Dirichlet forms and associated heat kernels on the Cantor set induced by random walks on trees

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Abstract

Transient random walk on a tree induces a Dirichlet form on its Martin boundary, which is the Cantor set. The procedure of the inducement is analogous to that of the Douglas integral on S^1 associated with the Brownian motion on the unit disk. In this paper, those Dirichlet forms on the Cantor set induced by random walks on trees are investigated. Explicit expressions of the hitting distribution (harmonic measure) ν and the induced Dirichlet form on the Cantor set are given in terms of the effective resistances. An intrinsic metric on the Cantor set associated with the random walk is constructed. Under the volume doubling property of ν with respect to the intrinsic metric, asymptotic behaviors of the heat kernel, the jump kernel and moments of displacements of the process associated with the induced Dirichlet form are obtained. Furthermore, relation to the noncommutative Riemannian geometry is discussed.

1 Introduction

Transient random walks eventually go to "infinity", which is not just a single point but a collection of possible behaviors of random walks as the time tends to the infinity. A rigorous way to describe this "infinity" is the Martin boundary, where all the boundary values of harmonic functions lie. In certain cases, a transient random walk naturally induces a (Hunt) process (or equivalently a Dirichlet form) on its Martin boundary. In this paper, we are going to study such an induced (Hunt) process in the case of a random walk on a tree, whose Martin boundary is known to coincide with the Cantor set.

A well-known example of such an induced Dirichlet form is the Douglas integral

$$D(\varphi, \psi) = \frac{\pi}{4}\int_0^{2\pi}\int_0^{2\pi} \frac{(\varphi(\theta) - \varphi(\theta'))(\psi(\theta) - \psi(\theta'))}{\sin^2\left(\frac{\theta-\theta'}{2}\right)}\nu(d\theta)\nu(d\theta')$$

on the circle $S^1 = \{z|z = e^{i\theta}, \theta \in \mathbb{R}\}$, where $\nu(d\theta) = d\theta/2\pi$ is the uniform distribution on S^1. In this case the process which induces the Douglas integral on its Martin boundary S^1 is (reflected) Brownian motion on the unit disk

$\mathbb{D} = \{z \in \mathbb{R}, |z| < 1\}$, which is not even a random walk. We use, however, this example to illustrate the procedure of inducement of Dirichlet forms on Martin boundaries. The Dirichlet from associated with the Brownian motion on $\mathbb{D}$ is the (half of) Dirichlet integral

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{\mathbb{D}}\Big(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\Big)dxdy$$

defined for $u, v \in H^1(\mathbb{D}) = \{u|u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in L^2(\mathbb{D})\}$, where $\mathbb{D}$ is identified with $\{(x,y)|x, y \in \mathbb{R}, x^2+y^2 < 1\}$ and the derivatives are in the sense of distributions. Note that the Brownian motion starting from $0 \in \mathbb{D}$ will hit the boundary S^1 uniformly due to the symmetry. In this case, the uniform distribution ν on S^1 is called the hitting distribution (or harmonic measure) of S^1 starting from 0. The key stone of the bridge between the Dirichlet integral on $\mathbb{D}$ and the Douglas integral on S^1 is the Poisson integral which gives harmonic functions on $\mathbb{D}$ from boundary values on S^1. Let H be the operation of applying the Poisson integral. Namely, we define $H(\varphi)$ by

$$(H(\varphi))(re^{i\theta}) = \int_0^{2\pi} \frac{1-r^2}{1-2r\cos{(\theta-\theta')}+r^2}\varphi(\theta')\nu(d\theta').$$

Then

$$\mathrm{Dom}(D) = \{\varphi|\varphi \in L^2(S^1,\nu), H\varphi \in H^1(\mathbb{D})\} \quad \text{and} \quad D(\varphi,\psi) = \mathcal{E}(H\varphi, H\psi)$$

for any $\varphi, \psi \in \mathrm{Dom}(D)$.

The above example shows us the essence of obtaining Dirichlet forms on Martin boundaries from random walks. Suppose that we have a Markov process or a random walk on a space X, that its Martin boundary M is well-defined and that we have the following ingredients (a), (b) and (c):

(a) the Dirichlet form $\mathcal{E}$ associated with the original process or the random walk

(b) the hitting distribution ν of M starting from certain point in X

(c) the map H which transforms functions on M into harmonic functions on X.

Then the induced form $\mathcal{E}_M$ on the Martin boundary M is given by

$$\mathcal{E}_M(\varphi,\psi) = \mathcal{E}(H\varphi, H\psi)$$

for $\varphi, \psi \in \mathcal{F}_M$, where $\mathcal{F}_M = \{\psi|\psi \in L^2(M,\nu), H\psi \in \mathrm{Dom}(\mathcal{E})\}$.

The theory of the Martin boundary for a transient random walk originated with the classical works of Doob [4] and Hunt [6]. Since then, it has been developed by many authors and, as a result, all the necessary ingredients for constructing an induced form $(\mathcal{E}_M, \mathcal{F}_M)$ are already well-established. For example, one can find them in [16].

In this paper, we are going to focus on the case of a transient random walk on a tree and study the induced form $(\mathcal{E}_M, \mathcal{F}_M)$ on its Martin boundary. The original motivation of this work comes from the study of traces of the Brownian motion on the Sierpinski gasket. By removing the line segment on the bottom of the Sierpinski gasket, one can associate a random walk on a binary tree with the Brownian motion. We will present the details on this idea in a separate paper[10].

Due to Cartier[2], the Martin boundary of a transient random walk on a tree is known to be (homeomorphic to) the Cantor set $\Sigma = \{1,2\}^{\mathbb{N}} = \{i_1 i_2 \ldots | i_j \in \{1,2\} \text{ for } j \in \mathbb{N}\}$. Hereafter we use Σ to denote the Martin boundary in place of M. Let us fix what we mean by a random walk and a tree at this point.

Random walk: Let (V, E) be a non-directed graph, i.e. V is the set of vertices and $E = \{(x,y) | x \neq y \in T, x \text{ and } y \text{ are connected by an edge}\}$ is the set of edges. We assume that $\{y | (x,y) \in E\}$ is a finite set for any $x \in V$. For each $(x,y) \in E$, we assign a conductance $C(x,y) > 0$ which satisfies $C(x,y) = C(y,x)$. Define $p(x,y) = C(x,z)/\sum_{(x,z)\in E} C(x,y)$. $p(x,y)$ gives the probability of transition from x to y in the unit time.

Tree: A non-directed graph (T, E) is a tree if and only if it is connected and there exists no non-trivial cyclic path. We always fix a reference point $\phi \in T$ and assign conductances on (T, E) as described above.

In this framework, assuming the transience of the random walk, we are going to obtain explicit expressions of the hitting distribution ν starting from the reference point $\phi \in T$ and the induced form $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ in terms of effective resistances, prove that $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ is a regular Dirichlet form, and construct an intrinsic metric on the Martin boundary Σ with respect to the random walk. Then, assuming the volume doubling property of ν with respect to the intrinsic metric, we are going to determine asymptotic behaviors of the heat kernel, the jump kernel and moments of displacements of the process generated by the regular Dirichlet form $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$.

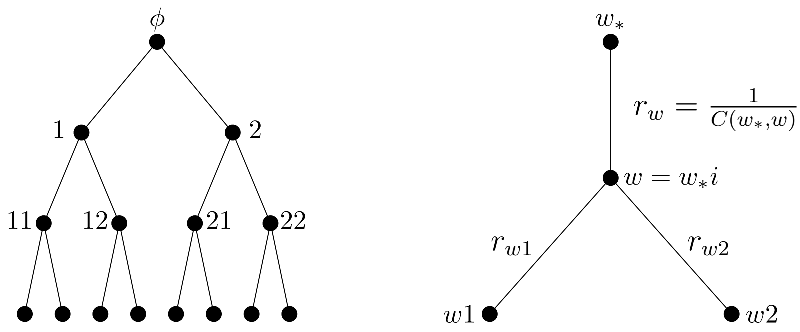
All the results in this paper are obtained in the generality of the above framework. In this introduction, however, we are going to present statements in the case of the binary tree for the sake of simplicity. Let $T = \cup_{m\geq 0}\{1,2\}^m$, where $\{1,2\}^0 = \{\phi\}$ and $\{1,2\}^m = \{w_1 \ldots w_m | w_1, \ldots, w_m \in \{1,2\}\}$ and let $E = \{(w, wi), (wi, w) | w \in T, i \in \{1,2\}\}$. (T, E) is called the (complete infinite) binary tree. See Figure 1. For $w \in T$, we set $r_{wi} = C(w, wi)^{-1}$ for $i = 1, 2$, $T_w = \{wv | v \in T\}$ and $\Sigma_w = \{wi_1 i_2 \ldots | i_j \in \{1,2\} \text{ for any } j \in \mathbb{N}\}$. T_w is a subtree of $T = T_\phi$ and Σ_w is a subset of the Martin boundary $\Sigma = \Sigma_\phi$. The quadratic form $\mathcal{E}_w$ associated with the subtree T_w is given by

$$\mathcal{E}_w(f,f) = \sum_{v\in T_w} \sum_{i=1,2} \frac{1}{r_{vi}} (f(v) - f(vi))^2.$$

The effective resistance between w and Σ_w with respect to the subtree T_w is given by

$$R_w = \big(\min\{\mathcal{E}_w(f,f) | f(w) = 1, \text{ the support of } f \text{ is a finite set}\}\big)^{-1}.$$

Under these notations, assuming the transiency of the random walk, we have obtained the following results.



The binary tree T Local structure of T

Figure 1: Random walk on the binary tree T

Explicit expression of the hitting distribution (Section 3): The hitting distribution ν is characterized by $\nu(\Sigma) = 1$ and

$$\nu(\Sigma_{wi}) = \frac{R_w}{r_{wi} + R_{wi}} \nu(\Sigma_w)$$

for any $w \in T$ and any $i \in \{1, 2\}$.
Explicit expression of $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ (Section 4): For $w \in T$, let $(u)_w$ be the average of u on Σ_w with respect to ν, i.e. $(u)_w = \int_{\Sigma_w} u(\omega)\nu(d\omega)/\nu(\Sigma_w)$. Then

$$\mathcal{E}_\Sigma(\varphi, \psi) = \sum_{w \in T} \frac{\big((\varphi)_{w1} - (\varphi)_{w2}\big)\big((\psi)_{w1} - (\psi)_{w2}\big)}{r_{w1} + R_{w1} + r_{w2} + R_{w2}} \tag{1.1}$$

for any $\varphi, \psi \in \mathcal{F}_\Sigma = \{\varphi | \mathcal{E}_\Sigma(\varphi, \varphi) < +\infty\}$. In particular, $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ is shown to be a regular Dirichlet form on $L^2(\Sigma, \nu)$.
Wavelet base consisting of eigenfunctions (Section 4): Define

$$\varphi_w = \frac{\nu(\Sigma_{w2})\chi_{\Sigma_{w1}} - \nu(\Sigma_{w1})\chi_{\Sigma_{w2}}}{\sqrt{\nu(\Sigma_{w1})^2\nu(\Sigma_{w2}) + \nu(\Sigma_{w2})^2\nu(\Sigma_{w1})}}$$

for any $w \in T$, where χ_A is the characteristic function of $A \subseteq \Sigma$. Then $\{1, \varphi_w | w \in T\}$ is a complete orthonormal system of $L^2(\Sigma, \nu)$ consisting of the eigenfunctions of the non-negative self-adjoint operator L_Σ associated with the Dirichlet form $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$. Furthermore, define $D_w = R_w \nu(\Sigma_w)$ for any $w \in T$. Then $L_\Sigma \varphi_w = (D_w)^{-1} \varphi_w$ for any $w \in T$. Such a "wavelet" base on the Cantor set consisting of eigenfunctions has been observed in Kozyrev[11] and Pearson-Bellissard[13]. In fact, the example in [13] is shown to be a special case of our framework in Section 12. Also it is noteworthy that D_w is expressed as the Gromov product of certain metric related to effective resistances. See (4.1).
Construction of intrinsic metric (Section 5): For $\omega, \tau \in \Sigma$ with $\omega \neq \tau$, the confluence $[\omega, \tau] \in T$ is $\omega_1 \ldots \omega_m$, where $\omega = \omega_1\omega_2 \ldots, \tau = \tau_1\tau_2 \ldots \in \Sigma$,

$m = \min\{k|\omega_k \neq \tau_k\} - 1$. Define $d(\omega, \tau) = D_{[\omega,\tau]}$. Then $d(\cdot,\cdot)$ is a metric on Σ and it is thought of as the intrinsic metric with respect to the random walk. Moreover, we show that ν has the volume doubling property with respect to d, i.e. $\nu(B(\omega, 2r)) \leq c\nu(B(\omega, r))$ for any $\omega \in \Sigma$ and any $r > 0$, where c is independent of ω and r and $B(\omega, r) = \{\tau|\tau \in \Sigma, d(\omega, \tau) < r\}$, if and only if there exist $c_* > 0$ and $\lambda \in (0, 1)$ such that $\nu(\Sigma_{wi}) \leq \lambda\nu(\Sigma_w)$ and $D_{wi_1\ldots i_m} \leq c_*\lambda^m D_w$ for any $w \in T$ and any $i, i_1, \ldots, i_m \in \{1, 2\}$.
Asymptotic behavior of the heat kernel (Section 6): Assume that ν has the volume doubling property with respect to d. Define

$$p(t, \omega, \tau) = \sum_{n\geq 0} \frac{e^{-\lambda_{[\omega]_{n-1}}t} - e^{-\lambda_{[\omega]_n}t}}{\nu(\Sigma_{[\omega]_n})} \chi_{\Sigma_{[\omega]_n}}(\tau)$$

for any $t > 0$ and any $\omega, \tau \in \Sigma$, where $[\omega]_n = \omega_1 \ldots \omega_n$ for $\omega = \omega_1\omega_2 \ldots \in \Sigma$. Then $p(t, \omega, \tau)$ is continuous on $(0, +\infty] \times \Sigma \times \Sigma$ and is the heat kernel (fundamental solution) of the heat equation $\frac{\partial u}{\partial t} = -L_\Sigma u$. Namely,

$$(e^{-L_\Sigma t}u_0)(\omega) = \int_\Sigma p(t, \omega, \tau)u_0(\tau)\nu(d\tau)$$

for any $u_0 \in L^2(\Sigma, \nu)$. Moreover,

$$p(t, \omega, \tau) \asymp \min\Big\{\frac{t}{d(\omega, \tau)\nu(\Sigma_{[\omega,\tau]})}, \frac{1}{\nu(B(\omega, t))}\Big\} \tag{1.2}$$

for any $t \in (0, 1]$ and any $\omega, \tau \in \Sigma$, where the notation "$\asymp$" is defined at the end of introduction. This heat kernel estimate is a variant of that studied in Chen-Kumagai[3] and satisfies a typical asymptotic behavior of jump processes. From (1.2), we also have asymptotic behaviors of moments of displacement:

$$E_\omega(d(\omega, X_t)^\gamma) \asymp \begin{cases} t & \text{if } \gamma > 1, \\ t(|\log t| + 1) & \text{if } \gamma = 1, \\ t^\gamma & \text{if } 0 < \gamma < 1 \end{cases} \tag{1.3}$$

for any $\omega \in \Sigma$ and any $t \in (0, 1]$, where $E_\omega(\cdot)$ is the expectation with respect to the Hunt process associated with the Dirichlet form $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ on $L^2(\Sigma, \nu)$. In Section 14, (1.3) is shown to hold in general for jump processes which satisfy the heat kernel estimate (1.2).
Generalization and inverse problem (Sections 9 and 10): We investigate the following class of quadratic forms $\mathcal{Q}$'s on $L^2(\Sigma, \mu)$ given by

$$\mathcal{Q}(\varphi, \psi) = \sum_{w\in T} a_w((\varphi)_{\mu,w1} - (\varphi)_{\mu,w2})((\psi)_{\mu,w1} - (\psi)_{\mu,w2}), \tag{1.4}$$

where $a_w > 0$ for any $w \in T$, μ is a Borel regular probability measure on Σ and $(u)_{\mu,v} = \mu(\Sigma_v)^{-1} \int_{\Sigma_v} u(\omega)\mu(d\omega)$. We show an equivalent condition for $\mathcal{Q}$ being a regular Dirichlet form. Moreover, we consider when a quadratic

form $\mathcal{Q}$ given by (1.4) is induced by a transient random walk on T. Let $\lambda_w = a_w\mu(\Sigma_w)/(\mu(\Sigma_{w1})\mu(\Sigma_{w2}))$ for $w \in T$. Define two conditions (A) and (B) as follows:
(A) $\lambda_w < \lambda_{wi}$ for any $(w, i) \in T \times \{1, 2\}$ and $\lim_{m\to\infty} \lambda_{[\omega]_m} = +\infty$ for any $\omega \in \Sigma$
(B) $\lambda_w < \lambda_{wi}$ for any $(w, i) \in T \times \{1, 2\}$ and $\lim_{m\to\infty} \lambda_{[\omega]_m} = +\infty$ for μ-a.e. $\omega \in \Sigma$.
Then we prove

$$\text{(A)} \;\; \Rightarrow \mathcal{Q} \text{ is induced by a random walk on } T \;\; \Rightarrow \;\; \text{(B)}.$$

Furthermore, we show that neither (A) nor (B) is equivalent to $\mathcal{Q}$ being induced by a random walk on T in Section 11.
Relation to noncommutative Riemannian geometry (Section 12): In [13], Pearson and Bellissard have constructed noncommutative Riemannian geometry on the Cantor set including Laplacians and Dirichlet forms from an ultra-metric. In fact, their Dirichlet forms belong to the class described by (1.4). As an application of the inverse problem, we prove that Dirichlet forms derived from self-similar ultra-metrics on the binary tree are induced by random walks and obtain heat kernel estimate (1.2) and moments of displacement (1.3). This extends the result of [13] where the authors have studied self-similar ultra-metrics whose similarity ratios are equal.

Behind all those results, the theory of resistance forms, which is briefly reviewed in Section 13, plays an important role. For example, if a random walk on a graph (V, E) is transient, we may add the infinity I to the vertices V and consider a natural resistance form on $V \cup \{I\}$ associated with the random walk on V. See Section 2 for details. By using this fact, the Martin kernel is expressed by the resistance metric (effective resistance) associated with the resistance form on $V \cup \{I\}$. This idea is essentially the key of obtaining the results of this paper.

Finally we introduce several conventions in the notation in this paper.
(1) Let f and g be real valued functions on a set A. We write $f(x) \asymp g(x)$ on A if and only if there exist $c_1, c_2 > 0$ such that $c_1 g(x) \le f(x) \le c_2 g(x)$ for any $x \in A$.
(2) Let X be a set. We define $\ell(X) = \{u | u : X \to \mathbb{R}\}$.

2 Weighted graphs as random walks

Although our main subject is a random walk on a tree, we introduce and study more general framework of weighted graphs in this section. The most important result is (2.3), where the Martin kernel is expressed by the resistance metric. This relation (2.3) is the foundation of all the theories in the following sections. Except that, most of the definitions and results are classical and can be found in [16] for example.

First we define weighted graphs and associated notions.

Definition 2.1. (1) A pair (V, C) is called a weighted graph if and only if V is a countable set and $C : V \times V \to [0, \infty)$ satisfies that $C(x, y) = C(y, x)$ for any $x, y \in V$ and that $C(x, x) = 0$ for any $x \in V$. The points in V are called vertices of the graph (V, C). Two vertices $x, y \in V$ are said to be connected if and only if $C(x, y) > 0$. The neighborhood $N_x(V, C)$ of $x \in V$ is defined by $N_x(V, C) = \{y | C(x, y) > 0\}$. For $x_0, x_1, \ldots, x_n \in V$, $\mathbf{p} = (x_0, x_1, \ldots, x_n)$ is called a path if and only if $C(x_i, x_{i+1}) > 0$ for any $i = 0, \ldots, n-1$. We define the length $|p|$ of a path $p = (x_0, x_1, \ldots, x_n)$ by n. If $x_0 = x$ and $x_n = y$, then a path $(x_0, \ldots, x_n)$ is called a path between x and y. A path $\mathbf{p} = (x_0, \ldots, x_n)$ is called simple if and only if $x_i \neq x_j$ for any $i \neq j$.
(2) A weighted graph (V, C) is called irreducible if and only if a path between x and y exists for any $x, y \in V$.
(3) A weighted graph (V, C) is called locally finite if and only if $N_x(V, C)$ is a finite set for any $x \in V$.

In this paper, we always assume that V is an infinite set and a weighted graph (V, C) is irreducible and locally finite. A weighted graph gives a reversible Markov chain on V.

Definition 2.2. Let $C(x) = \sum_{y \in V} C(x, y)$ and let $p(x, y) = C(x, y)/C(x)$. For $n = 0, 1, 2, \ldots$, we define $p^{(n)}(x, y)$ for $x, y \in V$ inductively by $p^{(0)}(x, y) = \delta_{xy}$, where δ_{xy} is the Dirac's delta, and

$$p^{(n+1)}(x, y) = \sum_{z \in V} p^{(n)}(x, z) p(z, y).$$

Define $G(x, y) = \sum_{n \geq 0} p^{(n)}(x, y)$, which may be infinite. $G(x, y)$ is called the Green function of (V, C). A weighted graph (V, C) is said to be transient if and if $G(x, y) < +\infty$ for any $x, y \in V$.

We regard $p(x, y)$ as the transition probability from x to y in the unit time. Let $(\{Z_n\}_{n \geq 0}, \{Q_x\}_{x \in V})$ be the associated random walk (or the Markov chain) on V. Then $p^{(n)}(x, y) = Q_x(Z_n = y)$, i.e. $p^{(n)}(x, y)$ is the probability of the transition from x to y at time n. Since $p(x, y)C(x) = p(y, x)C(y)$, this Markov chain is reversible. One can easily see that (V, C) is transient if $G(x_0, y_0) < +\infty$ for a single pair of points (x_0, y_0).

Next we define the notions of Laplacian, harmonic functions and resistance form associated with a (locally finite irreducible) weighted graph (V, C).

Definition 2.3. (1) The Laplacian $L : \ell(V) \to \ell(V)$ associated with (V, C) is defined by

$$(Lu)(x) = \sum_{y \in V} C(x, y)(u(y) - u(x))$$

for any $u \in \ell(V)$ and any $x \in X$. We say $u \in \ell(V)$ is harmonic on V with respect to (V, C) if $(Lu)(x) = 0$ for any $x \in V$. We use $\mathcal{H}(V, C)$, $\mathcal{H}^+(V, C)$ and $\mathcal{H}^\infty(V, C)$ to denote the collection of harmonic functions, non-negative harmonic

functions, bounded harmonic functions on V with respect to (V, C) respectively.
(2) Define

$$\mathcal{F} = \{u | u : V \to \mathbb{R}, \sum_{x,y \in V} C(x,y)(u(x) - u(y))^2 < +\infty\} \quad \text{and}$$

$$\mathcal{E}(u,v) = \frac{1}{2} \sum_{x,y \in V} C(x,y)(u(x) - u(y))(v(x) - v(y))$$

for any $u, v \in \mathcal{F}$. For a reference point $\phi \in V$, we define

$$\mathcal{E}_\phi(u,v) = \mathcal{E}(u,v) + u(\phi)v(\phi)$$

for any $u, v \in \mathcal{F}$. Define $C_0(V) = \{u | u \in \ell(V), \#(\mathrm{supp}(u)) < +\infty\}$, where $\mathrm{supp}(u)$ is the support of u and $\#(A)$ is the number of elements in A.

Proposition 2.4. (1) *$(\mathcal{E}, \mathcal{F})$ is a resistance form on V.*
(2) *If $u \in C_0(V)$ and $v \in \mathcal{F}$, then*

$$\mathcal{E}(u,v) = -\sum_{x \in V} u(x)(Lv)(x). \tag{2.1}$$

See Section 13 for the definition of resistance forms.

Definition 2.5. $(\mathcal{E}, \mathcal{F})$ defined in the last proposition is called the resistance form associated with the weighted graph (V, C).

The following theorem is one of the most essential results on the type problem of random walks. It has originated with Yamasaki in [17, 18]. See [15, Theorem 4.8] for details.

Theorem 2.6. *The following conditions are equivalent:*
(Tr1) *(V, C) is transient.*
(Tr2) *$1 \notin (C_0(V))_{\mathcal{E}_\phi}$, where $(C_0(V))_{\mathcal{E}_\phi}$ is the closure of $C_0(V)$ with respect to $\mathcal{E}_\phi$.*
(Tr3) *$(C_0(V))_{\mathcal{E}_\phi} \neq \mathcal{F}$.*
(Tr4) *$\sup\{u(x)^2/\mathcal{E}(u,u) | u \in C_0(V)\} < +\infty$ for any $x \in V$.*

By the above theorem, if (V, C) is transient, we may introduce a point I, which is thought of as the point of infinity, and construct a new resistance form on $V \cup \{I\}$. Later in (2.3), the Martin kernel of (V, C) is shown to be described by the resistance metric associated with this new resistance form on $V \cup \{I\}$.

Proposition 2.7. *Assume that (V, C) is transient. Define $\mathcal{F}_* = (C_0(V))_{\mathcal{E}_\phi} + \mathbb{R} = \{f + a | f \in (C_0(V))_{\mathcal{E}_\phi}, a \in \mathbb{R}\}$. Let $I \notin V$ and define $u(I) = a$ if $u = f + a$ for $f \in (C_0(V))_{\mathcal{E}_\phi}$ and $a \in \mathbb{R}$. Denote $V_* = V \cup \{I\}$. Then*
(1) *$(\mathcal{E}, \mathcal{F}_*)$ is a resistance form on V_*.*
(2) *There exists $g_* : V_* \times V_* \to [0, \infty)$ such that the following conditions* (a), (b) *and* (c) *are satisfied:*
(a) *$g_*(x,y) = g_*(y,x)$ for any $x, y \in V_*$.*

(b) *For any* $x \in V$, *let* $g_*^x(y) = g_*(x,y)$. *Then* g_*^x *is a unique element in* $(C_0(V))_{\mathcal{E}_\phi}$ *which satisfies* $\mathcal{E}(g_*^x, u) = u(x)$ *for any* $u \in (C_0(V))_{\mathcal{E}_\phi}$.
(c) *For any* $x, y \in V$,

$$g_*(x,y) = \frac{R_*(x,I) + R_*(y,I) - R_*(x,y)}{2}, \tag{2.2}$$

where $R_*(\cdot,\cdot)$ *is the resistance metric on* V_* *associated with* $(\mathcal{E}, \mathcal{F}_*)$ *given by*

$$R_*(x,y) = \max\left\{\frac{(u(x)-u(y))^2}{\mathcal{E}(u,u)}\middle| u \in \mathcal{F}_*, \mathcal{E}(u,u) \neq 0\right\}.$$

Note that the right-hand side of (2.2) is the Gromov product associated with the metric space $(V \cup \{I\}, R_*)$.

Remark. There is no weighted graph (V_*, C_*) whose associated resistance form is $(\mathcal{E}_*, \mathcal{F}_*)$. Otherwise we have $\psi_* \in \mathcal{F}_*$, where ψ_* is the characteristic function of the infinity I. Then there exists $u \in (C_0(V))_{\mathcal{E}_\phi}$ such that $\psi_* = u + 1$. Since $u \equiv -1$ on V and (T, C) is transient, this contradicts to the fact that $1 \notin (C_0(V))_{\mathcal{E}_\phi}$.

Remark. Let $R(\cdot,\cdot)$ be the resistance metric associated with the resistance form $(\mathcal{E}, \mathcal{F})$ on V. Then $R(\cdot,\cdot)$ and $R_*(\cdot,\cdot)$ are called the limit resistance and the minimal resistance respectively in [14]. Also, they are called the free resistance and the wired resistance respectively in [7]. For the use of the terminology "free" and "wired", see [12].

Proof. (1) We will verify the conditions (RFi) for $i = 1, \ldots, 5$ in Definition 13.4. (RF1) and (RF2) are immediate. Let χ_x be the characteristic function of a single point $x \in V$. Since $\chi_x \in C_0(V)$ for any $x \in V$, we have (RF3). For $x \neq y \in V$, since $\mathcal{F}_* \subset \mathcal{F}$, the supremum in (RF4) with $\mathcal{F} = \mathcal{F}_*$ is finite. If $x \in V$ and $y = I$, then the supremum is also finite by the condition (Tr4) of Theorem 2.6. Finally, let $u \in \mathcal{F}_*$. Then $u = f + a$ for some $f \in (C_0(V))_{\mathcal{E}_\phi}$ and $a \in \mathbb{R}$. Then $\overline{f+a} - \overline{a} \in (C_0(V))_{\mathcal{E}_\phi}$. Hence $\overline{u}(I) = (\overline{f+a})(I) = \overline{a} = \overline{u(I)}$. This shows (RF5) for $(\mathcal{E}, \mathcal{F}_*)$. Thus $(\mathcal{E}, \mathcal{F}_*)$ is a resistance form on V_*.

Applying Theorem 13.2 to the resistance form $(\mathcal{E}, \mathcal{F}_*)$ on V_* with $B = \{I\}$, we obtain the statements (2) and (3). Note that in this case, B is a single point. The remark after Theorem 13.2 shows that $R_B = R_*$. □

The element I is considered as the point of infinity.

Definition 2.8. Assume that (V, C) is transient. When it is necessary to specify (V, C), we use $I_{(V,C)}$ to denote the infinity I associated with $(\mathcal{E}, \mathcal{F}_*)$ and write $R_x(V, C)$ instead of $R_*(x, I)$, which is the resistance metric between x and $I_{(V,C)}$ with respect to $(\mathcal{E}, \mathcal{F}_*)$. If (V, C) is not transient, then we define $R_x(V, C) = \infty$.

According to the use of terminology in Section 13, $g_*(\cdot,\cdot)$ is the I-Green function associated with $(\mathcal{E}, \mathcal{F}_*)$. In light of the following theorem, we call $g_*(\cdot,\cdot)$ the symmetrized Green function of the weighted graph (V, C).

Theorem 2.9. *Assume that (V, C) is transient. Then, for any $x, y \in V$,*

$$g_*(x, y) = G(x, y)/C(y).$$

To prove the above theorem, we need the next lemma.

Lemma 2.10. *Assume that (V, C) is transient. Define $\varphi_y^n(x) = p^{(n)}(x, y)$.*
(1) $\varphi_y^n \in C_0(V)$.
(2) *For any $x, y \in V$*

$$(L\varphi_y^n)(x) = C(x)(p^{(n+1)}(x, y) - p^{(n)}(x, y)) = C(y)(p^{(n+1)}(y, x) - p^{(n)}(y, x)).$$

(3)

$$\begin{aligned}\mathcal{E}(\varphi_x^n, \varphi_y^m) &= C(y)(p^{(n+m)}(y, x) - p^{(n+m+1)}(y, x)) \\ &= C(x)(p^{(n+m)}(x, y) - p^{(n+m+1)}(x, y)).\end{aligned}$$

(4) *Define $G^y(x) = \sum_{n\geq 0} \varphi_y^n(x)$. Then $G^y \in (C_0(V))_{\mathcal{E}_\phi}$ and $\mathcal{E}(u, G^y) = C(y)u(y)$ for any $u \in (C_0(V))_{\mathcal{E}_\phi}$.*

Proof. Since (V, C) is locally finite, we verify (1). By using (2.1), routine calculations show (2) and (3).
(4) Let $G^{n,y}(x) = \sum_{i=0}^{n-1} \varphi_y^i$. For $n < m$, by (3),

$$\mathcal{E}(G^{m+1,y} - G^{n,y}, G^{m+1,y} - G^{n,y}) = C(y)\Big(\sum_{i=2n}^{n+m} p^{(i)}(y, y) - \sum_{i=n+m+1}^{2m+1} p^{(i)}(y, y)\Big).$$

Since $\sum_{i=0}^{\infty} p^{(i)}(y, y)$ is finite, we see that $\mathcal{E}(G^{m+1,y} - G^{n,y}, G^{m+1,y} - G^{n,y}) \to 0$ as $n, m \to \infty$. Moreover, $G^{n,y}(x) = \sum_{i=0}^{n-1} p^{(i)}(x, y) \to G(x, y)$ as $n \to \infty$ for $x \in V$. Hence $\{G^{n,y}\}_{n\geq 1}$ is a Cauchy sequence with respect to $\mathcal{E}_\phi$ and its limit is G^y. Note that $G^{n,y} \in C_0(V)$. Hence $G^y \in (C_0(V))_{\mathcal{E}_\phi}$. For any $u \in C(V_0)$,

$$\begin{aligned}\mathcal{E}(u, G^{n,y}) &= \sum_{i=0}^{n-1} \mathcal{E}(u, \varphi_y^i) = -\sum_{i=0}^{n-1}\sum_{x\in V} u(x)(L\varphi_y^i)(y) \\ &= C(y)\sum_{i=1}^{n-1}\sum_{x\in V} u(x)(p^{(i)}(y, x) - p^{(i+1)}(y, x)) \\ &= C(y)\sum_{x\in V} u(x)(p^{(0)}(y, x) - p^{(n)}(y, x)).\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\mathcal{E}(u, G^y) = C(y)\sum_{x\in V} u(x)p^{(0)}(y, x) = C(y)u(y).$$

□

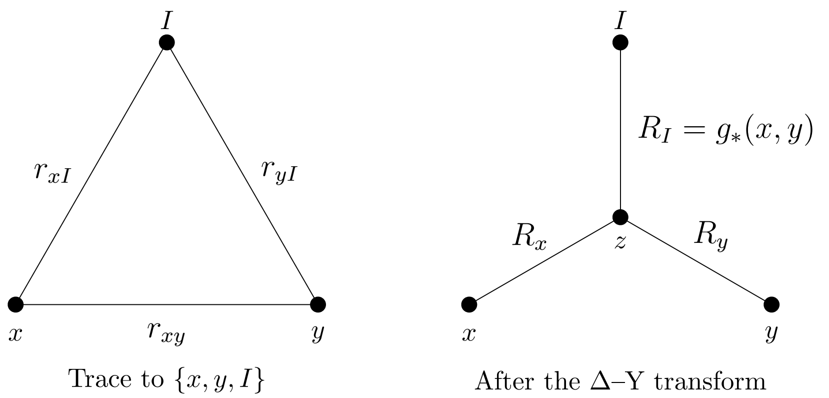


Figure 2: Calculation of $g_*(x, y)$ by means of Δ–Y transform

Proof of Theorem 2.9. By Lemma 2.10-(4), it follows that $G^y \in (C_0(V))_{\mathcal{E}_\phi}$ and $\mathcal{E}(u, G^y/C(y)) = u(y)$ for any $u \in (C_0(V))_{\mathcal{E}_\phi}$. Proposition 2.7-(b) shows that $G(x, y)/C(y) = G^y(x)/C(y) = g_*(x, y)$. □

The next lemma is technically useful in the following sections. One can refer to Theorem 13.3 for the definition of a trace of a resistance form.

Lemma 2.11. *Assume that (V, C) is transient. Set $U = \{x, y, I\}$ for $x, y \in X$ with $x \neq y$. Let $(\mathcal{E}|_U, \mathcal{F}_*|_U)$ be the trace of the resistance form $(\mathcal{E}, \mathcal{F}_*)$ on U. Suppose that*

$$\mathcal{E}|_U(u, u) = c_{xI}(u(x) - u(I))^2 + c_{yI}(u(y) - u(I))^2 + c_{xy}(u(x) - u(y))^2$$

for any $u : U \to \mathbb{R}$. Define $r_{xI} = 1/c_{xI}, r_{yI} = 1/c_{yI}$ and $r_{xy} = 1/c_{xy}$. Then

$$g_*(x, y) = \begin{cases} \frac{r_{xI}r_{yI}}{r_{xI}+r_{yI}+r_{xy}} & \text{if } c_{xI}c_{yI}c_{xy} > 0, \\ 0 & \text{if } c_{xy} = 0, \\ r_{yI} & \text{if } c_{xI} = 0, \\ r_{xI} & \text{if } c_{yI} = 0. \end{cases}$$

This lemma is proven by using the Δ–Y transform to $\mathcal{E}|_U$ as illustrated in Figure 2. See [9, Lemma 2.1.15] for the Δ–Y transform.

Proof. Assume $c_{xy}c_{yI}c_{xI} > 0$. Let $R_x = r_{xI}r_{xy}/R$ and $R_y = r_{yI}r_{xy}/R$ and $R_I = r_{xI}r_{yI}/R$, where $R = r_{xI} + r_{yI} + r_{xy}$. Define a resistance form $\widetilde{\mathcal{E}}$ on four points $\{x, y, I, z\}$ by

$$\widetilde{\mathcal{E}}(u, u) = \sum_{p=x,y,I} \frac{1}{R_p}(u(p) - u(z))^2.$$

Then by the Δ–Y transform, $(\mathcal{E}|_U, \mathcal{F}_*|_U)$ is equal to the trace of resistance form $\widetilde{\mathcal{E}}$ to U. Hence $R_*(x, I) = R_x + R_I, R_*(y, I) = R_y + R_I$ and $R_*(x, y) = R_x + R_y$.

Using (2.2), we immediately obtain $g_*(x,y) = R_I$. If any of c_{xy}, c_{xI} and c_{yI} is 0, then direct calculation easily shows the proposition. $\square$

In the rest of this section, we introduce the notion of the Martin boundary of a transient weighted graph and related results originally studied in 1960's. See [16, Section 24] for the references and details.

Definition 2.12. Assume (V,C) is transient. Define

$$K_{x_0}(x,y) = \frac{G(x,y)}{G(x_0,y)}$$

for any $x_0, x, y \in T$. $K_{x_0}(x,y)$ is called the Martin kernel of (V,C).

Using (2.2), we have

$$K_{x_0}(x,y) = \frac{g_*(x,y)}{g_*(x_0,y)} = \frac{R_*(x,I) + R_*(y,I) - R_*(x,y)}{R_*(x_0,I) + R_*(y,I) - R_*(x_0,y)}, \tag{2.3}$$

where R_* is the resistance metric with respect to $(\mathcal{E}, \mathcal{F}_*)$.

Proposition 2.13. *Assume (V,C) is transient. Then there exists a unique minimal compactification $\widetilde{V}$ of V (up to homeomorphism) such that K_{x_0} is extended to a continuous function from $V \times \widetilde{V}$ to $\mathbb{R}$. $\widetilde{V}$ is independent of the choice of x_0. Moreover, there exists a $\widetilde{V}\backslash V$-valued random variable Z_∞ such that*

$$Q_x\Big(\lim_{n\to\infty} Z_n = Z_\infty\Big) = 1$$

for any $x \in X$.

Definition 2.14. Assume that (V,C) is transient. $\widetilde{V}$ is called the Martin compactification of V. Define $M(V,C) = \widetilde{V}\backslash V$, which is called the Martin boundary of (V,C). Define a probability measure ν_x on $M(V,C)$ by

$$\nu_x(B) = Q_x(Z_\infty \in B)$$

for any Borel set $B \subseteq M(V,C)$. ν_x is called the hitting distribution on the Martin boundary $M(V,C)$ starting from x.

Note that ν_x's for $x \in V$ are mutually absolutely continuous and satisfy $\nu_x = \sum_{y \in V} p(x,y)\nu_y$. The following theorem is the fundamental result on the Martin boundary and representation of harmonic functions on a weighted graph.

Theorem 2.15. *Assume that (V,C) is transient.*
(1) *$K_{x_0}(\cdot,y) \in \mathcal{H}^+(V,C) \cap \mathcal{H}^\infty(V,C)$ for any $x_0 \in V$ and any $y \in \widetilde{V}$.*
(2) *For any $h \in \mathcal{H}^+(V,C)$, there exists a Borel regular measure ν^h on $M(V,C)$ such that*

$$h(x) = \int_{M(V,C)} K_{x_0}(x,y)\nu^h(dy).$$

(3) *For any $h \in \mathcal{H}^\infty(V,C)$, there exists $f \in L^\infty(M(V,C), \nu_{x_0})$ such that*

$$h(x) = \int_{M(V,C)} K_{x_0}(x,y)f(y)\nu_{x_0}(dy).$$

3 Transient trees and their Martin boundaries

In this section, we introduce the notion of a tree, which is a special case of weighted graphs. The main goals are Theorems 3.8 and 3.13, where the hitting distribution on the Martin boundary and the Martin kernel are expressed in terms of resistance metrics of sub-trees.

Definition 3.1. (1) A pair (T, C) is called a tree if and only if it satisfies the following conditions (Tree1) and (Tree2):
(Tree1) (T, C) is an irreducible and locally finite weighted graph.
(Tree2) For any $x, y \in T$, there exists a unique simple path between x and y.
(2) Let (T, C_1) and (T, C_2) be trees. Then (T, C_1) and (T, C_2) is said to have a common graph structure if and only if $C_1(x, y) = 0$ implies $C_2(x, y) = 0$, and vise versa for any $x, y \in T$. We write $(T, C_1) \underset{\mathrm{G}}{\sim} (T, C_2)$ if (T, C_1) and (T, C_2) have a common graph structure.

Fix a countably infinite set T. Then the relation $\underset{\mathrm{G}}{\sim}$ is an equivalence relation on $\{(T, C)|(T, C) \text{ is a tree}\}$. We use $G(T, C)$ to denote the equivalence class of a tree (T, C) with respect to $\underset{\mathrm{G}}{\sim}$. The equivalence class $G(T, C)$ determines the structure of T as a non-directed graph.

Hereafter in this section, (T, C) is always a tree.

Definition 3.2. (1) For $x, y \in T$, the unique simple path between x and y is called the geodesic between x and y and denoted by $\overline{xy}$. The path distance $|x, y|$ between x and y is defined by the length of the geodesic between x and y.
(2) For $x \in T$, define $\pi_x : T \to T$ by

$$\pi_x(y) = \begin{cases} x_{n-1} & \text{if } x \neq y \text{ and } \overline{xy} = (x_0, \ldots, x_{n-1}, x_n), \\ x & \text{if } y = x. \end{cases}$$

If $x \neq y$, $\pi_x(y)$ is called the predecessor of y with respect to x. Also we define $S_x(y) = N_y(T, C) \backslash \{\pi_x(y)\}$. The points in $S_x(y)$ are called successors of y with respect to x.
(3) An infinite path $(x_0, x_1, \ldots) \in T^{\mathbb{N}}$ is called a geodesic ray originated from $x \in V$ if and only if $x_0 = x$ and $(x_0, x_1, \ldots, x_n) = \overline{x_0 x_n}$ for any $n \geq 1$. Two geodesic rays $(x_0, x_1, \ldots)$ and $(y_0, y_1, \ldots)$ is equivalent if and only if $(x_m, x_{m+1}, \ldots) = (y_k, y_{k+1}, \ldots)$ for some m and k. Define $\Sigma(T, C)$ as the collection of the equivalence classes of geodesic rays. We write $\widehat{T} = T \cup \Sigma(T, C)$.

Easily, $\Sigma(T, C)$ is identified with the collections of geodesic rays originated from a fixed point $x \in T$.

Lemma 3.3. *Let $x \in T$. Define $\Sigma^x(T, C)$ by the collection of the geodesic rays originated from x. Then $\Sigma(T, C)$ is naturally identified with $\Sigma^x(T, C)$.*

We need the notion of sub-tree T^x_y and associated collection of geodesic rays $\Sigma^x_y(T, C)$ to describe finer structure of $\Sigma^x(T, C)$.

Definition 3.4. We write $z \in \overline{xy}$ if and only if $\overline{xy} = (x_0, \ldots, x_n)$ and $z = x_i$ for some $i = 0, \ldots, n$. Define $T_y^x = \{z | y \in \overline{xz}\}$ and

$$\Sigma_y^x(T, C) = \{(x_0, x_1, \ldots) | (x_0, x_1, \ldots) \in \Sigma^x(T, C), x_m = y \text{ for some } m \geq 0\}.$$

Define $\widehat{T}_y^x = T_y^x \cup \Sigma_y^x(T, C)$.

By Lemma 3.3, we always identify $\Sigma(T, C)$ as $\Sigma^x(T, C)$. If no confusion may occur, we write Σ, Σ^x and Σ_y^x in place of $\Sigma(T, C), \Sigma^x(T, C)$ and $\Sigma_y^x(T, C)$ respectively hereafter.

Proposition 3.5. *Define*

$$\mathcal{O} = \{U | U \subseteq \widehat{T}, \textit{for any } \omega \in \Sigma^x \cap U, \\ \textit{there exists } y \in T \textit{ such that } \omega \in \Sigma_y^x \textit{ and } \widehat{T}_y^x \subseteq U.\}$$

Then $\mathcal{O}$ gives a topology of $\widehat{T}$ and $(\widehat{T}, \mathcal{O})$ is compact. Moreover, the closure of T is $\widehat{T}$.

The restriction of $\mathcal{O}$ to T is the discrete topology on T. $(\widehat{T}, \mathcal{O})$ is called the end compactification of T.

Remark. Note that the notions introduced in this section so far only depend on the equivalence class $G(T, C)$. In particular, if (T, C_1) and (T, C_2) are trees and $G(T, C_1) = G(T, C_2)$, then $\Sigma(T, C_1)$ is naturally identified with $\Sigma(T, C_2)$ and the end compactifications of (T, C_1) and (T, C_2) are the same.

The following fundamental theorem on the Martin boundary of a tree is due to Cartier [2]. See [16] for details.

Theorem 3.6. *Assume (T, C) is transient. Then the Martin compactification $\widetilde{T}$ of T coincides with the end compactification $\widehat{T}$. For any $h \in \mathcal{H}^\infty(T, C)$, there exists a unique $\varphi \in L^\infty(M(T, C), \nu_{x_0})$ such that*

$$h(x) = \int_{M(T,C)} K_{x_0}(x, y)\varphi(y)\nu_{x_0}(dy)$$

for any $x \in V$. Moreover, $h(x_n) \to \varphi(\omega)$ as $n \to \infty$ ν_{x_0}-almost every $\omega \in \Sigma(T, C)$, where $(x_0, x_1, \ldots)$ is the geodesic ray originated from x_0 representing $\omega \in \Sigma(T, C)$.

By the above theorem, we identify the Martin boundary $M(T, C)$ with Σ hereafter. Let $\mathcal{O}_\Sigma$ be the relative topology of $\mathcal{O}$ on Σ. Then $(\Sigma, \mathcal{O}_\Sigma)$ is compact.

Recall that ν_x is the hitting distribution on Σ starting from $x \in V$ defined by $\nu_x(B) = Q_x(Z_\infty \in B)$ for a Borel set $B \subseteq \Sigma$. We are going to give an expression of $\nu_x(\Sigma_y^x)$ by means of resistance metrics of sub-trees (T_y^x, C_y^x) defined below.

Definition 3.7. Let $x, y \in T$ with $x \neq y$. Define $r_y^x = 1/C(\pi_x(y), y)$. Let C_y^x be the restriction of C onto T_y^x. We write $R_y^x = R_y(T_y^x, C_y^x)$ and $\rho_y^x = r_y^x + R_y^x$.

Remark. If (T^x_y, C^x_y) is not transient, then $R^x_y = \rho^x_y = \infty$. Moreover, (T^x_y, C^x_y) is not transient if and only if (T^x_z, C^x_z) is not transient for all $z \in S_x(y)$.

Theorem 3.8. *Assume that (T, C) is transient. Then*

$$\nu_x(\Sigma^x_y) = \begin{cases} \frac{R^x_{\pi_x(y)}}{\rho^x_y} \nu_x(\Sigma^x_{\pi_x(y)}) & \text{if } (T^x_{\pi_x(y)}, C^x_{\pi_x(y)}) \text{ is transient,} \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

for any $y \in T\backslash\{x\}$.

By the inductive use of this theorem, if $\overline{xy} = (x_0, \ldots, x_n)$, then

$$\nu(\Sigma^x_y) = \begin{cases} \frac{R^x_{x_0} R^x_{x_1} \cdots R^x_{x_{n-1}}}{\rho^x_{x_1} \rho^x_{x_2} \cdots \rho^x_{x_n}} & \text{if } (T^x_y, C^x_y) \text{ is transitive,} \\ 0 & \text{if } (T^x_y, C^x_y) \text{ is not transitive.} \end{cases}$$

Lemma 3.9. *Assume (T, C) is transient. Then*

$$\nu_x(\Sigma^x_y) = R_x(T, C)/\rho^x_y$$

for any $y \in N_x(T, C)$.

Note that $R_x(T, C) = (\sum_{y \in N_x(T,C)} 1/\rho^x_y)^{-1}$.

Proof. If (T^x_y, C^x_y) is not transient, then $\nu_x(\Sigma^x_y) = Q_x(Z_\infty \in \Sigma^x_y) = 0$. Assume that (T^x_y, C^x_y) is transient. Let $F(x, y) = Q_x(Z_n = y \text{ for some } n \geq 0)$. By [16, (1.13)-(b)] and Theorem 2.9,

$$F(x, y) = \frac{G(x, y)}{G(y, y)} = \frac{g_*(x, y)}{g_*(y, y)} \tag{3.2}$$

Using [16, (26.5)], we have

$$\nu_x(T^x_y) = \frac{F(x, y)(1 - F(x, y))}{1 - F(x, y)F(y, x)} = \frac{(g_*(x, x) - g_*(x, y))g_*(x, y)}{g_*(x, x)g_*(y, y) - g_*(x, y)^2}. \tag{3.3}$$

We consider the trace of $(\mathcal{E}, \mathcal{F}_*)$ to the three points $\{x, y, I\}$, where $I = I_{(T,C)}$ as in Lemma 2.11. By using the same notations,

$$g_*(x, x) = R_x(T, C) = \frac{r_{xI}(r_{xy} + r_{yI})}{R},$$
$$g_*(y, y) = R_y(T, C) = \frac{r_{yI}(r_{xy} + r_{xI})}{R},$$
$$g_*(x, y) = \frac{r_{xI} r_{yI}}{R},$$

where $R = r_{xy} + r_{xI} + r_{yI}$. (3.3) and these imply

$$\nu_x(T^x_y) = \frac{r_{xI}}{R} = \frac{R_x(T, C)}{r_{xy} + r_y} = \frac{R_x(T, C)}{\rho^x_y}.$$

□

Proof of Theorem 3.8. Let $\mathcal{J}_y^x = \{Z_n \in T_y^x \text{ for sufficiently large } n\}$. Then $\nu_x(\Sigma_y^x) = Q_x(\mathcal{J}_y^x)$. Set $\mathcal{I}_z^x = \{Z_n \in T_z^x \text{ for any } n \geq 0\}$, where $z = \pi_x(y)$. By the Markov property,

$$\nu_x(\Sigma_y^x) = Q_x(\mathcal{J}_y^x) = Q_x(\mathcal{J}_z^x)Q_z(\mathcal{J}_y^x \cap \mathcal{I}_z^x|\mathcal{I}_z^x) = \nu_x(\Sigma_z^x)Q_z(\mathcal{J}_y^x \cap \mathcal{I}_y^x|\mathcal{I}_z^x).$$

Let $(\{\widetilde{Z}_n\}_{n\geq 0}, \{\widetilde{Q}_w\}_{w \in T_z^x})$ be the Markov chain associated with (T_z^x, C_z^x) and let $\widetilde{\nu}_z$ is the associated hitting distribution. Then

$$Q_z(\mathcal{J}_y^x \cap \mathcal{I}_z^x|\mathcal{I}_z^x) = \widetilde{Q}_z(\widetilde{Z}_n \in T_y^x \text{ for sufficiently large } n) = \widetilde{\nu}_z(\Sigma_y^x).$$

Applying Lemma 3.9 to (T_z^x, C_z^x), we obtain $\widetilde{\nu}_z(\Sigma_y^x) = R_z^x/\rho_y^x$. Thus we have (3.1). $\square$

As an application of Theorem 3.8, we may identify the support of ν_x, which is the Poisson boundary of (T, C).

Definition 3.10. Define

$$T_*^x = \{y | y \in T,\ (T_y^x, C_y^x) \text{ is transient}\}$$

and let C_*^x be the restriction of C to T_*^x.

$\Sigma(T_*^x, C_*^x)$ is naturally identified with $\{(x_0, x_1, \ldots)|(x_0, x_1, \ldots) \in \Sigma^x, x_m \in T_*^x \text{ for any } m \geq 0\}$. In this sense, $\Sigma(T_*^x, C_*^x)$ is regarded as a subset of Σ.

Corollary 3.11. *The support of* ν_x *is* $\Sigma(T_*^x, C_*^x)$.

Next we give an expression of the Martin kernel in terms of resistance metrics of sub-trees.

Definition 3.12. Assume that (T, C) is transient. For $x \neq y \in T$, define

$$\eta_y^x = \begin{cases} \frac{R_y^x}{\rho_y^x} & \text{if } (T_y^x, C_y^x) \text{ is transient,} \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 3.13. *Let* $x_0, x \in T$ *with* $x_0 \neq x$ *and let* $\overline{x_0 x} = (x_0, x_1, \ldots, x_n)$, *where* $x_n = x$. *For any* $y \in T$, *define* $k(x_0, x, y)$ *as the unique* k *which satisfies* $y \in T_{x_k}^{x_0} \cap T_{x_k}^x$. *Then,*

$$K_{x_0}(x, y) = \left(\eta_{x_0}^x \eta_{x_1}^x \cdots \eta_{x_{k-1}}^x\right)^{-1} \eta_{x_{k+1}}^{x_0} \cdots \eta_{x_n}^{x_0}, \tag{3.4}$$

where $k = k(x_0, x, y)$.

The rest of this section is devoted to proving Theorem 3.13.

Lemma 3.14. *Assume that* (T, C) *is transient. Let* $z \in T$. *If* $x \in N_z(T, C)$ *and* $y \in T_x^z$, *then*

$$K_z(x, y) = 1/\eta_z^x. \tag{3.5}$$

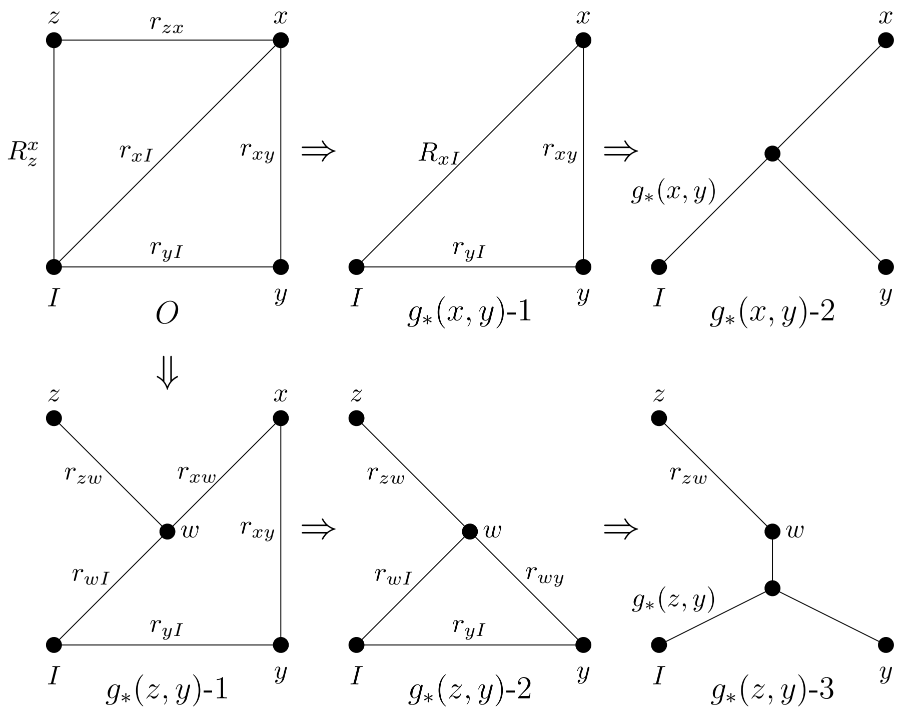


Figure 3: Calculation of $g_*(x,y)$ and $g_*(z,y)$

Proof. We have three cases, namely,
Case 1: both (T^x_z, C^x_z) and (T^z_x, C^z_x) are transient.
Case 2: (T^x_z, C^x_z) is transient while (T^z_x, C^z_x) is not.
Case 3: (T^z_x, C^z_x) is transient while (T^x_z, C^x_z) is not.
First we consider Case 1. Let

$$c_{xI}(u(x) - u(I))^2 + c_{yI}(u(y) - u(I))^2 + c_{xy}(u(x) - u(y)^2$$

be the trace of the resistance form associated with (T^z_x, C^z_z). Assume that $c_{xI}c_{yI}c_{xy} > 0$. Set $r_{xI} = 1/c_{xI}, r_{yI} = 1/c_{yI}$ and $r_{xy} = 1/r_{xy}$. Now our network corresponds to "O" at the upper left corner of Figure 3. We will follow the arrows in Figure 3 to calculate $g_*(x,y)$ and $g_*(z,y)$. To obtain $g_*(x,y)$, we calculate the combined resistance R_{xI} of $r_{xz} + R^x_z$ and r_{xI} and get $g_*(x,y)$-1 from O. Applying Δ–Y transform to $g_*(x,y)$-1, we have $g_*(x,y)$-2. Then Lemma 2.11 gives $g_*(x,y)$. After performing these processes,

$$g_*(x,y) = \frac{r_{xI}r_{yI}(R^x_z + r_{xz})}{(r_{yI} + r_{xy})(R^x_z + r_{xz} + r_{xI}) + (R^x_z + r_{xz})r_{xI}}.$$

To get $g_*(z,y)$, we first apply the Δ–Y transform to three points network $\{z, x, I\}$ in O and obtain $g_*(z,y)$-1. To set $r_{wy} = r_{xw} + r_{xy}$ gives $g_*(z,y)$-2. Finally applying the Δ–Y transform to three point network $\{I, w, y\}$, we get

$g_*(x,y)$-3. Then Lemma 2.11 gives $g_*(z,y)$. After performing these processes,

$$g_*(z,y) = \frac{r_{xI}r_{yI}R^x_z}{(r_{yI}+r_{xy})(R^x_z+r_{xz}+r_{xI})+(R^x_z+r_{xz})r_{xI}}.$$

Since $K_z(x,y) = g_*(x,y)/g_*(z,y)$, we have (3.5). If one of c_{xI}, c_{yI} and $c_{xy} = 0$, then the corresponding edge in O of Figure 3 is disconnected. The calculation is considerably easier and one can confirm (3.5) as well.

In Case 2, r_{xI} and r_{yI} are infinite and the corresponding edges in O of Figure 3 are disconnected. Hence it is easy to obtain (3.5).

In Case 3, R^x_z is infinite and the corresponding edge in O of Figure 3 is disconnected. Therefore $g_*(x,z) = g_*(x,y)$ and this implies (3.5). □

Proof of Theorem 3.13. Let $z = x_0$. We use induction in $|z,x|$. Assume $|z,x| = 1$. Then $y \in T^z_x$ or $y \in T^z_x$. If $y \in T^z_x$, then Lemma 3.14 shows (3.4). If $y \in T^x_z$, then $K_z(x,y) = (K_x(z,y))^{-1} = \eta^z_z$ and so (3.4) follows. Next assume that (3.4) holds if $|z,x| \leq m$. Let $|z,x| = m+1$ and let $\overline{zx} = (x_0, x_1, \ldots, x_m, x_{m+1})$. Then by Lemma 3.14,

$$K_z(x,y) = K_z(x_m,y)K_{x_m}(x,y) = K_z(x_m,y) \times \begin{cases} \left(\eta^x_{x_m}\right)^{-1} & \text{if } y \in T^{x_m}_x, \\ \eta^{x_m}_x & \text{if } y \in T^x_{x_m}. \end{cases}$$

Note that $\eta^{x_m}_x = \eta^z_x$ and $T^{x_m}_x = T^z_x$. Hence using the hypothesis of the induction, we obtain (3.4). Thus we have completed the proof. □

4 Induced form on the Martin boundary

In this section, we present a structure theorem (Theorem 4.4) of the Dirichlet form on the Cantor set induced by a transient random walk on a tree. Throughout this section, (T,C) is a transient tree and $\phi \in T$ is a reference point.

The trace on the Martin boundary $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ is defined as follows.

Definition 4.1. Fix a reference point $\phi \in T$. Write $T_\# = T\backslash\{\phi\}$. Define a linear map $H : L^1(\Sigma^\phi, \nu_\phi) \to \ell(T)$ by

$$H(f)(x) = \int_{\Sigma^x} K_\phi(x,y)f(y)\nu_\phi(dy)$$

for any $x \in X$. Moreover, define

$$\mathcal{F}_\Sigma = \{f | f \in L^2(\Sigma^\phi, \nu_\phi), H(f) \in \mathcal{F}\}$$

and

$$\mathcal{E}_\Sigma(f,g) = \mathcal{E}(H(f), H(g))$$

for any $f, g \in \mathcal{F}_\Sigma$.

Note that $H(f) \in \mathcal{H}(T, C)$ since $K_\phi(\cdot, y) \in \mathcal{H}(T, C)$ for any $y \in \Sigma$.

We are going to study the quadratic form $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$. Since the support of ν_ϕ is $\Sigma(T_*^\phi, C_*^\phi)$, we will consider (T_*^ϕ, C_*^ϕ) in place of (T, C). Equivalently, we assume that (T_y^x, C_y^x) is transient for any $x, y \in T$ hereafter. As a result, the Martin boundary is equal to the Poisson boundary and every edge in T must has at least one successor, i.e. $S_x(y) \neq \emptyset$ for any $x, y \in T$. (Note that if $S_x(y) = \emptyset$, then (T_y^x, C_y^x) is recurrent.) For ease of notation, we omit writing ϕ in notations. For example, we use $\Sigma, \nu, T_x, C_x, R_x, \rho_x$ and η_x instead of $\Sigma^\phi, \nu_\phi, T_x^\phi, C_x^\phi, R_x^\phi, \rho_x^\phi$ and η_x^ϕ.

The values D_x and λ_x defined below play an essential role in this paper. For example, $\{\lambda_x | x \in T\} \cup \{0\}$ will be identified with the collection of eigenvalues of the self-adjoint operator associated with $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$.

Definition 4.2. Define $D_x = \nu(\Sigma_x)R_x$ and $\lambda_x = 1/D_x$ for any $x \in T$. The map $x \to \lambda_x$ from T to $(0, \infty)$ is called the eigenvalue map associated with (T, C).

By (2.2) and Theorem 5.2-(1), we will see that

$$D_x = \frac{R_*(x, I) + R_*(\phi, I) - R_*(x, \phi)}{2}, \tag{4.1}$$

whose right-hand side coincides with the Gromov product of the metric R_*.

Lemma 4.3. *For any $x \in T$ and any $y \in S(x)$,*

$$\frac{D_y}{D_x} = \frac{R_y}{\rho_y} < 1 \tag{4.2}$$

and

$$D_x - D_y = r_y \nu(\Sigma_y) \tag{4.3}$$

In particular, $D_x > D_y$ and $\lambda_y > \lambda_x$ for any $x \in T$ and any $y \in S(x)$.

Proof. By Theorem 3.8,

$$\frac{D_y}{D_x} = \frac{R_x R_y}{\rho_y R_x} = \frac{R_y}{\rho_y} = \frac{R_y}{r_y + R_y} < 1$$

Again by Theorem 3.8, $\nu(\Sigma_y)\rho_y = \nu(\Sigma_x)R_x$. This immediately implies (4.3). Since every (T_y, C_y) is assumed to be transient, it follows that $\nu(\Sigma_y) > 0$. □

The next theorem shows that $\mathcal{E}|_\Sigma$ has a simple expression by means of $\{\lambda_x\}_{x \in T}$ and ν.

Theorem 4.4. *For any $f \in L^1(\Sigma, \nu)$,*

$$\begin{aligned} \mathcal{E}(H(f), H(f)) &= \sum_{x \in T} \frac{\lambda_x}{2\nu(\Sigma_x)} \sum_{y,z \in S(x)} \nu(\Sigma_y)\nu(\Sigma_z)\big((f)_y - (f)_z\big)^2 \\ &= \sum_{x \in T} \frac{R_x}{2} \sum_{y,z \in S(x)} \frac{1}{\rho_y \rho_z}\big((f)_y - (f)_z\big)^2 \end{aligned} \tag{4.4}$$

where $(f)_y = \nu(\Sigma_y)^{-1} \int_{\Sigma_y} f(x)\nu(dx)$.

Remark. By Theorem 3.8, we see that

$$\frac{\lambda_x}{2\nu(\Sigma_x)}\nu(\Sigma_y)\nu(\Sigma_z) = \frac{R_x}{2\rho_y\rho_z}$$

for any $x \in T$ and any $y, z \in S(x)$.

The statement of the above theorem includes $\mathcal{E}(H(f), H(f)) < +\infty$ if and only if the right-hand side of (4.4) is finite.

Definition 4.5. (1) Define $|x| = |\phi, x|$ for any $x \in T$. Let $\overline{\phi x} = (x_0, \ldots, x_n)$. Then define $[x]_m = x_m$ for any $m = 0, 1, \ldots, n = |x|$.
(2) Let $\omega = (x_0, x_1, x_2, \ldots) \in \Sigma$. Define $[\omega]_n = x_n$ for $n \geq 0$.
(3) For $\omega, \tau \in \Sigma$ with $\omega \neq \tau$, define $N(\omega, \tau) = \max\{n | [\omega]_n = [\tau]_n\}$ and $[\omega, \tau] = [\omega]_{N(\omega,\tau)}$. We use $[\omega, \tau]_m$ to denote $[[\omega, \tau]]_m$.

Using Theorem 4.4, we may realize aspects of the nature of the quadratic form $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ as follows.

Theorem 4.6. (1) *$(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ is a regular Dirichlet form on $L^2(\Sigma, \nu)$.*
(2) *Define $J : (\Sigma \times \Sigma)\backslash\Delta \to [0, \infty)$ by*

$$J(\omega, \tau) = \frac{1}{2}\left(\lambda_\phi + \sum_{m=0}^{N(\omega,\tau)-1} \frac{\lambda_{[\omega,\tau]_{m+1}} - \lambda_{[\omega,\tau]_m}}{\nu(\Sigma_{[\omega,\tau]_{m+1}})}\right) \tag{4.5}$$

for any $\omega, \tau \in \Sigma$ with $\omega \neq \tau$, where $\Delta = \{(\omega, \omega) | \omega \in \Sigma\}$. Then

$$\mathcal{F}_\Sigma = \left\{u \middle| u \in L^2(\Sigma, \nu), \int_{\Sigma\times\Sigma} J(\omega, \tau)(u(\omega) - u(\tau))^2 \nu(d\omega)\nu(d\tau) < +\infty\right\}$$

and, for any $u, v \in \mathcal{F}_\Sigma$,

$$\mathcal{E}_\Sigma(u, v) = \int_{\Sigma\times\Sigma} J(\omega, \tau)(u(\omega) - u(\tau))(v(\omega) - v(\tau))\nu(d\omega)\nu(d\tau).$$

$J(\omega, \tau)$ is called the jump kernel of $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$.
(3) *Let L_Σ be the non-negative self-adjoint operator on $L^2(\Sigma, \nu)$ associated with $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ on $L^2(\Sigma, \nu)$. Define*

$$E_x = \{\varphi | \varphi = \sum_{y \in S(x)} a_y \chi_{\Sigma_y}, \sum_{y \in S(x)} a_y/\rho_y = 0\}$$

for $x \in T$. Then E_x is contained in the domain of L_Σ and $L_\Sigma\varphi = \lambda_x\varphi$ for any $\varphi \in E_x$. Moreover, let $\{\varphi_{x,i}\}_{i=1,\ldots,\#(S(x))-1}$ be a orthonormal base of E_x with respect to $L^2(\Sigma, \nu)$-inner product. Then $\{\chi_\Sigma, \varphi_{x,i} | x \in T, i = 1, \ldots, \#(S(x))-1\}$ is a complete orthonormal system of $L^2(\Sigma, \nu)$.

Remark. By Lemma 4.3, $J(\omega, \tau) > 0$ for any $(\omega, \tau) \in (\Sigma \times \Sigma)\backslash\Delta$.

We will prove Theorem 4.6 in Section 9 as a special case of generalized framework. Let us forget (T, C) for the moment and think T as a non-direct graph. Namely, we fix $\phi \in T$ and choose an equivalence class with respect to $\underset{G}{\sim}$. Then we have $\pi : T \to T$, Σ, $S(x)$, Σ_x and so on as we remarked after Proposition 3.5. Let μ be a Borel regular probability measure on Σ which satisfies $\mu(\Sigma_x) > 0$ for any $x \in T$ and let $\lambda : T \to [0, \infty)$. Define a quadratic form

$$\mathcal{Q}(u, v) = \sum_{x \in T} \frac{\lambda(x)}{2\mu(\Sigma_x)} \sum_{y,z \in S(x)} \mu(\Sigma_y)\mu(\Sigma_z)((u)_{y,\mu} - (u)_{z,\mu})((v)_{y,\mu} - (v)_{z,\mu}),$$

where $(f)_{x,\mu} = \mu(\Sigma_x)^{-1} \int_{\Sigma_x} f(y)\mu(dy)$. Set $\mathcal{D} = \{f | f \in L^2(\Sigma, \mu), \mathcal{Q}(f, f) < +\infty\}$. We study this quadratic form $(\mathcal{Q}, \mathcal{D})$ in Section 9. For example, $(\mathcal{Q}, \mathcal{D})$ will be shown to be a regular Dirichlet form on $L^2(\Sigma, \mu)$ if the counterpart of $J(\omega, \tau)$ is non-negative for any $\omega, \tau \in \Sigma$ with $\omega \neq \tau$. Moreover, let L be the associated non-negative self-adjoint operator on $L^2(\Sigma, \mu)$. Define $E_{x,\mu} = \{\sum_{y \in S(x)} a_y \chi_{\Sigma_y} | \sum_{y \in S(x)} a_y \mu(\Sigma_y) = 0\}$. Then we will see that $L\varphi = \lambda(x)\varphi$ for any $\varphi \in E_x$. See Section 9 for details.

The rest of this section is devoted to proving Theorem 4.4.

Lemma 4.7. *Let $\overline{\phi x} = (x_0, \ldots, x_n)$, where $x_0 = \phi$ and $x_n = x$. Then*

$$\nu(\Sigma_x) = \eta^x_{x_0} \cdots \eta^x_{x_{n-2}} \times \frac{R^x_{x_{n-1}}}{R^x_{x_{n-1}} + R^{x_0}_x + r_{x_{n-1}x_n}} \tag{4.6}$$

Proof. We use induction in $|x|$. First if $|x| = 1$, then by Theorem 3.8,

$$\nu(\Sigma_x) = \frac{R_\phi}{\rho_x}.$$

Since $(R_\phi)^{-1} = (r_{\phi x} + R_x)^{-1} + (R^x_\phi)^{-1}$, we have $\nu(\Sigma_x) = R^x_\phi/(R^x_\phi + \rho_x)$. This implies (4.6) in this case. Now it is enough to show that

$$M_y = \frac{R_{\pi(y)}}{\rho_y} M_{\pi(y)} \tag{4.7}$$

for any $y \in T$ with $|y| \geq 2$, where M_y is the right-hand side of (4.6). Let $x = \pi(y)$ and $z = \pi(x)$. Then,

$$\frac{M_y}{M_x} = \frac{R^y_z}{R^y_z + r_{zx}} \frac{R^y_x}{R^y_x + R^x_y + r_{xy}} \times \frac{R^x_z + R^z_x + r_{zx}}{R^x_z}. \tag{4.8}$$

Note that $R^y_z = R^x_z$. Define $T_0 = \{x\} \cup T \backslash (T^z_x \cup T^y_x)$ and let C_0 be the restriction of C onto T_0. Set $\tau_x = R_x(T_0, C_0)$, i.e. τ_x is the resistance between x and the infinity $I_{(T_0,C_0)}$. Set $\tau_y = r_{xy} + R^x_y$ and $\tau_z = r_{zx} + R^x_z$. (See Figure 4. Three infinities $I_{(T_0,C_0)}, I_{(T^x_y,C^x_y)}$ and $I_{(T^x_z,C^x_z)}$ are identified as the infinity $I_{(T,C)}$.) Then it follows that $R^y_x = \tau_x\tau_z/(\tau_z + \tau_x)$ and $R^z_x = \tau_y\tau_x/(\tau_y + \tau_x)$. Applying these to (4.8), we have $M_y/M_x = \tau_x/(\tau_y + \tau_x)$. Since $R_x = \tau_x\tau_y/(\tau_x + \tau_y)$ and $\tau_y = \rho_y$, (4.7) follows. □

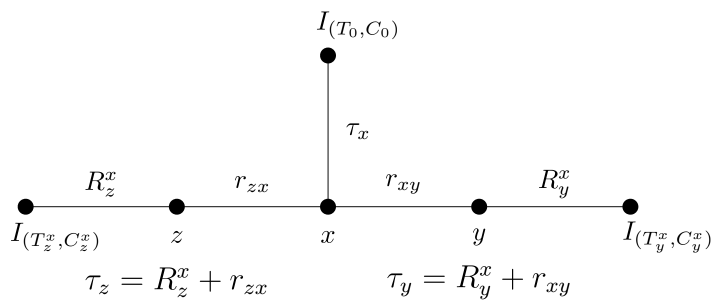


Figure 4: Calculation of M_y/M_x

Theorem 4.8. *For any* $(x, y) \in T_\# \times \widehat{T}$,

$$K(x, y) = \eta_x K(\pi(x), y) + (1 - \eta_x)\frac{1}{\nu(\Sigma_x)}\chi_{\widehat{T}_x}(y).$$

Proof. Let $\overline{\phi x} = (x_0, \ldots, x_{n-1}, x_n)$ where $x_0 = \phi$ and $x_n = x$. Write $z = x_{n-1}(= \pi(x))$. If $x \notin \widehat{T}_x$, then Theorem 3.13 yields $K(x, y) = \eta_x K(z, y)$. Suppose $x \in \widehat{T}_x$. Again by Theorem 3.13,

$$\begin{aligned} K(x, y) &= (\eta_z^x)^{-1} K(z, y) = \eta_x K(z, y) + \big((\eta_z^x)^{-1} - \eta_x\big)K(z, y) \\ &= \eta_x K(z, y) + \Big(\frac{R_z^x + r_{zx}}{R_z^x} - \frac{R_x^z}{R_x^z + r_{zx}}\Big)K(z, y) \\ &= \eta_x K(z, y) + (1 - \eta_x)\frac{R_z^x + R_x^z + r_{zx}}{R_z^x}(\eta_\phi^x \eta_{x_1}^x \cdots \eta_{x_{n-2}}^x)^{-1} \end{aligned}$$

Now Lemma 4.7 implies $K(x, y) = \eta_x K(z, y) + (1 - \eta_x)/\nu(\Sigma_x)$. ☐

Corollary 4.9. *For any* $f \in L^1(\Sigma, \nu)$,

$$H(f)(x) = \begin{cases} (f)_\phi = \int_\Sigma f d\nu & \textit{if } x = \phi, \\ \eta_x H(f)(\pi(x)) + (1 - \eta_x)(f)_x & \textit{if } x \neq \phi. \end{cases} \tag{4.9}$$

Moreover, let $f, u \in L^1(\Sigma, \nu)$. *If* $H(f)(x) = H(u)(x)$ *for any* $x \in T$, *then* $f(\omega) = u(\omega)$ *for* ν*-almost every* $\omega \in \Sigma$.

Proof. (4.9) is direct from Theorem 4.8. Using (4.9) inductively, we see that $(f)_x = (u)_x$ for any $x \in T$. Therefore, $f(\omega) = u(\omega)$ for ν-almost every $\omega \in \Sigma$. ☐

Definition 4.10. Define $W_m = \{x|x \in T, |x| = m\}$ for $m \geq 0$. For $f \in L^1(\Sigma, \nu)$, define $[f]_m = \sum_{x \in W_m} (f)_x \chi_{\Sigma_x}$, $\mathcal{E}_0(f) = 0$ and, for $m \geq 1$,

$$\mathcal{E}_m(f) = \sum_{x \in \cup_{k=1}^m W_k} \frac{1}{r_x} \big(H(f)(\pi(x)) - H(f)(x)\big)^2.$$

Corollary 4.9 implies the following lemma.

Lemma 4.11. *For any $m \geq 1$ and any $f \in L^1(\Sigma, \nu)$,*

$$\begin{aligned}\mathcal{E}_m(f) &= \mathcal{E}_{m-1}(f) + \sum_{x \in W_m} \frac{r_x}{(\rho_x)^2} \big(H(f)(\pi(x)) - (f)_x\big)^2 \\ &= \mathcal{E}_{m-1}(f) + \sum_{x \in W_{m-1}} \sum_{y \in S(x)} \frac{r_y}{(\rho_y)^2} \big(H(f)(x) - (f)_y\big)^2\end{aligned}$$

Definition 4.12. Let $(\mathcal{E}_x, \mathcal{F}_x)$ be the resistance form associated with (T_x, C_x) for $x \in T$. Define

$$R_{k,x} = \Big(\inf\{\mathcal{E}_x(u,u)|u : T_x \to \mathbb{R}, u(x) = 1, u(y) = 0 \text{ if } |y| \geq |x| + k + 1\}\Big)^{-1}$$

for $k \geq 1$ and $R_{0,x} = 0$.

Lemma 4.13. (1) $\lim_{k \to \infty} R_{k,x} = R_x$.
(2) *For any $k \geq 1$ and any $x \in T$,*

$$\frac{1}{R_{k,x}} = \sum_{y \in S(x)} \frac{1}{r_y + R_{k-1,y}}$$

Proof. (1) Note that (T_x, C_x) is transient. Define $\mathcal{F}_{*,x} = (C_0(T_x))_{\mathcal{E}_x} + 1$, where $\mathcal{E}_x(u,v) = \mathcal{E}(u,v) + u(x)v(x)$. Write $I_x = I_{(T_x, C_x)}$. Recall that $(\mathcal{E}_x, \mathcal{F}_{*,x})$ is a resistance form on $T_x \cup \{I_x\}$ and that

$$R_x = \Big(\min\{\mathcal{E}_x(u,u)|u \in \mathcal{F}_{*,x}, u(x) = 1, u(I_x) = 0\}\Big)^{-1}.$$

Hence $R_x \geq R_{k,x}$. There exists $\psi \in \mathcal{F}_{*,x}$ such that $\psi(x) = 1$ and $\psi(I_x) = 0$ and $\mathcal{E}_x(\psi, \psi) = 1/R_x$. Since $\psi(I_x) = 0$, $\psi \in (C_0(T_x))_{\mathcal{E}_x}$. Hence there exists $\{\psi_n\}_{n \geq 1} \in C_0(T_x)$ such that $\psi_n(x) = 1$ and $\mathcal{E}_x(\psi_n, \psi_n) \to 1/R_x$ as $n \to \infty$. Choose k_n so that $\text{supp}(\psi_n) \subseteq \{y|y \in T_x, |y| \geq |x| + k_n + 1\}$. Then $\mathcal{E}_x(\psi_n, \psi_n)^{-1} \leq R_{k_n,x}$. Since $k_n \to \infty$ as $n \to \infty$ and $R_{k,x} \leq R_{k+1,x}$ for any k, it follows that $R_x \leq \lim_{k \to \infty} R_{k,x} \leq R_x$.
(2) This follows by applying the formula of combined resistance. $\square$

Lemma 4.14. *Let $1 \leq m \leq n$.*

$$\min_{g: g \in L^1(\Sigma, \nu), [g]_m = [f]_m} \mathcal{E}_n(g) = \mathcal{E}_m(f) + \sum_{x \in W_m} \frac{R_{n-m,x}}{(R_x)^2} \big(H(f)(x) - (f)_x\big)^2 \quad (4.10)$$

for any $f \in L^1(\Sigma, \nu)$.

Proof. We use an inductive argument. If $[f]_n = [g]_n$, then $\mathcal{E}_n(f) = \mathcal{E}_n(g)$ by (4.9). Hence (4.10) holds for $n = m$. Assume that we have (4.10) for m. Choose $u \in L^1(\Sigma, \nu)$ so that $[u]_{m-1} = [f]_{m-1}$. By (4.9) and the induction hypothesis,

$$\min_{g:g\in L^1(\Sigma,\nu),[g]_m=[u]_m} \mathcal{E}_n(g) = \mathcal{E}_m(u) + \sum_{x\in W_{m-1}} \sum_{y\in S(x)} \frac{R_{n-m,y}}{(R_y)^2}\big(H(u)(y) - (u)_y\big)^2$$
$$= \mathcal{E}_{m-1}(u) + \sum_{x\in W_{m-1}} \sum_{y\in S(x)} \frac{r_y + R_{n-m,y}}{(\rho_y)^2}\big(H(u)(x) - (u)_y\big)^2$$
$$= \mathcal{E}_{m-1}(f) + \sum_{x\in W_{m-1}} \sum_{y\in S(x)} \frac{r_y + R_{n-m,y}}{(\rho_y)^2}\big(H(f)(x) - (u)_y\big)^2$$

Now, we consider the minimum of the last expression subject to the constraint $[u]_{m-1} = [f]_{m-1}$, i.e.

$$(f)_x = \sum_{y\in S(x)} \frac{R_x}{\rho_y}(u)_y$$

for any $x \in W_{m-1}$. The method of Lagrange multipliers along with Lemma 4.13-(2) shows

$$\min_{u:[u]_{m-1}=[f]_{m-1}} \Big(\min_{g:[g]_m=[u]_m} \mathcal{E}(g)\Big)$$
$$= \mathcal{E}_{m-1}(f) + \sum_{x\in W_{m-1}} \frac{1}{(R_x)^2}\Big(\sum_{y\in S(x)} \frac{1}{r_y + R_{n-m,y}}\Big)^{-1}\big(H(f)(x) - (f)_x\big)^2$$
$$= \mathcal{E}_{m-1}(f) + \sum_{x\in W_{m-1}} \frac{R_{n-m+1,y}}{(R_x)^2}\big(H(f)(x) - (f)_x\big)^2.$$

□

Lemma 4.15. *For any $f \in L^1(\Sigma, \nu)$,*

$$\frac{(f)_x}{R_x}(H(f)(x) - (f)_x) - \sum_{y\in S(x)} \frac{(f)_y}{R_y}(H(f)(y) - (f)_y)$$
$$= \frac{R_x}{2} \sum_{y,z\in S(x)} \frac{1}{\rho_y\rho_z}\big((f)_y - (f)_z\big)^2. \quad (4.11)$$

Proof. Note that $(f)_x = \sum_{y\in S(x)}(R_x/\rho_y)(f)_y$. Also by (4.9), if $y \in S(x)$, then

$H(f)(y) - (f)_y = \eta_y(H(f)(x) - (f)_y)$. By those relations,

$$
\begin{aligned}
&\frac{(f)_x}{R_x}(H(f)(x) - (f)_x) - \sum_{y \in S(x)} \frac{(f)_y}{R_y}(H(f)(y) - (f)_y) \\
=&\Big(\sum_{y \in S(x)} \frac{1}{\rho_y}(f)_y \Big)(H(f)(x) - (f)_x) - \sum_{y \in S(x)} \frac{(f)_y}{\rho_y}(H(f)(x) - (f)_y) \\
=& - R_x \Big(\sum_{y \in S(x)} \frac{1}{\rho_y}(f)_y \Big)^2 + \sum_{y \in S(x)} \frac{1}{\rho_y}((f)_y)^2 \\
=& \sum_{y \in S(x)} \Big(\frac{1}{\rho_y} - \frac{R_x}{(\rho_y)^2} \Big)((f)_y)^2 - R_x \sum_{y,z \in S(x), y \neq z} \frac{1}{\rho_y \rho_z}(f)_y(f)_z \\
=& R_x \sum_{y \in S(x)} \frac{1}{\rho_y} \Big(\sum_{z \in S(x), z \neq y} \frac{1}{\rho_z} \Big)((f)_y)^2 - R_x \sum_{y,z \in S(x), y \neq z} \frac{1}{\rho_y \rho_z}(f)_y(f)_z \\
=& \frac{R_x}{2} \sum_{y,z \in S(x)} \frac{1}{\rho_y \rho_z}\big((f)_y - (f)_z\big)^2.
\end{aligned}
$$

□

Lemma 4.16. *Let $f \in L^1(\Sigma, \nu)$. For any $m \geq 0$,*

$$
- \sum_{x \in W_m} \frac{(f)_x}{R_x}(H(f)(x) - (f)_x) = \sum_{x \in \cup_{j=0}^{m-1} W_j} \frac{R_x}{2} \sum_{y,z \in S(x)} \frac{1}{\rho_y \rho_z}\big((f)_y - (f)_z\big)^2. \tag{4.12}
$$

For $m = 0$, the right-hand side of (4.12) *is considered as* 0.

Proof. Since $H(f)(\phi) = (f)_\phi$, we have (4.12) for $m = 0$. Suppose that (4.12) holds for $m = 0, 1, \ldots, n-1$. Then Lemma 4.15 suffices to show (4.12) for $m = n$. □

Lemma 4.17. *Let $f \in L^1(\Sigma, \nu)$. Then*

$$
\mathcal{E}_m(f) + \sum_{x \in W_m} \frac{1}{R_x}\big(H(f)(x) - (f)_x\big)^2 = \sum_{x \in \cup_{j=0}^{m-1} W_j} \frac{R_x}{2} \sum_{y,z \in S(x)} \frac{1}{\rho_y \rho_z}\big((f)_y - (f)_z\big)^2.
$$

Proof. Define $V_m = (\cup_{j=1}^m W_j) \cup (\cup_{x \in W_m}\{I_x\})$, where $I_x = I_{(T_x, C_x)}$. For $u \in \ell(V_m)$, define

$$
\widetilde{\mathcal{E}}_m(u,u) = \sum_{x \in \cup_{j=1}^m W_m} \frac{1}{r_x}\big(u(x) - u(\pi(x))\big)^2 + \sum_{x \in W_m} \frac{1}{R_x}(u(x) - u(I_x))^2.
$$

Then $(\widetilde{\mathcal{E}}_m, \ell(V_m))$ is a resistance form on V_m. Let L_m be the associated Laplacian, i.e. non-positive definite symmetric operator from $\ell(V_m)$ to $\ell(V_m)$ which satisfies $\mathcal{E}(u,v) = -\sum_{x \in V_m} u(x)(L_m v)(x)$. Define $F : V_m \to \mathbb{R}$ by $F(x) =$

$H(f)(x)$ for $x \in \cup_{j=0}^{m} W_j$ and $F(I_x) = (f)_x$ for $x \in W_m$. By (4.9), for any $x \in W_m$,

$$\begin{aligned}(L_m F)(x) &= \frac{1}{R_x}(F(I_x) - F(x)) + \frac{1}{r_x}(F(\pi(x)) - F(x)) \\ &= \frac{1}{R_x}((f)_x - H(f)(x)) + \frac{1}{r_x}(H(f)(\pi(x)) - H(f)(x)) = 0.\end{aligned}$$

In the same manner, it follows that $(L_m F)(x) = 0$ for any $x \in \cup_{j=0}^{m} W_m$. Therefore,

$$\widetilde{\mathcal{E}}_m(F, F) = -\sum_{x \in W_m} F(I_x)(L_m F)(I_x) = -\sum_{x \in W_m} \frac{(f)_x}{R_x}(H(f)(x) - (f)_x),$$

which equals to the left-hand side of (4.12). On the other hand,

$$\widetilde{\mathcal{E}}_m(F, F) = \mathcal{E}_m(f) + \sum_{x \in W_m} \frac{1}{R_x}\big(H(f)(x) - (f)_x\big)^2$$

by definition. Combining these facts with Lemma 4.16, we immediately deduce the desired equation. □

Proof of Theorem 4.4. By Lemma 4.14,

$$\mathcal{E}_n(f) \geq \mathcal{E}_m(f) + \sum_{x \in W_m} \frac{R_{n-m,x}}{(R_x)^2}\big(H(f)(x) - (f)_x\big)^2$$

for any $n \geq m \geq 1$. Applying Lemma 4.13-(1) and Lemma 4.17, we have

$$\mathcal{E}(H(f), H(f)) \geq \sum_{x \in \cup_{j=0}^{m-1} W_j} \frac{R_x}{2} \sum_{y,z \in S(x)} \frac{1}{\rho_y \rho_z}\big((f)_y - (f)_z\big)^2 \geq \mathcal{E}_m(f)$$

for any $m \geq 1$. Since $\mathcal{E}_m(f) \to \mathcal{E}(H(f), H(f))$ as $m \to \infty$, we immediately verify the theorem. □

5 Intrinsic metric and volume doubling property

In the last section, we have introduced $\{D_x\}_{x \in T}$ and shown that D_{x_n} is monotonically decreasing if $\omega = (x_0, x_1, \dots) \in \Sigma$. In this section, we construct an ultra-metric d on Σ where the diameter of Σ_x equals to D_x for any $x \in T$ and provide a simple condition which is equivalent to the volume doubling property of ν with respect to this ultra-metric d. The ultra-metric d will turn out to be suitable for describing asymptotic behaviors of the Hunt process associated with the regular Dirichlet form $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ on $L^2(\Sigma, \nu)$ in the next section.

In this section, (T, C) is a transient tree. We assume that (T_y^x, C_y^x) is transient for any $x, y \in T$. We fix a reference point $\phi \in X$ and use the same notation as in the previous sections. To avoid nonessential complications, we further assume that $\#(S(x)) \geq 2$ for any $x \in T$. Even without this assumption, the statements in this and following sections, except Corollary 6.9, hold with minor modification. This assumption implies the following proposition.

Proposition 5.1. $(\Sigma, \mathcal{O}_\Sigma)$ *has no isolated point and is a Cantor set, i.e. compact, perfect and totally disconnected.*

To describe the next theorem, we use the notations from Definition 2.8. Recall that $\mathcal{F}_* = (C_0(T))_{\mathcal{E}_\phi} + \mathbb{R}$ and that g_* is the symmetrized Green function of (T, C).

Theorem 5.2. (1) $g_*(\phi, x) = D_x$ *for any* $x \in T$.
(2) $\sum_{x \in W_n} \nu(\Sigma_x) D_x \to 0$ *as* $n \to \infty$.
(3) $D_{[\omega]_n} \to 0$ *as* $n \to \infty$ *for* ν*-almost every* $\omega \in \Sigma$.

Notation. We write $D_{n,\omega} = D_{[\omega]_n}$.

We have an example of (T, C) where $lim_{n\to\infty} D_{[\omega]_n} > 0$ for some $\omega \in \Sigma$ in Section 11.

Proof. (1) Define $\psi(x) = g_*(\phi, x)$. Then ψ is a $\{\phi, I\}$-harmonic function with boundary value $\psi(\phi) = R_\phi$ and $\psi(I) = 0$. Let $(\mathcal{E}_m, \mathcal{F}_m)$ be the trace of $(\mathcal{E}, \mathcal{F}_*)$ on $T_m \cup \{I\}$, where $T_m = \cup_{n=0}^m W_m$ and let L_m be the associated discrete Laplacian on $W_m \cup \{I\}$. Note that $\psi|_{T_m \cup \{I\}}$ is also a $\{\phi, I\}$-harmonic function with respect to $(\mathcal{E}_m, \mathcal{F}_m)$. Hence, for $x \in W_m$,

$$L_m\psi(x) = \frac{\psi(\pi(x)) - \psi(x)}{r_x} + \frac{\psi(\phi) - \psi(x)}{R_x} = 0$$

Since $\psi(\phi) = 0$, we obtain

$$\psi(x)/\psi(\pi(x)) = D_x/D_{\pi(x)} \tag{5.1}$$

by (4.2). As $\psi(\phi) = R_\phi = D_\phi$, (5.1) implies $\psi(x) = D_x$ on W_m for any $m \geq 0$ inductively.
(2) Let $(E_m, \ell(T_m))$ be the resistance form on a finite set T_m associated with $(T_m, C|_{T_m})$ and let H_m be the associated (discrete) Laplacian. Then by (4.3)

$$\begin{aligned} E_m(\psi, \psi) &= -\sum_{x \in T_m} \psi(x)(H_m\psi)(x) \\ &= -R_\phi \sum_{y \in S(\phi)} \frac{D_y - D_\phi}{r_y} - \sum_{x \in T_m} D_x \frac{D_{\pi(x)} - D_x}{r_x} \\ &= R_\phi - \sum_{x \in T_m} D_x \nu(\Sigma_x). \end{aligned} \tag{5.2}$$

Note that $E_m(\psi,\psi) \to \mathcal{E}(\psi,\psi) = R_\phi$ as $m \to \infty$. Hence by (5.2), we have desired result.
(3) Define $\psi_n : \Sigma \to \mathbb{R}$ by $\psi_n = \sum_{x \in W_n} D_x \chi_{\Sigma_x}$. By (4.2), ψ_n is positive and monotonically decreasing as $n \to \infty$. By (2),

$$\int_\Sigma \psi_n(\omega)\nu(d\omega) = \sum_{x \in T_m} D_x \nu(\Sigma_x) \to 0$$

as $n \to \infty$. This immediately implies that $\psi_n(\omega) \to 0$ as $n \to \infty$ for ν-almost every $\omega \in \Sigma$. Since $\psi_n(\omega) = D_{n,\omega}$, we have completed our proof. □

Now we define an ultra-metric on the Cantor set Σ by means of D_x.

Definition 5.3. Define

$$d(\omega,\tau) = \begin{cases} D_{[\omega,\tau]} & \text{if } \omega \neq \tau, \\ 0 & \text{if } \omega = \tau. \end{cases}$$

Proposition 5.4. (1) *$d(\cdot,\cdot)$ is a metric on Σ. Moreover it is an ultra-metric, i.e.*

$$\max\{d(\omega,\tau), d(\tau,\eta)\} \geq d(\omega,\tau). \tag{5.3}$$

for any $\omega,\tau,\eta \in \Sigma$.
(2) *$\max_{\tau\in\Sigma} d(\omega,\tau) = R_\phi$ for any $\omega \in \Sigma$.*
(3) *Define $B(\omega,r) = \{\tau | d(\omega,\tau) < r\}$ for any $\omega \in \Sigma$ and $r > 0$. $B(\omega,r) = \Sigma_{[\omega]_n}$ if and only if $D_{[\omega]_n} < r \leq D_{[\omega]_{n-1}}$.*
(4) *The identity from (Σ,d) to $(\Sigma,\mathcal{O}_\Sigma)$ is continuous. Moreover, (Σ,d) is homeomorphic to $(\Sigma,\mathcal{O}_\Sigma)$ if and only if $D_{[\omega]_n} \to 0$ as $n \to \infty$ for any $\omega \in \Sigma$.*

Proof. (1) It is enough to show (5.3), which implies the triangle inequality. Other properties of metric are immediate. If $T_{[\omega,\eta]} \subseteq T_{[\omega,\tau]}$, then multiple applications of (4.2) show that $d(\omega,\eta) < d(\omega,\tau)$. Hence we have (5.3). Otherwise, $d(\omega,\eta) = d(\tau,\eta)$ and (5.3) holds.
(2) For any $\omega \in \Sigma$, we may choose $\tau \in \Sigma$ so that $[\omega,\tau] = \phi$. Then $d(\omega,\tau) = D_\phi = R_\phi$.
(3) Fix $\omega \in \Sigma$. Then $d(\omega,\tau) \in \{D_{m,\omega} | m \geq 0\}$. Moreover, $d(\omega,\tau) < D_{n-1,\omega} \Leftrightarrow d(\omega,\tau) \leq D_{n,\omega} \Leftrightarrow \tau \in \Sigma_{[\omega]_n}$. Therefore, $B(\omega,r) = \Sigma_{[\omega]_n}$ if and only if $D_{n,\omega} < r \leq D_{n-1,\omega}$.
(4) Note that $\{\Sigma_x\}_{x\in T}$ is a fundamental system of neighborhoods of $\mathcal{O}_\Sigma$. For any $x \in T_\#$, choose $\omega \in \Sigma_x$. If $r = D_x$, then $B(\omega,r) = \Sigma_x$. Hence Σ_x is open with respect to (Σ,d) for any $x \in T$. This implies the continuity of the identity from (Σ,d) to $(\Sigma,\mathcal{O}_\Sigma)$. Assume that $D_{n,\omega} \to 0$ as $n \to \infty$ for any $\omega \in \Sigma$. Let U be an open set with respect to (Σ,d). For any $\omega \in U$, $B(\omega,r) \subseteq U$ for some $r > 0$. By the assumption, there exists n such that $D_{n,\omega} < r$ and hence $\Sigma_{[\omega]_n} \subseteq U$. Therefore, U is an open set with respect to $\mathcal{O}_\Sigma$. This shows that (Σ,d) and $(\Sigma,\mathcal{O}_\Sigma)$ are homeomorphic.

Finally assume $D_{n,\omega} \to D > 0$ for some $\omega \in \Sigma$. If $0 < r < D$, then $B(\omega,r) = \{\omega\}$. Therefore $\{\omega\}$ is an open set with respect to (Σ,d). On the other hand, $\{\omega\} \notin \mathcal{O}_\Sigma$ by Proposition 5.1. Hence (Σ, d) is not homeomorphic to $(\Sigma, \mathcal{O}_\Sigma)$. $\square$

The following theorem gives necessary and sufficient condition for ν to have the volume doubling property with respect to the metric d.

Theorem 5.5. *ν has the volume doubling property with respect to d, i.e., there exists $c > 0$ such that $\nu(B(x, 2r)) \le c\nu(B(x, r))$ for any $x \in \Sigma$ and any $r > 0$, if and only if the following two conditions* $(\mathrm{EL})_\nu$ *and* (D) *hold:*
$(\mathrm{EL})_\nu$: *There exists $c_1 \in (0, 1)$ such that $c_1 \le \nu(\Sigma_x)/\nu(\Sigma_{\pi(x)})$ for any $x \in T_\#$.*
(D): *For any $n \ge 0$ and any $\omega \in \Sigma$,*

$$D_{[\omega]_{n+m}} \le \alpha D_{[\omega]_n},$$

where $m \ge 1$ and $\alpha \in (0, 1)$ are independent of n and ω.

By the condition (D), $D_{[\omega]_n} \to 0$ for any $\omega \in \Sigma$ if ν has the volume doubling property with respect to d. Also the number of neighboring vertices is shown to be uniformly bounded under the volume doubling property as follows.

Proposition 5.6. *If* $(\mathrm{EL})_\nu$ *hold, then* $\sup_{x \in T, y \in S(x)} \nu(\Sigma_y)/\nu(\Sigma_x) < 1$ *and* $\sup_{x \in T} \#(S(x)) < +\infty$.

Proof. If $\#(S(x)) > 1$, then $c_1 \le \sum_{z \in S(x), z \ne y} \nu(\Sigma_z)/\nu(\Sigma_x) = 1 - \nu(\Sigma_y)/\nu(\Sigma_x)$. Hence $\nu(\Sigma_y)/\nu(\Sigma_x) \le 1 - c_1$. Moreover, $c_1\#(S(x)) \le \sum_{y \in S(x)} \nu(\Sigma_y)/\nu(\Sigma_x) = 1$. This shows $\#(S(x)) \le 1/c_1$. $\square$

The following alternative definition of the volume doubling property is sometimes useful.

Proposition 5.7. *ν has the volume doubling property with respect to d if and only if there exist $c > 0$ and $\alpha \in (0, 1)$ such that $\nu(B(\omega, r)) \le c\nu(B(\omega, \alpha r)$ for any $\omega \in \Sigma$ and any $r \in (0, R_\phi]$.*

Proof of Theorem 5.5. Assume the volume doubling property. Then,

$$\nu(B(\omega, D_{n,\omega} + \epsilon)) \le c\nu(B(\omega, D_{n,\omega}/2 + \epsilon/2))$$

Choose ϵ so that $D_{n,\omega} + \epsilon < D_{n-1,\omega}$ and $D_{n,\omega}/2 + \epsilon/2 \le D_{n,\omega}$. Proposition 5.4-(3) shows $\nu(\Sigma_{[\omega]_n}) \le c\nu(B(\omega, D_{n,\omega})) = c\nu(\Sigma_{[\omega]_{n+1}})$. If $x \in T$ and any $y \in S(x)$, then $x = [\omega]_n$ and $y = [\omega]_{n+1}$ for some $\omega \in \Sigma$. Hence $1/c \le \nu(\Sigma_y)/\nu(\Sigma_x)$. Thus we have $(\mathrm{EL})_\nu$. By Proposition 5.6, there exists $\gamma \in (0, 1)$ such that $\nu(\Sigma_y)/\nu(\Sigma_x) \le \gamma$ for any $x \in T$ and any $y \in S(x)$. Choose $m \ge 1$ so that $\gamma^m < 1/c$. For any $\omega \in \Sigma$, we have

$$\nu(B(\omega, D_{n+m,\omega})) = \nu(\Sigma_{[\omega]_{n+m+1}}) \le \gamma^m \nu(\Sigma_{[\omega]_{n+1}}) = \gamma^m \nu(B(\omega, D_{n,\omega}).$$

Using the volume doubling property, we obtain

$$\frac{1}{c}\nu(B(\omega, 2D_{n+m,\omega})) \le \nu(B(\omega, D_{n+m,\omega})) \le \gamma^m \nu(B(\omega, D_{n,\omega}).$$

Hence $D_{n+m,\omega} \leq D_{n,\omega}/2$. Thus we have (D).

Conversely, we assume $(\mathrm{EL})_\nu$ and (D). If $D_{n,\omega} < r \leq D_{0,\omega} = R_\phi$, then we may choose $k \leq n$ such that $D_{k,\omega} < r \leq D_{k-1,\omega}$. Since $D_{k+m,\omega} \leq \alpha D_{k,\omega} < \alpha r$, we have $\nu(\Sigma_{[\omega]_{k+m}}) \leq \nu(B(\omega, \alpha r))$. On the other hand, by $(\mathrm{EL})_\nu$, $\nu(\Sigma_{[\omega]_{k+m}}) \geq (c_1)^m \nu(\Sigma_{[\omega]_k}) = (c_1)^m \nu(B(\omega, r))$. Thus

$$\nu(B(\omega, r) \leq c\nu(B(\omega, \alpha r)) \tag{5.4}$$

for any $r > D_{n,\omega}$, where $c = (c_1)^{-(m+1)}$. By Proposition 5.7, we have the volume doubling property of ν with respect to d. $\square$

6 Asymptotic behaviors of the process

In this section, we study the asymptotic behavior of the heat kernel (transition density), the jump kernel $J(\cdot, \cdot)$ and moments of displacement associated with the Dirichlet form $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ on $L^2(\Sigma, \nu)$ under the assumption that ν is volume doubling with respect to d. As in the last section, (T, C) is a transient tree, (T^x_y, C^x_y) is assumed to be transient for any $x, y \in T$. We fix a reference point ϕ. Moreover, we continue to assume that $\#(S(x)) \geq 2$ for any $x \in T$. All the statements except Corollary 6.9, however, will hold without this assumption.

Making use of Theorem 4.6, we have a formal expression of a heat kernel associated with the Dirichlet form $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ on $L^2(\Sigma, \nu)$ as follows:

$$p(t, \omega, \tau) = 1 + \sum_{x \in T} e^{-\lambda_x t} \sum_{j=1}^{\#(S(x))-1} \varphi_{x,j}(\omega)\varphi_{x,j}(\tau). \tag{6.1}$$

By Lemma 6.1 below,

$$\sum_{j=1}^{\#(S(x))-1} \varphi_{x,j}(\omega)\varphi_{x,j}(\tau) = \sum_{y \in S(x)} \frac{1}{\nu(\Sigma_y)} \chi_{\Sigma_y}(\omega)\chi_{\Sigma_y}(\tau) - \frac{1}{\nu(\Sigma_x)} \chi_{\Sigma_x}(\omega)\chi_{\Sigma_x}(\tau).$$

Combining this with (6.1), we obtain

$$p(t, \omega, \tau) = \begin{cases} 1 + \displaystyle\sum_{n=0}^{\infty} \Big(\frac{1}{\nu(\Sigma_{[\omega]_{n+1}})} - \frac{1}{\nu(\Sigma_{[\omega]_n})} \Big) e^{-\lambda_{[\omega]_n} t} & \text{if } \omega = \tau, \\ \displaystyle\sum_{n=0}^{N(\omega,\tau)} \frac{1}{\nu(\Sigma_{[\omega,\tau]_n})} \big(e^{-\lambda_{[\omega,\tau]_{n-1}} t} - e^{-\lambda_{[\omega,\tau]_n} t} \big) & \text{if } \omega \neq \tau, \end{cases} \tag{6.2}$$

where we define $\lambda_{[\omega]_{-1}} = 0$ and write $[\omega, \tau]_n = [[\omega, \tau]]_n$. If we allow ∞ as a value, $p(t, \omega, \tau)$ is well-defined on $(0, \infty) \times \Sigma^2$ through (6.2). The value ∞ may occur on the diagonal. Also, by (6.2), $p(t, \omega, \tau)$ is continuous on $(0, \infty) \times \big((\Sigma \times \Sigma) \backslash \Delta \big)$, where Δ is the diagonal.

Lemma 6.1. *Let $V = \{1, \ldots, n\}$ and let $\mu : V \to (0, +\infty)$. Define an inner product $(\cdot, \cdot)_\mu$ of $\ell(V)$ by $(u, v)_\mu = \sum_{k \in V} \mu(k)u(k)v(k)$ for any $u, v \in \ell(V)$. If $(\varphi_1, \ldots, \varphi_{n-1})$ is a orthonormal base of*

$$\mathcal{L}_\mu = \{u | u \in \ell(V), (u, \chi_V)_\mu = 0\}$$

with respect to $(\cdot, \cdot)_\mu$, then

$$\sum_{i=1}^{n-1} \varphi_i(k)\varphi_i(m) = \sum_{j=1}^{n} \frac{\chi_j(k)\chi_j(m)}{\mu(j)} - \frac{1}{\sum_{j=1}^n \mu(j)} = \frac{\delta_{km}}{\mu(k)} - \frac{1}{\sum_{j=1}^n \mu(j)},$$

where δ_{km} is the Kronecker delta.

Proof. We use induction in n. To distinguish between different n's, we write V_n and $\mathcal{L}_{n,\mu}$ instead of V and $\mathcal{L}_\mu$. If $n = 2$, then

$$\varphi_1 = \frac{1}{\sqrt{\mu(1)\mu(2)(\mu(1) + \mu(2))}} \begin{pmatrix} -\mu(2) \\ \mu(1) \end{pmatrix}$$

This implies $\varphi_1(k)\varphi_1(m) = \sum_{j=1}^2 \chi_j(k)\chi_j(m)/\mu(j) - (\mu(1) + \mu(2))^{-1}$.

Suppose the lemma is true for n. Let $\{\varphi_1, \ldots, \varphi_{n-1}\}$ be a orthonormal base of $\mathcal{L}_{n,\mu}$. We extend the domain of φ_i to $\{1, \ldots, n, n+1\}$ by setting $\varphi_i(n+1) = 0$ for any $i = 1, \ldots, n-1$. Then $(\varphi_1, \ldots, \varphi_{n-1})$ is a orthonormal system of $\mathcal{L}_{n+1,\mu}$ as well. Moreover, by the induction hypothesis,

$$\sum_{j=1}^{n-1} \varphi_j(k)\varphi_j(m) = \sum_{j=1}^{n} \frac{\chi_j(k)\chi_j(m)}{\mu(j)} - \frac{1}{\sum_{j=1}^n} \sum_{j=1}^{n} \chi_j(k) \sum_{j=1}^{n} \chi_j(m) \tag{6.3}$$

Define $\varphi_n \in \mathcal{L}_{n+1,\mu}$ by

$$\varphi_n(k) = \sqrt{\frac{\mu(n+1)}{\sum_{j=1}^n \mu(j) \sum_{j=1}^{n+1} \mu(j)}} \times \begin{cases} 1 & \text{if } k \in \{1, \ldots, n\}, \\ -\frac{\sum_{j=1}^n \mu(j)}{\mu(n+1)} & \text{if } k = n+1. \end{cases}$$

Then $(\varphi_1, \ldots, \varphi_n)$ is a orthonormal base of $\mathcal{L}_{n+1,\mu}$. Adding $\varphi_n(k)\varphi_n(m)$ to (6.3), we obtain the statement of the lemma for $n + 1$. □

In fact, the "formal" heat kernel $p(t, \omega, \tau)$ is shown to be a transition density of a Hunt process associated with the regular Dirichlet form $(\mathcal{E}|_\Sigma, \mathcal{F}|_\Sigma)$ under a suitable assumption.

Proposition 6.2. *Assume that $\lim_{n\to\infty} D_{[\omega]_n} = 0$ for any $\omega, \in \Sigma$. Define $p^{t,\omega}(\tau) = p(t, \omega, \tau)$ for any $\omega, \tau \in \Sigma$ and any $t > 0$. Then*

$$\int_\Sigma p^{t,\omega} d\nu = 1 \quad \textit{and} \quad \int_\Sigma p^{t,\omega} p^{s,\tau} d\nu = p(t + s, \omega, \tau) \tag{6.4}$$

for any $\omega, \tau \in \Sigma$ with $\omega \neq \tau$ and any $t, s > 0$. In particular, define $(p_t u)(\omega) = \int_\Sigma p^{t,\omega} u d\nu$ for any Borel measurable bounded function $u : \Sigma \to \mathbb{R}$. Then $\{p_t\}_{t>0}$ is a Markovian transition function in the sense of [5, Section 1.4].

Proof. If $\omega \neq \tau$, then

$$p(t,\omega,\tau) = \sum_{n\geq 0} \frac{e^{-\lambda_{[\omega]_{n-1}}t} - e^{-\lambda_{[\omega]_n}t}}{\nu(\Sigma_{[\omega]_n})} \chi_{\Sigma_{[\omega]_n}}(\tau) \tag{6.5}$$

Note that this is an infinite sum of non-negative functions. Using (6.5), we obtain (6.4) by a routine but careful calculation. The fact that $\{p_t\}_{t>0}$ is a Markovian transition function is immediate from (6.4). □

By this proposition, if $\lim_{n\to\infty} D_{[\omega]_n} = 0$ for any $\omega \in \Sigma$, then $(p_t u)(\omega) = (T_t u)(\omega)$ for ν-a.e. $\omega \in \Sigma$, where $\{T_t\}_{t>0}$ is the strongly continuous semigroup on $L^2(\Sigma,\nu)$ associated with the Dirichlet form $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ on $L^2(\Sigma,\nu)$. Moreover, we have the following theorem.

Theorem 6.3. *If $\lim_{n\to\infty} D_{[\omega]_n} = 0$ for any $\omega \in \Sigma$, then there exists a Hunt process $(\{X_t\}_{t>0}, \{P_\omega\}_{\omega\in\Sigma})$ on Σ whose transition density is $p(t,\omega,\tau)$, i.e.*

$$E_\omega(f(X_t)) = \int_\Sigma p(t,\omega,\tau) f(\tau)\nu(d\tau)$$

for any $\omega \in \Sigma$ and any Borel measurable bounded function $f : \Sigma \to \mathbb{R}$, where $E_\omega(\cdot)$ is the expectation with respect to P_ω.

Note that the Hunt process $(\{X_t\}_{t>0}, \{P_\omega\}_{\omega\in\Sigma})$ is naturally associated with the Dirichlet form $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ on $L^2(\Sigma,\nu)$.

The essence of the proof of Theorem 6.3 is to show that the transition function $\{p_t\}_{t>0}$ is a Feller transition function. Namely we are going to prove that $p_t(C(\Sigma)) \subseteq C(\Sigma)$ and $||p_t u - u||_\infty \to 0$ as $t \downarrow 0$ for any $u \in C(\Sigma)$, where $C(\Sigma)$ is the collection of continuous functions on Σ.

Lemma 6.4. *Let $\mathcal{C}_m = \{\sum_{x\in W_m} a_x \chi_{\Sigma_x} | a_x \in \mathbb{R}$ for any $x \in W_m\}$ and let $\mathcal{C} = \cup_{m\geq 0}\mathcal{C}_m$. Then $||p_t u - u||_\infty \to 0$ as $t \downarrow 0$ for any $u \in \mathcal{C}$.*

Proof. For any $u \in \mathcal{C}_m$, by using an inductive argument, it follows that

$$u = c + \sum_{k=1}^{m} \sum_{x\in W_k} \sum_{i=1}^{\#(S(x))-1} b_{x,i}\varphi_{x,i}.$$

(See Lemma 9.2 for details.) This implies

$$p_t u = c + \sum_{k=1}^{m} \sum_{x\in W_k} \sum_{i=1}^{\#(S(x))-1} e^{-\lambda_x t} b_{x,i}\varphi_{x,i}.$$

Hence there exists $c > 0$ such that $||p_t u - u||_\infty \leq c(1 - e^{-\lambda_\phi t})$. □

Proof of Theorem 6.3. Let $\omega, \xi \in \Sigma$ and let $N = N(\omega, \xi)$. Then by (6.5),

$$|p(t,\omega,\tau) - p(t,\xi,\tau)| \\ = \sum_{n>N} \left(\frac{e^{-\lambda_{[\omega]_{n-1}}t} - e^{-\lambda_{[\omega]_n}t}}{\nu(\Sigma_{[\omega]_n})} \chi_{\Sigma_{[\omega]_n}}(\tau) + \frac{e^{-\lambda_{[\xi]_{n-1}}t} - e^{-\lambda_{[xi]_n}t}}{\nu(\Sigma_{[\xi]_n})} \chi_{\Sigma_{[\xi]_n}}(\tau) \right)$$

Since $\lim_{n\to\infty} D_{[\omega]_n} = 0$ for any $\omega \in \Sigma$, we have

$$\int_\Sigma |p^{t,\omega} - p^{t,\xi}| d\nu = 2e^{-\lambda_{[\omega,\xi]}t}$$

Hence if u is a bounded Borel measurable function on Σ, then

$$|(p_t u)(\omega) - (p_t u)(\tau)| \le 2e^{-\lambda_{[\omega,\xi]}t} ||u||_\infty.$$

Again by the fact that $\lim_{n\to\infty} D_{[\omega]_n} = 0$ for any $\omega \in \Sigma$, we see $p_t u \in C(\Sigma)$. In particular $p_t(C(\Sigma)) \subseteq C(\Sigma)$.

Let $u \in C(\Sigma)$ and fix $\epsilon > 0$. Then there exist $m \ge 0$ and $u_m \in \mathcal{C}_m$ such that $||u - u_m||_\infty < \epsilon/3$. By Lemma 6.4,

$$||p_t u - u||_\infty \le ||p_t u - p_t u_m||_\infty + ||p_t u_m - u_m||_\infty + ||u - u_m||_\infty < \epsilon$$

for sufficiently small $t > 0$. Hence $||p_t u - u||_\infty \to 0$ as $t \downarrow 0$. Thus $\{p_t\}_{t>0}$ is a Feller transition function. Then by [5, Theorem A.2.2], (see also [1, Theorem I.9.4]), we obtain the desired statement. □

Notation. We write $\nu(m, \omega) = \nu(\Sigma_{[\omega]_m})$ for any $\omega \in \Sigma$ and any $m \ge 0$.

Without any further assumptions, $p(t, \omega, \tau)$ satisfies the following estimates.

Proposition 6.5. (1) *For any $\omega \in \Sigma$ and any $t > 0$,*

$$p(t,\omega,\omega) \ge \frac{1}{e} \frac{1}{\nu(B(\omega,t))} \tag{6.6}$$

(2) *If $0 < t \le d(\omega, \tau)$, then*

$$p(t,\omega,\tau) \le \frac{t}{d(\omega,\tau)\nu(\Sigma_{[\omega,\tau]})}. \tag{6.7}$$

Proof. (1) If $D_{n,\omega} < t \le D_{n-1,\omega}$ for some $n \ge 1$, then $t/D_{m,\omega} \le 1$ for $m = 0, 1, \ldots, n-1$. Hence

$$p(t,\omega,\omega) \ge 1 + \sum_{m=0}^{n-1} \Big(\frac{1}{\nu(m+1,\omega)} - \frac{1}{\nu(m,\omega)} \Big) e^{-1} \ge \frac{1}{e\nu(n,\omega)}.$$

By Proposition 5.4-(3), it follows that $\nu(n, \omega) = \nu(B(\omega, t))$. Hence we have (6.6) for $t \in (0, R_\phi]$. For $t \ge R_\phi$, $B(\omega, t) = \Sigma$ and hence $p(t, \omega, \omega) \ge 1 \ge e^{-1} = e^{-1}/\nu(B(\omega, t))$.

(2) Write $N = N(\omega, \tau), \lambda_n = \lambda_{[\omega]_n}, D_n = D_{n,\omega}$ and $\nu_n = \nu(\Sigma_{[\omega]_n})$. Then $d(\omega, \tau) = D_N$. By letting $f(t) = p(t, \omega, \tau)$, (6.2) implies

$$f(t) = \sum_{n=0}^{N} \frac{e^{-\lambda_{n-1}t} - e^{-\lambda_n t}}{\nu_n},$$

$$f'(t) = \sum_{n=0}^{N} \frac{\lambda_n e^{-\lambda_n t} - \lambda_{n-1} e^{-\lambda_{n-1}t}}{\nu_n},$$

$$f''(t) = \sum_{n=0}^{n} \frac{(\lambda_{n-1})^2 e^{-\lambda_{n-1}t} - (\lambda_n)^2 e^{-\lambda_n t}}{\nu_n},$$

where $\lambda_{-1} = 0$. Since $\lambda_{n-1}t \leq \lambda_n t \leq D_N/D_n \leq 1$, we see that $f'(t) \geq 0$ and $f''(t) \leq 0$ for any $t \in [0, D_N]$. Hence

$$f'(D_N)t \leq f(t) \leq f'(0)t \tag{6.8}$$

for any $t \in [0, D_N]$. (6.8) along with

$$f'(0) = \sum_{n=0}^{N-1} \lambda_n \Big(\frac{1}{\nu_n} - \frac{1}{\nu_{n+1}}\Big) + \frac{\lambda_N}{\nu_N} \leq \frac{1}{D_N \nu_N} = \frac{1}{d(\omega, \tau)\nu(\Sigma_{[\omega,\tau]})}$$

shows (6.7) for any $t \in (0, d(\omega, \tau)]$. □

The volume doubling property of ν leads us to two-sided estimates of $p(t, \omega, \tau)$ and $J(\omega, \tau)$. Note that under the volume doubling property of ν, we have $\lim_{n\to\infty} D_{[\omega]_n} = 0$ for any $\omega \in \Sigma$ by Theorem 5.5.

Theorem 6.6. *Suppose ν has the volume doubling property with respect to d.*
(1) *$p(t, \omega, \tau)$ is continuous on $(0, \infty) \times \Sigma \times \Sigma$. Define*

$$q(t, \omega, \tau) = \begin{cases} \dfrac{t}{d(\omega, \tau)\nu(\Sigma_{[\omega,\tau]})} & \text{if } 0 < t \leq d(\omega, \tau), \\ \dfrac{1}{\nu(B(\omega, t))} & \text{if } t > d(\omega, \tau). \end{cases}$$

Then

$$p(t, \omega, \tau) \asymp q(t, \omega, \tau) \tag{6.9}$$

on $(0, \infty) \times \Sigma \times \Sigma$.
(2) *For any $(\omega, \tau) \in (\Sigma \times \Sigma)\backslash\Delta$,*

$$J(\omega, \tau) \asymp \frac{1}{d(\omega, \tau)\nu(\Sigma_{[\omega,\tau]})}. \tag{6.10}$$

The proof of this theorem is given at the end of this section.

The heat kernel estimate (6.9) can be thought of as a generalized version of the counterpart in [3], where ν is supposed to satisfy the uniform volume doubling property: i.e. $\nu(B(x, r)) \asymp f(r)$ for any $r > 0$ and $f(r)$ has the doubling property. In this paper, however, we do not require the uniform volume doubling property.

The following proposition gives an alternative expression for $q(t, \omega, \tau)$.

Proposition 6.7. *For any $t > 0$ and any $\omega, \tau \in \Sigma$,*

$$q(t, \omega, \tau) = \min\Big\{\frac{t}{d(\omega, \tau)\nu(\Sigma_{[\omega,\tau]})}, \frac{1}{\nu(B(\omega, t))}\Big\}.$$

The above proposition is immediate from the following lemma.

Lemma 6.8. $t\nu(B(\omega, t)) \leq d(\omega, \tau)\nu(\Sigma_{[\omega,\tau]})$ *if and only if* $t \leq d(\omega, \tau)$.

Proof. Assume $0 < t \leq D_\phi = R_\phi$. Then $D_{n,\omega} < t \leq D_{n-1,\omega}$ for some $n \geq 1$. By Proposition 5.4-(3), $B(\omega, t) = \Sigma_{[\omega]_n}$. If $t \leq d(\omega, \tau)$, then $N(\omega, \tau) \leq n - 1$ and hence $t\nu(\Sigma_{[\omega]_n}) \leq d(\omega, \tau)\nu(\Sigma_{[\omega,\tau]})$. Otherwise, $t > d(\omega, \tau)$ implies $N(\omega, \tau) \leq n$. Therefore, $t\nu(\Sigma_{[\omega]_n}) > d(\omega, \tau)\nu(\Sigma_{[\omega,\tau]})$.

Next assume $t > D_\phi$. Then $B(\omega, t) = \Sigma$ and $t > d(\omega, \tau)$. Hence $t\nu(\Sigma_{[\omega]_n}) > d(\omega, \tau)\nu(\Sigma_{[\omega,\tau]})$. □

Next we have estimate of moments of the displacement. In the next corollary, the assumption that $\#(S(x)) \geq 2$ for any $x \in T$ is essential.

Corollary 6.9. *Suppose that ν has the volume doubling property with respect to d. Then*

$$E_\omega(d(\omega, X_t)^\gamma) \asymp \begin{cases} t & \text{if } \gamma > 1, \\ t(|\log t| + 1) & \text{if } \gamma = 1 \\ t^\gamma & \text{if } 0 < \gamma < 1. \end{cases} \tag{6.11}$$

for any $\omega \in \Sigma$ and any $t \in (0, 1]$.

Note that we have an extra log-term when $\gamma = 1$. This is due to the slow polynomial decay of the off-diagonal part of $p(t, \omega, \tau)$ with respect to the space variable ω. See the discussion after Theorem 14.1 for details. Corollary 6.9 is a special case of Theorem 14.1, which shows that the above behavior of moments of displacement occurs whenever $p(t, \omega, \tau)$ enjoys certain type of heat kernel estimate which is typical in certain class of jump processes.

Proof. Let us verify all the assumptions of Theorem 14.1 with $X = \Sigma, \mu = \nu$ and $\phi(r) = r$. Theorem 6.3 suffices to show the assumption (1) in Section 14. By Proposition 5.6, there exist $c_1, c_2, c_3 \in (0, 1)$ and $R > 0$ such that $c_1\nu(B(\omega, r)) \leq \nu(B(\omega, c_3r) \leq c_2\nu(B(\omega, r))$ for any $\omega \in \Sigma$ and any $r \in (0, R]$. Hence we have (14.1). Theorem 6.6 implies (14.2). Therefore, we may apply Theorem 14.1 and obtain the above corollary. □

The rest of this section is devoted to showing Theorem 6.6.

Proof of Theorem 6.6-(1). Combining (6.1) and Lemma 6.1, we have

$$p(t, \omega, \tau) = 1 + \sum_{n \geq 0} e^{-\lambda_{[\omega]_n} t}\Big(\frac{1}{\nu(\Sigma_{[\omega]_{n+1}})}\chi_{\Sigma_{[\omega]_{n+1}}}(\tau) - \frac{1}{\nu(\Sigma_{[\omega]_n})}\chi_{\Sigma_{[\omega]_n}}(\tau)\Big). \tag{6.12}$$

Note that every term in the sum is continuous on $(0, \infty) \times \Sigma \times \Sigma$.

On the other hand, by Theorem 5.5, the volume doubling property implies that there exist $\alpha \in (0,1)$ and $c > 0$ such that

$$\nu(\Sigma_{[\omega]_n}) \geq \alpha^n \quad \text{and} \quad D_{n,\omega} \leq c\alpha^n$$

for any $\omega \in \Sigma$ and any $n \geq 0$. Hence the absolute value of each term in (6.12) is no greater than $c'\alpha^{-n}e^{-c^{-1}\alpha^{-n}T}$ on $[T,\infty)\times\Sigma\times\Sigma$. Since the infinite sum of these terms is convergent, the infinite sum in (6.12) is uniformly convergent on $[T,\infty)\times\Sigma\times\Sigma$ by the Weierstrass M-test. Therefore, $p(t,\omega,\tau)$ is continuous on $(0,\infty)\times\Sigma\times\Sigma$. □

Proof of Theorem 6.6-(2). We use the notations in the proof of Proposition 6.5. By Theorem 5.5, there exist $\alpha \in (0,1)$ and $M \in \mathbb{N}$ such that

$$\alpha^m \nu_n \leq \nu_{n+m} \tag{EL$_\nu$}$$

$$D_{n+M} \leq \alpha D_n \tag{D}$$

for any n and m. Choosing suitable $\alpha \in (0,1)$ and $c > 0$, we also have

$$D_{n+m} \leq c\alpha^m D_n \tag{D'}$$

for any n and m. Note that α, c and M are independent of $\omega \in \Sigma$.

The upper estimate for $0 < t \leq d(\omega,\tau)$ is immediate by Proposition 6.5-(2).

Lower estimate for $\mathbf{0} < \mathbf{t} \leq \mathbf{d}(\omega,\tau)$: Assume that $M \leq N+1$. By $(\mathrm{EL})_\nu$,

$$\begin{aligned} f'(D_N) &\geq \sum_{n=N-M+1}^{N} \frac{\lambda_n e^{-\lambda_n D_N} - \lambda_{n-1}e^{-\lambda_{n-1}D_N}}{\nu_n} \\ &\geq \sum_{n=N-M+1}^{N} \alpha^{N-n}\frac{\lambda_n e^{-\lambda_n D_N} - \lambda_{n-1}e^{-\lambda_{n-1}D_N}}{\nu_N} \\ &\geq \frac{\alpha^{M-1}}{\nu_N D_N}\Big(\frac{1}{e} - \frac{D_N}{D_{N-M}}e^{-\frac{D_N}{D_{N-M}}}\Big). \end{aligned} \tag{6.13}$$

By (D), we see $D_N/D_{N-M} \leq \alpha < 1$. Hence

$$f'(D_N) \geq \frac{\alpha^{M-1}(e^{-1} - \alpha e^{-\alpha})}{\nu_N D_N} \geq \frac{C}{\nu_N D_N}. \tag{6.14}$$

Next we consider the case where $N+1 \leq M$. Since $D_{m+1,\omega} < D_{m,\omega}$ for any $\omega \in \Sigma$ and any $m \geq 0$, there exists $\beta \in (0,1)$ such that $D_{m+1,\omega} \leq \beta D_{m,\omega}$ for any $\omega \in \Sigma$ and any $m \leq M$. Note that $N-1 \leq M$ and hence $D_N \leq \beta D_{N-1}$. Therefore

$$\begin{aligned} f'(D_N) &\geq \frac{\lambda_N e^{-1} - \lambda_{N-1}e^{-\lambda_{N-1}D_N}}{\nu_N} \\ &\qquad\qquad \geq \frac{1}{\nu_N D_N}\Big(\frac{1}{e} - \frac{D_N}{D_{N-1}}e^{-\frac{D_N}{D_{N-1}}}\Big) \geq \frac{e^{-1} - \beta e^{-\beta}}{\nu_N D_N}. \end{aligned}$$

Using this and (6.14), we have the desired lower estimate for $0 < t \le d(\omega, \tau)$ from (6.8).
Upper estimate for $\mathbf{d}(\omega,\tau) < \mathbf{t}$: By (6.2), $p(t,\omega,\tau) \le p(t,\omega,\omega)$. Hence it is enough to show that $p(t,\omega,\omega) \le c/\nu(B(\omega,t))$. Suppose $D_m < t \le D_{m+1}$. Then

$$1+\sum_{n=0}^{m-1}\Big(\frac{1}{\nu_{n+1}}-\frac{1}{\nu_n}\Big)e^{-\lambda_n t} \le 1+\sum_{n=0}^{m-1}\Big(\frac{1}{\nu_{n+1}}-\frac{1}{\nu_n}\Big) = \frac{1}{\nu_m} = \frac{1}{\nu(B(\omega,t))} \quad (6.15)$$

Let $n \ge m$. Then by (D)', $t/D_n \ge D_m/D_n \ge c^{-1}\alpha^{m-n}$. Also by $(\mathrm{EL})_\nu$, $1/\nu_{n+1} - 1/\nu_n \le 1/\nu_{n+1} \le \alpha^{-(n+1-m)}/\nu_m$. These imply

$$\sum_{n=m}^{\infty}\Big(\frac{1}{\nu_{n+1}}-\frac{1}{\nu_n}\Big)e^{-\lambda_n t} \le \frac{\alpha}{\nu_m}\sum_{k=0}^{\infty}\alpha^{-k}e^{-c^{-1}\alpha^{-k}} \le \frac{C}{\nu(B(\omega,t))}. \quad (6.16)$$

Since C is independent of ω and m, (6.15) and (6.16) yield the upper estimate for $d(\omega,\tau) < t \le D_0$. If $t > D_0$, then $\nu(B(\omega,t)) = 1$. Since

$$p(t,\omega,\omega) = 1+\sum_{n=0}^{\infty}\Big(\frac{1}{\nu_{n+1}}-\frac{1}{\nu_n}\Big)e^{-\lambda_n t} \le 1+\sum_{n=0}^{\infty}\alpha^{-(n+1)}e^{-c^{-1}\alpha^{-n}},$$

we have the upper estimate in this case.
Lower estimate for $\mathbf{d}(\omega,\tau) < t$: If $D_0 < t$, then $B(\omega,t) = \Sigma$ and

$$p(t,\omega,\tau) \ge \frac{1}{\nu_0}(1-e^{-\lambda_0 t}) \ge 1-e^{-1} = \frac{1-e^{-1}}{\nu(B(\omega,t))}.$$

Assume $d(\omega,\tau) < t \le D_0$. Then $D_m < t \le D_{m-1}$ for some $m \in \{1,2,\dots,N\}$. Suppose $M \le N$. Then we may choose $k \in \{0,\dots,m-1\}$ so that $m \le k+M \le N$. By $(\mathrm{EL})_\nu$, it follows that $\nu_m \ge \alpha^M \nu_n$ for any $n = k, k+1, \dots, k+M$. Hence

$$p(t,\omega,\tau) \ge \sum_{n=k+1}^{k+M}\frac{e^{-\lambda_{n-1}t}-e^{-\lambda_n t}}{\nu_n} \ge \alpha^M\frac{e^{-\lambda_k t}-e^{-\lambda_{k+M}t}}{\nu_m} \ge \alpha^M\frac{e^{-b}-e^{-a}}{\nu_m},$$

where $a = t/D_{k+M}$ and $b = t/D_k$. Using (D), we see that $b \le 1 \le a$ and $b \le \alpha a$. Since $\min\{e^{-b}-e^{-a} | b \le 1 \le a, b \le \alpha a\} > 0$, it follows that $p(t,\omega,\tau) \ge C'/\nu(B(\omega,t))$, where C' is independent of ω and t.

Finally assume $N+1 \le M$. As in the proof of the lower estimate for $0 < t \le d(\omega,\tau)$, we have $D_m \le \beta D_{m-1}$, where $\beta \in (0,1)$ is independent of ω and t. Hence

$$p(t,\omega,\tau) \ge \frac{e^{-\lambda_{m-1}t}-e^{\lambda_m t}}{\nu_m}.$$

Since $t/D_{m-1} \le 1 \le t/D_m$ and $t/D_{m-1} \le \beta t/D_m$, the similar argument as above shows $p(t,\omega,\tau) \ge C''/\nu(B(\omega,t))$. □

Proof of Theorem 6.6-(3). We continue to use the notations in the proof of Proposition 6.5. By (4.5),

$$J(\omega,\tau) = \frac{1}{2}\Big(\lambda_0 + \sum_{m=0}^{N-1} \frac{\lambda_{m+1}-\lambda_m}{\nu_{m+1}}\Big),$$

where $N = |x|$. Note that $(\mathrm{EL})_\nu$, (D) and (D)' in the proof of Theorem 6.6-(2) hold under the volume doubling property of ν. First we show the lower estimate. Assume that $N \geq M+1$, where M is the constant appearing in (D). By $(\mathrm{EL})_\nu$, $\nu_{N-i} \leq \alpha^{-M}\nu_N$ for $0 \leq i \leq M$. Hence, by (D),

$$\begin{aligned} J(\omega,\tau) &\geq \frac{\alpha^M}{2} \sum_{m=N-M}^{N-1} \frac{\lambda_{m+1}-\lambda_m}{\nu_N} \geq \frac{\alpha^M}{2}\frac{\lambda_N - \lambda_{N-M}}{\nu_N} \\ &\geq \frac{\alpha^M(1-\alpha)\lambda_N}{2\nu_N} = \frac{\alpha^M(1-\alpha)}{2d(\omega,\tau)\nu(\Sigma_{[\omega,\tau]})}. \end{aligned}$$

In case $N \leq M+1$, we may apply the same discussion as in the lower estimate for $d(\omega,\tau) < t$ in the proof of Theorem 6.6-(2) and obtain the lower estimate in this case.

Next by (D)', it follows that $\lambda_{N-i} \leq c\alpha^i\lambda_N$. This implies

$$J(\omega,\tau) \leq \frac{1}{2}\sum_{m=0}^{N} \frac{\lambda_m}{\nu_N} \leq \frac{c(1+\alpha+\ldots+\alpha^N)\lambda_N}{2\nu_N} \leq \frac{c}{2(1-\alpha)d(\omega,\tau)\mu(\Sigma_{[\omega,\tau]})}.$$

Thus we have shown the upper estimate. □

7 Completion with respect to resistance metric

As in the last section, (T, C) is a transient tree, $(\mathcal{E}, \mathcal{F})$ and R are the associated resistance form and resistance metric on T respectively.

Let $\overline{T}$ be the completion of T with respect to the resistance metric. Then by [9, Theorem 2.3.10], $(\mathcal{E}, \mathcal{F})$ can be thought of as a resistance form on $\overline{T}$. Write $T_R = \overline{T}\backslash T$. Then by Theorem 13.3, we have a resistance form $(\mathcal{E}|_{T_R}, \mathcal{F}|_{T_R})$ on T_R which is the trace of the resistance from $(\mathcal{E}, \mathcal{F})$ on T_R. The natural question is if $T_R = \Sigma$ and $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma) = (\mathcal{E}|_{T_R}, \mathcal{F}|_{T_R})$ or not. In particular, if this is true, then $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ is a resistance form on Σ. In the followings, we establish a sufficient condition in Theorem 7.1 and a necessary condition in Theorem 7.2 for being $T_R = \Sigma$ and $(\mathcal{E}|_{T_R}, \mathcal{F}|_{T_R}) = (\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$. Using these theorems, we will show that both $T_R = \Sigma$ and $T_R \neq \Sigma$ can occur by examples in the next section.

Theorem 7.1. *Let $r_x = C(x, \pi(x))^{-1}$. If $\sum_{m\geq 1}(\max_{x\in W_m} r_x) < +\infty$, where $W_m = \{x|x \in T, |x| = m\}$, then there exists a homeomorphism between $\overline{T}$ and $T \cup \Sigma$ which is the identity on T. Moreover, $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ coincides with $(\mathcal{E}|_\Sigma, \mathcal{F}|_\Sigma)$ which is the trace of $(\mathcal{E}, \mathcal{F})$ on Σ, where Σ is identified with $\overline{T}\backslash T$ through the homeomorphism. In particular, $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ is a resistance form on Σ.*

Proof of Theorem 7.1. Define $r_m = \max_{x \in W_m} r_x$. Let $\omega \in \Sigma$. For $m > n$,

$$R([\omega]_n, [\omega]_m) = \sum_{i=n+1}^{m} r_{[\omega]_i} \leq \sum_{i=n+1}^{m} r_i$$

Therefore, $\{[\omega]_n\}_{n \geq 0}$ is an R-Cauchy sequence. Denote the R-limit of $\{[\omega]_n\}_{n \geq 0}$ by $\Psi(\omega) \in \overline{T}$. Since $R([\omega]_n, [\tau]_n) \geq r_{[\omega]_{N(\omega,\tau)+1}}$ for $n > N(\omega, \tau)$, $\Psi : \Sigma \to \overline{T}$ is injective. Similar argument shows that $\Psi(\Sigma) \subseteq \overline{T} \backslash T$. Let $\{x_n\}_{n \geq 0}$ be an R-Cauchy sequence and let x_* be its R-limit in $\overline{T}$. Assume $x \notin T$. Then there exists a subsequence $\{y_n\}_{n \geq 0}$ of $\{x_n\}_{n \geq 0}$ such that $|y_n| \to \infty$ as $n \to \infty$. For any $m \geq 0$, there exists $z_m \in W_m$ such that $y_n \in T_{z_m}$ for sufficiently large n. Note that $\omega = (z_0, z_1, \ldots)$ is a geodesic ray and that $R(\omega, z) \leq \sum_{i=m+1}^{\infty} r_i$ for any $z \in T_{z_m}$. Those facts imply $x_* = \omega$. Hence Ψ is bijective. Now, extend $\Psi : T \cup \Sigma \to \overline{T}$ by $\Psi(x) = x$ for $x \in T$. It is routine to see that Ψ is continuous. Thus Ψ is a homeomorphism between $T \cup \Sigma$ and $\overline{T}$ and $\Psi|_T$ is identity. Hereafter, we identify Σ with $\overline{T} \backslash T$.

Now, let $f \in \mathcal{F}_\Sigma$. Then $H(f) \in \mathcal{F}$. Note that $H(f)$ is extended to a continuous function on $\overline{T} = T \cup \Sigma$. Hence Theorem 3.6 shows that $H(f)(x) = H(u)(x)$ for any $x \in T$, where $u = H(f)|_\Sigma$. Making use of Corollary 4.9, we have $(u)_x = (f)_x$ for any $x \in T$. This implies $u = f$ for ν-almost everywhere. Hence $f \in \mathcal{F}|_\Sigma = \{h|_\Sigma | h \in \mathcal{F}\}$.

Let $f \in \mathcal{F}|_\Sigma$. Let $h(f)$ be the Σ-harmonic function with boundary value f, i.e. $h(f) \in \mathcal{F}$ attains the following minimum:

$$\min\{\mathcal{E}(v, v) | v \in \mathcal{F}, v|_\Sigma = f\}.$$

Since $h(f)$ is harmonic on T and is bounded, Theorem 3.6 shows that $h(f)(x) = H(u)(x)$ for some $u \in L^\infty(\Sigma, \nu)$ and $h(f)([\omega]_n) \to u(x)$ for ν-almost every $\omega \in \Sigma$. Note that $h(f)$ is continuous on $\overline{T}$ and $h(f)|_\Sigma = f$. It follows that $u = f$ and hence $h(f) = H(f)$. Therefore, $f \in \mathcal{F}_\Sigma$ and hence $\mathcal{F}|_\Sigma = \mathcal{F}_\Sigma$. Finally $\mathcal{E}|_\Sigma(f, f) = \mathcal{E}(h(f), h(f)) = \mathcal{E}(H(f), H(f)) = \mathcal{E}_\Sigma(f, f)$ for any $f \in \mathcal{F}_\Sigma = \mathcal{F}|_\Sigma$. Thus $(\mathcal{E}|_\Sigma, \mathcal{F}|_\Sigma)$ is identified with $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$. □

Next we state a necessary condition.

Theorem 7.2. *If $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ is a resistance form on Σ and the associated resistance metric gives the same topology as $\mathcal{O}_\Sigma$, then $R_{[\omega]_n} \to 0$ as $n \to \infty$ for any $\omega \in \Sigma$.*

Proof of Theorem 7.2. Using [8, Theorem 9.4], we see that $p(t, \omega, \tau)$ is jointly continuous and hence it is bounded. (Since $p(t, \omega, \tau)$ is continuous except the diagonal Δ, the heat kernel given in [8, Theorem 9.4] is equal to $p(t, \omega, \tau)$.) By [8, Corollary 9.6], $tp(t, \omega, \omega) \to 0$ as $t \downarrow 0$ for any $\omega \in \Sigma$. Using Proposition 6.5-(1), we have $t/\nu(B(\omega, t)) \to 0$ as $t \downarrow 0$. If $D_{n,\omega} < t \leq D_{n-1,\omega}$, then $B(\omega, t) = \Sigma_{[\omega]_n}$. Hence $t/\nu(B(\omega, t)) \geq D_{n,\omega}/\nu(\Sigma_{[\omega]_n}) = R_{[\omega]_n}$. Since $n \to \infty$ as $t \downarrow 0$, we have $R_{[\omega]_n} \to 0$ as $n \to \infty$. □

8 Example: binary tree

In this section, we are going to illustrate the results obtained in the previous sections by examples which are (infinite complete) binary trees. Our examples are divided into two classes. The first class is the collection of self-similar trees, where the volume doubling property is automatic under the assumption of transience. The other class is homogeneous trees, through which we will explore various phenomena when the volume doubling property fails.

Definition 8.1. Let $W_m = \{1,2\}^m$ for $m \geq 0$, where $W_0 = \{\phi\}$. Define $W_* = \cup_{m\geq 0} W_m$. We denote $(w_1,\ldots,w_m)$ as $w_1\ldots w_m$. For $w_1\ldots w_m \in W_*\backslash W_0$, define $\pi(w_1\ldots w_m) = w_1\ldots w_{m-1}$ and $S(w_1\ldots w_m) = \{w_1\ldots w_m 1, w_1\ldots w_m 2\}$. Assume that $C : W_*\times W_* \to [0,\infty)$ satisfies $C(w,v) = C(v,w)$ and $C(w,v) > 0$ if and only if $\pi(w) = v$ or $\pi(v) = w$. (W_*, C) is called the (infinite complete) binary tree.

For binary trees, we always choose ϕ as the reference point. Then, the notions $\pi(w), S(w)$ and W_m are consistent with those defined in Sections 3 and 4. For any (W_*, C), the collection of infinite geodesic rays originated from ϕ, Σ^ϕ, is identified with the Cantor set $\{1,2\}^{\mathbb{N}}$. As a standard metric on Σ, we introduce $d_*(\cdot,\cdot)$.

Definition 8.2. Define $d_*(\omega,\tau) = 2^{-N(\omega,\tau)}$ for any $\omega \neq \tau \in \Sigma$ and $d_*(\omega,\tau) = 0$ if $\omega = \tau$.

It is easy to see that $d_*(\cdot,\cdot)$ is a distance on Σ.

First we consider a kind of self-similar binary tree (W_*, C_S).

Definition 8.3. Let $r_1, r_2 > 0$. For $w \in W_*$, define $C_S(w, wi) = (r_w r_i)^{-1}$, where $r_w = r_{w_1}\cdots r_{w_m}$ for any $w = w_1\ldots w_m \in W_*$.

Theorem 8.4. *(W_*, C_S) is transient if and only if $r_1r_2/(r_1+r_2) < 1$. In particular, if $r_1 = r_2 = r$, then (W_*, C_S) is transient if and only if $0 < r < 2$.*

Proof. Let $(\{X_n\}_{n\geq 0}, \{P_w\}_{w\in W_*})$ be the random walk associated with (W_*, C_S). Then $P_\phi(|X_{n+1}| = |X_n| + 1 \,|\, |X_n| \geq 1) = (r_1+r_2)/(r_1+r_2+r_1r_2) = p_1$ and $P_\phi(|X_{n+1}| = |X_n| - 1 \,|\, |X_n| \geq 1) = r_1r_2/(r_1+r_2+r_1r_2) = p_2$. Therefore, we may associate a random walk $(\{Z_n\}_{n\geq 0}, \{Q_k\}_{k\in\mathbb{N}_*})$ on $\mathbb{N}_* = \{0,1,\ldots\}$ such that $Q_k(Z_{n+1} = Z_n + 1 | Z_n \geq 1) = p_1$ and $Q_k(Z_{n+1} = Z_n - 1 | Z_n \geq 1) = p_2$ if $Z_n \geq 1$ and $Q_k(Z_{n+1} = 1 | Z_n = 0) = 1$. This random walk is transient if and only if $p_1 > p_2 \Leftrightarrow r_1r_2/(r_1+r_2) < 1$. □

By applying the results in the previous sections, we obtain the following statements.

Lemma 8.5. *Assume that (W_*, C_S) is transient.*

(1) $R_\phi = \Big(\dfrac{r_1+r_2}{r_1r_2} - 1\Big)^{-1}$.

(2) $R_w = r_w R_\phi$ *for any* $w \in W_*$.

(3) *Let* $\nu_1 = r_2/(r_1+r_2)$ *and* $\nu_2 = r_1/(r_1+r_2)$*. Then* $\nu(\Sigma_w) = \nu_{w_1}\cdots\nu_{w_m}$ *for any* $w = w_1 \ldots w_m \in W_*$*.*
(4) $D_w = \nu(\Sigma_w)R_w = (r_1r_2/(r_1+r_2))^{|w|}R_\phi$ *for any* $w \in W_*$*.*
(5) ν *has the volume doubling property with respect to* $d(\cdot,\cdot)$*, where* $d(\cdot,\cdot)$ *has been given in Definition 5.3.*

Proof. (1) Considering the self-similarity, R_ϕ satisfies

$$\frac{1}{r_1(R_\phi+1)} + \frac{1}{r_2(R_\phi+1)} = \frac{1}{R_\phi}.$$

(2) This is obvious from the self-similarity.
(3) Use Theorem 3.8.
(4) Combine (2) and (3).
(5) One can easily verify the conditions $(\mathrm{EL})_\nu$ and (D) in Theorem 5.5. □

The above lemma shows that $d(\omega,\tau) = D_{[\omega,\tau]} = (r_1r_2/(r_1+r_2))^{N(\omega,\tau)}R_\phi$. Hence we have $d(\omega,\tau) = d_*(\omega,\tau)^\delta R_\phi$ for any $\omega,\tau \in \Sigma$, where

$$\delta = \log\Big(\frac{r_1+r_2}{r_1r_2}\Big)/\log 2.$$

Combining those with Theorem 6.6, we have the following heat kernel estimate.

Theorem 8.6. *Assume that* (W_*, C_S) *is transient. Let* $p(t,\omega,\tau)$ *be the associated heat kernel. Define*

$$q_*(t,\omega,\tau) = \begin{cases} \dfrac{t}{d_*(\omega,\tau)^\delta \nu(\Sigma_{[\omega,\tau]})} & \text{if } 0 < t \le d_*(\omega,\tau)^\delta, \\ \dfrac{1}{\nu(B_*(\omega,t^{1/\delta}))} & \text{if } t > d_*(\omega,\tau)^\delta. \end{cases}$$

Then

$$p(t,\omega,\tau) \asymp q_*(t,\omega,\tau) \tag{8.1}$$

on $(0,\infty)\times\Sigma\times\Sigma$*, where* $B_*(\omega,r) = \{\tau | d_*(\omega,\tau) < r\}$*. In particular, if* $r_1 = r_2$*, then* $\delta = 1 - \log r/\log 2$ *and we may replace* q_* *in* (8.1) *by*

$$\widetilde{q}(t,\omega,\tau) = \begin{cases} \dfrac{t}{d_*(\omega,\tau)^{\delta+1}} & \text{if } 0 < t \le d_*(\omega,\tau)^\delta, \\ t^{-1/\delta} & \text{if } t > d_*(\omega,\tau)^\delta. \end{cases}$$

Theorem 8.7. $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ *is a resistance form on* Σ *and the associated resistance metric gives the same topology as* $\mathcal{O}_\Sigma$ *if and only if* $0 < r_1 < 1$ *and* $0 < r_2 < 1$*.*

Proof. Assume that $0 < r_1 < 1$ and $0 < r_2 < 1$. Let $r = \max\{r_1, r_2\}$. Then $0 < r < 1$ and $\max_{x\in W_m} r_x \le r^m$. By Theorem 7.1, $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ is a resistance form on Σ.

Conversely say $r_1 \ge 1$. Let $\omega = 1111\ldots$. Then $R_{[\omega]_n} = (r_1)^n R_\phi$. Now Theorem 7.2 yields the desired conclusion. □

Next we study a class of homogeneous binary trees.

Definition 8.8. Let $r(i) > 0$ for any $i \geq 0$. Define $C_H(w, wi) = r(|w|)^{-1}$ for any $w \in W_*$ and any $i \in \{1, 2\}$.

Theorem 8.9. *(W_*, C_H) is transient if and only if $\sum_{n \geq 0} 2^{-(n+1)} r(n) < +\infty$.*

Proof. Let us reduce W_m to an single point $m \in \mathbb{N} \cup \{0\}$. Then the resulting weighted graph is $(\mathbb{N} \cup \{0\}, C)$, where $C(m, m+1) = 2^{m+1}/r(m)$. Hence $R_\phi = R_0 = \sum_{n \geq 0} r(n)/2^{(n+1)}$. Since the transience is equivalent to the condition that $R_0 < +\infty$, we have the claim of the theorem. □

By the homogeneity of the tree, we can easily see that $\nu(\Sigma_x) = \nu(\Sigma_y)$ for any $x, y \in W_m$. Also the same method as in the proof of the above theorem gives R_w. As a consequence, we have the followings.

Lemma 8.10. *Assume that (W_*, C_H) is transient. Then for any $w \in W_*$,*

$$\nu(\Sigma_w) = 2^{-|w|}, \quad R_w = \sum_{n \geq 0} \frac{r(|w| + n)}{2^{n+1}} \quad \textit{and} \quad D_w = \sum_{n \geq |w|} \frac{r(n)}{2^{n+1}}. \tag{8.2}$$

Hereafter, we always assume that (W_*, C_H) is transient. By (8.2), D_w only depends on $|w|$. For $n \geq 0$, we define $D_n = D_w$ and $\lambda_n = 1/D_n$ for $w \in W_n$. Note that $\{D_n\}_{n \geq 0}$ is strictly decreasing and $\lim_{n \to \infty} D_n = 0$. Conversely, given a strictly decreasing sequence $\{D_n\}_{n \geq 0}$ with $\lim_{n \to \infty} D_n = 0$, we may construct an associated $\{r(n)\}_{n \geq 0}$ by letting $r(n) = 2^{n+1}(D_n - D_{n+1})$. By (6.2), we have

$$p(t, \omega, \omega) = 1 + \sum_{n=0}^{\infty} 2^n e^{-\lambda_n t} \tag{8.3}$$

and for $\omega \neq \tau$,

$$\begin{aligned} p(t, \omega, \tau) &= \sum_{n=0}^{N(\omega,\tau)} 2^n (e^{-\lambda_{n-1} t} - e^{-\lambda_n t}) \\ &= 1 + \sum_{n=0}^{N(\omega,\tau)-1} 2^n e^{-\lambda_n t} - 2^{N(\omega,\tau)} e^{-\lambda_{N(\omega,\tau)} t} \end{aligned} \tag{8.4}$$

From these, if $p(T, \omega, \omega) < +\infty$ for some $T > 0$, then $p(t, \omega, \tau)$ is continuous on $[T, \infty) \times \Sigma \times \Sigma$. Choosing an appropriate decreasing sequence $\{D_n\}_{n \geq 0}$, we may obtain a variety of heat kernels with interesting behaviors. For instance, we have an example where $p^{t,\omega}(\cdot) = p(t, \omega, \cdot)$ is getting more and more regular as $t \uparrow \log 2$ as follows.

Example 8.11. Let $D_n = (n+1)^{-1}$ for $n \geq 0$. Then

$$p(t, \omega, \tau) = \begin{cases} (1 - e^{-t}) \dfrac{2^{N+1} e^{-(N+1)t} - 1}{2e^{-t} - 1} & \text{if } t \neq \log 2, \\ (N+1)(1 - e^{-t}) & \text{if } t = \log 2, \end{cases} \tag{8.5}$$

where $N = N(\omega, \tau)$. Define $p^{t,\omega}$ by $p^{t,\omega}(\tau) = p(t, \omega, \tau)$. Then by (8.2) and (8.5), for $q \geq 1$, $p^{t,\omega} \in L^q(\Sigma, \nu)$ if and only if $(1 - 1/q) \log 2 < t$. (We regard $1/q$ as 0 if $q = \infty$.) In other words, if $0 < t \leq \log 2$, then $p^{t,\omega} \in L^q(\Sigma, \nu)$ for $1 \leq q < \log 2/(\log 2 - t)$ and $p(t, \omega, \tau)$ is finite and continuous on $(\log 2, \infty) \times \Sigma \times \Sigma$.

To describe the diagonal part $p(t, \omega, \omega)$, we introduce the eigenvalue counting function $\mathcal{N}(\lambda)$ of the non-negative definite self-adjoint operator associated with the Dirichlet from $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$ on $L^2(\Sigma, \nu)$. The eigenvalue counting function $\mathcal{N}(\lambda)$ is defined as the number of eigenvalues of L which are no greater than λ. By Theorem 4.6, λ_n is an eigenvalue with multiplicity 2^n and 0 is an eigenvalue with multiplicity 1. Hence,

$$\mathcal{N}(\lambda) = 1 + \sum_{n:\lambda_n \leq \lambda} 2^n = 2^{F(\lambda)},$$

where F is defined by $F(\lambda) = n$ if and only if $\lambda_{n-1} \leq \lambda < \lambda_n$. (Recall that $\lambda_{-1} = 0$.) (8.3) yields the next proposition.

Proposition 8.12. *For the homogeneous* (W_*, C_H)*,*

$$p(t, \omega, \omega) = t \int_0^\infty e^{-st} \mathcal{N}(s) ds = \int_0^\infty e^{-s} \mathcal{N}\Big(\frac{s}{t}\Big) ds$$

Furthermore, if $f : [0, \infty) \to [0, \infty)$ *is a monotonically non-decreasing function and* $f(\lambda_n) = n + 1$ *for any* $n \geq -1$*, then for any* $t > 0$*,*

$$\frac{1}{2} \int_0^\infty e^{-s} 2^{f(s/t)} ds \leq p(t, \omega, \omega) \leq \int_0^\infty e^{-s} 2^{f(s/t)} ds.$$

By using the above proposition, an asymptotic behavior of $p(t, \omega, \omega)$ as $t \downarrow 0$ may be determined even if the volume doubling property fails as in the next example.

Example 8.13. Let $D_n = (n + 1)^{-2}$ for $n \geq 0$. Then $\lambda_n = (n + 1)^2$. In this case, ν does not have the volume doubling property with respect to d. Proposition 8.12 implies that

$$p(t, \omega, \omega) \asymp \int_0^\infty e^{-s} 2^{\sqrt{s/t}} ds.$$

on $(0, \infty) \times \Sigma$. Now, for any $c > 0$,

$$\int_0^\infty e^{c\sqrt{s/t} - s} ds = 1 + \frac{c}{\sqrt{t}} \exp\Big(\frac{c^2}{4t}\Big) \int_{-\frac{c}{2\sqrt{t}}}^\infty e^{-y^2} dy.$$

Hence

$$p(t, \omega, \omega) \asymp 1 + \frac{1}{\sqrt{t}} \exp\Big(\frac{(\log 2)^2}{4t}\Big)$$

on $(0, \infty) \times \Sigma$. Since $\nu(B(\omega, t))^{-1} \asymp 2^{\sqrt{1/t}}$, the degree of divergence of $p(t, \omega, \omega)$ as $t \downarrow 0$ is actually much higher than what is expected by the formula which holds with the volume doubling property.

9 Generalization and jump kernel

In this section, as a generalization of $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$, we study a class of Dirichlet forms and/or closed forms on the space of infinite geodesic rays Σ of a tree as a non-directed graph.

Let T be a countably infinite set. To give an a priori structure of a tree as a non-directed graph, we fix an equivalence class with respect to $\underset{G}{\sim}$. This is equivalent to choose $\phi \in T$ and $\pi : T \to T$ such that $\pi(\phi) = \phi$ and, for any $x \in T\backslash\{\phi\}$, $\pi^{(n)}(x) = \phi$ for some $n \geq 1$, where $\pi^{(n)} = \underbrace{\pi \circ \ldots \circ \pi}_{n\text{-times}}$. All the notions associated with the structure of a tree as a non-directed graph can be derived from ϕ and π, for example, $T_\# = T\backslash\{\phi\}$,

$$\begin{aligned} S(x) &= \{y|\pi(y) = x\}, \\ T_x &= \{y|y \in T, \pi^{(n)}(y) = x \text{ for some } n \geq 0\}, \\ \Sigma &= \{(x_i)_{i\geq 0}|x_0 = \phi, \pi(x_{i+1}) = x_i \text{ for any } i \geq 0\}, \\ \Sigma_x &= \{(x_i)_{i\geq 0}|(x_i)_{i\geq 0} \in \Sigma, x_m = x \text{ for some } m \geq 0\}. \end{aligned}$$

To avoid unnecessary technical complexity, we assume that $2 \leq \#(S(x)) < +\infty$ for all $x \in T$. The space Σ is equipped with the canonical topology $\mathcal{O}_\Sigma$ which generated by the basis of open sets $\{\Sigma_x|x \in T\}$. Note that $(\Sigma, \mathcal{O}_\Sigma)$ is compact.

Definition 9.1. (1) Let $\lambda : T \to [0, \infty)$ and let $\mu \in \mathcal{M}_P(\Sigma)$, where

$$\mathcal{M}_P(\Sigma) = \{\mu|\mu \text{ is a Borel regular probability measure on } \Sigma \quad \mu(\Sigma_x) > 0 \text{ for any } x \in T\}.$$

Then for $\Gamma = (\lambda, \mu)$, we define

$$\mathcal{D}^\Gamma = \Big\{u\Big|u \in L^2(\Sigma, \mu), \sum_{x\in T} \frac{\lambda(x)}{2\mu(\Sigma_x)} \sum_{y,z\in S(x)} \mu(\Sigma_y)\mu(\Sigma_z)\big((u)_{\mu,y} - (u)_{\mu,z}\big)^2 < +\infty\Big\},$$

where $(u)_{\mu,x} = \mu(\Sigma_x)^{-1} \int_{\Sigma_x} u d\mu$, and

$$\mathcal{Q}^\Gamma(u, v) = \sum_{x\in T} \frac{\lambda(x)}{2\mu(\Sigma_x)} \sum_{y,z\in S(x)} \mu(\Sigma_y)\mu(\Sigma_z)\big((u)_{\mu,y} - (u)_{\mu,z})((v)_{\mu,y} - (v)_{\mu,z})$$

for any $u, v \in \mathcal{D}^\Gamma$.
(2) Define

$$E_{x,\mu} = \Big\{f\Big|f = \sum_{y\in S(x)} a_y \chi_{\Sigma_y}, \int_\Sigma f(y)\mu(dy) = 0\Big\}.$$

Comparing with (4.4), it is apparent that $\mathcal{Q}^\Gamma$ is a generalization of $\mathcal{E}_\Sigma$. We are going to show that $(\mathcal{Q}^\Gamma, \mathcal{D}^\Gamma)$ has properties which are analogous to $(\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)$.

Lemma 9.2. *Let $\varphi_0(\omega) = 1$ for any $\omega \in \Sigma$. For any $x \in T$, choose an $L^2(\Sigma, \mu)$-orthonormal base $(\varphi_{x,1}, \ldots, \varphi_{x,M(x)})$ of $E_{x,\mu}$, where $M(x) = \#(S(x)) - 1$. Then $\{\varphi_0, \varphi_{x,n} | x \in T, 1 \leq n \leq M(x)\}$ is a complete orthonormal system of $L^2(\Sigma, \mu)$.*

Proof. Let $\mathcal{C}_m = \{a_0\varphi_0 + \sum_{x\in T_m}\sum_{n=1}^{M(x)} a_{x,n}\varphi_{x,n} | a_0, a_{x,n} \in \mathbb{R}\}$ and let $\mathcal{C} = \cup_{m\geq 0}\mathcal{C}_m$. Then $\mathcal{C} = \{\sum_{i=1}^k \alpha_k \chi_{\Sigma_{x_k}} | k = 0, 1, \ldots, x_1, \ldots, x_k \in T, \alpha_1, \ldots, \alpha_k \in \mathbb{R}\}$. Hence $\mathcal{C}$ is dense in $C(\Sigma)$ with respect to the supremum norm. This implies that $\mathcal{C}$ is dense in $L^2(\Sigma, \mu)$. Since $\{\varphi_0, \varphi_{x,n} | x \in T, 1 \leq n \leq M(x)\}$ is orthonormal, it is a complete orthonormal system. □

Theorem 9.3. *Let $\lambda : T \to [0, \infty)$ and let $\mu \in \mathcal{M}_P(\Sigma)$. Let $\Gamma = (\lambda, \mu)$. Then*

$$\mathcal{D}^\Gamma = \Big\{u \Big| u \in L^2(\Sigma, \mu), u = a_0\varphi_0 + \sum_{x\in T}\sum_{n=1}^{M(x)} a_{x,n}\varphi_{n,x}, \\ \sum_{x\in T}\sum_{n=1}^{M(x)} \lambda(x)(a_{x,n})^2 < +\infty\Big\}. \tag{9.1}$$

Moreover, if $u, v \in \mathcal{D}^\Gamma$, $u = a_0\varphi_0 + \sum_{x\in T}\sum_{n=1}^{M(x)} a_{x,n}\varphi_{x,n}$ and $v = b_0\varphi_0 + \sum_{x\in T}\sum_{n=1}^{M(x)} b_{x,n}\varphi_{x,n}$, then

$$\mathcal{Q}^\Gamma(u, v) = \sum_{x\in T}\lambda(x)\sum_{n=1}^{M(x)} a_{x,n}b_{x,n}. \tag{9.2}$$

In particular, $(\mathcal{Q}^\Gamma, \mathcal{D}^\Gamma)$ is a closed quadratic form on $L^2(\Sigma, \mu)$ and

$$L_\Gamma u = \lambda_x u$$

for any $x \in T$ and any $u \in E_{x,\mu}$, where L_Γ is the nonnegative self-adjoint operator on $L^2(\Sigma, \mu)$ associated with $(\mathcal{Q}^\Gamma, \mathcal{D}^\Gamma)$.

We can also obtain a (formal) expression on the integral kernel $p(t, x, y)$ of the semigroup e^{-tL_Γ} by the same formula as (6.2).

Proof. Let $T_m = \cup_{n=0}^m W_n$ and let

$$\mathcal{Q}_m(u, v) = \sum_{x\in T_m}\frac{\lambda(x)}{2\mu(\Sigma_x)}\sum_{y,z\in S(x)}\mu(\Sigma_y)\mu(\Sigma_z)((u)_{y,\mu} - (u)_{z,\mu})((v)_{y,\mu} - (v)_{z,\mu})$$

for any $u, v \in L^2(\Sigma, \mu)$. If $u = a_0\varphi_0 + \sum_{x\in T}\sum_{n=1}^{M(x)} a_{x,n}\varphi_{x,n}$ and $v = b_0\varphi_0 + \sum_{x\in T}\sum_{n=1}^{M(x)} b_{x,n}\varphi_{x,n}$, then

$$\mathcal{Q}_m(u, v) = \sum_{x\in T_m}\lambda(x)\sum_{n=1}^{M(x)} a_{x,n}b_{x,n}.$$

Since $u \in \mathcal{D}^\Gamma$ if and only if $\mathcal{Q}_m(u, u)$ is convergent as $m \to \infty$, we have (9.1). The rest follows immediately. □

To obtain an alternative expression of $\mathcal{Q}^\Gamma$ by using an integral kernel, we define a transformation Φ_μ and Λ_μ.

Definition 9.4. For any $x \in T$, define $[x]_n = x_n$ for $n = 1, 2, \ldots, M$, where $M = |x|$ and $(x_0, x_1, \ldots, x_{M-1}, x_M)$ is the geodesic between ϕ and x. (Hence $x_0 = \phi$ and $x_M = x$.) Let μ be a Borel regular probability measure on Σ. Define a linear map $\Phi_\mu : \ell(T) \to \ell(T)$ and $\Lambda_\mu : \ell(T) \to \ell(T)$ by

$$(\Phi_\mu(\lambda))(x) = \frac{1}{2}\left(\lambda([x]_0) + \sum_{m=0}^{|x|-1} \frac{\lambda([x]_{m+1}) - \lambda([x]_m)}{\mu(\Sigma_{[x]_{m+1}})}\right) \tag{9.3}$$

for any $\lambda \in \ell(T)$ and any $x \in T$, and

$$(\Lambda_\mu(J))(x) = 2J(x)\mu(\Sigma_x) + 2\sum_{m=0}^{|x|-1} J([x]_m)\big(\mu(\Sigma_{[x]_m}) - \mu(\Sigma_{[x]_{m+1}})\big) \tag{9.4}$$

for any $J \in \ell(T)$ and any $x \in T$.

Simple calculation shows that Λ_μ is the inverse of Φ_μ.

Lemma 9.5. $\Lambda_\mu \circ \Phi_\mu = \Phi_\mu \circ \Lambda_\mu =$ Identity on $\ell(T)$.

Definition 9.6. For $J : T \to [0, \infty)$, define $L_J : (\Sigma \times \Sigma)\backslash\Delta \to [0, \infty)$ by $L_J(\omega, \tau) = J([\omega, \tau])$ for any $\omega, \tau \in \Sigma$ with $\omega \neq \tau$,

$$\mathcal{D}_{J,\mu} = \left\{u \middle| u \in L^2(\Sigma, \mu), \int_{\Sigma^2} L_J(\omega, \tau)(u(\omega) - u(\tau))^2 \mu(d\omega)\mu(d\tau) < +\infty\right\}$$

and, for any $u, v \in \mathcal{D}_{J,\mu}$,

$$\mathcal{Q}_{J,\mu}(u, v) = \int_{\Sigma^2} L_J(\omega, \tau)(u(\omega) - u(\tau))(v(\omega) - v(\tau))\mu(d\omega)\mu(d\tau).$$

Theorem 9.7. *Let $\lambda : T \to [0, \infty)$ and let $\mu \in \mathcal{M}_P(\Sigma)$. Write $\Gamma = (\lambda, \mu)$. Then $(\mathcal{Q}^\Gamma, \mathcal{D}^\Gamma)$ is a regular Dirichlet from on $L^2(\Sigma, \mu)$ if and only if $(\Phi_\mu(\lambda))(x) \geq 0$ for any $x \in T$. Moreover, assume that $(\Phi_\mu(\lambda))(x) \geq 0$ for any $x \in T$. Then $\mathcal{D}^\Gamma = \mathcal{D}_{\Phi_\mu(\lambda),\mu}$ and $\mathcal{Q}^\Gamma(u, v) = \mathcal{Q}_{\Phi_\mu(\lambda),\mu}(u, v)$ for any $u, v \in \mathcal{D}^\Gamma$.*

Remark. Define $\ell_+(T) = \{u | u : T \to [0, \infty)\}$. Then $\Phi_\mu(\lambda)(x) \geq 0$ for any $x \in T$ if and only if $\lambda \in \Lambda_\mu(\ell_+(T))$.

Before proving Theorem 9.7, we state an immediate corollary which follows by (9.4) and Lemma 9.5.

Corollary 9.8. *Let $J : T \to [0, \infty)$ and let $\mu \in \mathcal{M}_p(\Sigma)$. Then $(\mathcal{Q}_{J,\mu}, \mathcal{D}_{J,\mu})$ is a regular Dirichlet from on $L^2(\Sigma, \mu)$. Moreover, $(\mathcal{Q}_{J,\mu}, \mathcal{D}_{J,\mu}) = (\mathcal{Q}^\Gamma, \mathcal{D}^\Gamma)$, where $\Gamma = (\Lambda_\mu(J), \mu)$.*

Next two lemmas are needed to prove Theorem 9.7.

Lemma 9.9. *Let $J : T \to [0, \infty)$, let $\mu \in \mathcal{M}_P(\Sigma)$ and let $x \in T$. Then for any $\varphi \in E_{x,\mu}$ and any $u \in L^2(\Sigma, \mu)$, $L_J(\omega, \tau)(\varphi(\omega) - \varphi(\tau))(u(\omega) - u(\tau))$ is $\mu \times \mu$-integrable on $\Sigma \times \Sigma$ and*

$$\int_{\Sigma\times\Sigma} L_J(\omega,\tau)(\varphi(\omega)-\varphi(\tau))(u(\omega)-u(\tau))\mu(d\omega)\mu(d\tau) = \lambda_x(\varphi,u)_\mu, \tag{9.5}$$

where $\lambda_x = (\Lambda_\mu(J))(x)$ and $(\varphi, u)_\mu = \int_\Sigma \varphi(\omega)u(\omega)\mu(d\omega)$.

Proof. Define $Y_x = \cup_{y,z\in S(x),y\neq z}\Sigma_y \times \Sigma_z$. Then $L_J = \sum_{x\in T} J(x)\chi_{Y_x}$. Let $K_{u,v}(\omega, \tau) = L_J(\omega, \tau)(u(\omega) - u(\tau))(v(\omega) - v(\tau)$ and let $\varphi = \sum_{y\in S(x)} a_y\chi_{\Sigma_y}$. Since $\int_\Sigma \varphi(\omega)\mu(d\omega) = 0$, it follows that $\sum_{y\in S(x)} a_y\mu(\Sigma_y) = 0$. We devide $\{(\omega, \tau)|\varphi(\omega) \neq \varphi(\tau)\}$ into three regions Y_x, $\Sigma_x \times (\Sigma\backslash\Sigma_x)$ and $(\Sigma\backslash\Sigma_x) \times \Sigma_x$. For the first part, since $L_J(\omega, \tau) = J(x)$ on Y_x, $K_{\varphi,u}(\omega, \tau)$ is integrable on Y_x and

$$\begin{aligned}
&\int_{Y_x} K_{\varphi,u}(\omega,\tau)\\
&= \sum_{y,z\in S(x)} (a_y - a_z)\Big(\mu(\Sigma_z)\int_{\Sigma_y} u(\omega)\mu(d\omega) - \mu(\Sigma_y)\int_{\Sigma_z} u(\tau)\mu(d\tau)\Big)\\
&= 2\mu(\Sigma_x)J(x)(\varphi,u)_\mu.
\end{aligned}$$

For the second region, let $U_{x,m} = \Sigma_{[x]_m}\backslash\Sigma_{[x]_{m+1}}$. Then $\Sigma\backslash\Sigma_x = \cup_{m=0}^{|x|-1} U_{x,m}$. Note that $L_J(\omega, \tau) = J([x]_m)$ on $\Sigma_x \times U_{x,m}$. Hence $K_{\varphi,u}(\omega, \tau)$ is integrable on each $\Sigma_x \times U_{x,m}$. Now

$$\begin{aligned}
\int_{\Sigma_x\times U_{x,m}} K_{\varphi,u}(\omega,\tau)\mu(d\omega)\mu(d\tau) &= \int_{\Sigma_x\times U_{x,m}} J([x]_m)\varphi(\omega)u(\omega)\mu(d\omega)\mu(\delta\tau)\\
&= J([x]_m)(\mu(\Sigma_{[x]_m}) - \mu(\Sigma_{[x]_{m+1}})(\varphi,u)_\mu.
\end{aligned}$$

Hence

$$\int_{\Sigma_x\times\Sigma\backslash\Sigma_x} K_{\varphi,u}(\omega,\tau)\mu(d\omega)\mu(d\tau) = \sum_{m=0}^{|x|-1} J([x]_m)(\mu(\Sigma_{[x]_m}) - \mu(\Sigma_{[x]_{m+1}}))(\varphi,u)_\mu.$$

The third part is the same as the second part. As a whole, we obtain (9.5). □

Lemma 9.10. *Let $J : T \to [0, \infty)$ and let $\mu \in \mathcal{M}_P(\Sigma)$. Then $(\mathcal{Q}_{J,\mu}, \mathcal{D}_{J,\mu})$ is a regular Dirichlet form on $L^2(\Sigma, \mu)$. Moreover,*

$$\mathcal{D}_{J,\mu} = \Big\{a_0\varphi_0 + \sum_{x\in T}\sum_{n=1}^{M(x)} a_{x,n}\varphi_{x,n}\Big|(a_0)^2 + \sum_{x\in T}(1+\lambda_x)\sum_{n=1}^{M(x)}(a_{x,n})^2 < +\infty\Big\}, \tag{9.6}$$

where $\lambda_x = (\Lambda_\mu(J))(x)$, and

$$\mathcal{Q}_{J,\mu}(u,v) = \sum_{x\in T}\lambda_x\sum_{n=1}^{M(x)} a_{x,n}b_{x,n} \tag{9.7}$$

for any $u = a_0\varphi_0 + \sum_{x \in T}\sum_{n=1}^{M(x)} a_{x,n}\varphi_{x,n}, v = b_0\varphi_0 + \sum_{x\in T}\sum_{n=1}^{M(x)} b_{x,n}\varphi_{x,n} \in \mathcal{D}_{J,\mu}$.

Proof. By [5, Example 1.2.4], it follows that $(\mathcal{Q}_{J,\mu}, \mathcal{D}_{J,\mu})$ is a Dirichlet form. Note that $\{\varphi_0, \varphi_{x,n} | x \in T, n = 1, \ldots, M(x)\}$ is a complete orthonormal system of $L^2(\Sigma, \mu)$. Moreover, Lemma 9.9 implies that $\varphi_{x,n} \in \mathcal{D}_{J,\mu}$ and $\mathcal{Q}_{J,\mu}(\varphi_{x,n}, u) = \lambda_x(\varphi_{x,n}, u)_\mu$ for any $u \in \mathcal{D}_{J,\mu}$. These facts yield (9.6) and (9.7). Let $\mathcal{C}$ be the same as in the proof of Lemma 9.2. Then $\mathcal{C}$ is dense in $C(\Sigma)$ with respect to the supremum norm and in $L^2(\Sigma, \mu)$ as well. By (9.6) and (9.7), it follows that the $(\mathcal{Q}_{J,\mu})_1$-closure of $\mathcal{C}$ is $\mathcal{D}_{J,\mu}$. Hence $\mathcal{C}$ is a core and $(\mathcal{Q}_{J,\mu}, \mathcal{D}_{J,\mu})$ is regular. □

Proof of Theorem 9.7. Let $J(x) = (\Phi_\mu(\lambda))(x)$ for any $x \in T$.

First assume that $J(x) \geq 0$ for any $x \in T$. Then, by Theorem 9.3 and Lemma 9.10, we have all the desired statements.

Conversely, assume $(\mathcal{Q}^\Gamma, \mathcal{D}^\Gamma)$ is a regular Dirichlet form on $L^2(\Sigma, \mu)$. In particular, it has the Markov property. If $u = \sum_{y \in W_{m+1}} a_y \chi_{\Sigma_y}$, then

$$\begin{aligned} \mathcal{Q}^\Gamma(u, u) &= \sum_{x \in T_m} \frac{\lambda(x)}{\mu(\Sigma_x)} \sum_{y,z \in S(x)} \mu(\Sigma_y)\mu(\Sigma_z)((u)_{y,\mu} - (u)_{z,\mu})^2 \\ &= \sum_{x \in T_m} J(x) \int_{Y_x} (u(\omega) - u(\tau))^2 \mu(d\omega)\mu(d\tau) \end{aligned} \tag{9.8}$$

For any $x \in T$, choose $y \neq z \in S(x)$. Define $u_{\alpha,\beta} = \alpha\chi_{\Sigma_y} + \beta\chi_{\Sigma_z}$. Then by (9.8),

$$\begin{aligned} \mathcal{Q}^\Gamma(u_{\alpha,\beta}, u_{\alpha,\beta}) &= \sum_{n=0}^{m} 2J([y]_n)\big(\mu(\Sigma_{[y]_n}) - \mu(\Sigma_{[y]_{n+1}})\big)\alpha^2 \\ &\quad + \sum_{n=0}^{m} 2J([z]_n)\big(\mu(\Sigma_{[z]_n}) - \mu(\Sigma_{[z]_{n+1}})\big)\beta^2 - 4J(x)\mu(\Sigma_y)\mu(\Sigma_z)\alpha\beta \end{aligned} \tag{9.9}$$

Since $\mathcal{Q}^\Gamma$ is non-negative, (9.9) can be summarized as

$$\mathcal{Q}^\Gamma(u_{\alpha,\beta}, u_{\alpha,\beta}) = A\alpha^2 + B\alpha\beta + C\beta^2,$$

where A, B and C are constants, $A, C > 0$ and $B = -4J(x)\mu(\Sigma_y)\mu(\Sigma_z)$. Applying the unit contraction to $u_{1,\beta}$ for $\beta < 0$, we have

$$\mathcal{Q}^\Gamma(u_{1,\beta}, u_{1,b}) \geq \mathcal{Q}^\Gamma(u_{1,0}, u_{1,0}).$$

This implies that $B\beta + C\beta^2 \geq 0$ for any $\beta < 0$. Therefore, $B \leq 0$ and so $J(x) \geq 0$. □

Recall that if $(\lambda_x)_{x \in T}$ is the eigenvalue map associated with a transient tree (T, C), then $\lambda_{[\omega]_n}$ is monotonically increasing for any $\omega \in \Sigma$. Next we consider such an class of $\lambda : T \to (0, \infty)$.

Lemma 9.11. *Define*

$$\ell^\pi(T) = \{u|u : T \to (0,\infty), u(x) > u(\pi(x)) \text{ for any } x \in T_\#\}.$$

If $\mu \in \mathcal{M}_P(\Sigma)$*, then* $\Phi_\mu(\ell^\pi(T)) = \ell^\pi(T)$ *and* $\Lambda_\mu(\ell^\pi(T)) = \ell^\pi(T)$*.*

Proof. Let $\lambda \in \ell(T)$ and let $J = \Phi_\mu(\lambda)$. Then (9.3) and (9.4) imply

$$\lambda(x) - \lambda(\pi(x)) = 2\mu(\Sigma_{\pi(x)})(J(x) - J(\pi(x))) \tag{9.10}$$

for any $x \in T_\#$ and $\lambda(\phi) = 2\mu(\Sigma_\phi)J(\phi)$. This equality suffices for the proof. □

The above lemma and Theorem 9.3 imply the following corollary.

Corollary 9.12. *Let* $\mu \in \mathcal{M}_P(\Sigma)$*. If* $\lambda \in \ell^\pi(T)$*, then* $(\mathcal{Q}^\Gamma, \mathcal{D}^\Gamma)$ *is a regular Dirichlet form on* $L^2(\Sigma,\mu)$*, where* $\Gamma = (\lambda,\mu)$*. Moreover,* $\mathcal{D}^\Gamma = \mathcal{D}_{J,\mu}$ *and* $\mathcal{Q}^\Gamma = \mathcal{Q}_{J,\mu}$*, where* $J = \Phi_\mu(\lambda)$*.*

Finally we give a proof of Theorem 4.6.

Proof of Theorem 4.6. Lemma 4.3 shows that $(\lambda_x)_{x\in T} \in \ell^\pi(T)$. Hence by the above corollary, we have the statements (1) and (2) of the theorem. The statement (3) is immediate from (9.2). □

10 Inverse problem

As in the last section, T is assumed to have a structure of a tree as a non-directed graph by specifying $\phi \in T$ and $\pi : T \to T$. In other words, we have fixed an equivalence class G with respect to $\underset{G}{\sim}$.

Let $\mathcal{TR}(G)$ be the collection of transient trees whose structure as a non-direct graph are G, i.e.

$$\mathcal{TR}(G) = \{(T,C)|(T,C) \text{ is a tree}, G(T,C) = G \text{ and } (T^x_y, C^x_y) \text{ is transient for any } x,y \in T\}.$$

Then by the results of the previous sections, we have a map from $\mathcal{TR}(G)$ to $\ell^\pi(T) \times \mathcal{M}_P(T)$ defined by $(T,C) \to ((\lambda_x)_{x\in T}, \nu)$, where $(\lambda_x)_{x\in T}$ is the eigenvalue map of (T,C) and ν is the hitting distribution starting from ϕ associated with (T,C). Let us call this map Θ, which is actually injective because $(\lambda_x)_{x\in T}$ and ν give the values of $(r_x)_{x\in T_\#}$ by (4.3). Note that $D_x = 1/\lambda_x$. In this section, we are interested in the inverse of $\Theta : \mathcal{TR}(G) \to \ell^\pi(T) \times \mathcal{M}_P(\Sigma)$. In particular, we try to understand the image of Θ. Define

$$\ell^\pi_{\infty,\mu}(T) = \{\lambda|\lambda \in \ell^\pi(T), \lim_{n\to\infty} \lambda([\omega]_n) = +\infty \text{ for } \mu\text{-a.e. } \omega \in \Sigma\}$$

and

$$\ell^\pi_\infty(T) = \{\lambda|\lambda \in \ell^\pi(T), \lim_{n\to\infty} \lambda([\omega]_n) = +\infty \text{ for all } \omega \in \Sigma\}.$$

Then by Theorem 5.2,

$$\Theta(\mathcal{TR}(G)) \subseteq \bigcup_{\mu \in \mathcal{M}_P(\Sigma)} \big(\ell^{\pi}_{\infty,\mu}(T) \times \{\mu\}\big). \tag{10.1}$$

On the other hand, the next theorem implies that

$$\ell^{\pi}_{\infty}(T) \times \mathcal{M}_P(\Sigma) \subseteq \Theta(\mathcal{TR}(G)). \tag{10.2}$$

In the next section, we have examples which show that the equality does not holds any of (10.1) and (10.2).

Theorem 10.1. *For any $(\lambda, \mu) \in \ell^{\pi}_{\infty}(T) \times \mathcal{M}_P(\Sigma)$, there exists $(T, C) \in \mathcal{TR}(G)$ such that the hitting distribution starting from ϕ is μ and the eigenvalue map is $(\lambda(x))_{x \in T}$, where $(r_x)_{x \in T_\#}$ is given by*

$$r_x = \frac{1}{\mu(\Sigma_x)}\Big(\frac{1}{\lambda(\pi(x))} - \frac{1}{\lambda(x)}\Big) \tag{10.3}$$

for any $x \in T_\#$ for any $y \in T$. In particular, (10.2) holds.

Remark. By (4.3), the value given by (10.3) is the only possible $(r_x)_{x \in T_\#}$ under which λ is the eigenvalue map and μ is the hitting distribution starting from ϕ.

The rest of this section is devoted to a proof of Theorem 10.1. First we prove the following key lemma.

Lemma 10.2. *Let (T, C) be a tree. Set $r_x = C(\pi(x), x)^{-1}$ for any $x \in T_\#$. Let $(\mathcal{E}, \mathcal{F})$ be the corresponding resistance form on T. Assume that $u : T \to \mathbb{R}$ and that there exists a Borel regular probability measure μ on Σ such that*

$$\frac{u(\pi(x)) - u(x)}{r_x} = \mu(\Sigma_x) \tag{10.4}$$

for any $x \in T_\#$. Then
(1) $u \in \mathcal{F}$ if and only if $\sum_{x \in W_m} u(x)\mu(\Sigma_x)$ is convergent as $n \to \infty$.
(2) Assume $u \in \mathcal{F}$. Then for any $v \in \mathcal{F}$,

$$\mathcal{E}(u, v) = v(\phi) - \lim_{m \to \infty} \sum_{x \in W_m} v(x)\mu(\Sigma_x). \tag{10.5}$$

In particular $\mathcal{E}(u, v) = v(\phi)$ for any $v \in (C_0(T))_{\mathcal{E}_\phi}$ and $(r_x)_{x \in T_\#}$ is transient.
(4) Assume $u \in \mathcal{F}$. Let $g_*(x, y)$ be the symmetrized Green function of (T, C). Also let ν be the hitting distribution starting from ϕ. If $u(\phi) = \mathcal{E}(u, u)$ and $\sum_{x \in W_m} u(x)\nu(\Sigma_x) \to 0$ as $n \to \infty$, then $u(x) = g_*(\phi, x)$ for any $x \in T$. Moreover, $\mu = \nu$.

Proof. Let L be the (discrete) Laplacian associated with $(\mathcal{E}, \mathcal{F})$, i.e.

$$(Lf)(x) = \frac{f(\pi(x)) - f(x)}{r_x} + \sum_{y \in S(x)} \frac{f(y) - f(x)}{r_y}.$$

for any f and any $x \in T$. Then by (10.4), for any $x \in T_\#$,

$$\begin{aligned}(Lu)(x) &= \frac{u(\pi(x)) - u(x)}{r_x} + \sum_{y \in S(x)} \frac{u(y) - u(x)}{r_y} \\ &= \mu(\Sigma_x) - \sum_{y \in \mathcal{S}(x)} \mu(\Sigma_y) = 0, \\ (Lu)(\phi) &= \sum_{y \in S(\phi)} \frac{u(y) - u(\phi)}{r_y} = -1.\end{aligned} \tag{10.6}$$

Define $\mathcal{E}_m(f,g) = \sum_{x \in T_m} (f(\pi(x)) - f(x))(g(\pi(x)) - g(x))/r_x$. Note that $\mathcal{E}_m(f,f) \to \mathcal{E}(f,f)$ as $m \to \infty$ for any $f : T \to \mathbb{R}$ and $\mathcal{E}_m(f,g) \to \mathcal{E}(f,g)$ as $m \to \infty$ for any $f, g \in \mathcal{F}$. By (10.6),

$$\begin{aligned}\mathcal{E}_m(u,v) &= -\sum_{x \in T_{m-1}} v(x)(Lu)(x) - \sum_{x \in W_m} v(x) \frac{u(\pi(x)) - u(x)}{r_x} \\ &= v(\phi) - \sum_{x \in W_m} v(x)\mu(\Sigma_x).\end{aligned} \tag{10.7}$$

(1) If $\sum_{x \in W_m} u(x)\mu(\Sigma_x)$ is convergent as $m \to \infty$, then (10.7) shows

$$\lim_{m \to \infty} \mathcal{E}_m(u,u) = u(\phi) - \lim_{m \to \infty} u(x)\mu(\Sigma_x).$$

Hence $u \in \mathcal{F}$. Conversely, if $u \in \mathcal{F}$, then $\mathcal{E}_m(u,u)$ is convergent as $m \to \infty$. Considering (10.7), we see that $\sum_{x \in W_m} u(x)\mu(\Sigma_x)$ is convergent as $m \to \infty$.
(2) Taking $m \to \infty$ in (10.7), we obtain (10.5). If $v \in C_0(T)$, this immediately imply that $\mathcal{E}(u,v) = v(\phi)$. Let $v \in (C_0(T))_{\mathcal{E}_\phi}$. Then there exists $\{v_n\}_{n \geq 1} \subseteq C_0(T)$ such that $\mathcal{E}(v - v_n, v - v_n) \to 0$ and $v_n(\phi) \to v(\phi)$ as $n \to \infty$. Since $\mathcal{E}(u, v_n) = v_n(\phi)$, it follows that $\mathcal{E}(u,v) = v(\phi)$ by taking the limit as $n \to \infty$. Note that $\mathcal{E}(u,1) = 0 \neq 1$. This implies that $1 \notin (C_0(T))_{\mathcal{E}_\phi}$. By Theorem 2.6, the corresponding Markov chain (T, C), i.e. $\{r_x\}_{x \in T_\#}$, is transient.
(3) Write $\psi(x) = g_*(\phi, x)$ for any $x \in T$. Then

$$\begin{aligned}\mathcal{E}_m(u,\phi) &= -u(\phi) \sum_{x \in S(\phi)} \frac{\psi(x) - \psi(\phi)}{r_x} - \sum_{x \in W_m} u(x) \frac{\psi(\pi(x)) - p(x)}{r_x} \\ &= u(\phi) \sum_{x \in S(\phi)} \nu(\Sigma_x) - \sum_{x \in W_m} u(x)\nu(\Sigma_x) \\ &= u(\phi) - \sum_{x \in W_m} u(x)\nu(\Sigma_x).\end{aligned}$$

If $\sum_{x \in W_m} u(x)\nu(\Sigma_x) \to 0$ as $n \to \infty$, then we obtain $\mathcal{E}(u,\phi) = u(\phi)$. Also we have $\mathcal{E}(u,\psi) = \psi(\phi)$ as $\psi \in (C_0(T))_{\mathcal{E}_\phi}$. Assume $\mathcal{E}(u,u) = u(\phi)$. Then $\mathcal{E}(u - \psi, u - \psi) = 0$ and hence $u - \psi$ is a constant. Since $u(\phi) = \psi(\phi)$, it follows that $u = \psi$. By (4.3), Theorem 5.2-(1) and (10.3),

$$\mu(\Sigma_x) = \frac{u(\pi(x)) - u(x)}{r_x} = \frac{g_*(\phi, \pi(x)) - g_*(\phi, x)}{r_x} = \nu(\Sigma_x)$$

for any $x \in T_{\#}$. Therefore $\mu = \nu$. $\square$

For $(\lambda, \mu) \in \ell^{\pi}(T) \times \mathcal{M}_P(\Sigma)$, we define $(r_x)_{x \in T_{\#}}$ by (10.3) and the corresponding tree (T, C). Let $(\mathcal{E}, \mathcal{F})$ be the associated resistance form on T.

Lemma 10.3. *Define $\delta : T \to \mathbb{R}$ by $\delta(x) = \lambda(x)^{-1}$. Assume $\lambda \in \ell^{\pi}_{\infty,\mu}(T)$.*
(1) $\sum_{x \in W_m} \delta(x)\mu(\Sigma_x) \to 0$ as $m \to \infty$.
(2) $\delta \in \mathcal{F}$ and $\mathcal{E}(\delta, \delta) = \delta(\phi)$. For any $v \in (C_0(T))^{\mathcal{E}_\phi}$, $\mathcal{E}(\delta, v) = v(\phi)$.
(3) $(r_x)_{x \in T_{\#}}$ is transient.

Proof. (1) Define $\delta_m : \Sigma \to \mathbb{R}$ by $\delta_m = \sum_{x \in W_m} \delta(x)\chi_{\Sigma_x}$. Since $\lambda \in \ell^{\pi}(T)$, δ_m is monotonically decreasing. Moreover, $\delta_m(\omega) = \delta([\omega]_m) \to 0$ as $m \to \infty$ for μ-almost every $\omega \in \Sigma$. Hence $\sum_{x \in W_m} \delta(x)\mu(\Sigma_x) = \int_\Sigma \delta_m(\omega)\mu(\omega) \to 0$ as $m \to \infty$.
(2), (3), (4) Let $u = \delta$. Then by (10.3), u satisfies (10.4). Using (1) and applying Lemma 10.2, we immediately obtain (2), (3) and (4). $\square$

Finally we give a proof of Theorem 10.1.

Proof of Theorem 10.1. Assume $\lambda \in \ell^{\pi}_{\infty}(T)$. Let $\delta_m = \sum_{x \in W_m} \delta(x)\chi_{\Sigma_x}$. Then, $\delta_m(\omega) = \delta([\omega]_m) \to 0$ as $m \to \infty$. Since δ_m is monotonically non-increasing, $\int_\Sigma \delta_m(\omega)\nu(\omega) = \sum_{x \in W_m} \delta(x)\nu(\Sigma_x) \to 0$ as $m \to \infty$. Combining this fact with Lemma 10.3, we can verify all the assumptions in Lemma 10.2 with $u = \delta$. Therefore, $\delta(x) = g_*(\phi, x)$ for any $x \in T$, where $g_*(x, y)$ be the symmetrized Green function of (T, C). By Theorem 5.2-(1), $(g_*(\phi, x)^{-1})_{x \in T}$ is the eigenvalue map of $\{r_x\}_{x \in T_{\#}}$. Note that we have defined $\delta(x) = \lambda(x)^{-1}$. Hence $(\lambda(x))_{x \in T}$ is the eigenvalue map of $(r_x)_{x \in T_{\#}}$ and μ is the hitting distribution starting from ϕ. $\square$

11 Examples: the inverse problem

Let $T = W_*$, where W_* is the (infinite complete) binary tree defined in Section 8. Let G represents the structure of the binary tree as a non-directed graph. We define $\sigma : \Sigma \to \Sigma$ by $\sigma(i_1 i_2 \ldots) = i_2 i_3 \ldots$.

First we describe an example $(\lambda^{(1)}, \mu)$ where $\lambda^{(1)} \notin \ell^{\pi}_{\infty}(T)$ but $(\lambda^{(1)}, \mu) \in \Theta(\mathcal{TR}(G))$.

Lemma 11.1. *Define $x_m = 22\ldots2 \in W_m$ for $m \geq 0$, where $x_0 = \phi$, and $y_m = x_{m-1}1$ for $m \geq 1$. Then $\Sigma = (\cup_{m \geq 1} \Sigma_{y_m}) \cup \{222\ldots\}$.*

Definition 11.2. (1) Define μ as the Bernoulli measure on Σ with weight $(1/2, 1/2)$, that is, μ is a Borel regular probability measure and $\mu(\Sigma_x) = 2^{-|x|}$.
(2) We define $(\delta_1(x))_{x \in T}$ by $\delta_1(x_m) = 1 + 2^{-m}$ for any $m \geq 0$ and $\delta_1(y_m w) = 2^{-(|w|+1)}\delta_1(x_{m-1})$ for any $m \geq 1$ and $w \in T$. Set $\lambda^{(1)}(x) = \delta_1(x)^{-1}$ for any $x \in T$.

By the above lemma, $\delta_1([\omega]_n) \to 0$ as $n \to \infty$ if and only if $\omega \neq 222\ldots$. Hence we have the following proposition.

Proposition 11.3. $\lambda^{(1)} \in \ell^\pi_{\infty,\mu}(T)\backslash\ell^\pi_\infty(T)$.

By (4.3) and (10.3), it follows that the only candidate for $(r_x)_{x\in T_\#}$ associated with $(\lambda^{(1)}(x), \mu)$ is

$$r^{(1)}_x = \begin{cases} 1 & \text{if } x = x_m \text{ for } m \geq 1, \\ 2^{m-1} + 1 & \text{if } x = y_m w \text{ for } m \geq 1 \text{ and } w \in T. \end{cases}$$

Let $(T, C^{(1)})$ be the tree corresponding to $(r^{(1)}_x)_{x\in T_\#}$. Lemma 10.3 shows that $(T, C^{(1)}) \in \mathcal{TR}(G)$.

Theorem 11.4. $\Theta((T, C^{(1)})) = (\lambda^{(1)}, \mu)$.

Proof. By the definition of $\delta^{(1)}$, $\{\delta^{(1)}([\omega]_n)\}_{n\geq 0}$ does not converge to 0 as $n \to \infty$ if and only if $\omega = 222\ldots$. Let ν be the hitting distribution starting from ϕ of $(r_x)_{x\in T_\#}$. Then by (4.3), we have $\nu(\Sigma_{x_m}) = (D_{x_{m-1}} - D_{x_m})/r^{(1)}_{x_m} = D_{x_{m-1}} - D_{x_m}$. Since $\{D_{x_m}\}_{m\geq 0}$ is convergent as $m \to \infty$, it follows that $\nu(\Sigma_{x_m}) \to 0$ as $m \to \infty$. This shows that $\nu(\{222\ldots\}) = 0$. Hence $\delta^{(1)}([\omega]_n) \to 0$ as $n \to \infty$ for ν-almost every $\omega \in \Sigma$. Define $\delta^{(1)}_m = \sum_{x\in W_m} \delta^{(1)}(x)\chi_{\Sigma_x}$. Then $\delta^{(1)}_m(\omega) = \delta([\omega]_m) \to 0$ as $m \to \infty$ for ν-almost every $\omega \in \Sigma$. Hence $\sum_{x\in W_m} \delta^{(1)}(x)\nu(\Sigma_x) = \int_\Sigma \delta^{(1)}_m(\omega)\nu(d\omega) \to 0$ as $m \to \infty$. Lemma 10.2 implies that $\lambda^{(1)} = (\delta^{(1)})^{-1}$ is equal to the hitting distribution of $(r_x)_{x\in T_\#}$ and that $\mu = \nu$. □

Next example $(\lambda^{(2)}, \mu)$ does not belong to $\Theta(\mathcal{TR}(G))$ but satisfies that $\lambda^{(2)} \in \ell^\pi_{\infty,\mu}(T)$, where μ is the same as in Definition 11.2.

Definition 11.5. (1) Define $V_{2m} \subseteq W_{2m}$ by $V_{2m} = \{11, 22\}^m$ for $m \geq 0$. Also define $V_{2m-1} = \{wi | w \in V_{2m-2}, i \in \{1, 2\}\}$ and $U_{2m} = \{wij | w \in V_{2m-2}, i, j \in \{1, 2\}, i \neq j\}$ for $m \geq 1$.
(2) Let

$$\delta^{(2)}(x) = \begin{cases} 1 + 2^{-n} & \text{if } x \in V_n \text{ for } n \geq 0, \\ 2^{-(|w|+1)}(1 + 2^{-(2m-1)}) & \text{if } x = yw \text{ for } y \in U_{2m} \text{ and } w \in T. \end{cases}$$

Define $\lambda^{(2)}(x) = (\delta^{(2)}(x))^{-1}$.

Note that $\Sigma = \{11, 22\}^{\mathbb{N}} \cup (\cup_{m\geq 1}(\cup_{y\in U_{2m}} \Sigma_y))$. Since $\mu(\{11, 22\}^{\mathbb{N}}) = 0$, we have the next proposition.

Proposition 11.6. $\lambda^{(2)} \in \ell^\pi_{\infty,\mu}(T)\backslash\ell^\pi_\infty(T)$.

As in the case of $\delta^{(1)}$, the only candidate for $(r_x)_{x\in T_\#}$ is given by (10.3). We use $(r^{(2)}_x)_{x\in T_\#}$ to denote this candidate whose values are

$$r^{(2)}_x = \begin{cases} 1 & \text{if } x \in V_n \text{ for some } n \geq 1, \\ 1 + 2^{2m-1} & \text{if } x = yw \text{ for } y \in V_{2m} \text{ and } w \in T. \end{cases}$$

We use $(T, C^{(2)})$ to denote the corresponding tree.

Theorem 11.7. *Let $\Theta((T, C^{(2)})) = ((\lambda_x)_{x\in T}, \nu)$. Then $(\lambda_x)_{x\in T} \in \ell^\pi_\infty(T)$ and $\nu(\{11, 22\}^{\mathbb{N}}) > 0$, where $\{11, 22\}^{\mathbb{N}} = \{i_1 i_2 \ldots | i_{2m-1} = i_{2m}$ for any $m \geq 1\}$. In particular, $(\lambda^{(2)}(x))_{x\in T} \neq (\lambda_x)_{x\in T}$ and ν is not absolutely continuous with respect to μ. In particular, $(\lambda^{(2)}, \mu) \notin \Theta(\mathcal{TR}(G))$.*

Constructing $\Theta((T, C^{(2)}))$, we prove Theorem 11.7 in the rest of this section.

Lemma 11.8. *Define $\{\beta_m\}_{m\geq 0}$ by*

$$\beta_m = (-1)^{m-1}\Big(1 - \Big(\frac{1}{2}\Big)^{2m+1}\Big)^{-1}\Big(1 - \Big(\frac{1}{2}\Big)^{2m-1}\Big)^{-1} \tag{11.1}$$

and let $F(z) = \sum_{m\geq 0} \beta_m z^{2m+1}$. Then

$$F(2z) - F(z) + 2\Big(F\Big(\frac{z}{2}\Big) - F(z)\Big) = \frac{6z^2}{4z^2 + 1} F(z) \tag{11.2}$$

for any $|z| < 1/2$ and $F(t)$ is strictly monotonically increasing on $[0, 1/2]$.

Proof. Let $F(z) = \sum_{m\geq 0} \alpha_m z^{2m+1}$. Then (11.2) yields

$$\sum_{m\geq 1}\Big(X_m + \frac{32}{X_m} - 12\Big)\alpha_m z^{2m+1} + \sum_{m\geq 0}\Big(X_m + \frac{2}{X_m} - 3\Big)\alpha_{m+1} z^{2m+1} = 6\sum_{m\geq 0}\alpha_m z^{2m+1}$$

where $X_m = 2^{2m+3}$. This implies $\alpha_1 = 8\alpha_0/7$ and

$$(2^{2m+3} - 16)\alpha_m + (2^{2m+3} - 1)\alpha_{m+1} = 0.$$

for $m \geq 1$. In fact, by (11.1), these equalities do hold if $\alpha_m = \beta_m$ for any $m \geq 0$. Hence we have (11.2).

Note that $\beta_0 = 2, \beta_1 = 16/7$ and that $0 < -\beta_{2n} < \beta_{2n-1}$ for $n \geq 1$. Hence for $t \geq 0$,

$$\begin{aligned} F'(t) &= 2 + \sum_{n\geq 1}(4n+1)\beta_{2n} t^{4n} + \sum_{n\geq 1}(4n-1)\beta_{2n-1} t^{4n-2} \\ &\geq 2 + \sum_{n\geq 1}\big((4n-1)\beta_{2n-1} - (4n+1)|\beta_{2n}|t^2\big)t^{4n-2}. \end{aligned}$$

Since $(4n-1) > (4n+1)\times 1/4$ for $n \geq 1$, it follows that $F'(t) > 2$ for $t \in [0, 1/2]$ and hence $F(t)$ is strictly monotonically increasing on $[0, 1/2]$. □

Using (11.2), we have the following lemma.

Lemma 11.9. *Define $t_m = 2^{-m}/\sqrt{2}$. Then for $m \geq 2$,*

$$F(t_{m-1}) - F(t_m) + 2(F(t_{m+1}) - F(t_m)) = \frac{\frac{3}{2}\big(\frac{1}{2}\big)^{2m-1}}{\big(\frac{1}{2}\big)^{2m-1} + 1} F(t_m) \tag{11.3}$$

Routine calculations using (11.3) show the next lemma.

Lemma 11.10. *Let* $\alpha = 3(5F(t_1) - 4F(t_2))^{-1}$. *Define* $(\delta_*(x))_{x\in T}$ *by*

$$\delta_*(x) = \begin{cases} \alpha F(t_1) + \frac{1}{2} & \text{if } x = \phi, \\ \alpha F(t_m) & \text{if } x \in V_{2m-1} \text{ for } m \geq 1, \\ \alpha(2F(t_{m+1}) + F(t_m))/3 & \text{if } x \in V_{2m} \text{ for } m \geq 1, \\ \alpha 2^{-(|w|+1)} F(t_m) & \text{if } x = yw \text{ for } y \in U_{2m} \text{ and } w \in T. \end{cases}$$

Then

$$(L^{(2)}\delta_*)(\phi) = -1 \quad \text{and} \quad (L^{(2)}\delta_*)(x) = 0 \tag{11.4}$$

for any $x \in T_\#$, *where* $L^{(2)}$ *is the Laplacian associated with* $(r_x^{(2)})_{x\in T_\#}$.

Proof of Theorem 11.7. By Lemma 11.8, if $\lambda_*(x) = \delta_*(x)^{-1}$, then $\lambda_* \in \ell_\infty^\pi(T)$. Define $\nu_*(\Sigma) = 1$ and $\nu_*(\Sigma_x) = (\delta_*(\pi(x)) - \delta_*(x))/r_x^{(2)}$ for any $x \in T_\#$. Then (11.4) shows that $\nu_*(\Sigma_x) = \nu_*(\Sigma_{x1}) + \nu_*(\Sigma_{x2})$. Hence ν_* is identified as a Borel regular probability measure on Σ and $\nu_* \in \mathcal{M}_P(\Sigma)$. Applying Lemma 10.2 with $u = \delta_*$ and $\mu = \nu_*$, we have that $(\lambda_*, \nu_*) = \Theta((T, C^{(2)}))$. Obviously $\lambda_* \neq \lambda^{(2)}$. If $\nu_*(\{11, 22\}^{\mathbb{N}}) = 0$, then $\delta^{(2)}([\omega]_n) \to 0$ as $n \to \infty$ for ν_*-almost every $\omega \in \Sigma$. Then Lemma 10.2 implies that $\lambda^{(2)}$ is the eigenvalue map of $(r^{(2)})_{x\in T_\#}$. This contradiction shows that $\nu_*(\{11, 22\}^{\mathbb{N}}) > 0$. □

12 Relation to noncommutative Riemannian geometry

In [13], Pearson and Bellissard have constructed a framework for noncommutative Riemannian geometry on the Cantor set. Starting from an ultra-metric, they have obtained a probability measure μ, Dirichlet forms depending on a parameter s and associated Laplacians. From our point of view, they have defined Dirichlet forms by giving $J : T \to \Sigma$ and μ as in Section 9.

Let us give a quick introduction to their construction of a measure and a Dirichlet form. Unfortunately, any flavor of noncommutative geometry, which was the spirit of the original paper [13], does not remain in our brief review. Let fix a structure of a tree T as a non-directed graph as in Section 9. Namely, we fix $\phi \in T$ and $\pi : T \to T$. Let $\rho : T \to (0, \infty)$ satisfy that $\rho(x) > \rho(y)$ for any $x \in T$ and any $y \in S(x)$ and that $\lim_{n\to\infty} \rho([\omega]_n) = 0$ for any $\omega \in \Sigma$. We define an ultra-metric $d_\rho(\cdot, \cdot)$ on Σ by $d_\rho(\omega, \tau) = \rho([\omega, \tau])$. It is easy to see that d_ρ is a metric on Σ which gives the same topology as the original one. Define the zeta function $\zeta(z)$ associated with the ultra-metric d_ρ by

$$\zeta(z) = \sum_{x\in T} \rho(x)^z$$

and assume that there exists $s_0 > 0$ such that $\zeta(z)$ has a singularity at $z = s_0$ and is holomorphic on $\{z | z \in \mathbb{C}, \mathrm{Re}(z) > s_0\}$.

Theorem 12.1 (Pearson and Bellissard)**.** *With some regularity assumptions on ζ, there exists a unique Borel regular probability measure μ on Σ which satisfies*

$$\mu(\Sigma_x) = \lim_{s \downarrow s_0} \frac{\sum_{y \in T_x} \rho(y)^s}{\sum_{y \in T} \rho(y)^s}$$

for any $x \in T$.

Definition 12.2. For $s \in \mathbb{R}$, define $J_s : T \to [0, \infty)$ by

$$J_s(x) = \frac{\rho(x)^{s-2}}{\sum_{(w,v) \in Y_x} \mu(\Sigma_w)\mu(\Sigma_v)},$$

where $Y_x = \cup_{y,z \in S(x), y \neq z} \Sigma_y \times \Sigma_z$. Also define $\lambda_s : T \to \mathbb{R}$ by $\lambda_s = \Lambda_\mu(J_s)$.

The Dirichlet form $(Q_s, \mathrm{Dom}(Q_s))$ on $L^2(\Sigma, \mu)$ defined in [13] coincides with $(\mathcal{Q}_{J_s,\mu}, \mathcal{D}_{J_s,\mu})$. Note that $J_s(x) > 0$ and hence $\lambda_s(x) > 0$ for any $s \in \mathbb{R}$ and any $x \in T$. Combining Theorem 9.3 and Corollary 9.8, we obtain the following fact which was originally obtained in [13].

Theorem 12.3. *Let L_s be the non-negative definite self-adjoint operator associated with the Dirichlet form $(\mathcal{Q}_{J_s,\mu}, \mathcal{D}_{J_s,\mu})$ on $L^2(\Sigma, \mu)$. Then the spectrum of L_s is pure point.*

By Theorem 10.1, if $\lambda_s \in \ell^\pi_\infty(T)$, then $(\lambda_s, \mu) = \Theta((T, C))$ for some transient tree (T, C). Unfortunately, it is difficult to tell whether $\lambda_s \in \ell^\pi_\infty(T)$ or not in general. From now on, we discuss this problem in the case of a self-similar ρ on the (complete infinite) binary tree W_* which has been introduced in Section 8. We let $T = W_*$ for the rest of this section.

Definition 12.4. Let $\rho_1, \rho_2 \in (0, 1)$. Define $\rho : T \to (0, 1)$ by $\rho(w_1 \ldots w_m) = \rho_{w_1} \cdots \rho_{w_m}$ for $w_1 \ldots w_m \in T$.

The case when $\rho_1 = \rho_2$ has been studied in [13].

Proposition 12.5. (1) *The zeta function $\zeta(z)$ associated with the ultra-metric d_ρ is given by*

$$\zeta(z) = \sum_{m=0}^{\infty} \big((\rho_1)^z + (\rho_2)^z\big)^m = \frac{1}{1 - \big((\rho_1)^z + (\rho_2)^z\big)}$$

and the singularity s_0 is the unique real number which satisfies

$$(\rho_1)^{s_0} + (\rho_2)^{s_0} = 1.$$

(2) *Let $\mu_i = (\rho_i)^{s_0}$ for $i = 1, 2$. Then the measure μ given in Theorem 12.1 is the self-similar measure with weight (μ_1, μ_2), i.e. $\mu(\Sigma_{w_1 \ldots w_m}) = \mu_{w_1} \cdots \mu_{w_m}$ for any $w = w_1 \ldots w_m \in T$.*

By the above proposition, it follows that $\mu(\Sigma_w) = \rho(w)^{s_0}$. Hence

$$J_s(w) = \frac{\rho(w)^{s-2-2s_0}}{2\mu_1\mu_2} \tag{12.1}$$

for any $w \in T$. Lemma 9.11 implies that $\lambda_s \in \ell^\pi(T)$ if and only if $s < 2 + 2s_0$. By (12.1),

$$\lambda_s(w) = \frac{1}{\mu_1\mu_2}\Big(\rho(w_1 \dots w_m)^{s-2-s_0} + \sum_{n=0}^{m-1}(1-\mu_{w_{n+1}})\rho(w_1 \dots w_n)^{s-2-s_0}\Big) \tag{12.2}$$

for any $w = w_1 \dots w_m \in W_*$. (12.2) has been obtained in [13] for the case when $\rho_1 = \rho_2$.

Proposition 12.6. *$\lambda_s \in \ell^\pi_\infty(T)$ if and only if $s \le 2+s_0$. Moreover, if $s \le 2+s_0$, then*

$$\lambda_s(w) \asymp \begin{cases} \rho(w)^{s-2-s_0} & \text{if } s < 2+s_0, \\ |w| & \text{if } s = 2+s_0 \end{cases}$$

for any $w \in T$.

Corollary 12.7. *There exists a transient tree (T, C_s) such that $\Theta((T, C_s)) = (\lambda_s, \mu)$ if and only if $s \le 2 + s_0$.*

Let $d^{(s)}$ be the metric associated with (T, C_s) defined in Definition 5.3. Since $d^{(s)}(\omega, \tau) = \lambda_s([\omega, \tau])^{-1}$, Proposition 12.6 yields

$$d^{(s)}(\omega, \tau) \asymp \begin{cases} d_\rho(\omega, \tau)^{2+s_0-s} & \text{if } s < 2+s_0, \\ N(\omega, \tau)^{-1} & \text{if } s = 2+s_0 \end{cases} \tag{12.3}$$

for any $\omega, \tau \in \Sigma$.

By (12.3), μ has the volume doubling property with respect to $d^{(s)}$ if and only if $s < 2 + s_0$. Applying the results in Section 6, we have the following asymptotic estimates of the heat kernel and moments of displacement.

Theorem 12.8. *Assume $s < 2+s_0$. There exists a jointly continuous transition density $p(t, \omega, \tau)$ on $(0, \infty) \times \Sigma \times \Sigma$ for the Hunt process associated with the Dirichlet form $(\mathcal{Q}_{J_s,\mu}, \mathcal{D}_{J_s,\mu})$ on $L^2(\Sigma, \mu)$. Define*

$$q_1(t, \omega, \tau) = \begin{cases} \dfrac{t}{d_\rho(\omega, \tau)^{2+2s_0-s}} & \text{if } d_\rho(\omega, \tau)^{2+s_0-s} > t, \\ \dfrac{1}{\mu(B_{d_\rho}(\omega, t^{1/(2+s_0-s)}))} & \text{if } d_\rho(\omega, \tau)^{2+s_0-s} \le t. \end{cases}$$

Then, $p(t, \omega, \tau) \asymp q_1(t, \omega, \tau)$ for any $(t, \omega, \tau) \in (0, \infty) \times \Sigma \times \Sigma$. Moreover,

$$E_\omega\big(d_\rho(\omega, X_t)^{(2+s_0-s)\gamma}\big) \asymp \begin{cases} t & \text{if } \gamma > 1, \\ t(|\log t| + 1) & \text{if } \gamma = 1, \\ t^\gamma & \text{if } 0 < \gamma < 1 \end{cases} \tag{12.4}$$

for any $\omega \in \Sigma$ and any $t \in (0, 1]$.

In the case when $\rho_1 = \rho_2$, Pearson and Bellissard have obtained an averaged version of (12.4) with an exact expression of the leading term in [13]. See Theorem 14.1 and the following discussion for an observation on the appearance of the $|\log t|$ term in the case of $\gamma = 1$.

13 Appendix A: transience of resistance forms

In this appendix, we first introduce several basic notions on resistance forms which are needed in this paper. Then, we will study the "transience" of general resistance forms as an extension of the case of weighted graphs in Section 2.

Definition 13.1 (Resistance form)**.** Let X be a set. A pair $(\mathcal{E}, \mathcal{F})$ is called a resistance form on X if it satisfies the following conditions (RF1) through (RF5).

(RF1) $\mathcal{F}$ is a linear subspace of $\ell(X)$ containing constants and $\mathcal{E}$ is a non-negative symmetric quadratic form on $\mathcal{F}$. $\mathcal{E}(u,u) = 0$ if and only if u is constant on X.

(RF2) Let $\sim$ be an equivalent relation on $\mathcal{F}$ defined by $u \sim v$ if and only if $u - v$ is constant on X. Then $(\mathcal{F}/{\sim}, \mathcal{E})$ is a Hilbert space.

(RF3) For any $x, y \in X$, if $x \neq y$, then there exists $u \in \mathcal{F}$ such that $u(x) \neq u(y)$.

(RF4) For any $p, q \in X$,

$$\sup\left\{\frac{|u(p) - u(q)|^2}{\mathcal{E}(u,u)}\middle| u \in \mathcal{F}, \mathcal{E}(u,u) > 0\right\}$$

is finite. The above supremum is denoted by $R_{(\mathcal{E},\mathcal{F})}(p,q)$.

(RF5) $\overline{u} \in \mathcal{F}$ and $\mathcal{E}(\bar{u}, \bar{u}) \leq \mathcal{E}(u,u)$ for any $u \in \mathcal{F}$, where

$$\overline{u}(p) = \begin{cases} 1 & \text{if } u(p) \geq 1, \\ u(p) & \text{if } 0 < u(p) < 1, \\ 0 & \text{if } u(p) \leq 0. \end{cases}$$

Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a set X. We call $R_{(\mathcal{E},\mathcal{F})}(\cdot,\cdot)$ the resistance metric on X associated with $(\mathcal{E}, \mathcal{F})$. (It is known that $R_{(\mathcal{E},\mathcal{F})}(\cdot,\cdot)$ is a metric on X. See [9] for example.)

Hereafter in this section, $(\mathcal{E}, \mathcal{F})$ is always a resistance from on X and $R(\cdot,\cdot)$ is the associated resistance metric.

We now introduce two important notions, the Green function and trace. Let B be a nonempty subset of X. The following result enables us to define the B-Green function $g_B(\cdot,\cdot)$. See [8, Section 4] for details and the proof.

Theorem 13.2 (Green function)**.** *Let $\mathcal{F}(B) = \{u | u \in \mathcal{F}, u|_B \equiv 0\}$. Assume that $\cap_{u \in \mathcal{F}(B)} u^{-1}(0) = B$.*

(1) *There exists a unique $g_B : X \times X \to [0, \infty)$ satisfying that $g_B^x \in \mathcal{F}(B)$ and $\mathcal{E}(g_B^x, u) = u(x)$ for any $u \in \mathcal{F}(B)$ and any $x \in X$, where $g_B^x(y) = g_B(x,y)$. Moreover, $g_B(x,y) = g_B(y,x)$ and $g_B(x,y) > 0$ if and only if $x, y \in X \backslash B$.*

(2) *Let* $b \in B$ *and define* $u(B) = u(b)$ *for* $u \in \mathcal{F}(B) + \mathbb{R}$. *Then* $(\mathcal{E}, \mathcal{F}(B) + \mathbb{R})$ *can be regarded as a resistance form on* $X_B = (X \backslash B) \cup \{B\}$ *Let* $R_B(\cdot, \cdot)$ *be the associated resistance metric, then*

$$g_B(x, y) = \frac{R_B(x, B) + R_B(y, B) - R_B(x, y)}{2}$$

for any $x, y \in X$.

The function $g_B(\cdot, \cdot)$ is called the B-Green function of the resistance form $(\mathcal{E}, \mathcal{F})$ on X.

Remark. If B is a single point, then the assumption of Theorem 13.2 is satisfied by the definition of the resistance form and the next theorem shows that $\mathcal{F}(B) + \mathbb{R} = \mathcal{F}$. Hence $R_B = R$ in this case.

The next theorem is from [8, Section 7].

Theorem 13.3 (Trace)**.** *Set* $\mathcal{F}|_B = \{u|_B : u \in \mathcal{F}\}$.
(1) *There exists a linear map* $h_B : \mathcal{F}|_B \to \mathcal{F}$ *such that* $h_B(f)|_B = f$ *and*

$$\mathcal{E}(h_B(f), h_B(f)) = \min\{\mathcal{E}(u, u) | u \in \mathcal{F}, u|_B = f\}$$

for any $f \in \mathcal{F}|_B$. *Moreover,* $h_B(f)$ *is the unique element in* $\{u | u \in \mathcal{F}, u|_B = f\}$ *that attains the above minimum and* $\mathcal{F} = \mathcal{F}(B) \oplus \mathrm{Im}(h_B)$, *where "*$\oplus$*" means that* $\mathcal{E}(u, v) = 0$ *for any* $u \in \mathcal{F}(B)$ *and any* $v \in \mathrm{Im}(h_B)$.
(2) *Define* $\mathcal{E}|_B(f, f) = \mathcal{E}(h_B(f), h_B(f))$ *for* $f \in \mathcal{F}|_B$. *Then* $(\mathcal{E}|_B, \mathcal{F}|_B)$ *is a resistance form on* B *and the associated resistance metric on* B *is equal to the restriction of* R *onto* B.

$(\mathcal{E}|_B, \mathcal{F}|_B)$ is called the trace of $(\mathcal{E}, \mathcal{F})$ on B and $h_B(f)$ is called the B-harmonic function with the boundary value f.

To extend the notion of "transience" for general resistance forms, we need to study the collection of resistance forms $(\mathcal{E}', \mathcal{F}')$ on X which are restrictions of $(\mathcal{E}, \mathcal{F})$ i.e. $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{E}'(u, v) = \mathcal{E}(u, v)$ for any $u, v \in \mathcal{F}'$.

Definition 13.4. Let $\mathcal{A}$ be a subset of $\mathcal{F}$.
(1) $\mathcal{A}$ is said to have the Markov property if and only if $\overline{u} \in \mathcal{A}$ for any $u \in \mathcal{A}$.
(2) $\mathcal{A}$ is said to separate points in X if and only if, for any $x, y \in X$ with $x \neq y$, there exists $u \in \mathcal{A}$ such that $u(x) = 1$ and $u(y) = 0$.
(3) Define $\mathcal{N}$ and $\mathcal{A}^{(\mathcal{E}, \mathcal{F})}$ by

$$\mathcal{N} = \{\mathcal{F}' | \mathcal{F}' \subseteq \mathcal{F}, (\mathcal{E}, \mathcal{F}') \text{ is a resistance form on } X\}$$

and

$$\mathcal{A}^{(\mathcal{E}, \mathcal{F})} = \bigcap_{\mathcal{F}' \in \mathcal{N}, \mathcal{A} \subseteq \mathcal{F}'} \mathcal{F}'.$$

The above definition of "separation of points" is slightly stronger than the usual one, which only requires the existence of function u with $u(x) \neq u(y)$.

Proposition 13.5. *Let $p \in X$. Define $\mathcal{E}_p(u,v) = \mathcal{E}(u,v) + u(p)v(p)$ for any $u, v \in \mathcal{F}$. Then $\mathcal{E}_p$ is an inner product on $\mathcal{F}$ and $(\mathcal{F}, \mathcal{E}_p)$ is a Hilbert space. The topology induced by $\mathcal{E}_p$ is independent of the choice of $p \in X$.*

Let 1 be the characteristic function of X. For a linear subspace $\mathcal{A}$ of $\mathcal{F}$, we use $\mathcal{A} + 1$ to denote the direct sum of $\mathcal{A}$ and the space of constants on X.

Proposition 13.6. *Let $\mathcal{A}$ be a linear subspace of $\mathcal{F}$. If $\mathcal{A}$ separates points on X, then $\mathcal{A}^{(\mathcal{E},\mathcal{F})} \in \mathcal{N}$. Moreover, if $\mathcal{A} + 1$ has the Markov property, then $\mathcal{A}^{(\mathcal{E},\mathcal{F})} = (\mathcal{A})_{\mathcal{E}_p} + 1$, where $(\mathcal{A})_{\mathcal{E}_p}$ is the closure of $\mathcal{A}$ with respect to the inner product $\mathcal{E}_p$.*

Proof. It is immediate to verify that $(\mathcal{E}, \mathcal{A}^{(\mathcal{E},\mathcal{F})})$ is a resistance form on X. Assume that $\mathcal{A} + 1$ has the Markov property. Then we can easily show that $(\mathcal{A})_{\mathcal{E}_p} + 1$ is a resistance form on X. Since every $\mathcal{F}' \in \mathcal{N}$ contains $(\mathcal{A})_{\mathcal{E}_p} + 1$, it follows that $\mathcal{A}^{(\mathcal{E},\mathcal{F})} = (\mathcal{A})_{\mathcal{E}_p} + 1$. □

Definition 13.7. Let $\mathcal{A}$ be a subset of $\mathcal{F}$. For any $U \subseteq X$, define $\mathcal{A}_U = \{u | u \in \mathcal{A}, u(x) \geq 1 \text{ for any } x \in U\}$ and

$$\mathrm{Cap}_{\mathcal{A}}(\mathrm{U}) = \begin{cases} +\infty & \text{if } \mathcal{A}_U = \emptyset, \\ \inf_{u \in \mathcal{A}_U} \mathcal{E}(u,u) & \text{if } \mathcal{A}_U \neq \emptyset. \end{cases}$$

The following result and the method of its proof are analogous to those of Theorem 2.6.

Theorem 13.8. *Let $\mathcal{A}$ be a subspace of $\mathcal{F}$. Assume that $1 \notin \mathcal{A}$. Then the following conditions are equivalent:*
(T1) $1 \notin (\mathcal{A})_{\mathcal{E}_p}$.
(T2) *For any $x \in X$, there exists $c > 0$ such that $u(x)^2 \leq c\mathcal{E}(u,u)$ for any $u \in \mathcal{A}$.*
(T3) $\mathrm{Cap}_{\mathcal{A}}(\mathrm{K}) > 0$ *for any non-empty compact subset $K \subseteq X$.*
(T4) $\mathrm{Cap}_{\mathcal{A}}(\mathrm{K}) > 0$ *for some non-empty compact subset $K \subseteq X$.*

Proof. (T1) ⇒ (T3): Assume that $\mathrm{Cap}_{\mathcal{A}}(\mathrm{K}) = 0$ for a non-empty compact set K. Then $\mathcal{A}_K \neq \emptyset$. Choose $x \in K$. There exists $u_n \in \mathcal{A}$ such that $u_n(x) = 1$ for any n and $\mathcal{E}(u_n, u_n) \to 0$ as $n \to \infty$. Define $v_n = u_n - u_n(p)$. Then $\{v_n\}$ is a Cauchy sequence with respect to $\mathcal{E}_p$. Let v be the limit of $\{v_n\}$. Since $v_n(x) = u_n(x) - u_n(p) = 1 - u_n(p) \to v(x)$ as $n \to \infty$, $\{u_n(p)\}$ converges to some $\alpha \in \mathbb{R}$ as $n \to \infty$. Hence $u_n = v_n + u_n(p)$ converges to $v + \alpha$ with respect to $\mathcal{E}_p$. Now $\mathcal{E}(v + \alpha, v + \alpha) = \lim_{n\to\infty} \mathcal{E}(u_n, u_n) = 0$. Therefore $v + \alpha$ is a constant. Note that $v(x) + \alpha = 1$, we see that $\{u_n\}$ converges to 1 as $n \to \infty$. Hence $1 \in (\mathcal{A})_{\mathcal{E}_p}$.
(T3) ⇒ (T4): This is obvious.
(T4) ⇒ (T1): Assume $1 \in (\mathcal{A})_{\mathcal{E}_p}$. Then there exists $\{u_n\}_{n\geq 1} \subset \mathcal{A}$ such that $\mathcal{E}_p(1 - u_n, 1 - u_n) \to 0$ as $n \to \infty$. Let $x \in K$. Then $u_n(x) \to 1$ as $n \to \infty$. Let $M = \mathrm{diam}((K, R))$. Then

$$|u_n(x) - u_n(y)|^2 \leq M\mathcal{E}(u_n, u_n)$$

for any $y \in K$. Hence we may choose $N > 0$ such that $2u_n \in \mathcal{A}_K$ for any $n \geq N$. Since $\mathcal{E}(2u_n, 2u_n) \to 0$ as $n \to \infty$, we have $\mathrm{Cap}_{\mathcal{A}}(\mathrm{K}) = 0$.
(T1) $\Rightarrow$ (T2): If (T2) does not hold, then there exists $\{u_n\} \subset \mathcal{A}$ which has exactly the same properties as in the implication of (T1) $\Rightarrow$ (T3). The rest of the argument is entirely the same.
(T2) $\Rightarrow$ (T4): This is obvious. □

Definition 13.9. Let $\mathcal{A}$ be a subspace of $\mathcal{F}$. We say that $(\mathcal{E}, \mathcal{F})$ is $\mathcal{A}$-transient if and only if $1 \notin (\mathcal{A})_{\mathcal{E}_p}$ and denote $(\mathcal{A})_{\mathcal{E}_p} + 1$ by $\mathcal{F}_*(\mathcal{A})$. In particular, we say $(\mathcal{E}, \mathcal{F})$ is transient if it is $\mathcal{F} \cap C_0(X)$-transient, where $C_0(X)$ is the collection of continuous functions with compact supports with respect to the resistance metric. We write $\mathcal{F}_* = \mathcal{F}_*(\mathcal{F} \cap C_0(X))$.

In Example13.12, we have non-trivial example of $\mathcal{A}$ other than $\mathcal{F} \cap C_0(X)$. If $(\mathcal{E}, \mathcal{F})$ is the resistance form associated with a weighted graph (V, C), then $C_0(V) \subseteq \mathcal{F}$ and hence (V, C) is transient if and only if $(\mathcal{E}, \mathcal{F})$ is transient.

If $(\mathcal{E}, \mathcal{F})$ is $\mathcal{A}$-transient, it is natural to expect that $(\mathcal{E}, \mathcal{F}_*(\mathcal{A}))$ is a resistance form on $X \cup \{I_{\mathcal{A}}\}$, where $I_{\mathcal{A}}$ should correspond to the "infinity". To realize such a statement, we need to assume an additional property of $\mathcal{A}$ which is an extension of the Markov property.

Definition 13.10. Let $\mathcal{A}$ be a subspace of $\mathcal{F}$. $\mathcal{A}$ is said to have the extended Markov property if and only if $\overline{u+a} - \overline{a} \in \mathcal{A}$ for any $u \in \mathcal{A}$ and any $a \in \mathbb{R}$.

It is easy to see that $\mathcal{F} \cap C_0(X)$ has the extended Markov property. Note that if $\mathcal{A}$ has the extended Markov property, then both $\mathcal{A}$ and $\mathcal{A} + 1$ have the Markov property.

Theorem 13.11. *Let $\mathcal{A}$ be a linear subspace of $\mathcal{F}$ which separates points in X and has the extended Markov property. Assume that $(\mathcal{E}, \mathcal{F})$ is $\mathcal{A}$-transient. Let $I_{\mathcal{A}} \notin X$ and extend $u \in \mathcal{F}_*(A)$ to $u : X \cup \{I_{\mathcal{A}}\} \to \mathbb{R}$ by setting $u(I_{\mathcal{A}}) = a$, where $u = f + a$ for $f \in (A)_{\mathcal{E}_p}$ and $a \in \mathbb{R}$. Then $(\mathcal{E}, \mathcal{F}_*(\mathcal{A}))$ is a resistance form on $X \cup \{I_{\mathcal{A}}\}$.*

Proof. Write $I = I_{\mathcal{A}}$. Since $\mathcal{A} + 1$ has the Markov property, $(\mathcal{E}, \mathcal{F}_*(\mathcal{A}))$ is a resistance form on X by Proposition 13.6. We show that $(\mathcal{E}, \mathcal{F}_*(\mathcal{A}))$ is a resistance form on $X \cup \{I\}$. (RF1), (RF2) and (RF3) are immediate. Since $u - u(I) \in (\mathcal{A})_{\mathcal{E}_p}$ for any $u \in \mathcal{F}_*(A)$, (T2) of Theorem 13.8 implies that

$$\sup\left\{\frac{|u(x) - u(I)|^2}{\mathcal{E}(u, u)}\middle| u \in \mathcal{F}_*(\mathcal{A}), \mathcal{E}(u, u) > 0\right\} < +\infty.$$

This yields (RF4). Let $u \in \mathcal{F}_*(\mathcal{A})$ and set $v = u - u(I)$. Then $v \in (\mathcal{A})_{\mathcal{E}_p}$ and $\overline{u} - \overline{u(I)} = \overline{v + u(I)} - \overline{u(I)} \in (\mathcal{A})_{\mathcal{E}_p}$. Hence $\pi(\overline{u}) = \overline{u(I)}$ and we have (RF5). □

Example 13.12. Let (T, C) be the (complete infinite) binary tree defined in Section 8 and let $(\mathcal{E}, \mathcal{F})$ be the associated resistance form on T. Define $\mathcal{A} = \{u | u \in \mathcal{F}, \#\{w | w = w_1 \ldots w_m \in T, w_1 = 1, u(w) \neq 0\} < +\infty\}$. Then $\mathcal{A}$ satisfies the extended Markov property and separates points in T. Let $C = C_S$, where C_S is the self-similar weight defined in Definition 8.3. Then by the similar arguments as in the proof of Theorem 8.4, it follows that $(\mathcal{E}, \mathcal{F})$ is $\mathcal{A}$-transient if and only if $r_1 r_2/(r_1 + r_1) < 1$. Also we have $\mathcal{F}_* = \mathcal{F}_*(C_0(V)) \subseteq \mathcal{F}_*(\mathcal{A}) \subseteq \mathcal{F}$ and none of the equalities hold.

14 Appendix B: moments of displacement

In this section, we will consider general Hunt processes on a compact metric space (X, d). Let μ be a Borel regular finite measure on (X, d). Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X, \mu)$. We use $(\{X_t\}_{t>0}, \{P_x\}_{x\in X})$ to denote the associated Hunt process on X.

Assumptions (1) There exists a jointly continuous transition density $p(t, x, y)$ on $(0, \infty) \times X \times X$ such that

$$E_x(f(X_t)) = \int_X p(t, x, y)f(y)\mu(dy)$$

for any $x \in X$, any $t > 0$ and any bounded μ-measurable function $f : X \to \mathbb{R}$.
(2) There exist $c_1, c_2, c_3 \in (1, \infty)$ and $R > 0$ such that

$$c_1\mu(B_d(x, r)) \le \mu(B_d(x, c_3 r)) \le c_2\mu(B_d(x, r)) \tag{14.1}$$

for any $x \in X$ and $r \in (0, R]$.
(3) There exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that the following transition density estimate holds, i.e. if we define $q_*(t, x, y)$ by

$$q_*(t, x, y) = \begin{cases} \dfrac{t}{\phi\big(d(x, y)\big)\mu\big(B_d(x, d(x, y))\big)} & \text{if } t \le \phi(d(x, y)), \\ \dfrac{1}{\mu\big(B_d(x, \phi^{-1}(t))\big)} & \text{if } 0 < \phi(d(x, y)) \le t, \end{cases}$$

then

$$p(t, x, y) \asymp q_*(t, x, y) \tag{14.2}$$

for any $x, y \in X$ and any $t \in (0, 1]$.
(4) There exist $c_4, c_5, c_6 \in (1, \infty)$ such that

$$c_4\phi(r) \le \phi(c_5 r) \le c_6\phi(r) \tag{14.3}$$

for any $r \ge 0$.

Remark. (14.2) may be thought of as one of typical heat kernel estimates for jump processes. In [3], Chen and Kumagai have studied a Hunt process associated with a Dirichlet form $(\mathcal{E}, \mathcal{F})$ defined by

$$\mathcal{E}(u, u) = \int_X (u(x) - u(y))^2 J(x, y)\mu(dx)\mu(dy),$$

where $J(x,y)$ is a given jump kernel. Assuming that $\mu(B_d(x,r)) \asymp V(r)$ for some strictly monotone function $V : [0,\infty) \to [0,\infty)$, they have shown that (14.2) holds if $J(x,y)^{-1} \asymp \phi(d(x,y))V(d(x,y))$.

Theorem 14.1. *Suppose that the above four assumptions hold. Then*

$$E_x(\phi(d(x,X_t))^\gamma) \asymp \begin{cases} t & \text{if } \gamma > 1, \\ t(|\log t| + 1) & \text{if } \gamma = 1, \\ t^\gamma & \text{if } 0 < \gamma < 1, \end{cases} \tag{14.4}$$

for any $t \in (0,1]$ and any $x \in X$.

The $|\log t| + 1$ term in the case of $\gamma = 1$ is due to the (relatively) slow decay of off-diagonal part $t/(\phi(d(x,y))\mu(B_d(x,d(x,y)))$ of the heat kernel estimate with respect to the space variable x. Note that the decay of the off-diagonal part is exponential if a heat kernel enjoys (sub-)Gaussian asymptotic behavior which is typical in many cases of diffusion processes, for example, the Brownian motion on Euclidean domain and the Sierpinski gasket. In such a case, there appears no $|\log t| + 1$ term in moments of displacement.

Remark. Suppose that (X,d) is not bounded. If Assumption (1) holds, (14.1) holds with $R = \infty$, (14.2) holds for any $x,y \in X$ and any $t > 0$ and (14.4) hold, then an analogous argument as the proof of Theorem 14.1 shows

$$E_x(\phi(d(x,X_t))^\gamma) = \begin{cases} +\infty & \text{if } \gamma \geq 1, \\ t^\gamma & \text{if } 0 < \gamma < 1. \end{cases}$$

We use $V(x,r) = \mu(B_d(x,r))$.

Lemma 14.2. *Suppose that the above four assumptions hold. Then there exists $\alpha, \beta, c \in (1,\infty)$ and $T > 0$ such that*

$$\beta V(x,\phi^{-1}(s)) \leq V(x,\phi^{-1}(\alpha s)) \leq cV(x,\phi^{-1}(s))$$

for any $x \in X$ and any $s \in (0,T]$.

Proof of Theorem 14.1. Note that

$$E_x(\phi(d(x,X_t))^\gamma) = \int_X p(t,x,y)\phi(d(x,y))^\gamma \mu(dy).$$

Since $p(t,x,y)$ is jointly continuous, it is enough to show (14.4) for $t \in (0,T]$. We divide the domain X of the above integral into $X_1 = \{y|\phi(d(x,y)) > t\}$ and $X_2 = \{y|\phi(d(x,y)) \leq t\}$.

Define $N = \max\{n|n \in \mathbb{Z}, \alpha^n t < T\}$. Set $Y_n = \{y|\alpha^n t \leq \phi(d(x,y)) < \alpha^{n+1}t\}$ for $n < N$ and $Y_N = \{y|\alpha^N t \leq \phi(d(x,y))\}$. By (14.2),

$$\begin{aligned} \int_{X_1} p(t,x,y)\phi(d(x,y))^\gamma \mu(dy) &\asymp \int_{X_1} \frac{t\phi(d(x,y))^{\gamma-1}}{V(x,d(x,y))}\mu(dy) \\ &= \sum_{n=0}^{N} \int_{Y_n} \frac{t\phi(d(x,y))^{\gamma-1}}{V(x,d(x,y))}\mu(dy) \end{aligned} \tag{14.5}$$

Now, for $n = 0, 1, \ldots, N-1$,

$$t\frac{(\alpha^n t)^{\gamma-1}(V(x,\phi^{-1}(\alpha^{n+1}t)) - V(x,\phi^{-1}(\alpha^n t)))}{V(x,\phi^{-1}(\alpha^{n+1}t))} \le \int_{Y_n} \frac{\phi(d(x,y))^{\gamma-1}}{V(x,d(x,y))}\mu(dy)$$
$$\le t\frac{(\alpha^{n+1}t)^{\gamma-1}(V(x,\phi^{-1}(\alpha^{n+1}t)) - V(x,\phi^{-1}(\alpha^n t)))}{V(x,\phi^{-1}(\alpha^n t))}. \quad (14.6)$$

By Lemma 14.2, we have

$$\int_{Y_n} \frac{\phi(d(x,y))^{\gamma-1}}{V(x,d(x,y))}\mu(dy) \asymp t(\alpha^n t)^{\gamma-1} \quad (14.7)$$

for $n = 0, 1, \ldots, N-1$. Using the similar arguments, we confirm that (14.7) is valid for $n = N$. Hence by (14.5), (14.6) and (14.7),

$$\int_{X_1} p(t,x,y)\phi(d(x,y))^{\gamma}\mu(dy) \asymp \sum_{n=0}^{N} t(\alpha^n t)^{\gamma-1} \asymp \begin{cases} t & \text{if } \gamma > 1, \\ t(|\log t| + 1) & \text{if } \gamma = 1, \\ t^{\gamma} & \text{if } 0 < \gamma < 1. \end{cases} \quad (14.8)$$

Next let $Z_n = \{y | \alpha^{-n} t > \phi(d(x,y)) \le \alpha^{-(n+1)} t\}$ for $n \ge 0$. Then

$$\begin{aligned} &\int_{X_2} p(t,x,y)\phi(d(x,y))^{\gamma}\mu(dy) \\ &\asymp \int_{X_2} \frac{\phi(d(x,y))^{\gamma}}{V(x,\phi^{-1}(t))}\mu(dy) = \sum_{n=0}^{\infty}\int_{Z_n} \frac{\phi(d(x,y))^{\gamma}}{V(x,\phi^{-1}(t))}\mu(dy) \\ &\asymp \sum_{n=0}^{\infty} \frac{(\alpha^{-n}t)^{\gamma}(V(x,\phi^{-1}(\alpha^{-n}t)) - V(x,\phi^{-1}(\alpha^{-(n+1)}t))}{V(x,\phi^{-1}(t))} \\ &\le \sum_{n=0}^{\infty} (\alpha^{-n}t)^{\gamma} \le ct^{\gamma}. \end{aligned} \quad (14.9)$$

Combining (14.8) and (14.9), we obtain (14.4). □

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