)) = \emptyset$ if $i \neq j$ and there exists $(u(j_n), \ldots, u(j_n + k_n)) \in \mathcal{C}_{v_*(n),l}(0, M, J)$. Choose $x_i \in \Omega_{|w|+k,v}$ for each $i = 1, \ldots, n$. Since $g_{v_*(n)} \in \mathcal{Q}_{v_*(n),l}(0, M, \Omega_*^{(J)})$, we have

$$\sum_{i=j_n}^{j_n+k_n} h(u(i)) \geq \sum_{i=j_n}^{j_n+k_n} f(x_n)g_{v_*(n)}(u(i)) \geq f(x_n).$$

This and the fact that $(x_1, \ldots, x_n) \in \mathcal{C}_{k,w}(0, M, \overline{\Omega}^{2M+J})$ yield

$$\sum_{i=1}^{m} h(u(i)) \geq \sum_{j=1}^{n_*} f(x_j) \geq 1.$$

Thus Claim 1 has been verified. $\square$
Set $C_0 = C_h(p, (L_*)^{N+1}L_0)$. Then by Lemma 21.3 and Lemma 14.5,

$$h(u)^p \leq \left(\sum_{v\in\Gamma_M(\pi^l(u))} \sum_{x\in\Omega_{|w|+k,v}} f(x)g_v(u)\right)^p$$
$$\leq C_0 \sum_{v\in\Gamma_M(\pi^l(u))} \sum_{x\in\Omega_{|w|+k,v}} f(x)^p g_v(u)^p.$$

Set $M_{p,j,w'} = \mathrm{Mod}_p((T)_{|w'|+j}, J^h_{J,|w'|+j}, S^j(w'), (S^j(\Gamma_M(w')))^c)$. The above inequality yields

$$M_{p,k+l,w} \leq \sum_{u\in(T)_{|w|+k+l}} h(u)^p \leq C_0 \sum_{v\in(T)_{|w|+k}} \sum_{x\in\Omega_{|w|+k,v}} \sum_{u\in(T)_{|w|+k+l}} f(x)^p g_v(u)^p.$$

Hence following the process for getting $\mathcal{M}_{p,k}(N_1, N_2, \Omega)$, we have

$$\begin{aligned}
M_{p,k+l,w} &\leq C_0 \sum_{v\in(T)_{|w|+k}} \sum_{x\in\Omega_{|w|+k,v}} f(x)^p M_{p,l,v} \\
&\leq C_0 \sum_{v\in(T)_{|w|+k}} \sum_{x\in\Omega_{|w|+k,v}} f(x)^p \mathcal{M}_{p,l}(0, M, \Omega_*^{(J)}) \\
&\leq C_0 L_* \mathcal{M}_{p,k}(0, M, \Omega^{2M+J}) \mathcal{M}_{p,l}(0, M, \Omega_*^{(J)}).
\end{aligned}$$

Continuing the process, we finally obtain

$$\mathcal{M}_{p,k+l}(0, M, \Omega_*^{(J)}) \leq C\mathcal{M}_{p,k}(0, M, \overline{\Omega}^{2M+J})\mathcal{M}_{p,l}(0, M, \Omega_*^{(J)}),$$

where $C = C_0 L_*$. $\square$

Proof of Theorem 22.1. Write $\mathcal{M}_{p,j} = \mathcal{M}_{p,j}(0, M, \Omega_*^{(J)})$ and $\mathcal{M}'_{p,j} = \mathcal{M}_{p,j}(0, M, \overline{\Omega}^{2M+J})$. By Lemma 22.3,

$$\mathcal{M}_{p,mk+l} \leq (C\mathcal{M}'_{p,k})^m \mathcal{M}_{p,l}. \tag{22.4}$$

Assume that $C\mathcal{M}'_{p,k} < 1 - \epsilon$ for some $\epsilon \in (0, 1)$. Then for any $w \in T$, there exists $f_w \in \mathcal{Q}_{w,k}(0, M, \overline{\Omega}^{2M+J})$ such that

$$C \sum_{x\in\Omega_{|w|+k}(S^k(\Gamma_M(w)))} f_w(x)^p < 1 - \epsilon.$$

Since

$$\lim_{\delta\to 0} \max_{x\in[0,1]} (x^{p-\delta} - x^p) = 0,$$

there exists $\delta_* > 0$ such that $x^{p-\delta} \leq x^p + C_2^{-1}C^{-1}\epsilon/2$ for any $\delta \in (0, \delta_*]$ and $x \in [0, 1]$, where $C_2 = (N_*)^k (L_*)^{M+N} L_0$. By Lemma 22.2-(1),

$$\begin{aligned}
\#(\Omega_{|w|+k}(S^k(\Gamma_M(w)))) &\leq \#(S^k(\Gamma_M(w))) \max_{v\in S^k(\Gamma_M(w))} \#(\Omega_{|w|+k,v}) \\
&\leq (L_*)^M (N_*)^k (L_*)^N L_0 = C_2.
\end{aligned}$$

This implies

$$C \sum_{v\in S^k(\Gamma_M(w))} f_w(v)^{p-\delta} \leq C \sum_{x\in\Omega_{|w|+k}(S^k(\Gamma_M(w)))} f_w(x)^{p-\delta} + \frac{\epsilon}{2} \leq 1 - \frac{\epsilon}{2}.$$

Therefore $\mathcal{M}_p(0, M, \Omega_*^{(J)}) = 0$. By Theorem 21.8, if $M \geq M_*$, it follows that $\dim_{AR}(X, d) < p$. Consequently, $C\mathcal{M}'_{p,k} \geq 1$ for any $k \geq 1$. Therefore, if $M \geq M_*$, then

$$C^{-1} \leq \underline{\mathcal{M}}_{p_*}(0, M, \overline{\Omega}^{2M+J}).$$

Using Lemma 21.7, we see that $0 < \underline{\mathcal{E}}_{p_*}(0, M, \overline{\Omega}^{2M+J})$. Then the inequality (22.1) shows that $0 < \underline{\mathcal{E}}_{p_*}(0, M, \Omega)$. Since $\underline{\mathcal{E}}_p(0, M, \Omega) \leq \underline{\mathcal{E}}_p(M', M, \Omega)$ for any $M' \in \{0, 1, \ldots, M - M_*\}$, we conclude that $0 < \underline{\mathcal{E}}_{p_*}(N_1, N_2, \Omega)$ for any $N_1, N_2 \geq 0$ with $N_2 \geq N_1 + M_*$. Again by Lemma 21.7, it follows that $0 < \underline{\mathcal{M}}_{p_*}(N_1, N_2, \Omega)$ as well. $\square$

Appendix

A Fact from measure theory

Proposition A.1. *Let* $(X, \mathcal{M}, \mu)$ *be measurable space and let* $N \in \mathbb{N}$*. If* $U_i \in \mathcal{M}$ *for any* $i \in \mathbb{N}$ *and*

$$\#(\{i | i \in \mathbb{N}, x \in U_i\}) \leq N \tag{A.1}$$

for any $x \in X$*, then*

$$\sum_{i=1}^{\infty} \mu(U_i) \leq N\mu\biggl(\bigcup_{i\in\mathbb{N}} U_i\biggr).$$

Proof. Set $U = \cup_{i\in\mathbb{N}} U_i$. Define $U_{i_1 \ldots i_m} = \cap_{j=1,\ldots,m} U_{i_j}$. By (A.1), if $m > N$, then $U_{i_1\ldots i_m} = \emptyset$. Fix $m \geq 0$ and let rearrange $\{U_{i_1\ldots i_m} | i_1 < i_2 < \ldots < i_m\}$ so that

$$\{Y_j^m\}_{j\in\mathbb{N}} = \{U_{i_1\ldots i_m} | i_1 < i_2 < \ldots < i_m\}.$$

Define

$$X_j^m = Y_j^m \backslash (\bigcup_{i\in\mathbb{N}, i\neq j} Y_i^m).$$

Then

$$U = \bigcup_{m=0}^{N} \Bigl(\bigcup_{j\in\mathbb{N}} X_j^m\Bigr)$$

and $X_j^m \cap X_l^k = \emptyset$ if $(m, j) \neq (k, l)$. This implies

$$\mu(U) = \sum_{m=0}^{N} \sum_{j\in\mathbb{N}} \mu(X_j^m).$$

Set $I_j = \{(k, l) | U_j \supseteq X_l^k \neq \emptyset\}$. Then by (A.1), we have $\#(\{j | (k, l) \in I_j\}) \leq N$ for any (k, l). This implies

$$\sum_{j=1}^{\infty} \mu(U_i) \leq N \sum_{m=0}^{N} \sum_{j\in\mathbb{N}} \mu(X_j^m) = N\mu(U).$$

□

B List of definitions, notations and conditions

Definitions

adapted – Definition 6.1, Definition 6.7
Ahlfors regular – Definition 8.18
Ahlfors regular conformal dimension – Definition 19.1
Ahlfors regular metric – Definition 15.1

Notations

$|w, v|$ – Definition 12.3
$\overline{wv}$: the geodesic between w and v of a tree, – Definition 3.1
$(w|v)_{((T,\mathcal{B}),\phi)}$, $(w|v)$: Gromov product – Definition 7.6
$|w|$ – Definition 3.2
$|w|_{(T,\mathcal{A},\phi)}$ – Remark after Definition 3.2
$w \wedge v$ – Definition 3.6
$[\omega]_m$ – Definition 3.6
$\langle\cdot\rangle_M$ – Definition 17.3

Equivalence relations
$\underset{\mathrm{AC}}{\sim}$ – Definition 8.3
$\underset{\mathrm{BL}}{\sim}$ relation on weight functions – Definition 8.1
$\underset{\mathrm{BL}}{\sim}$ relation on metrics – Definition 8.9
$\underset{\mathrm{GE}}{\sim}$ – Definition 10.1
$\underset{\mathrm{QS}}{\sim}$ – Definition 13.1

Conditions
(ADa), $(\mathrm{ADb})_{\mathrm{M}}$ – Theorem 6.5
(BF1), (BF2) – Section 16
(BL), (BL1), (BL2), (BL3) – Theorem 8.8
$(\mathrm{EV})_M$, $(\mathrm{EV2})_M$, $(\mathrm{EV3})_M$, $(\mathrm{EV4})_M$, $(\mathrm{EV5})_M$ – Theorem 6.12
(G1), (G2), (G3) – Definition 5.1
(N1), (N2), (N3), (N4), (N5) – Definition 19.5
(P1), (P2) – Definition 4.1
(SQ1), (SQ2), (SQ3) – Section 11
(SF) – Strongly finite, (2.3)
(TH) – Few lines before Definition 2.1
(TH1), (TH2), (TH3), (TH4) – Theorem 9.3
(VD1), (VD2), (VD3), (VD4) – Theorem 10.9

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