Harmonic Analysis for Resistance Forms

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Abstract

In this paper, we define the Green functions for a resistance form by using effective resistance and harmonic functions. Then the Green functions and harmonic functions are shown to be uniformly Lipschitz continuous with respect to the resistance metric. Making use of this fact, we construct the Green operator and the (measure valued) Laplacian. The domain of the Laplacian is shown to be a subset of uniformly Lipschitz continuous functions while the domain of the resistance form in general consists of uniformly 1/2-Hölder continuous functions.

1 Introduction

The theory of resistance forms has been developed as the foundation of analysis on post critically finite self-similar sets. See [16] for example. It should correspond to a part of potential theory where each point has a positive capacity. In this paper, for a resistance form, we give a simple definition of the Green function associated with a boundary consisting of any finite number of points and show that the Green function is always uniformly Lipschitz continuous with respect to the distance given by the effective resistance. Then we will follow ramifications of this fact.

More precisely, let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a set X. Then there exists a natural distance R on X associated with $(\mathcal{E}, \mathcal{F})$. R is called the resistance metric. See Section 2 for details. In Section 4, we will define the Green function $g_B : X \times X \to [0, +\infty)$, where B is a non-empty finite subset of X. In Proposition 4.3, g_B is characterized as a reproducing kernel of $(\mathcal{E}, \mathcal{F}_B)$, where $\mathcal{F}_B = \{u | u \in F, u|_B \equiv 0\}$: define $g_B^x : X \to [0, +\infty)$ by $g_B^x(y) = g_B(x, y)$, then

$$\mathcal{E}(u, g_B^x) = u(x)$$

for any $u \in \mathcal{F}_B$ and any $x \in X$. Next in Theorem 4.5, the Green function g_B is shown to be uniformly Lipschitz continuous with respect to the resistance metric R. Precisely,

$$|g_B(x, y) - g_B(x, z)| \leq R(y, z)$$

for any x, y and z. This also implies that harmonic functions are uniformly Lipschitz continuous with respect to the resistance metric. In Section 5, we will define the Green operator G_B from measures on X to $\mathcal{F}_B$. (Assuming that (X, R) is compact for simplicity, we mean the dual space of the continuous functions on (X, R) by measures.) Then we will define the domain of the Laplacian in the generalized (or universal) sense, $\mathcal{D}^L$, by $\mathcal{D}^L = \mathrm{Im}(G_B) \oplus \mathcal{H}_B$, where $\mathcal{H}_B$ is the collection of harmonic functions on X with respect to the boundary B. (We use the word "generalized" (or universal) sense because the image of the Laplacian is measures in general. In [22], we can find an idea of the measure valued Laplacian in the case of post critically finite self-similar sets.) In fact, $\mathcal{D}^L$ is shown to be independent of B in Theorem 5.5. Moreover we will see that every element of $\mathcal{D}^L$ is uniformly Lipschitz continuous with respect the resistance metric. Also in Section 6, any $u \in \mathcal{D}^L$ is shown to have the Neumann derivative $(du)_x$ for any $x \in X$. These facts will lead us to the definition of Laplacians in the generalized sense and we will have the Dirichlet Laplacian with boundary B, L_B and the Neumann Laplacian L. $\mathcal{D}^L$ is the domain of both L_B and L. Furthermore, in Theorem 6.8, we have the following expression of the resistance form:

$$\mathcal{E}(u, v) = \sum_{p \in B} u(p)(dv)_p - (L_B v)(u)$$

for any $u \in \mathcal{F}$ and any $v \in \mathcal{D}^L$. (We also obtain the counterpart of this for the Neumann Laplacian L.) Note that all the notions (i.e. the Green function g_B, the Green operator G_B, the domain of Laplacians $\mathcal{D}^L$, the Neumann derivative $(du)_p$ and the Laplacians L_B and L) are independent of measures. In this sense, these are "universal" objects.

In Section 8, we will introduce a measure μ on X and consider measures which are absolutely continuous with respect to μ. Then the Green operator G_B is realized as $G_{B,\mu} : L^1(X, \mu) \to \mathcal{F}_B$. In the course of discussions, we will finally show that the restriction of the "universal" Laplacians are the self-adjoint operator coming from the Dirichlet form (the closed form) on $L^2(X, \mu)$ associated with the resistance form $(\mathcal{E}, \mathcal{F})$.

In Section 9, we will apply the results in the previous sections to a self-similar resistance form given by a regular harmonic structure on a post critically finite (p. c. f. for short) self-similar structure. (Such a resistance form is discussed in [16] in detail.) Indeed, by using probabilistic method, it has already shown that the Green function, harmonic functions and the elements in the domain of the Laplacian are uniformly Lipschitz continuous with respect to the resistance metric. See [6], [18], [8] and [13] for example. They first establish a detailed short time offdiagonal estimate of the heat kernel with respect to a special self-similar measure ν, which is determined by the harmonic structure, and then show the uniform Lipschitz continuity of the above mentioned functions. Since this method depends on the special measure ν, one can only know that the elements in the domain of the ν-Laplacian are uniformly Lipschitz continuous. In contrast, our method in this paper do not require any measure and hence the discussions are more direct and simple. Moreover the elements in $\mathcal{D}^L$ (the domain of universal Laplacians) are shown to be uniformly Lipschitz continuous.

Also in Section 9, we will obtain relations between $\mathcal{F}$, $\mathcal{D}^L$ and C_L, where C_L is the collection of uniformly Lipschitz continuous functions. In particular, we show that $\mathcal{D}^L \subset \mathcal{F} \cap \mathcal{C}_L$ but $\mathcal{D}^L \neq \mathcal{F} \cap C_L$.

We can apply the results in this paper to other classes of fractals: the Sierpinski carpets in $\mathbb{R}^2$ studied by Barlow-Bass [2, 3, 4, 5] and Kusuoka-Zhou [19], the randomaized self-similar sets studied by Hambly [10, 11, 12] and the Markov (graph directed) p. c. f. self-similar sets by Hambly-Nyberg [14] and Kigami-Strichartz-Walker [17]. For these three classes of fractals, one can construct a regular local Dirichlet form with a certain kind of self-similarity and those forms are known to be resistance forms.

Although we only consider finite sets as boundaries in this paper, it is interesting to study the general case where the boundary can be an infinite set. For the Sierpinski gasket, such a case has been studied partly in [21]. The first question should be to determine a proper class of sets which can be thought of as boundaries. This problem is worth exploring in the future.

2 Resistance form and harmonic functions

In this section, we will briefly review the theory of Dirichlet forms and Laplacians on finite sets and resistance forms. See [16, Chapter 2] for details and complete proofs.

Notation. For a set V, we define $\ell(V) = \{f | f : V \to \mathbb{R}\}$. If V is a finite set, $\ell(V)$ is considered to be equipped with the standard inner product $(\cdot, \cdot)_V$ defined by $(u, v)_V = \sum_{p \in V} u(p)v(p)$ for any $u, v \in \ell(V)$. Also $|u|_V = \sqrt{(u, u)_V}$ for any $u \in \ell(V)$.

First we give a definition of Dirichlet forms on a finite set V.

Definition 2.1 (Dirichlet forms). Let V be a finite set. A symmetric bilinear form on $\ell(V)$, $\mathcal{E}$ is called a Dirichlet form on V if it satisfies

(DF1) $\mathcal{E}(u, u) \geq 0$ for any $u \in \ell(V)$,

(DF2) $\mathcal{E}(u, u) = 0$ if and only if u is constant on V

and

(DF3) For any $u \in \ell(V)$, $\mathcal{E}(u, u) \geq \mathcal{E}(\bar{u}, \bar{u})$, where $\bar{u}$ is defined by

$$\bar{u}(p) = \begin{cases} 1 & \text{if } u(p) \geq 1, \\ u(p) & \text{if } 0 < u(p) < 1, \\ 0 & \text{if } u(p) \leq 0. \end{cases}$$

We use $\mathcal{DF}(V)$ to denote the collection of Dirichlet forms on V.

Condition (DF3) is called the Markov property.

Notation. Let V be a finite set. The characteristic function χ_U^V of a subset $U \subseteq V$ is defined by

$$\chi_U^V(q) = \begin{cases} 1 & \text{if } q \in U, \\ 0 & \text{otherwise.} \end{cases}$$

If no confusion can occur, we write χ_U instead of χ_U^V. If $U = \{p\}$ for a point $p \in V$, we write χ_p instead of $\chi_{\{p\}}$. If $H : \ell(V) \to \ell(V)$ is a linear map, then we set $H_{pq} = (H\chi_q)(p)$ for $p, q \in V$. For $f \in \ell(V)$, $(Hf)(p) = \sum_{q \in V} H_{pq} f(q)$.

Definition 2.2 (Laplacians). A symmetric linear operator $H : \ell(V) \to \ell(V)$ is called a Laplacian on V if it satisfies

(L1) H is non-positive definite,

(L2) $Hu = 0$ if and only if u is a constant on V,

and

(L3) $H_{pq} \geq 0$ for all $p \neq q \in V$.

We use $\mathcal{LA}(V)$ to denote the collection of Laplacians on V.

For $H \in \mathcal{LA}(V)$, define a bilinear form $\mathcal{E}_H$ on $\ell(V)$ by $\mathcal{E}_H(u, v) = -(u, Hv)$. Then $\mathcal{E}_H$ is a Dirichlet form on V. This map from $\mathcal{LA}(V)$ to $\mathcal{DF}(V)$ gives a natural bijective correspondence between $\mathcal{LA}(V)$ and $\mathcal{DF}(V)$.

We may also associate an electrical network on V consisting of resistances to a Laplacian $H \in \mathcal{LA}(V)$. Let $H \in \mathcal{LA}(V)$. For any $p, q \in V$ with $p \neq q$, set $R_{pq} = (H_{pq})^{-1}$ and attach a resistor of resistance R_{pq} between terminals p and q. If electrical potentials of p and q are $v(p)$ and $v(q)$ respectively, then the current from q to p is $c_{pq} = (R_{pq})^{-1}(v(q) - v(p))$. Hence the total current at p is $\sum_{q \in V} H_{pq}(v(q) - v(p)) = (Hv)(p)$. (Note that $H_{pp} = -\sum_{q \in V \setminus p} H_{pq}$ for any Laplacian.)

Definition 2.3 (Effective resistance). Let $H \in \mathcal{LA}(V)$. For any $p, q \in V$ with $p \neq q$, define

$$R_H(p, q) = \big(\min\{\mathcal{E}_H(u, u) | u \in \ell(V), u(p) = 1, u(q) = 0\}\big)^{-1}.$$

Also define $R_H(p, p) = 0$ for any $p \in V$. $R_H(p, q)$ is called the effective resistance between p and q with respect to H.

$R_H(p, q)$ is the actual resistance between p and q considering all the resistors associated with a Laplacian H. The remarkable fact is that $R_H(\cdot, \cdot)$ is a distance on V.

Proposition 2.4. *Let $H \in \mathcal{LA}(V)$, then $R_H(\cdot, \cdot)$ is a distance on V.*

Definition 2.5. (1) Let V_1 and V_2 be finite sets and let $H_i \in \mathcal{LA}(V_i)$ for $i = 1, 2$. We write $(V_1, H_1) \leq (V_2, H_2)$ if and only if $V_1 \subseteq V_2$ and, for any $u \in \ell(V_1)$,

$$\mathcal{E}_{H_1}(u, u) = \min\{\mathcal{E}_{H_2}(v, v) | v \in \ell(V_2), v|_{V_1} = u\}.$$

(2) Let V_i be a finite set for $i = 0, 1, 2, \cdots$ and let $H_i \in \mathcal{LA}(V_i)$ for any $i \geq 0$. $\mathcal{S} = \{(V_m, H_m)\}_{m \geq 0}$ is called a compatible sequence if and only if $(V_m, H_m) \leq (V_{m+1}, H_{m+1})$ for any $m \geq 0$.

If $(V_1, H_1) \leq (V_2, H_2)$, then it is easy to see that $R_{H_1}(p,q) = R_{H_2}(p,q)$ for any $p, q \in V_1$. In fact, the converse is also true.

Proposition 2.6. *Let V_1 and V_2 be finite sets and let $H_i \in \mathcal{LA}(V_i)$ for $i = 1, 2$. Assume that $V_1 \subseteq V_2$. Divide H_2 into four parts:*

$$H_2 = \begin{pmatrix} T & {}^tJ \\ J & X \end{pmatrix},$$

where $T : \ell(V_1) \to \ell(V_1), J : \ell(V_1) \to \ell(V_2\backslash V_1)$ and $X : \ell(V_2\backslash V_1) \to \ell(V_2\backslash V_1)$. Then the following three conditions are equivalent.
(1) $(V_1, H_1) \leq (V_2, H_2)$.
(2) $H_1 = T - {}^tJX^{-1}J$.
(3) $R_{H_2}|_{V_1\times V_1} = R_{H_1}$.

Remark. X in the above proposition is known to be negative definite. See [16, Lemma 2.1.5] for details.

Definition 2.7. Let $\mathcal{S} = \{(V_m, H_m)\}_{m\geq 0}$ be a compatible sequence. Then, define $V_* = \cup_{m\geq 0} V_m$,

$$\mathcal{F}(\mathcal{S}) = \{u | u : V_* \to \mathbb{R}, \lim_{m\to\infty} \mathcal{E}_{H_m}(u|_{V_m}, u|_{V_m}) < \infty\}.$$

For any $u, v \in \mathcal{F}(\mathcal{S})$, define

$$\mathcal{E}_{\mathcal{S}}(u, v) = \lim_{m\to\infty} \mathcal{E}_{H_m}(u|_{V_m}, u|_{V_m}).$$

Also for any $p, q \in V_*$, define

$$R_{\mathcal{S}}(p, q) = R_{H_m}(p, q),$$

where m is chosen so that $p, q \in V_m$.

Remark. Since $\mathcal{S}$ is a compatible sequence, $\mathcal{E}_{H_m}(u|_{V_m}, u|_{V_m})$ is monotonically increasing. So the limit appearing in the definition of $\mathcal{F}(\mathcal{S})$ does exists if we allow ∞ as the value of the limit.

By Proposition 2.6, the definition of $R_{\mathcal{S}}$ is well-defined. Also Proposition 2.4, implies that $R_{\mathcal{S}}(\cdot, \cdot)$ is a distance on V_*. Note that V_* is merely a countable set. Considering the completion of a metric space $(V_*, R_{\mathcal{S}})$, however, we may get an uncountable set. In fact, $(\mathcal{E}_{\mathcal{S}}, \mathcal{F}(\mathcal{S}))$ is a resistance form on V_* defined below. Hence Theorem 2.12 justifies the completion of $(V_*, R_{\mathcal{S}})$.

Definition 2.8 (Resistance form). Let X be a set. A pair $(\mathcal{E}, \mathcal{F})$ is called a resistance form on X if it satisfies the following conditions (RF1) through (RF5).
(RF1) $\mathcal{F}$ is a linear subspace of $\ell(X)$ containing constants and $\mathcal{E}$ is a non-negative symmetric quadratic form on $\mathcal{F}$. $\mathcal{E}(u, u) = 0$ if and only if u is constant on X.
(RF2) Let $\sim$ be an equivalent relation on $\mathcal{F}$ defined by $u \sim v$ if and only if

$u - v$ is constant on X. Then $(\mathcal{F}/\sim, \mathcal{E})$ is a Hilbert space.
(RF3) For any finite subset $V \subset X$ and for any $v \in \ell(V)$, there exists $u \in \mathcal{F}$ such that $u|_V = v$.
(RF4) For any $p, q \in X$,

$$\sup\{\frac{|u(p) - u(q)|^2}{\mathcal{E}(u,u)} : u \in \mathcal{F}, \mathcal{E}(u,u) > 0\}$$

is finite. The above supremum is denoted by $R_{(\mathcal{E},\mathcal{F})}(p,q)$.
(RF5) If $u \in \mathcal{F}$, then $\bar{u} \in \mathcal{F}$ and $\mathcal{E}(\bar{u},\bar{u}) \leq \mathcal{E}(u,u)$, where $\bar{u}$ is defined in the same manner as (DF3) in Definition 2.1.

We use $\mathcal{RF}(X)$ to denote the collection of resistance forms on X.

Condition (RF5) is called the Markov property. By (RF5), we obtain the following lemma.

Lemma 2.9. *For real valued functions u and v on X, define $u \vee v$ and $u \wedge v$ by*

$$(u \vee v)(x) = \max\{u(x), v(x)\} \quad \textit{and} \quad (u \wedge v)(x) = \min\{u(x), v(x)\}$$

for any $x \in X$. Let $(\mathcal{E},\mathcal{F})$ be a resistance form on X. Then $u \vee v$ and $u \wedge v$ belong to $\mathcal{F}$ for any $u, v \in \mathcal{F}$.

Proposition 2.10. *Let $(\mathcal{E},\mathcal{F})$ be a resistance form on a set X. Then, for any $p, q \in X$, the supremum in* (RF4) *is the maximum. Moreover, for any finite set $V \subseteq X$, there exists a unique $H_V \in \mathcal{LA}(V)$ such that $R_{H_V} = R_{(\mathcal{E},\mathcal{F})}|_{V\times V}$. In particular, $R_{(\mathcal{E},\mathcal{F})}$ is a distance of X.*

Definition 2.11. Let $(\mathcal{E},\mathcal{F})$ be a resistance form on a set X. $R_{(\mathcal{E},\mathcal{F})}$ is called the resistance metric on X associated with the resistance form $(\mathcal{E},\mathcal{F})$ on X.

If no confusion can occur, we write $R_{(\mathcal{E},\mathcal{F})} = R$.

Let $(\mathcal{E},\mathcal{F})$ be a resistance form on X and let R be the associated resistance metric on X. Then by (RF4), for any $u \in \mathcal{F}$ and any $p, q \in X$,

$$R(p,q)\mathcal{E}(u,u) \geq |u(p) - u(q)|^2. \tag{2.1}$$

Hence every $u \in \mathcal{F}$ is uniformly 1/2-Hölder continuous with respect to R. So, if Ω is the completion of X with respect to R, then any $u \in \mathcal{F}$ is naturally extended to a continuous function on Ω. Using this extension, we may always regard $\mathcal{F}$ as the collection of functions on Ω.

Theorem 2.12. *Let $(\mathcal{E},\mathcal{F})$ be a resistance form on X and let R be the associated resistance metric on X. If Ω is the completion of X with respect to R, then $(\mathcal{E},\mathcal{F})$ is a resistance form on Ω. Moreover, the resistance metric associated with $(\mathcal{E},\mathcal{F})$ on Ω is the natural extension of the resistance metric R associated with $(\mathcal{E},\mathcal{F})$ on X.*

By the virtue of this theorem, if $(\mathcal{E},\mathcal{F})$ is a resistance form on X and R is the associated resistance metric, then (X, R) may be assumed to be complete.

Theorem 2.13. *Let $\mathcal{S} = \{(V_m, H_m)\}_{m\geq 0}$ be a compatible sequence. Then $(\mathcal{E}_\mathcal{S}, \mathcal{F}(\mathcal{S}))$ is a resistance form on V_* and the associated resistance metric coincides with $R_\mathcal{S}$. Moreover, if (Ω, R) is the completion of $(V_*, R_\mathcal{S})$, then (Ω, R) is separable and R is the resistance metric associated with the resistance form $(\mathcal{E}_\mathcal{S}, \mathcal{F}(\mathcal{S}))$ on Ω.*

By Theorem 2.13, from a compatible sequence, we can construct a resistance form on a set which is complete and separable under the associated resistance metric. The next theorem shows that the converse is also true.

Theorem 2.14. *Let $(\mathcal{E}, \mathcal{F})$ is a resistance form on X and let R be the associated resistance metric on X. Assume that (X, R) is separable. If $\{V_m\}_{m\geq 0}$ is a increasing sequence of finite subsets of X, then $\mathcal{S} = \{(V_m, H_{V_m})\}_{m\geq 0}$, where H_{V_m} is defined in Proposition 2.10, is a compatible sequence and $R_\mathcal{S} = R$ on $\cup_{m\geq 0} V_m$. In particular, if $\cup_{m\geq 0} V_m$ is dense in X, then $R_\mathcal{S} = R$ and $(\mathcal{E}, \mathcal{F}) = (\mathcal{E}_\mathcal{S}, \mathcal{F}(\mathcal{S}))$.*

Next we will define the notion of harmonic functions.

Proposition 2.15. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X and let V be a finite subset of X. Then for any $\rho \in \ell(V)$, there exists a unique $u \in \mathcal{F}$ such that $u|_V = \rho$ and*

$$\mathcal{E}(u, u) = \mathcal{E}_{H_V}(\rho, \rho) = \min\{\mathcal{E}(v, v) | v \in \mathcal{F}, v|_V = \rho\}.$$

Moreover, u is the unique element of $\mathcal{F}$ that satisfies, for any finite set $U \subseteq X$ containing V,

$$\begin{cases} H_U u|_{U\setminus V} & = 0 \\ u|_V & = \rho \end{cases}. \tag{2.2}$$

Denoting u appearing in the above theorem by $h_V(\rho)$, we see that $h_V : \ell(V) \to \mathcal{F}$ is linear.

Definition 2.16. Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X and let V be a finite subset of X. We define $\mathcal{H}_V = \mathrm{Im}(h_V)$. An element of $\mathcal{H}_V$ is called a V-harmonic function with respect to $(\mathcal{E}, \mathcal{F})$. More precisely, if $u = h_V(\rho)$ for $\rho \in \ell(V)$, then u is called the V-harmonic function with boundary value ρ with respect to $(\mathcal{E}, \mathcal{F})$. Also, for any $p \in V$, $h_V(\chi_p^V)$ is denoted by ψ_p^V.

It is easy to see that $\mathcal{H}_V$ is spanned by $\{\psi_p^V\}_{p\in V}$. In fact, $u = \sum_{p\in V} u(p)\psi_p^V$ for any $u \in \mathcal{H}_V$.

The second characterization of harmonic functions, (2.2), immediately implies the following proposition.

Proposition 2.17. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X. If U and V are finite subsets of U and $V \subseteq U$, then $\mathcal{H}_V \subseteq \mathcal{H}_U$.*

We also obtain the following maximum principle.

Proposition 2.18. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X and let V be a finite subset of X. If u is a V-harmonic function with respect to $(\mathcal{E}, \mathcal{F})$, then*

$$\min_{p \in V} u(p) = \min_{x \in X} u(x) \leq \max_{x \in X} u(x) = \max_{p \in V} u(p).$$

Proposition 2.19. *Let $V \neq \emptyset$ be a finite subset of X. Define $\mathcal{F}_V = \{u | u \in \mathcal{F}, u|_V \equiv 0\}$. Then $\mathcal{E}$ is an inner product on $\mathcal{F}_V$ and $(\mathcal{F}_V, \mathcal{E})$ is a Hilbert space.*

$(\mathcal{E}_V, \mathcal{F}_V)$ may be regarded as a resistance form imposed Dirichlet boundary condition on V.

Proof. If $u \in \mathcal{F}_V$ and u is constant on X, then $u = 0$ on X. Hence $\mathcal{F}_V$ can be thought of as an closed subspace of $(\mathcal{F}/\sim, \mathcal{E})$. By (RF2), $(\mathcal{F}_V, \mathcal{E})$ is complete. □

It follows that $\mathcal{H}_V \oplus \mathcal{F}_V = \mathcal{F}$. In fact, defining $P_V : \mathcal{F} \to \mathcal{H}_V$ by $P_V(u) = h_V(u|_V)$, we see that, for any $u \in \mathcal{F}, u = P_V u + (u - P_V u)$, where $P_V u \in \mathcal{H}_V$ and $u - P_V u \in \mathcal{F}_V$. Although $\mathcal{E}$ is not an inner product on $\mathcal{F}$, the following lemma says that each of $\mathcal{H}_V$ and $\mathcal{F}_V$ may be thought of as the "orthogonal complement" of the other with respect to $\mathcal{E}$.

Lemma 2.20. *Let $V \neq \emptyset$ be a finite subset of X.*
(1) *For any $u \in \mathcal{H}_V$ and any $v \in \mathcal{F}_V$,*

$$\mathcal{E}(u, v) = 0$$

(2) *For any $u, v \in \mathcal{F}$,*

$$\begin{aligned}\mathcal{E}(u, v) &= \mathcal{E}(u - P_V u, v - P_V v) + \mathcal{E}(P_V u, P_V v) \\ &= \mathcal{E}(u - P_V u, v - P_V v) + \mathcal{E}_{H_V}(u|_V, v|_V).\end{aligned}$$

where H_V is the Laplacian on V associated with $(\mathcal{E}, \mathcal{F})$.

By this lemma, if $u \in \mathcal{H}_V$ and $v \in \mathcal{F}$, then $\mathcal{E}(u, v) = \mathcal{E}_{H_V}(u|_V, v|_V)$.

Proof. (1) Since $P_V(\alpha u + v) = \alpha u$, we have $\mathcal{E}(\alpha u + v, \alpha u + v) \geq \mathcal{E}(\alpha u, \alpha u)$ for any $\alpha \in \mathbb{R}$. Therefore $\mathcal{E}(u, v) = 0$.
(2) By (1), $\mathcal{E}(u, v) = \mathcal{E}(u - P_V u, v - P_V(v)) + \mathcal{E}(P_V u, P_V v)$. Now, Proposition 2.15 implies that $\mathcal{E}(P_V u, P_V v) = \mathcal{E}_{H_V}(u|_V, v|_V)$. □

3 Resistance between a point and a set

In this section, we study resistance between a point and a set and introduce the notion of shorted resistance form.

Throughout this section, $(\mathcal{E}, \mathcal{F})$ is a resistance form on a set X and R is the associated resistance metric.

Proposition 3.1. *Let $V \neq \emptyset$ be a finite subset of X and let $p \in X \backslash V$. Define $R(p, V) = \mathcal{E}(\psi_p^{V \cup p}, \psi_p^{V \cup p})^{-1}$. Then*

$$R(p, V) = \big(\min\{\mathcal{E}(u, u) | u \in \mathcal{F}, u(p) = 1, u|_V \equiv 0\}\big)^{-1}. \tag{3.1}$$

Moreover, if $H = H_{V \cup p}$, then

$$R(p, V) = -(H_{pp})^{-1} = (\sum_{q \in V} H_{pq})^{-1}. \tag{3.2}$$

$R(p, V)$ is called the effective resistance between p and V. If $p \in V$, then we set $R(p, V) = 0$.

Proof. (3.1) is immediate by Proposition 2.15 and Definition 2.16. Also by Proposition 2.15,

$$\mathcal{E}(\psi_p^{V \cup p}, \psi_p^{V \cup p}) = \mathcal{E}_H(\chi_p^V, \chi_p^V).$$

This implies (3.2). □

Next we state three useful lemmas. The first lemma is used to prove the following two lemmas. It says that the effective resistance between two terminals is no larger than the resistance of the resistor directly attached between them.

Lemma 3.2. *Let U be a finite subset of X and let $H = H_U$ be the Laplacian on U associated with $(\mathcal{E}, \mathcal{F})$. Define $R_{pq} = (H_{pq})^{-1}$ for any $p \neq q \in U$. Then $R_{pq} \geq R(p, q)$.*

Proof. Let $W = \{p, q\}$. Then by Proposition 2.15, there exists a W-harmonic function $\psi \in \mathcal{F}$ such that $\psi(p) = 1, \psi(q) = 0$ and $R(p, q) = \mathcal{E}(\psi, \psi)^{-1}$. Note that Proposition 2.17 implies that ψ is also a U-harmonic function. Hence $\mathcal{E}(\psi, \psi) = \mathcal{E}_H(\psi|_U, \psi|_U) = \sum_{r,s \in U} H_{rs}(\psi(r) - \psi(s))^2/2 \geq H_{pq}$. Therefore, $R_{pq} \geq R(p, q)$. □

Lemma 3.3. *Let $V \neq \emptyset$ be a finite subset of X. Then, for any $p \in X$,*

$$(\#V)^{-1} \min_{q \in V} R(p, q) \leq R(p, V) \leq \min_{q \in V} R(p, q),$$

where $\#V$ is the number of elements of V.

Proof. Let $A = \{u | u \in \mathcal{F}, u(p) = 1, u|_V \equiv 0\}$. Also let $A_q = \{u | u \in \mathcal{F}, u(p) = 1, u(q) = 0\}$ for each $q \in V$. Since $A_q \supseteq A$, $R(p, q)^{-1} = \min_{u \in A_q} \mathcal{E}(u, u) \leq \min_{u \in A} \mathcal{E}(u, u) = R(p, V)^{-1}$. Hence $R(p, V) \leq \min_{q \in V} R(p, q)$.

Let $H = H_{V \cup p}$ and let $R_{pq} = (H_{pq})^{-1}$ for any $q \in V$. Then by (3.2), $R(p, V) = (\sum_{q \in V} (R_{pq})^{-1})^{-1} \geq (\#V)^{-1} \min_{q \in V} R_{pq}$. Now the required inequality follows immediately by Lemma 3.2. □

Lemma 3.4. *Let $V \neq \emptyset$ be a finite subset of X. For any $p \in V$ and $q \in X \backslash p$,*

$$0 \leq \psi_p^V(q) \leq \frac{R(q, V)}{R(p, q)}.$$

Proof. Since $\psi_p^V(q) = 0$ for $q \in V$, we may assume that $q \notin V$. Let $H = H_{V \cup q}$. By (2.2),

$$0 = (H\psi_p^V)(q) = \sum_{r \in V} H_{rq}(\psi_p(r) - \psi_p^V(q)) = H_{pq} - (\sum_{r \in V} H_{rq})\psi_p^V(q).$$

Using (3.2) and Lemma 3.2, we obtain $\psi_p^V(q) = R(q,V)H_{pq} \leq R(q,V)/R(p,q)$. □

Now we consider an electrical network shorted on a finite set.

Definition 3.5. Let $V \neq \emptyset$ be a finite subset of X. Set

$$\mathcal{F}^V = \{u | u \in \mathcal{F}, u \text{ is constant on } V.\}$$

and define a quadratic form on $\mathcal{F}^V$, $\mathcal{E}^V$ by $\mathcal{E}^V(u,v) = \mathcal{E}(u,v)$ for any $u, v \in \mathcal{F}^V$. Let $X^V = (X \backslash V) \cup b$, where b is an element of V.

X_V is the set where V is retracted to one point b. If no confusion can occur, we denote $b \in X^V$ by V. Note that $\mathcal{F}^V$ is identified with a collection of real valued functions on X^V.

Proposition 3.6. (1) *$(\mathcal{E}^V, \mathcal{F}^V)$ is a resistance form on X^V.*
(2) *Let U be a finite subset of X which contains V. Then $u \in \mathcal{F}^V$ is a U-harmonic function with respect to $(\mathcal{E}, \mathcal{F})$ if and only if u is a U^V-harmonic function with respect to $(\mathcal{E}^V, \mathcal{F}^V)$, where $U^V \subset X^V$ is defined by $U^V = U \backslash V \cup \{V\}$.*
(3) *Let R^V be the resistance metric associated with $(\mathcal{E}^V, \mathcal{F}^V)$. Then for any $x, y \in X \backslash V$, $R^V(x,y) \leq R(x,y)$. Also, if $x \in X \backslash V$, $R^V(x,V) = R(x,V)$.*

$(\mathcal{E}^V, \mathcal{F}^V)$ is called the V-shorted resistance form of $(\mathcal{E}, \mathcal{F})$.

Proof. (1) It is straightforward to show (RF1) through (RF5).
(2) Suppose that $\rho \in \ell(U)$ is constant on V. Note that ρ is naturally identified with an element in $\ell(U^V)$. Hence

$$\min\{\mathcal{E}(v,v) | v \in \mathcal{F}, v|_U = \rho\} = \min\{\mathcal{E}^V(v,v) | v \in \mathcal{F}_V, v|_{U^V} = \rho\}.$$

This immediately implies (2).
(3)

$$R(x,y)^{-1} = \min\{\mathcal{E}(u,u) | u \in \mathcal{F}, u(p) = 1, u(q) = 0\} \leq \min\{\mathcal{E}^V(u,u) | u \in \mathcal{F}^V, u(p) = 1, u(q) = 0\} = R^V(x,y)^{-1}.$$

The rest is obvious by definitions of $R^V(x,V)$ and $R(x,V)$. □

4 Green function

In this section, we will define the Green function associated with a resistance form with Dirichlet boundary condition and show that the Green function is the reproducing kernel of the form. Then, the Green function will be shown to be uniformly Lipschitz continuous with respect to the resistance metric. Also we will see that harmonic functions are uniformly Lipschitz continuous with respect to the resistance metric as well.

Throughout this section, we assume that $(\mathcal{E}, \mathcal{F})$ is a resistance form on a set X, that R is the associated effective resistance on X and that B is a non-empty finite subset of X.

Definition 4.1 (Green function). For any $x \in X$, define $g_B^x = R(x, B)\psi_x^{B\cup x}$. Also define $g_B(x, y) = g_B^x(y)$ for any $x, y \in X$. g_B is called the Green function of the resistance form $(\mathcal{E}, \mathcal{F})$ associated with the boundary B or the B-Green function of $(\mathcal{E}, \mathcal{F})$.

By the above definition, $g_B^x \in \mathcal{F}^B \subset \mathcal{F}$. Also recalling Proposition 3.1, we have $\mathcal{E}(\psi_x^{B\cup x}, \psi_x^{B\cup x}) = R(x, B)^{-1}$ for any $x \in X\backslash B$. These facts immediately imply the following proposition.

Proposition 4.2. (1) *$g_B(x, y) \geq 0$ for any $x, y \in X$. Also $g_B(x, y) = 0$ if $x \in B$ or $y \in B$.*
(2) *$\mathcal{E}(g_B^x, g_B^x) = R(x, B) = g_B(x, x)$ for any $x \in X$.*
(3) *$g_B(x, x) \geq g_B(x, y)$ for any $x, y \in X$.*

In fact, g_B is symmetric.

Proposition 4.3. *For any $u \in \mathcal{F}_B$, $\mathcal{E}(g_B^x, u) = u(x)$. In particular, for any $x, y \in x$, $\mathcal{E}(g_B^x, g_B^y) = g_B(x, y) = g_B(y, x)$.*

This fact shows that g_B is the reproducing kernel of the form $(\mathcal{E}, \mathcal{F}_B)$. Recall that $(\mathcal{F}_B, \mathcal{E})$ is a Hilbert space as we have shown in Proposition 2.19.

Also by this fact and Definition 4.1, we immediately have a relation between the effective resistance and the hitting time (when a stochastic process is associated with the resistance form). See Appendix B for details.

Proof. Let $V = B \cup x$ and let $D = H_V \in \mathcal{LA}(V)$ be the Laplacian on V associated with $(\mathcal{E}, \mathcal{F})$. Then since g_B^x is a V-harmonic function with respect to $(\mathcal{E}, \mathcal{F})$,

$$\mathcal{E}(g_B^x, u) = \mathcal{E}_D(g_B^x|_V, u|_V) = -g_B^x(x)(Du)(x) = -g_B(x, x)D_{xx}u(x). \quad (4.1)$$

Also by (3.2), $R(x, B)^{-1} = -D_{xx}$. This along with (4.1) implies that $\mathcal{E}(g_B^x, u) = u(x)$. □

Next we give an alternative expression of the Green function. Let V be a finite subset of X containing B and let $H = H_V$ be the Laplacian on V

associated with $(\mathcal{E}, \mathcal{F})$. Then there exist linear maps $T : \ell(B) \to \ell(B), J : \ell(B) \to \ell(V\backslash B)$ and $X : \ell(V\backslash B) \to \ell(V\backslash B)$ such that

$$H = \begin{pmatrix} T & {}^tJ \\ J & X \end{pmatrix}.$$

It is know that X is invertible. See [16, Lemma 2.1.5]. Set $G = (-X)^{-1}$.

Proposition 4.4. *In the above situation, $g_B(p, q) = G_{pq}$ for any $p, q \in V\backslash B$. Also $g_B^p = \sum_{q \in V\backslash B} G_{pq}\psi_q^V$ for any $p \in V\backslash B$.*

Proof. For any $u \in \mathcal{H}_V \cap \mathcal{F}_B$ and any $p \in V\backslash B$,

$$u(p) = \mathcal{E}(u, g_B^p) = \mathcal{E}_H(u, g_B^p) = -(u|_{V\backslash B}, X(g_B^p|_{V\backslash B}))_{V\backslash B},$$

where $(\cdot, \cdot)_{V\backslash B}$ is the standard inner-product defined in Section 2. This immediately implies that $-X(g_B^p|_{V\backslash B}) = \chi_p^{V\backslash B}$. Hence $g_B(p, q) = G_{pq}$ for any $p, q \in V\backslash B$. Set $f = \sum_{q \in V\backslash B} G_{pq}\psi_q^V$ for $p \in V\backslash B$. Then both g_B^p and f are V-harmonic functions and $g_B^p|_V = f|_V$. Therefore $g_B^p = f$. □

It is remarkable that the Green function is uniformly Lipschitz continuous with respect to the resistance metric as follows.

Theorem 4.5. *For any $x, y, z \in X$,*

$$|g_B(x, y) - g_B(x, z)| \leq R(y, z).$$

Corollary 4.6. *Let $V \neq \emptyset$ be a finite subset of X and let $D = H_V \in \mathcal{LA}(V)$ be the Laplacian on V associated with $(\mathcal{E}, \mathcal{F})$. If u is a V-harmonic function with respect to $(\mathcal{E}, \mathcal{F})$, then, for any $x, y \in X$,*

$$|u(x) - u(y)| \leq (-\mathrm{tr}(D))(\max_{p,q \in V} |u(p) - u(q)|)R(x, y),$$

where $\mathrm{tr}(\cdot)$ is the trace of matrices.

Proof. Set $V_p = V\backslash p$ for any $p \in V$. Then $g_{V_p}^p = R(p, V_p)\psi_p^V$. Hence Theorem 4.5 implies that

$$|\psi_p^V(x) - \psi_p^V(y)| \leq R(p, V_p)^{-1}R(x, y)$$

for any $x, y \in X$. Since $u = \sum_{p \in V} u(p)\psi_p^V$,

$$|u(x) - u(y)| \leq \sum_{p \in V} \frac{|u(p)|}{R(p, V_p)} R(x, y).$$

Replacing u by $u - \alpha$, we obtain

$$|u(x) - u(y)| \leq \sum_{p \in V} \frac{|u(p) - \alpha|}{R(p, V_p)} R(x, y)$$

Note that $R(p, V_p)^{-1} = \mathcal{E}(\psi_p^V, \psi_p^V) = -D_{pp}$. If $\alpha = u(p_*)$ for some $p_* \in V$, then

$$\sum_{p\in V} \frac{|u(p) - u(p_*)|}{R(p, V_p)} \leq \sum_{p\in V} -D_{pp} \max_{p,q\in V} |u(p) - u(q)|.$$

□

The rest of this section is devoted to proving Theorem 4.5.

Lemma 4.7. (1) *For any $x, y \in X$,*

$$|g_B(x, x) - g_B(y, y)| \leq R(x, y).$$

(2) *For any $x, y \in X$,*

$$0 \leq g_B(x, x) + g_B(y, y) - 2g_B(x, y) \leq R(x, y).$$

Proof. (1) Consider the shorted resistance form $(\mathcal{E}^B, \mathcal{F}^B)$ on X^B and the shorted effective resistance R^B. Since R^B is a metric on X^B, $|R^B(x, B) - R^B(y, B)| \leq R^B(x, y)$. By Proposition 3.6, $|R(x, B) - R(y, B)| \leq R(x, y)$. This immediately implies the required inequality.
(2) Let $h = g_B^x - g_B^y$. Then $\mathcal{E}(h, h) = h(x) - h(y) = g_B(x, x) + g_B(y, y) - 2g_B(x, y)$. Since $\mathcal{E}(h, h)R(x, y) \geq |h(x)-h(y)|^2$, it follows that $0 \leq h(x)-h(y) \leq R(x, y)$. □

Lemma 4.8. *For any $x, y \in X$,*

$$|g_B(x, x) - g_B(x, y)| \leq R(x, y).$$

Proof. By Lemma 4.7, $2|g(x, x) - g(x, y)| \leq |g(x, x) + g(y, y) - 2g(x, y)| + |g(x, x) - g(y, y)| \leq 2R(x, y)$.

□

Lemma 4.9. *For any $x, y, z \in X$,*

$$g_B(y, y)g_B(z, x) \leq g_B(y, x)g_B(y, z).$$

Proof. Fix y and z. Let $u(x) = g_B(y, y)g_B(z, x)$ and let $v(x) = g_B(y, x)g_B(y, z)$. Then $u(x)$ and $v(x)$ are harmonic functions with respect to $(X, \mathcal{E}, \mathcal{F}, B \cup y \cup z)$. Since $u(y) = v(y), u|_B = v_B \equiv 0$ and $u(z) = g_B(y, y)g_B(z, z) \geq g_B(y, z)^2 = v(z)$, the maximum principle implies that $u(x) \geq v(x)$ for any $x \in X$. □

Proof of Theorem 4.5. By Lemma 4.9,

$$\begin{aligned} g_B(y, y) &\geq \frac{g_B(x, y)}{g_B(y, y)}(g_B(y, y) - g_B(y, z)) \\ &\geq g_B(x, y) - \frac{g_B(x, y)g_B(y, z)}{g_B(y, y)} \geq g(x, y) - g(x, z). \end{aligned}$$

Exchanging y and z,

$$g_B(z,z) - g_B(z,y) \geq g_B(x,z) - g_B(y,z).$$

Hence by Lemma 4.8,

$$|g_B(x,y) - g_B(x,z)| \leq \max\{g_B(z,z) - g_B(z,y), g_B(y,y) - g_B(y,z)\} \leq R(y,z).$$

□

5 Green operators

By making use of the Green function $g_B(x,y)$, the associated Green operator G_B is formally given by

$$(G_B f)(x) = \int_X g_B(x,y) f(y) \mu(dy), \tag{5.1}$$

where μ is a measure on X In this section, we will define Green operators in a rather universal way in Theorem 5.5. Our Green operators coincide with the integral operator given by (5.1) in restricted situations. Through Green operators, we will finally obtain the universal domain of Laplacians, $\mathcal{D}^L$, in Definition 5.11.

As in the last section, $(\mathcal{E}, \mathcal{F})$ is a resistance form on a set X and R is the associated resistance metric on X. Also we assume that (X, R) is separable in this section.

Definition 5.1. For any $p \in X$ and any $u : X \to \mathbb{R}$, we define

$$||u||_{p,\frac{1}{2}} = \sup_{x \in X} \frac{|u(x)|}{\sqrt{1 + R(x,p)}}. \tag{5.2}$$

and

$$C_{\frac{1}{2}}(X,R) = \{u | u : X \to \mathbb{R}, u \text{ is continuous on } X, ||u||_{p,\frac{1}{2}} < \infty\}$$

In (5.2), we allow ∞ as a value of the supremum.

It is easy to see that the definition of $C_{\frac{1}{2}}(X,R)$ does not depend on $p \in X$. In fact, $||\cdot||_{p,\frac{1}{2}}$ is a norm on $C_{\frac{1}{2}}(X,R)$ and

$$(1 + R(p,q))^{-\frac{1}{2}} ||u||_{q,\frac{1}{2}} \leq ||u||_{p,\frac{1}{2}} \leq (1 + R(p,q))^{\frac{1}{2}} ||u||_{q,\frac{1}{2}}$$

for any $q \in X$ and any $u : X \to \mathbb{R}$. Moreover, $(C_{\frac{1}{2}}(X,R), ||\cdot||_{p,\frac{1}{2}})$ is a Banach space.

Proposition 5.2. *$\mathcal{F} \subset C_{\frac{1}{2}}(X,R)$. Moreover, let $B \neq \emptyset$ be a finite subset of X. Then the natural inclusion map from $(\mathcal{F}_B, \mathcal{E})$ to $(C_{\frac{1}{2}}(X,R), ||\cdot||_{p,\frac{1}{2}})$ is continuous. In particular, if $p \in B$, then $\sqrt{\mathcal{E}(u,u)} \geq ||u||_{p,\frac{1}{2}}$.*

Proof. By (2.1),

$$\sqrt{\mathcal{E}(u,u)} + \frac{|u(p)|}{\sqrt{1+R(x,p)}} \geq \frac{|u(x)|}{\sqrt{1+R(x,p)}}$$

for any $u \in \mathcal{F}$. Hence $\sqrt{\mathcal{E}(u,u)} + |u(p)| \geq ||u||_{p,\frac{1}{2}}$. □

Hereafter, when no confusion can occur, we write $||\cdot||_{\frac{1}{2}}$ or, simply, $||\cdot||$ instead of $||\cdot||_{p,\frac{1}{2}}$.

Definition 5.3. Let $\widetilde{C}_{\frac{1}{2}}(X,R)$ be the completion of $\mathcal{F}$ with respect to $||\cdot||_{p,\frac{1}{2}}$ in $C_{\frac{1}{2}}(X,R)$. The dual space of $(\widetilde{C}_{\frac{1}{2}}(X,R), ||\cdot||_{p,\frac{1}{2}})$ is denoted by $M(X,R)$. For any $\varphi \in M(X,R)$, $||\varphi||_{M(X,R)}$ is the dual norm:

$$||\varphi||_{M(X,R)} = \sup\{|\varphi(u)| : u \in \widetilde{C}_{\frac{1}{2}}(X,R), ||u||_{p,\frac{1}{2}} = 1\}.$$

For ease of notation, we sometimes use $||\cdot||$ in place of $||\cdot||_{M(X,R)}$.
Using Lemma 2.9, we immediately see the following lemma.

Lemma 5.4. *For any $u, v \in \widetilde{C}_{\frac{1}{2}}(X,R)$, $u \vee v$ and $u \wedge v$ belong to $\widetilde{C}_{\frac{1}{2}}(X,R)$.*

Let $\varphi \in M(X,R)$. Then by Proposition 5.2, $\varphi|_{\mathcal{F}_B} : \mathcal{F}_B \to \mathbb{R}$ is continuous with respect to the inner product $\mathcal{E}$ on $\mathcal{F}_B$. Since $(\mathcal{F}_B, \mathcal{E})$ is a Hilbert space, the dual space of $(\mathcal{F}_B, \mathcal{E})$ can be identified with $(\mathcal{F}_B, \mathcal{E})$ itself. Therefore, we have the following theorem.

Theorem 5.5. *Let $B \neq \emptyset$ be a finite subset of X. Then there exists a unique continuous linear map $G_B : M(X,R) \to \mathcal{F}_B$ such that*

$$\mathcal{E}(G_B\varphi, u) = \varphi(u) \tag{5.3}$$

for any $\varphi \in M(X,R)$ and any $u \in \mathcal{F}_B$. In particular,

$$(G_B\varphi)(x) = \varphi(g_B^x). \tag{5.4}$$

for any $\varphi \in M(X,R)$ and any $x \in X$. Moreover, for any $\varphi \in M(X,R)$, $G_B\varphi$ is uniformly Lipschitz continuous with respect to R on X.

Proof. Existence of $G_B\varphi$ satisfying (5.3) follows from the arguments above. By Proposition 5.2,

$$\mathcal{E}(G_B\varphi, G_B\varphi) = \varphi(G_B\varphi) \leq ||\varphi||||G_B\varphi||_{p,\frac{1}{2}} \leq ||\varphi||\sqrt{\mathcal{E}(G_B\varphi, G_B\varphi)}.$$

This implies $\sqrt{\mathcal{E}(G_B\varphi, G_B\varphi)} \leq ||\varphi||$. Hence G_B is continuous. Making use of the B-Green function, we obtain (5.4). By Theorem 4.5, $|(G_B\varphi)(x) - (G_B\varphi)(y)| \leq |\varphi(g_B^x - g_B^y)| \leq ||\varphi||R(x,y)$. □

Definition 5.6. Let $B \neq \emptyset$ be a finite subset of X. Then G_B is called the Green operator associated with boundary B with respect to $(\mathcal{E}, \mathcal{F})$ or the B-Green operator with respect to $(\mathcal{E}, \mathcal{F})$.

The Green operator G_B is, indeed, the universal version of the integral operator given by (5.1). See Section 8 for details.

Proposition 5.7. *Let $B \neq \emptyset$ be a finite subset of X. Then, for any $\varphi \in M(X,R)$ and any $u \in \mathcal{F}$,*

$$\mathcal{E}(G_B\varphi, u) = \varphi(u) - \sum_{p\in B} \varphi(\psi_p^B)u(p). \tag{5.5}$$

In particular, $\ker G_B = \{\sum_{p\in B} \alpha_p \delta_p\}$*, where δ_p is the Dirac's delta at p : $\delta_p(u) = u(p)$ for any $u \in C_{\frac{1}{2}}(X,R)$.*

Proof. Let $u_B = P_B u = \sum_{p\in B} u(p)\psi_p^B$. By Lemma 2.20 and (5.3)

$$\mathcal{E}(G_B\varphi, u) = \mathcal{E}(G_B\varphi, u - u_B) = \varphi(u - u_B).$$

This immediately implies (5.5). □

Lemma 5.8. *Let $B \neq \emptyset$ be a finite subset of X. Then $G_B(\delta_p) = g_B^p$ for any $p \in X$.*

Proof. Let $u = G_B(\delta_p)$. Then both u and g_B^P belong to $\mathcal{F}_B$. Also, for any $v \in \mathcal{F}_B$, $\mathcal{E}(u,v) = \delta_p(v) = v(p) = \mathcal{E}(g_B^p, u)$. Then $u = g_B^p$. □

In the rest of this section, we will study relations between the Green operators with different boundaries.

Lemma 5.9. *Let B_1 and B_2 be non-empty finite subsets of X satisfying $B_1 \subseteq B_2$.*
(1) $\{g_{B_1}^p, \psi_q^{B_1} | p \in B_2\backslash B_1, q \in B_1\}$ *is a base of $\mathcal{H}_{B_2}$. In fact, for any $u \in \mathcal{H}_{B_2}$,*

$$u = \sum_{p\in B_2\backslash B_1} \mathcal{E}(u, \psi_p^{B_2})g_{B_1}^p + \sum_{q\in B_1} u(q)\psi_q^{B_1}.$$

(2) *For any $\varphi \in M(X,R)$,*

$$G_{B_2}\varphi = G_{B_1}\varphi - \sum_{p\in B_2\backslash B_1} \varphi(\psi_p^{B_2})g_{B_1}^p = G_{B_1}(\varphi - \sum_{p\in B_2\backslash B_1} \varphi(\psi_p^{B_2})\delta_p)$$

Proof. (1) Set $u_1 = u - \sum_{q\in B_1} u(q)\psi_q^{B_1}$. Then $u_1 = u - P_{B_1}u$. Also since $P_{B_1}u \in \mathcal{H}_{B_1} \subseteq \mathcal{H}_{B_2}$, we see that $u_1 \in \mathcal{H}_{B_2}$. Hence by Lemma 2.20, for any $v \in \mathcal{F}_{B_1}$,

$$\mathcal{E}(u_1, v) = \mathcal{E}(u_1, P_{B_2}v) = \mathcal{E}(u_1, \sum_{p\in B_2\backslash B_1} v(p)\psi_p^{B_2}). \tag{5.6}$$

Note that $P_{B_2}v \in \mathcal{F}_{B_1}$. Again by $\mathcal{E}(P_{B_1}u, P_{B_2}v) = 0$. This along with (5.6) implies $\mathcal{E}(u_1, v) = \sum_{p\in B_2\backslash B_1} \mathcal{E}(u, \psi_p^{B_2})v(p)$. Hence $u_1 = \sum_{p\in B_2\backslash B_1} \mathcal{E}(u, \psi_p^{B_2})g_{B_1}^p$.
(2) By Proposition 5.7, $\mathcal{E}(G_{B_2}\varphi, u) = \varphi(u) - \sum_{p\in B_2\backslash B_1} \varphi(\psi_p^{B_2})u(p)$ for any $u \in \mathcal{F}_{B_1}$. Hence $\mathcal{E}(G_{B_1}\varphi - G_{B_2}\varphi, u) = \sum_{p\in B_2\backslash B_1} \varphi(\psi_p^{B_2})u(p)$ for any $u \in \mathcal{F}_{B_1}$. Since $G_{B_2}\varphi \in \mathcal{F}_{B_2} \subseteq \mathcal{F}_{B_1}$, we see that $G_{B_1}\varphi - G_{B_2}\varphi = \sum_{p\in B_2\backslash B_1} \varphi(\psi_p^{B_2})g_{B_1}^p$. Now combining this with Lemma 5.8, we obtain the desired equality. □

The next theorem is the principal relation between the images of the Green operators.

Theorem 5.10. (1) *Let B_1 and B_2 be non-empty finite subsets of X satisfying $B_1 \subseteq B_2$. Then $\mathcal{D}^L_{B_1,0} \supseteq \mathcal{D}^L_{B_2,0}$, where $\mathcal{D}^L_{B,0} = \mathrm{Im}(G_B)$ for any non-empty finite subset of X, B.*
(2) *$\mathcal{D}^L_{B,0} + \mathcal{H}_B$ is independent of B: $\mathcal{D}^L_{B_1,0} + \mathcal{H}_{B_1} = \mathcal{D}^L_{B_2,0} + \mathcal{H}_{B_2}$ for any non-empty finite subsets of X, B_1 and B_2.*

Proof. (1) By Lemma 5.9-(2), $G_{B_2}\varphi \in \mathrm{Im}(G_{B_1})$ for any $\varphi \in M(X,R)$.
(2) Define $\mathcal{D}^L_B = \mathcal{D}^L_{B,0} + \mathcal{H}_B$ for any non-empty finite subset of X, B. It is enough to show that $\mathcal{D}^L_{B_1} = \mathcal{D}^L_{B_2}$ if $B_1 \subseteq B_2$. Note that $\mathcal{H}_{B_1} \subseteq \mathcal{H}_{B_2}$. Since $g^p_{B_1} \in \mathcal{H}_{B_2}$ for any $p \in B_2 \backslash B_1$, Lemma 5.9-(2) implies that $G_{B_1}\varphi + u \in \mathcal{D}^L_{B_2}$ for any $\varphi \in M(X,R)$ and any $u \in \mathcal{H}_{B_1}$. Therefore $\mathcal{D}^L_{B_1} \subseteq \mathcal{D}^L_{B_2}$.

On the other hand, by Lemma 5.8 and Lemma 5.9, we see that, for any $u \in \mathcal{H}_{B_2}$, $u = \sum_{p \in B_2 \backslash B_1} \mathcal{E}(u, \varphi^{B_2}_p) G_{B_1}\delta_p + \sum_{q \in B_1} u(q)\psi^{B_1}_q \in \mathcal{D}^L_{B_1}$. Combining this with (1), it follows that $G_{B_1}\varphi + u \in \mathcal{D}^L_{B_1}$ for any $\varphi \in M(X,R)$ and $u \in \mathcal{H}_{B_2}$. Therefore $\mathcal{D}^L_{B_1} \supseteq \mathcal{D}^L_{B_2}$. □

Definition 5.11. Define $\mathcal{D}^L \subset \mathcal{F}$ by $\mathcal{D}^L = \mathcal{D}^L_{B,0} + \mathcal{H}_B$, where B is a non-empty finite subset of X.

In the next section, we will define Laplacians on $\mathcal{D}^L$, which may be thought of as the universal domain of Laplacians. In 7, we give an characterization of $\mathcal{D}^L$ with respect to discrete Laplacians $\{H_V | V \text{ is a finite subset of } X\}$ associated with $(\mathcal{E}, \mathcal{F})$.

If $B = \{p\}$ for $p \in X$, then $\mathcal{H}_B$ is the collection of constants on X. So, in such a case $\mathcal{D}^L = \mathcal{D}^L_{p,0} + \mathbb{R}$.

Proposition 5.12. *For any $u \in \mathcal{D}^L$, u is uniformly Lipschitz continuous with respect to R on X.*

Proof. Let $p \in X$ and let $B = \{p\}$. Then, for any $u \in \mathcal{D}^L$, there exist $\varphi \in M(X,R)$ and a constant c such that $u = G_B\varphi + c$. As $G_B\varphi$ is uniformly Lipschitz continuous by Theorem 5.5, so is u. □

Define

$$C_L(X,R) = \{u | u \text{ is uniformly Lipschitz continuous with respect to } R \text{ on } X\}. \tag{5.7}$$

Knowing the above proposition, we might expect that $C_L(X,R) \subseteq \mathcal{F}$ and that $\mathcal{D}^L = C_L(X,R) \cap \mathcal{F}$. Both conjecture are not true however even if (X,R) is compact. See Corollary 9.14.

6 Laplacians

In this section, we continue to assume that $(\mathcal{E}, \mathcal{F})$ is a resistance form on a set X, that R is the associated resistance metric on X and that (X, R) is separable.

To define Laplacians, we need to know more about the space $M(X, R)$. If (X, R) is locally compact and $\mathcal{F}$ is dense in $C_{\frac{1}{2}}(X, R)$, then the Riesz theorem (see [20] for example) implies that

$$M(X, R) = \{\mu | \mu = \mu_+ - \mu_-, \text{where } \mu_\pm \text{ are Borel regular measures on } X \text{ that satisfy } \int_X \sqrt{1 + R(x, p)}\mu_\pm(dx) < \infty\}.$$

Although (X, R) may not be locally compact in general, we can still divide $\varphi \in M(X, R)$ into the positive part φ_+ and the negative part φ_- by similar arguments to those in the proof of the Riesz theorem.

Definition 6.1. (1) Let u and v be real valued function on X. We write $u \leq v$ if and only if $u(x) \leq v(x)$ for any $x \in X$. Define $\widetilde{C}^+_{\frac{1}{2}}(X, R) = \{u | u \in \widetilde{C}_{\frac{1}{2}}(X, R), u \geq 0\}$.
(2) We say $\varphi \in M(X, R)$ is non-negative if $\varphi(u) \geq 0$ for any $u \in \widetilde{C}^+_{\frac{1}{2}}(X, R)$. Define $M^+(X, R) = \{\varphi | \varphi \in M(X, R), \varphi \text{ is non-negative.}\}$

Theorem 6.2. *For any $\varphi \in M(X, R)$, there exists a unique pair $(\varphi_+, \varphi_-) \in M^+(X, R)^2$ satisfying $\varphi = \varphi_+ - \varphi_-$ and*

$$\varphi_+(u) + \varphi_-(u) = \sup_{\substack{h \in \widetilde{C}_{\frac{1}{2}}(X,R) \\ 0 \leq |h| \leq u}} |\varphi(h)| \tag{6.1}$$

for any $u \in \widetilde{C}^+_{\frac{1}{2}}(X, R)$.

By analogy with the Riesz theorem, $\varphi_+ + \varphi_-$ corresponds to the "total variation" of φ.

Remark. We see $\nu(|f|) \geq |\nu(f)|$ for any $f \in \widetilde{C}_{\frac{1}{2}}(X, R)$ and any $\nu \in M^+(X, R)$. Therefore, (6.1) implies $||\varphi|| = ||\varphi_+ + \varphi_-|| \geq ||\varphi_\pm||$.

Proof. First, for $u \in \widetilde{C}^+_{\frac{1}{2}}(X, R)$, define $\mathcal{U}_u = \{h | h \in \widetilde{C}^+_{\frac{1}{2}}(X, R), 0 \leq h \leq u\}$ and

$$\varphi_+(u) = \sup_{h \in \mathcal{U}_u} \varphi(h).$$

Since $\varphi(0) = 0$, $\varphi_+(u) \geq 0$. Let u and v belong to $\widetilde{C}^+_{\frac{1}{2}}(X, R)$. Then,

$$\varphi_+(u + v) = \sup_{h \in \mathcal{U}_{u+v}} \varphi(h) \geq \sup_{h_1 \in \mathcal{U}_u, h_2 \in \mathcal{U}_v} \varphi(h_1 + h_2) = \varphi_+(u) + \varphi_+(v). \tag{6.2}$$

On the other hand, for any $h \in \mathcal{U}_{u+v}$, define h_1 and h_2 by $h_1 = h \wedge u$ and $h_2 = h - h_1$. By Lemma 5.4, $h_1 \in \mathcal{U}_u$ and $h_2 \in \mathcal{U}_v$. This immediately implies that equality holds in (6.2). So we have obtained

$$\varphi_+(u+v) = \varphi_+(u) + \varphi_+(v) \tag{6.3}$$

for any $u, v \in \widetilde{C}^+_{\frac{1}{2}}(X,R)$. Also it is easy to see that $\varphi_+(\alpha u) = \alpha\varphi_+(u)$ for any $u \in \widetilde{C}^+_{\frac{1}{2}}(X,R)$ and $\alpha \geq 0$. Next we define $\varphi_+(u)$ for any $u \in \widetilde{C}_{\frac{1}{2}}(X,R)$. Note that, for any $u \in \widetilde{C}_{\frac{1}{2}}(X,R)$, there exist u_1 and u_2 such that $u_1, u_2 \in \widetilde{C}^+_{\frac{1}{2}}(X,R)$ and $u = u_1 - u_2$. In fact, we may let $u_1 = u \vee 0$ and $u_2 = u_1 - u$. So, if $u = u_1 - u_2$ for $u_1, u_2 \in \widetilde{C}^+_{\frac{1}{2}}(X,R)$, then we set $\varphi_+(u) = \varphi_+(u_1) - \varphi_+(u_2)$. By (6.3), $\varphi_+(u)$ is well-defined: if $u = u_1 - u_2 = v_1 - v_2$ for $u_1, u_2, v_1, v_2 \in \widetilde{C}^+_{\frac{1}{2}}(X,R)$, then $\varphi_+(u_1) - \varphi_+(u_2) = \varphi_+(v_1) - \varphi_+(v_2)$. It is routine to show that $\varphi_+ \in M^+(X,R)$. Defining $\varphi_- = \varphi_+ - \varphi$, we see that $\varphi_- = (-\varphi)_+$. Obviously $\varphi = \varphi_+ - \varphi_-$ and $\varphi_\pm \in M^+(X,R)$. Next we show (6.1). Let $u \in \widetilde{C}^+_{\frac{1}{2}}(X,R)$. Note that $|h_1 - h_2| \leq u$ for any $h_1, h_2 \in \mathcal{U}_u$. On the other hand, for any $h \in \widetilde{C}_{\frac{1}{2}}(X,R)$ with $0 \leq |h| \leq u$, define $h_+ = h \vee 0$ and $h_- = (-h) \vee 0$. Then $h_\pm \in \mathcal{U}_u$ and $h = h_+ - h_-$. Therefore, by the fact that $\varphi_- = (-\varphi)_+$, we have

$$\varphi_+(u) + \varphi_-(u) = \sup_{h_1, h_2 \in \mathcal{U}_u} \varphi(h_1 - h_2) = \sup_{\substack{h \in \widetilde{C}_{\frac{1}{2}}(X,R) \\ |h| \leq u}} |\varphi(h)|. \tag{6.4}$$

Hence (6.1) holds.

The remaining part is the uniqueness. Let $\varphi_\pm$ be the ones defined above. Assume that there exist $\nu_\pm \in M^+(X,R)$ satisfying that $\varphi = \nu_+ - \nu_-$ and $\nu_+(u) + \nu_-(u) = \sup_{h \in \widetilde{C}_{\frac{1}{2}}(X,R), 0 \leq |h| \leq u} |\varphi(h)|$. Then $\nu_+(u) + \nu_-(u) = \varphi_+(u) - \varphi_-(u)$. Since $\varphi_+ - \varphi_- = \nu_+ - \nu_-$, it follows that $\nu_+ = \varphi_+$ and $\nu_- = \varphi_-$. □

Definition 6.3. Let $\{V_m\}_{m \geq 0}$ be a family of finite subsets of X. We say that $\{V_m\}_{m \geq 0}$ is an admissible sequence of (X,R) if and only if $V_m \subseteq V_{m+1}$ for any $m \geq 0$ and V_* is dense in (X,R), where V_* is defined by $V_* = \cup_{m \geq 0} V_m$.

Lemma 6.4. *Let $\{V_m\}_{m \geq 0}$ be an admissible sequence of (X,R). Then for any $p \in V_*$ and any $\varphi \in M(X,R)$, $\varphi(\psi_p^{V_m})$ converges as $m \to \infty$. Moreover the limit $\lim_{m \to \infty} \varphi(\psi_p^{V_m})$ does not depend on the choice of $\{V_m\}_{m \geq 0}$: if $\{U_m\}_{mge0}$ is an admissible sequence of (X,R) and $p \in U_* \cap V_*$, then $\lim_{m \to \infty} \varphi(\psi_p^{U_m}) = \lim_{m \to \infty} \varphi(\psi_p^{V_m})$.*

Proof. By Theorem 6.2, we may assume that φ is non-negative without loss of generality. Note that both $\psi_p^{V_m}$ and $\psi_p^{V_{m+1}}$ are V_{m+1}-harmonic functions. Hence the maximum principle implies that $\psi_p^{V_m} \geq \psi_p^{V_{m+1}} \geq 0$. Hence $\{\varphi(\psi_p^{V_m})\}_{m \geq 0}$ is monotonically decreasing and uniformly bounded. Hence it converges as $m \to \infty$. Let $a = \lim_{m \to \infty} \varphi(\psi_p^{V_m})$ and let $b = \lim_{m \to \infty} \varphi(\psi_p^{U_m})$. Assume that $a > b$.

Then there exists $m \geq 0$ such that $\varphi(\psi_p^{U_m}) < a$. Note that $\varphi(p_p^{V_m}) \geq a$ and hence $V_m \neq U_m$. So, $V = (V_m \cup U_m) \backslash V_m$ is not empty. Now by the maximum principle and Lemma 3.4,

$$\sup_{x \in X} |\psi_p^{V_k}(x) - \psi_p^{V_k \cup V}(x)| \leq \max_{q \in V} \psi_p^{V_k}(q) \leq \max_{q \in V} \frac{R(q, V_k)}{R(p, q)} \to 0$$

as $k \to \infty$. Therefore $||\psi_p^{V_k} - \psi_p^{V_k \cup V}||_{p, \frac{1}{2}} \to 0$ as $k \to \infty$. This implies the following contradiction:

$$a = \lim_{k \to \infty} \varphi(\psi_p^{V_k}) = \lim_{k \to \infty} \varphi(\psi_p^{V_k \cup V}) \leq \varphi(\psi_p^{V_m \cup U_m}) \leq \varphi(\psi_p^{U_m}) < a.$$

Hence $a = b$. □

Definition 6.5. Let $p \in X$ and let $\{V_m\}_{m \geq 0}$ be an admissible sequence of (X, R) with $p \in V_0$. For any $\varphi \in M(X, R)$, define $\varphi(p) = \lim_{m \to \infty} \varphi(\psi_p^{V_m})$.

By Lemma 6.4, $\varphi(p)$ does not depend on a choice of $\{V_m\}_{m \geq 0}$.
Next we define Neumann derivatives of $u \in \mathcal{D}^L$ at $p \in X$.

Theorem 6.6. *Let $p \in X$ and let $\{V_m\}_{m \geq 0}$ be an admissible sequence of (X, R) with $p \in V_0$. Then, for any $u \in \mathcal{D}^L$, $-(H_{V_m} u)(p)$ converges as $m \to \infty$, where H_{V_m} is the Laplacian on V_m associated with $(\mathcal{E}, \mathcal{F})$. Moreover, define*

$$(du)_p = -\lim_{m \to \infty} (H_{V_m} u)(p).$$

Then $(du)_p$ is independent of a choice of $\{V_m\}_{m \geq 0}$. $(du)_p$ is called the Neumann derivative of u at p. In particular, if $B \neq \emptyset$ is a finite subset of X and $u = f - G_B \varphi$, where $f \in \mathcal{H}_B$ and $\varphi \in M(X, R)$, then, for any $x \in X$,

$$(du)_x = \begin{cases} -(H_B f)(x) - \varphi(x) + \varphi(\psi_x^B) & \text{if } x \in B, \\ -\varphi(x) & \text{if } x \notin B, \end{cases}$$

where H_B is the Laplacian on B associated with $(\mathcal{E}, \mathcal{F})$.

In Proposition 8.6, we will see that $(du)_p$ is "usually" equal to zero for $p \notin B$.

Proof. First note that $-(H_{V_m} u)(p) = \mathcal{E}(u, \psi_p^{V_m})$ by Lemma 2.20-(2). Let $B \neq \emptyset$ be a finite subset of X. Choose $\{V_m\}_{m \geq 0}$ so that $V_0 = B$, that $V_m \subseteq V_{m+1}$ and that $\cup_{m \geq 0} V_m$ is dense in (X, R). Suppose $u = f - G_B \varphi$ for $f \in \mathcal{H}_B$ and $\varphi \in M(X, R)$. Then, for $x \in B$, Lemma 2.20 along with (5.5) implies

$$\mathcal{E}(u, \psi_x^m) = \mathcal{E}(f, \psi_x^B) - \mathcal{E}(G_B \varphi, \psi_x^m) = -(H_B f)(x) - \varphi(\psi_x^m) + \varphi(\psi_x^B),$$

where $\psi_p^m = \psi_p^{V_m}$. By Lemma 6.4, $\lim_{m \to \infty} \mathcal{E}(u, \psi_x^m) = -(H_B f)(x) - \varphi(x) + \varphi(\psi_x^B)$. When $x \notin B$, we assume that $x \in V_1$. Then Lemma 2.20 along with (5.5) implies

$$\mathcal{E}(u, \psi_x^m) = \mathcal{E}(f, P_B \psi_x^m) - \varphi(\psi_x^m) + \sum_{q \in B} \varphi(\psi_q^B) \psi_x^m(q).$$

Using Lemma 3.4, we see that $\psi_x^m(q) \to 0$ as $m \to \infty$ for any $q \in B$. Hence by Lemma 6.4, we obtain $\mathcal{E}(u, \psi_x^m) \to -\varphi(x)$ as $m \to \infty$.

Choosing $B = \{p\}$ and using Lemma 6.4, we also verify that $(du)_p$ is independent of a choice of $\{V_m\}_{m \geq 0}$. □

Now we define "Laplacians". First we consider a Laplacian with boundary condition on a finite set B.

Definition 6.7. Let B be a non-empty finite subset of X.
(1) Define $M_B^{NA}(X,R) = \{\varphi | \varphi \in M(X,R), \varphi(p) = 0 \text{ for any } p \in B\}$.
(2) Define $L_B : \mathcal{D}^L \to M_B^{NA}(X,R)$ by $L_B u = \varphi - \sum_{p \in B} \varphi(p)\delta_p$, where $u = u_B - G_B\varphi$ for $u_B \in \mathcal{H}_B$ and $\varphi \in M(X,R)$. L_B is called the B-Laplacian on X associated with the resistance form $(\mathcal{E}, \mathcal{F})$.

Using Proposition 5.7, we see that L_B is well-defined.

Theorem 6.8. *For a non-empty finite set B, define* $G_B^* = G_B|_{M_B^{NA}(X,R)}$. *Then* $G_B^* : M_B^{NA}(X,R) \to \mathcal{D}_{B,0}^L$ *is bijective and* $(G_B^*)^{-1} = -L_B|_{\mathcal{D}_{B,0}^L}$. *Moreover,*
(1) *For any* $u \in \mathcal{F}$ *and any* $v \in \mathcal{D}^L$,

$$\mathcal{E}(u,v) = \sum_{p \in B} u(p)(dv)_p - (L_B v)(u). \tag{6.5}$$

(2) *For any* $f \in \ell(B)$ *and any* $\varphi \in M_B^{NA}(X,R)$,

$$L_B u = \varphi \qquad \text{and} \qquad u|_B = f \tag{6.6}$$

if and only if

$$u = \sum_{p \in B} f(p)\psi_p^B - G_B^*\varphi. \tag{6.7}$$

(6.5) and (6.6) are counterparts of the Gauss-Green formula and the solution to the Dirichlet problem for Poisson's equation, respectively. (6.7) is equivalent to

$$u(x) = \sum_{p \in B} f(p)\psi_p^B(x) - \varphi(g_B^x)$$

for any $x \in X$.

Proof. Define $\mathrm{pr}_B^{NA} : \mathrm{M}(\mathrm{X},\mathrm{R}) \to \mathrm{M}_B^{NA}(\mathrm{X},\mathrm{R})$ by $\mathrm{pr}_B^{NA}(\varphi) = \varphi - \sum_{p \in B} \varphi(p)\delta_p$. Then Proposition 5.7 implies that $\ker G_B = \ker \mathrm{pr}_B^{NA}$. Hence $\mathrm{Im}(G_B^*) = \mathcal{D}_{B,0}^L$ and $\ker G_B^* = \{0\}$. Therefore G_B^* is bijective. By the definition of L_B, it follows that $(G_B^*)^{-1} = -L_B$. This fact immediately implies that (6.6) is equivalent to (6.7).

To show (6.5), assume that $v = v_B - G_B\varphi$ for $v_B \in \mathcal{H}_B$ and $\varphi \in M(X, R)$. Then by (5.5) and Theorem 6.6,

$$\mathcal{E}(u, v) = \mathcal{E}(u, v_B) - \mathcal{E}(u, G_B\varphi)$$
$$= -\sum_{p\in B} u(p)(H_B v_B)(p) - (\varphi(u) - \sum_{p\in B} u(p)\varphi(\psi_p^V)) = \sum_{p\in V} u(p)(dv)_p - (L_B v)(u).$$

□

Corollary 6.9. *Let B be a non-empty finite subset of X. Then, for $v \in \mathcal{D}^L$, define $Lv \in M(X, R)$ by*

$$Lv = L_B v - \sum_{p\in B} (dv)_p \delta_p. \tag{6.8}$$

Then L is independent of a choice of B and

$$\mathcal{E}(u, v) = -(Lv)(u) \tag{6.9}$$

for any $u \in \mathcal{F}$.

L is called the Neumann Laplacian (or the N-Laplacian for short) on X associated with $(\mathcal{E}, \mathcal{F})$. Comparing (6.9) with (6.5), we might regard L as L_B with $B = \emptyset$.

Proof. Let B_1 and B_2 be non-empty finite subsets of X. Then, by (6.5),

$$\sum_{p\in B_2} u(p)(dv)_p - (L_{B_2} v)(u) = \sum_{p\in B_1} u(p)(dv)_p - (L_{B_1} v)(u)$$

for any $u \in \mathcal{F}$. Note that $\mathcal{F}$ is dense in $\widetilde{C}_{\frac{1}{2}}(X, R)$. Therefore,

$$\sum_{p\in B_2} (dv)_p \delta_p - L_{B_2} v = \sum_{p\in B_1} (dv)_p \delta_p - L_{B_1} v.$$

Hence L is independent of B. (6.9) is obvious by (6.5). □

The following proposition express the basic property of L, that it is Fredholm with index zero.

Proposition 6.10. *Let $\varphi \in M(X, R)$. There exists $v \in \mathcal{D}^L$ such that $Lv = \varphi$ if and only if $\varphi(1) = 0$. Moreover, if $Lv = \varphi$, then $L(v + c) = \varphi$ for any constant $c \in \mathbb{R}$. In particular, $\ker L = \{u | u$ is constant on $X\}$ and $\mathrm{Im}(L) = \{\varphi | \varphi \in M(X, R), \varphi(1) = 0\}$.*

Proof. Since $\mathcal{E}(v, 1) = -(Lv)(1) = 0$, $Lv = \varphi$ implies $\varphi(1) = 0$. Conversely, suppose $\varphi(1) = 0$. Let $v = -G_p\varphi$ for $p \in X$. Then by (5.5), $\mathcal{E}(u, v) = -\varphi(u) + \varphi(1)u(p) = -\varphi(u)$. Hence $Lv = \varphi$. The rest is obvious. □

By (6.5) and (6.8), we immediately deduce the following relations between Laplacians.

Lemma 6.11. (1) *Let B be non-empty finite subset of X. Then, for any $v \in \mathcal{D}^L$, $Lv = L_B v$ if and only if $(dv)_p = 0$ for any $p \in B$.*
(2) *Let B_1, B_2 be non-empty finite subsets of X satisfying $B_1 \subseteq B_2$. Then, for any $v \in \mathcal{D}^L$, $L_{B_1} v = L_{B_2} v$ if and only if $(dv)_p = 0$ for any $p \in B_2 \backslash B_1$.*

Lemma 6.11-(1) may be thought of as a special case of Lemma 6.11-(2) with $B_2 = B$ and $B_1 = \emptyset$.

7 Characterization of the domain of the Laplacian

As in the previous sections, $(\mathcal{E}, \mathcal{F})$ is a resistance form on X, R is the resistance metric on X associated with $(\mathcal{E}, \mathcal{F})$. (X, R) is assumed to be separable.

Definition 7.1. (1) $C_b(X, R)$ is the collection of continuous and bounded functions on X equipped with the supremum norm $||\cdot||_\infty$. Also $\widetilde{C}_b(X, R)$ is the completion of $\mathcal{F} \cap C_b(X, R)$ with respect to the norm $||\cdot||_\infty$
(2) For any $\varphi \in M(X, R)$, the total variation of φ, $||\varphi||_{TV}$, is defined by

$$||\varphi||_{TV} = \varphi_+(1) + \varphi_-(1),$$

where $\varphi_\pm$ is defined in Theorem 6.2.

By Theorem 6.2, the total variation of φ is given by

$$||\varphi||_{TV} = \sup_{\substack{h \in \widetilde{C}_{\frac{1}{2}}(X,R) \\ 0 \leq |h| \leq 1}} |\varphi(h)|. \tag{7.1}$$

Theorem 7.2. *For any $u : X \to \mathbb{R}$, define*

$$||u||_{\mathcal{D}} = \sup\{\sum_{p \in V} |(H_V u)(p)| : V \text{ is a non-empty finite subset of } X\}.$$

Then $||u||_{\mathcal{D}} = ||Lu||_{TV} < +\infty$ for any $u \in \mathcal{D}^L$. Moreover, let $\{V_m\}_{m \geq 0}$ be an admissible sequence of (X, R). Then $\lim_{m\to\infty} \sum_{p \in V_m} |(H_{V_m} u)(p)| = ||u||_{\mathcal{D}}$ for any $u \in \mathcal{D}^L$.

Lemma 7.3. *For any $u \in \mathcal{D}^L$ and any non-empty finite set $V \subseteq X$,*

$$\sum_{p \in \in V} |(H_V u)(p)| \leq ||Lu||_{TV}$$

Proof. Define $\alpha(p) = 1$ if $(H_V u)(p) \geq 0$ and $\alpha(p) = -1$ if $|(H_V u)(p) < 0$. Set $f = \sum_{p \in V} \alpha(p) \psi_p^V$. Since $|(H_V u)(p)| = |\mathcal{E}(u, \psi_p^V)|$, it follows that $\sum_{p \in V} |(H_V u)(p)| = |\mathcal{E}(u, f)| = |(Lu)(f)|$. By the maximum principle (Proposition 2.18), $|f| \leq 1$. Hence by the definition of $||\cdot||_{TV}$, we obtain the desired inequality. □

Proof of Theorem 7.2. Let $u \in \mathcal{D}^L$. For any $\epsilon > 0$, there exists $f \in \widetilde{C}_{\frac{1}{2}}(X,R) \cap C_b(X,R)$ such that $||f||_\infty \leq 1$ and

$$||Lu||_{TV} - \epsilon/3 \leq |(Lu)(f)| \leq ||Lu||_{TV}. \tag{7.2}$$

Note that $\mathcal{F}$ is dense in $\widetilde{C}_{\frac{1}{2}}(X,R)$. Hence we may choose $h \in \mathcal{F}$ such that $||Lu||||f-h|| \leq \epsilon/3$. Set $g = (h \wedge 1) \vee (-1)$. Then, by Lemma 2.9, $g \in \mathcal{F} \cap C_b(X,R)$ and $||g||_\infty \leq 1$. Moreover, since $||f||_\infty \leq 1$, it follows that $|f(x) - h(x)| \geq |f(x) - g(x)|$ for any $x \in X$. This implies $|(Lu)(f-g)| \leq ||Lu||||f-g|| \leq ||Lu||||f-h|| \leq \epsilon/3$. By (7.2),

$$||Lu||_{TV} - 2\epsilon/3 \leq |(Lu)(g)| \leq ||Lu||_{TV}. \tag{7.3}$$

Now let $\{V_m\}_{m\geq 0}$ be an admissible sequence of (X,R). Then, for sufficiently large m,

$$|(Lu)(g) - (Lu)(g_m)| = |\mathcal{E}(u,g) - \mathcal{E}(u,g_m)| \leq \epsilon/3,$$

where $g_m = P_{V_m} g = \sum_{p\in V_m} g(p)\psi_p^{V_m}$. Hence by (7.3),$||Lu||_{TV} - \epsilon \leq |\mathcal{E}(u,g_m)|$. On the other hand, the fact that $||g||_\infty \leq 1$ along with Lemma 7.3 implies

$$|\mathcal{E}(u,g_m)| = |\sum_{p\in V_m} -g(p)(H_{V_m}u)(p)| \leq \sum_{p\in V_m} |(H_{V_m}u)(p)| \leq ||Lu||_{TV}. \tag{7.4}$$

Therefore, $||Lu||_{TV} - \epsilon \leq \sum_{p\in V_m} |(H_{V_m}u)(p)| \leq ||Lu||_{TV}$. □

It is noteworthy that the sequence $\sum_{p\in V_m} |(H_{V_m})u(p)|$ in Theorem 7.2 is monotonically nondecreasing by the following lemma.

Lemma 7.4. *Let U and V be non-empty finite subsets of X with $V \subseteq U$. Then for any $u : X \to \mathbb{R}$,*

$$(H_V u)(p) = \sum_{q\in U} \psi_p^V(q)(H_U u)(q), \tag{7.5}$$

where H_V and H_U are Laplacians on V and U, respectively, associated with $(\mathcal{E},\mathcal{F})$. In particular, $\sum_{p\in V} |(H_V u)(p)| \leq \sum_{p\in U} |(H_U u)(p)|$.

Proof. Divide H_U into four parts as in Proposition 2.6:

$$H_U = \begin{pmatrix} T & {}^tJ \\ J & X \end{pmatrix},$$

where $T : \ell(V) \to \ell(V)$, $J : \ell(V) \to \ell(U\backslash V)$ and $X : \ell(U\backslash V) \to \ell(U\backslash V)$. Since $(V,H_V) \leq (U,H_U)$, Proposition 2.6 implies that $H_V = T - {}^tJX^{-1}J$. For any $u \in lU$, set $u_0 = u|_V$ and $u_1 = u|_{U\backslash V}$. Then,

$$\begin{aligned} H_V u_1 &= (T - {}^tJX^{-1}J)u_1 = (Tu_0 + {}^tJu_1) - {}^tJX^{-1}(Ju_0 + Xu_1) \\ &= (H_U u)|_V - {}^tJX^{-1}(H_U u)|_{U\backslash V}. \end{aligned} \tag{7.6}$$

Now, ψ_p^V is the V-harmonic function with boundary value χ_p^V. Hence $\psi_p^V|_{U\backslash V} = -X^{-1}J\chi_p^V$. Therefore $(-{}^tJX^{-1})_{pq} = \psi_p^V(q)$ for any $p \in V$ and any $q \in U\backslash V$. Combining this with (7.6), we immediately verify (7.5) and hence

$$|(H_V u)(p)| \leq \sum_{q \in U} \psi_p^V(q)|(H_U u)(q)|.$$

The rest of the statement follows by summing this for all $p \in V$. □

Theorem 7.5. *Suppose that (X, R) is bounded. Then $u \in \mathcal{D}^L$ if and only if $u \in C_b(X, R)$ and $||u||_{\mathcal{D}} < +\infty$. Moreover, $\mathcal{D}^L$ is a Banach space with respect to the norm $||\cdot||_\infty + ||\cdot||_{\mathcal{D}}$.*

Proof. Suppose $u \in C_b(X, R)$ and $||u||_{\mathcal{D}} < +\infty$. Let $\{V_m\}_{m\geq 0}$ be a sequence of finite subsets of X satisfying that $V_m \subseteq V_{m+1}$ for any $m \geq 0$ and that $\cup_{m\geq 0} V_m$ is dense in (X, R). Note that $\mathcal{F} \subseteq \widetilde{C}_b(X, R) = \widetilde{C}_{\frac{1}{2}}(X, R)$ because (X, R) is bounded. For any $v \in \mathcal{F}$,

$$|\mathcal{E}_{H_{V_m}}(v|_{V_m}, u|_{V_m})| \leq ||v||_\infty \sum_{p \in V_m} |(H_{V_m} u)(p)|.$$

Therefore, $u \in \mathcal{F}$ and $|\mathcal{E}(v, u)| \leq ||v||_\infty ||u||_{\mathcal{D}}$ for any $v \in \mathcal{F}$. Since $\mathcal{F}$ is dense in $\widetilde{C}_b(X, R)$, $\mathcal{E}(\cdot, u)$ can be extended to be a bounded linear functional on $\widetilde{C}_b(X, R)$. Hence there exists $\varphi \in M(X, R)$ such that $\mathcal{E}(v, u) = \varphi(v)$ for any $v \in \mathcal{F}$. This implies that $u \in \mathcal{D}^L$ and $\varphi = -Lu$.

Obviously $||\cdot||_\infty + ||\cdot||_{\mathcal{D}}$ is a norm on $\mathcal{D}^L$. We also see that $\mathcal{D}^L$ is complete under this norm. □

8 Realization of Green operator and Laplacian

Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on a set X and let R be the associated resistance metric on R. We assume that (X, R) is separable and locally compact and that, for any $f \in C_0(X, R)$, there exists a sequence $\{f_n\}_{n\geq 0} \subset \mathcal{F} \cap C_b(X, R)$ such that $||f - f_n||_\infty \to 0$ as $n \to \infty$, where $C_0(X, R)$ is the collection of continuous functions with compact support. Also in this section, μ is a σ-finite Radon measure on (X, R): μ is a σ-finite Borel regular measure on (X, R) and $\mu(K) < +\infty$ for any compact subset $K \subseteq X$. Under those assumptions, we obtain

Proposition 8.1. *$\mathcal{F} \cap C_0(X, R)$ is dense in $C_0(X, R)$.*

Proof. Let $f \in C_0(X, R)$ and let K be the support of f. Define $f_+ = f \vee 0$ and $f_- = (-f) \vee 0$. Then $f_\pm \in C_0(X, R)$ and $f = f_+ - f_-$. Now, for any $\epsilon \geq 0$, there exists $g \in \mathcal{F} \cap C_b(X, R)$ such that $||f_+ - g||_\infty < \epsilon/2$. Set $h = (g - \epsilon/2) \vee 0$. The Markov property of $(\mathcal{E}, \mathcal{F})$ implies that $h \in \mathcal{F}$. Also if $x \notin K$, then $|f_+(x) - g(x)| = |g(x)| < \epsilon/2$. Hence $h \in C_0(X, R)$. Also we see that$||f_+ - h|| < \epsilon$. The same discussion implies that $||f_- - u|| < \epsilon$ for some $u \in \mathcal{F} \cap C_0(X, R)$. Therefore, $\mathcal{F} \cap C_0(X, R)$ is dense in $C_0(X, R)$. □

Definition 8.2. If (X, R) is not bounded, then, for $p \in X$, define $\mu_{p,\frac{1}{2}}$ by $\mu_{p,\frac{1}{2}}(A) = \int_A \sqrt{1 + R(x, p)}\mu(dx)$ for any Borel set $A \subseteq X$. If (X, R) is bounded then we set $\mu_{p,\frac{1}{2}} = \mu$ for any $p \in X$.

Note that if (X, R) is bounded, then $C_{\frac{1}{2}}(X, R) = C_b(X, R)$ and $\widetilde{C}_{\frac{1}{2}}(X, R) = \widetilde{C}_b(X, R)$. Hereafter, if (X, R) is bounded, $C_{\frac{1}{2}}(X, R)$ is regarded as equipped with the supremum norm. Accordingly, we modify the definition of the norm $|| \cdot ||_{M(X,R)}$ as follows:

$$||\varphi||_{M(X,R)} = \sup\{|\varphi(u)| : u \in \widetilde{C}_b(X, R), ||u||_\infty = 1\}.$$

Proposition 8.3. *For any $f \in L^1(X, \mu_{p,\frac{1}{2}})$ and any $u \in \widetilde{C}_{\frac{1}{2}}(X, R)$, define $\varphi_f(u) = \int_X f(x)u(x)\mu(dx)$. Then $\varphi_f \in M(X, R)$ and $||\varphi_f||_{M(X,R)} = ||f||_1$, where $||f||_1$ is the L^1-norm with respect to $\mu_{p,\frac{1}{2}}$.*

Proof. It is easy to see that $|\varphi_f(u)| \leq ||u||||f||_1$ for any $u \in \widetilde{C}_{\frac{1}{2}}(X, R)$. The equality $||f||_1 = ||\varphi_f||_{M(X,R)}$ is shown by routine arguments using the facts that μ is Borel regular and that $\widetilde{C}_{\frac{1}{2}}(X, R)$ contains $C_0(X, R)$. (Note that $C_0(X, R) \subseteq \widetilde{C}_{\frac{1}{2}}(X, R)$ if $\mathcal{F} \cap C_0(X, R)$ is dense in $C_0(X, R)$ with respect to the supremum norm.) □

Define $\Phi : L^1(X, \mu_{p,\frac{1}{2}}) \to M(X, R)$ by $\Phi(f) = \varphi_f$. Then, by the above theorem, Φ is an isometric embedding from $L^1(X, \mu_{p,\frac{1}{2}})$ to $M(X, R)$. Hereafter, through Φ, we regard $L^1(X, \mu_{p,\frac{1}{2}})$ as a subset of $M(X, R)$.

Theorem 8.4. *Let $B \neq \emptyset$ be a finite subset of X. If $G_{B,\mu} = G_B|_{L^1(X,\mu_{p,\frac{1}{2}})}$, then*

$$(G_{B,\mu}f)(x) = \int_X g_B(x, y)f(y)\mu(dy) \tag{8.1}$$

for any $f \in L^1(X, \mu_{p,\frac{1}{2}})$.

Proof. By (5.4),

$$(G_{B,\mu}f)(x) = (G_B\varphi_f)(x) = \varphi_f(g_B^x)$$

for any $x \in X$. This immediately implies (8.1). □

Definition 8.5. Let $B \neq \emptyset$ be a finite subset of X. Define $\mathcal{D}^L_{B,\mu,0} = \mathrm{Im}(G_{B,\mu})$ and $\mathcal{D}^L_{B,\mu} = \mathcal{D}^L_{B,\mu,0} \oplus \mathcal{H}_B$.

Proposition 8.6. *Assume that μ is non-atomic: $\mu(p) = 0$ for any $p \in X$.*
(1) Let $B \neq \emptyset$ be a finite subset of X. Then $(du)_p = 0$ for any $u \in \mathcal{D}^L_{B,\mu}$ and any $p \in X \backslash B$.
(2) Let B_1 and B_2 be non-empty finite subsets of X with $B_1 \subseteq B_2$. Then

$$\mathcal{D}^L_{B_1,\mu} = \{u | u \in \mathcal{D}^L_{B_2,\mu}, (du)_p = 0 \text{ for any } p \in B_2 \backslash B_1\}.$$

In particular, $\mathcal{D}^L_{B_1,\mu} \subseteq \mathcal{D}^L_{B_2,\mu}$.

Proof. (1) By Theorem 6.6, $(du)_x = -\varphi_f(x) = -f(x)\mu(x) = 0$ for any $x \in X \backslash B$.
(2) Lemma 5.9-(2) implies that $G_{B_1,\mu}f \in \mathcal{D}^L_{B_2,\mu}$. Hence $\mathcal{D}^L_{B_1,\mu} \subseteq \mathcal{D}^L_{B_2,\mu}$. Moreover, by (1), $(du)_x = 0$ for any $u \in \mathcal{D}^L_{B_1}$ and any $x \in B_2 \backslash B_1$.

Conversely let $u = u_{B_2} - G_{B_2,\mu}f$ for $u_{B_2} \in \mathcal{H}_{B_2}$ and $f \in L^1(X, \mu_{p,\frac{1}{2}})$. Assume that $(du)_p = 0$ for any $p \in B_2 \backslash B_1$. So using Theorem 6.6, we obtain

$$(du)_p = -(H_{B_2}u_{B_2})(p) + \varphi_f(\psi_p^{B_2}) = 0 \tag{8.2}$$

for any $p \in B_2 \backslash B_1$. By Lemma 5.9,

$$\begin{aligned} u &= u_{B_2} - G_{B_1,\mu}f + \sum_{p \in B_2 \backslash B_1} \varphi_f(\psi_p^{B_2}) g^p_{B_1} \\ &= \sum_{p \in B_2 \backslash B_1} (\mathcal{E}(u, \psi_p^{B_2}) + \varphi_f(\psi_p^{B_2})) g^p_{B_1} + \sum_{q \in B_1} u(q)\psi_q^{B_1} - G_{B_1,\mu}f. \end{aligned} \tag{8.3}$$

Since $\mathcal{E}(u, \psi_p^{B_2}) = -(H_{B_2}u_{B_2})(p)$ by Lemma 2.20, (8.3) along with (8.2) implies $u = \sum_{q \in B_1} u(q)\psi_q^{B_1} - G_{B_1,\mu}f$. Hence $u \in \mathcal{D}^L_{B_1,\mu}$. □

Proposition 8.7. *Suppose μ is non-atomic. Let B be a non-empty finite subset of X. Define $\Delta_{B,\mu} : \mathcal{D}^L_{B,\mu} \to L^1(X, \mu_{p,\frac{1}{2}})$ by $\Delta_{B,\mu}u = f$ for $u = u_B - G_{B,\mu}f$ where $u_B \in \mathcal{H}_B$ and $f \in L^1(X, \mu_{p,\frac{1}{2}})$. Then,*
(1) *$\Delta_{B,\mu}u = L_B u$ for any $u \in \mathcal{D}^L_{B,\mu}$. In particular,*

$$\mathcal{E}(v, u) = \sum_{p \in B} v(p)(du)_p - \int_X v\Delta_{B,\mu}u d\mu \tag{8.4}$$

for any $v \in \mathcal{F}$.
(2) *For any $f \in L^1(X, \mu_{p,\frac{1}{2}})$ and any $h \in \ell(B)$,*

$$\Delta_{B,\mu}u = f \quad \textit{and} \quad u|_B = h.$$

if and only if

$$u(x) = \sum_{p \in B} h(p)\psi_p^B(x) - \int_X g_B(x, y)f(y)\mu(dy)$$

for any $x \in X$. In particular, $G_{B,\mu}$ is invertible and $\Delta_{B,\mu}|_{\mathcal{D}^L_{B,\mu,0}} = -(G_{B,\mu})^{-1}$.
(3) *$||u||_{\mathcal{D}} = \sum_{p \in B} |(du)_p| + \int_X |\Delta_{B,\mu}f| d\mu$.*
(4) *Let B_1 and B_2 be non-empty finite subsets of X with $B_1 \subseteq B_2$. Then $\Delta_{B_1,\mu} = \Delta_{B_2,\mu}|_{\mathcal{D}^L_{B_1,\mu}}$.*

Proof. Theorem 6.8 immediately implies (1) and (2).
(3) By Theorem 7.2, (7.1) and (8.4) imply

$$||u||_{\mathcal{D}} = \sup_{\substack{h \in \widetilde{C}_{\frac{1}{2}}(X,R) \\ 0 \le |h| \le 1}} \Big| \sum_{p \in B} h(p)(du)_p - \int_X h\Delta_{B,\mu}u d\mu \Big|.$$

Hence we see that $||u||_{\mathcal{D}} \leq \sum_{p\in B} |(du)_p| + \int_X |\Delta_{B,\mu} u| d\mu$. Now by Proposition 8.1, $\widetilde{C}_{\frac{1}{2}}(X, R)$ contains $C_0(X, R)$. This shows the equality.
(4) Combining Proposition 8.6-(2) and (8.4), we obtain

$$\int_X v\Delta_{B_1,\mu} u d\mu = \int_X v\Delta_{B_2,\mu} u d\mu$$

for any $u \in \mathcal{D}^L_{B_1,\mu}$ and and $v \in \mathcal{F}$. Since $C_0(X, R) \cap \mathcal{F}$ is dense in $C_0(X, R)$, it follows that $\Delta_{B_1,\mu} u = \Delta_{B_2,\mu} u$. □

By (8.4), it follows that $\Delta_{B,\mu} u$ is the unique element in $L^1(X, \mu_{p,\frac{1}{2}})$ that satisfies

$$\mathcal{E}(u, v) = -\int_X (\Delta_{B,\mu} u) v d\mu$$

for any $v \in \mathcal{F}_B$.

Definition 8.8. Suppose μ is non-atomic. Define $\mathcal{D}^L_\mu$ by

$$\mathcal{D}^L_\mu = \{u | u \in \mathcal{D}^L_{B,\mu}, (du)_p = 0 \text{ for any } p \in B\},$$

where B is a non-empty finite subset of X.

By Proposition 8.6, $\mathcal{D}^L_\mu$ is independent of a choice of B. In fact,

$$\mathcal{D}^L_\mu = \bigcap_{\substack{B: B \neq \emptyset \\ B \text{ is a finite set}}} \mathcal{D}^L_{B,\mu}.$$

Proposition 8.9. *Suppose μ is non-atomic. $Lu \in L^1(X, \mu_{p,\frac{1}{2}})$ for any $u \in \mathcal{D}^L_\mu$. Let $\Delta_\mu = L|_{\mathcal{D}^L_\mu}$. Then $\Delta_\mu = \Delta_{B,\mu}|_{\mathcal{D}^L_\mu}$ for any non-empty finite set $B \subseteq X$ and $\Delta_\mu u$ is the unique element in $L^1(X, \mu_{p,\mu})$ that satisfies*

$$\mathcal{E}(u, v) = -\int_X (\Delta_\mu u) v d\mu \tag{8.5}$$

for any $v \in \mathcal{F}$. Moreover $||u||_{\mathcal{D}} = \int_X |\Delta_\mu u| d\mu$ for any $u \in \mathcal{D}^L_\mu$.

By Proposition 6.10, we see that $\ker \Delta_\mu =$ constants and $\mathrm{Im}(\Delta_\mu) = \{f | f \in L^1(X, \mu), \int_X f d\mu = 0\}$.

Proof. Let B be a non-empty finite subset of X. For any $u \in \mathcal{D}^L_\mu$, since $(du)_p = 0$ for any $p \in X$, (8.4) implies that

$$\mathcal{E}(u, v) = -\int_X v\Delta_{B,\mu} u d\mu \tag{8.6}$$

for any $v \in \mathcal{F}$. On the other hand, $\mathcal{E}(u, v) = -(Lu)(v)$. Therefore, $Lu = \Delta_{B,\mu} u$. □

Next, we identify $\Delta_{B,\mu}$ and Δ_μ with the non-negative self-adjoint operators on $L^2(X,\mu)$ associated with $(\mathcal{E}, \mathcal{F}_B \cap L^2(X,\mu))$ and $(\mathcal{E}, \mathcal{F} \cap L^2(X,\mu))$. Using the results in [16, Section 2.4] (in particular, Theorem 2.4.1 and 2.4.2), we immediately have the following theorem.

Theorem 8.10. *Suppose μ is non-atomic.*
(1) *Let B be a non-empty finite subset of X. Then $(\mathcal{E}, \mathcal{F}_B \cap L^2(X\backslash B,\mu))$ is a regular Dirichlet form on $L^2(X\backslash B,\mu)$.*
(2) *$(\mathcal{E}, \mathcal{F} \cap L^2(X,\mu))$ is a regular Dirichlet form on $L^2(X,\mu)$.*

Remark. Ordinarily, one assumes that the space is locally compact for a Dirichlet form. In the above theorem, however, (X,R) may not be locally compact in general.

If μ is non-atomic, then $L^2(X\backslash B,\mu)$ may be identified with $L^2(X,\mu)$. Hence we regard $(\mathcal{E}, \mathcal{F}_B)$ as a Dirichlet form on $L^2(X,\mu)$ hereafter.

The next proposition gives direct relations between the non-negative self-adjoint operators associated with the Dirichlet forms and the Laplacians $\Delta_{B,\mu}$ and Δ_μ.

Proposition 8.11. *Assume that μ is non-atomic.*
(1) *Let B be a non-empty finite subset of X and let $\Gamma_{B,\mu}$ be the non-negative self-adjoint operator on $L^2(X,\mu)$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{F}_B)$. Then $L^2(X,\mu) \cap G_{B,\mu}(L^2(X,\mu) \cap L^1(X,\mu_{p,\frac{1}{2}})) \subseteq \mathrm{Dom}(\Gamma_{B,\mu})$ and $\Gamma_{B,\mu}u = -\Delta_{B,\mu}u$ for any $u \in L^2(X,\mu) \cap G_{B,\mu}(L^2(X,\mu) \cap L^1(X,\mu_{p,\frac{1}{2}}))$.*
(2) *Let Γ_μ be the non-negative self-adjoint operator on $L^2(X,\mu)$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$. If $u \in \mathcal{D}^L_\mu \cap L^2(X,\mu)$ and $\Delta_\mu u \in L^2(X,\mu)$, then $u \in \mathrm{Dom}(\Gamma_\mu)$ and $\Gamma_\mu u = -\Delta_\mu u$.*

Proof. (1) By the definition of the non-negative self-adjoint operator associated with a closed form (see [16, Appendix B.1] for example), $u \in \mathrm{Dom}(\Gamma_{B,\mu})$ and $\Gamma_{B,\mu} = f$ if and only if

$$\mathcal{E}(u,v) = \int_X f v d\mu$$

for any $v \in \mathcal{F} \cap L^2(X,\mu)$. This along with (8.6) immediately implies the desired statement.
(2) Using (8.5), we immediately see the desired statement by similar arguments as in (1). □

We have a simpler statement under a restricted situation.

Lemma 8.12. *Suppose that μ is non-atomic. If $\int_X (1+R(x,p))\mu(dx) < \infty$, then $\mathcal{F} \subset L^2(X,\mu) \subset L^1(X,\mu_{p,\frac{1}{2}})$.*

Proof. Let $u \in \mathcal{F}$. Then $|u(x)-u(p)|^2 \leq R(x,p)\mathcal{E}(u,u)$. This implies that $|u(x)|^2 \leq 2R(x,p)\mathcal{E}(u,u) + 2|u(p)|^2$. Hence $u \in L^2(X,\mu)$. Next let $u \in L^2(X,\mu)$. Then

$$\Big|\int_X \sqrt{1+R(x,p)}u(x)\mu(dx)\Big| \leq \int_X (1+R(x,p))\mu(dx) \int_X |u(x)|^2 \mu(dx) < \infty.$$

□

Note that if $\int_X (1 + R(x,p))\mu(dx) < \infty$, then $\Gamma_{B,\mu}$ and Γ_μ have compact resolvents. (See [16, Theorem 2.4.2] for a proof.)

Theorem 8.13. *Assume that μ is non-atomic. Suppose $\int_X (1+R(p,x))\mu(dx) < \infty$.*
(1) *Let B be a non-empty finite subset of X. Then $\mathrm{Dom}(\Gamma_{B,\mu}) \subset \mathcal{D}^L_{B,\mu,0}$ and $\Gamma_{B,\mu}u = -\Delta_{B,\mu}u$ for any $u \in \mathrm{Dom}(\Gamma_{B,\mu})$. Moreover, $(\Gamma_{B,\mu})^{-1}$ is a compact operator and $(\Gamma_{B,\mu})^{-1} = G_{B,\mu}|_{L^2(X,\mu)}$*
(2) *$\mathrm{Dom}(\Gamma_\mu) \subseteq \mathcal{D}^L_\mu$ and $\Gamma_\mu u = -\Delta_{B,\mu}u$ for any $u \in \mathrm{Dom}(\Gamma_\mu)$.*

Proof. (1) Since $\Gamma_{B,\mu}$ has compact resolvent and 0 is not an eigenvalue of $\Gamma_{B,\mu}$, $\Gamma_{B,\mu}$ is invertible and $(\Gamma_{B,\mu})^{-1}$ is a compact operator. In particular, for any $f \in L^2(X,\mu)$, there exists a unique $u \in \mathrm{Dom}(\Gamma_{B,\mu})$ such that $\Gamma_{B,\mu}u = f$. For any $v \in \mathcal{F}$,

$$\mathcal{E}(u,v) = \int_X fv = \mathcal{E}(G_{B,\mu}f, v).$$

Hence $u = G_{B,\mu}f$. Therefore $\mathrm{Dom}(\Gamma_{B,\mu}) = G_{B,\mu}(L^2(X,\mu))$. By Proposition 8.11, it follows that $\Gamma_{B,\mu}u = -\Delta_{B,\mu}u$ for any $u \in \mathrm{Dom}(\Gamma_{B,\mu})$.
(2) Let $u \in \mathrm{Dom}(\Gamma_\mu)$. Then

$$\mathcal{E}(u,v) = \int_X v\Gamma_\mu u d\mu$$

for any $v \in \mathcal{F}$. By Lemma 8.12, $\Gamma_\mu u \in L^1(X, \mu_{p,\frac{1}{2}})$. Since $\int_X \Gamma_\mu u d\mu = \mathcal{E}(1,u) = 0$, we see that $\Gamma_\mu u \in \mathrm{Im}(\Delta_\mu)$. Therefore there exists $h \in \mathcal{D}^L_\mu$ such that $-\Delta_\mu h = \Gamma_\mu u$. By (8.5), $\mathcal{E}(h,v) = \mathcal{E}(u,v)$ for any $u \in \mathcal{F}$. Therefore $u - h$ is a constant and hence $u \in \mathcal{D}^L_\mu$. Finally, by Proposition 8.11-(2), $\Gamma_\mu u = -\Delta_\mu u$. □

9 P. c. f. self-similar sets

In this section, we apply the results in the previous sections to self-similar resistance forms (coming from harmonic structures) on post critically finite self-similar structures. In particular, we show relations between the domain of resistance forms, $\mathcal{F}$, the domain of Laplacian in generalized sense, $\mathcal{D}^L$, and uniformly Lipschitz continuous functions.

First we give a quick review of the theory of analysis on post critically finite self-similar sets. See [16, Chapter 3].

Definition 9.1. Let K be a compact metrizable topological space and let S be a finite set. Also, let F_i, for $i \in S$, be a continuous injection from K to itself. Then, $(K, S, \{F_i\}_{i \in S})$ is called a self-similar structure if there exists a continuous surjection $\pi : \Sigma \to K$ such that $F_i \circ \pi = \pi \circ i$ for every $i \in S$, where $\Sigma = S^{\mathbb{N}}$ is the one-sided shift space and $i : \Sigma \to \Sigma$ is defined by $i(w_1w_2w_3\cdots) = iw_1w_2w_3\cdots$ for each $w_1w_2w_3\cdots \in \Sigma$.

Note that if $(K, S, \{F_i\}_{i\in S})$ is a self-similar structure, then K is self-similar in the following sense:

$$K = \bigcup_{i\in S} F_i(K). \tag{9.1}$$

Notation. $W_m = S^m$ is the collection of words with length m. For $w = w_1w_2\cdots w_m \in W_m$, we define $F_w : K \to K$ by $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$ and $K_w = F_w(K)$. In particular, $W_0 = \{\emptyset\}$ and $F_\emptyset$ is the identity map. Also we define $W_* = \cup_{m\geq 0} W_m$.

Definition 9.2. Let $(K, S, \{F_i\}_{i\in S})$ be a self-similar structure. We define the critical set $\mathcal{C} \subset \Sigma$ and the post critical set $\mathcal{P} \subset \Sigma$ by

$$\mathcal{C} = \pi^{-1}(\bigcup_{i\neq j}(K_i \cap K_j)) \quad \text{and} \quad \mathcal{P} = \bigcup_{n\geq 1} \sigma^n(\mathcal{C}),$$

where σ is the shift map from Σ to itself defined by $\sigma(\omega_1\omega_2\cdots) = \omega_2\omega_3\cdots$. A self-similar structure is called post critically finite (p. c. f. for short) if and only if $\#(\mathcal{P})$ is finite.

Now, we fix a connected p. c. f. self-similar structure $(K, S, \{F_i\}_{i\in S})$.

Definition 9.3. Let $V_0 = \pi(\mathcal{P})$. For $m \geq 1$. Also set

$$V_m = \bigcup_{w\in W_m} F_w(\pi(\mathcal{P})) \quad \text{and} \quad V_* = \bigcup_{m\geq 0} V_m.$$

It is easy to see that $V_m \subset V_{m+1}$ and that K is the closure of V_*.

Next we explain how to construct Laplacians on a p. c. f. self-similar set. First we define a Laplacian on a finite set.

Proposition 9.4. *Let $D \in \mathcal{LA}(V_0)$ and let $\mathbf{r} = (r_1, r_2, \cdots, r_N)$, where $r_i > 0$ for $i \in S$. Define a symmetric bilinear form $\mathcal{E}^{(m)}$ on $\ell(V_m)$ by $\mathcal{E}^{(m)}(u, v) = \sum_{w\in W_m} {r_w}^{-1}\mathcal{E}_D(u \circ F_w, v \circ F_w)$, where $r_w = r_{w_1}\cdots r_{w_m}$ for $w = w_1w_2\cdots w_m \in W_m$. Then $\mathcal{E}^{(m)} \in \mathcal{DF}(V_m)$. We denote the Laplacian on V_m corresponding $\mathcal{E}^{(m)}$ by H_m.*

Definition 9.5. $(D, \mathbf{r})$ is called a harmonic structure if and only if the sequence $\{(V_m, H_m)\}_{m\geq 0}$ is a compatible sequence. Furthermore, a harmonic structure $(D, \mathbf{r})$ is called regular if and only if $0 < r_i < 1$ for any $i \in S$.

It is know that $(D, \mathbf{r})$ is a harmonic structure if and only if $(V_0, D) \leq (V_1, H_1)$. See [16, Proposition 3.1.3] for details. Hereafter we fix a regular harmonic structure $(D, \mathbf{r})$ on $(K, S, \{F_i\}_{i\in S})$.

If $(D, \mathbf{r})$ is a harmonic structure, then by Theorem 2.13, we have a resistance form $(\mathcal{E}, \mathcal{F})$ on V_* associated with the compatible sequence $\{(V_m, H_m)\}_{m\geq 0}$. Let R be the resistance metric on V_* corresponding to $(\mathcal{E}, \mathcal{F})$. Since $(D, \mathbf{r})$ is assumed to be regular, we have the following fact.

Theorem 9.6. *Let (Ω, R) be the completion of (V_*, R). Then Ω is naturally identified with K. Through this identification, R gives the same topology to K as the original distance of K. In particular, (K, R) is compact and $\mathcal{F}$ is a dense subset of $C(K, R)$ with respect to the supremum norm.*

See [16, Section 3.3] for the proof of the above theorem.

By this theorem, we may naturally think of $C(K, R)$ as a subset of $\ell(V_*)$. Also, if follows that $C_{\frac{1}{2}}(K, R) = \widetilde{C}_{\frac{1}{2}}(K, R) = C(K, R)$. Hence $M(K, R)$ is the dual space of $C(K, R)$ (i.e. measures on (K, R)). The following fact is a immediate corollary of Theorems 6.6 and 7.5. Recall that $\mathcal{D}^L \subset \mathcal{F} \cap C_L(K, R)$, where $C_L(K, R)$ is the collection of uniformly Lipschitz continuous functions on K with respect to R.

Proposition 9.7. *For any $u \in \ell(V_*)$, $||u||_{\mathcal{D}} = \lim_{m\to\infty} \sum_{p\in V_m} |(H_m u)(p)|$. Also $u \in \mathcal{D}^L$ if and only if $||u||_{\mathcal{D}} < \infty$. Moreover, for any $u \in \mathcal{D}^L$ and any $p \in K$, $\mathcal{E}(u, \psi_p^{V_m \cup p}) = -(H_{V_m \cup p} u)(p)$ converges as $m \to \infty$. The limit is denoted by $(du)_p$.*

The Green function g_B coincides with the one defined in [16, Appendix A.2] when B is a finite subset of V_*. T. Watanabe has studied the case where B is a general finite subset of K in [23]. He has obtained the Green function, harmonic functions and Laplacians and extended the results in [16]. He has also considered the case where the harmonic structure is not regular.

Let μ be a Borel regular measure on K with $\mu(K) = 1$. Also we assume that μ is non-atomic. Then $\mu_{p,\frac{1}{2}} = \mu$ and we may apply all the results in Section 8. $\mathcal{D}_\mu^B$ defined in [16, Appendix A.2] is equal to $G_{B,\mu}(C(K, R)) \oplus \mathcal{H}_B$ in our context, while $\mathcal{D}_{B,\mu}^L = G_{B,\mu}(L^1(K, R))$. So, the Laplacian $\Delta_{B,\mu}$ in this paper maps $\mathcal{D}_{B,\mu}^L$ to $L^1(K, \mu)$ and $-\Delta_{B,\mu}|_{G_{B,\mu}(L^2(K,\mu))}$ is the non-negative self adjoint operator on $L^2(K, \mu)$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{F}_B)$ on $L^2(K, \mu)$.

We will study relations between $\mathcal{F}, \mathcal{D}^L$ and $C_L(K, R)$ in the rest of this section.

Definition 9.8. (1) Define $\psi_p^m = \psi_p^{V_m}$ for $p \in V_m$. Set $i(p) = m$ if $p \in V_m \backslash V_{m-1}$ for any $p \in V_*$. (We think of V_{-1} as $\emptyset$.) Then, define $\psi_p = \psi_p^{i(p)}$.
(2) Define $u_m = \sum_{p\in V_m} u(p)\psi_p^m$ for any $u \in \ell(V_*)$ and any $m \geq 0$.

Any function $u \in \ell(V_*)$ has an expansion with respect to the basis $\{\psi_p\}_{p\in V_*}$.

Proposition 9.9. *Let $u \in \ell(V_*)$. If $\alpha_p(u) = u_{i(p)}(p) - u_{i(p)-1}(p)$ for any $p \in V_*$, then*

$$u(x) = \sum_{p\in V_*} \alpha_p(u)\psi_p(x) \tag{9.2}$$

for any $x \in V_*$. (In the definition of $\alpha_p(u)$, we set $u_{-1} = 0$ when $i(p) = 0$.)

Note that the converse of the above proposition is also true: for given $\{\alpha_p\}_{p\in V_*}$, letting $u = \sum_{p\in V_*} \alpha_p \psi_p$, we see that there exists $u \in \ell(V_*)$ such that $\alpha_p(u) = \alpha_p$ for any $p \in V_*$.

Definition 9.10. Let $u \in \ell(V_*)$ and let $w \in W_*$. Define $\alpha^w(u) \in \ell(V_1\backslash V_0)$ by $(\alpha^w(u))_q = \alpha_{F_w(q)}(u)$ for any $q \in V_1\backslash V_0$. Also define $H_w u \in \ell(V_1\backslash V_0)$ by $(H_w u)_q = (H_{|w|+1}u)(F_w(q))$ for any $q \in V_1\backslash V_0$. Let $a_w(u) = u_{|w|+1} \circ F_w - u_{|w|} \circ F_w$.

It is easy to see that $a_w(u) = \sum_{q \in V_1\backslash V_0} \alpha_{F_w(q)}(u)\psi_q$. In particular, $\alpha^w(u) = a_w(u)|_{V_1\backslash V_0}$.

Lemma 9.11. *For any $u \in \ell(V_*)$ and any $w \in W_*$,*

$$\alpha^w(u) = r_w X^{-1} H_w u.$$

Proof. Let $m = |w|$. Then, for any $q \in V_1\backslash V_0$,

$$(H_{m+1}u)(F_w(q)) = (r_w)^{-1}(H_1(u \circ F_w))(q) = (r_w)^{-1}(H_1(u_{m+1} \circ F_w)(q).$$

Since $(H_1 u_m \circ F_w)(q) = 0$ for any $q \in V_1\backslash V_0$, we have $H_w u = (r_w)^{-1} H_1 a_w(u) = (r_w)^{-1} X \alpha^w(u)$. □

Theorem 9.12. *Let $u \in \ell(V_*)$.*
(1) *$u \in \mathcal{F}$ if and only if*

$$\sum_{m \geq 0} \sum_{w \in W_m} r_w (|H_w u|_{V_1\backslash V_0})^2 < \infty. \tag{9.3}$$

(2) *$u \in C(K)$ if*

$$\sum_{m \geq 0} (\max_{w \in W_m} r_w |H_w u|_{V_1\backslash V_0}) < \infty. \tag{9.4}$$

(3) *If $u \in \mathcal{D}^L$, then*

$$\sup_{m \geq 0} (\sum_{w \in W_m} |H_w u|_{V_0\backslash V_1}) < \infty. \tag{9.5}$$

(4) *$u \in C_L(K, R)$ if*

$$\sum_{m \geq 0} (\max_{w \in W_m} |H_w u|_{V_1\backslash V_0}) < \infty. \tag{9.6}$$

This theorem will be proven at the end of this section. Meanwhile, applying the theorem to a special class of functions, we show relations between $\mathcal{F}$, $\mathcal{D}^L$ and $C_L(K, R)$.

Corollary 9.13. *Let $(\beta_i)_{i \in S} \in \ell(S)$ and let $\mathbf{c} \in \ell(V_1\backslash V_0)$. Assume that $\mathbf{c} \neq 0$. Define $u \in \ell(V_*)$ by*

$$\begin{aligned} \alpha_p(u) &= 0 \quad \text{for any } p \in V_0 \\ \alpha^w(u) &= r_w \beta_w \mathbf{c} \quad \text{for any } w \in W_*, \end{aligned} \tag{9.7}$$

where $\beta_w = b_{w_1} \cdots b_{w_m}$ *for* $w = w_1 w_2 \cdots w_m \in W_*$.
(1) $u \in \mathcal{F}$ *if and only if* $\sum_{i\in S} r_i|\beta_i|^2 < 1$.
(2) *If* $\max_{i\in S} r_i|\beta_i| < 1$, *then* $u \in C(K)$.
(3) *If* $u \in \mathcal{D}^L$, *then* $\sum_{i\in S} |\beta_i| \le 1$.
(4) *If* $\max_{i\in S} |\beta_i| < 1$, *then* $u \in C_L(K, R)$.

Proof. By Lemma 9.11, $H_w u = \beta_w X\mathbf{c}$. Hence the equations (9.3) - (9.6) is immediately translated into the corresponding conditions on β. □

By Proposition 5.12, it follows that $\mathcal{D}^L \subseteq \mathcal{F} \cap C_L(K, R)$. The next corollary says that the converse is not true.

Corollary 9.14. (1) $\mathcal{F} \cap C_L(K, R) \cap (\mathcal{D}^L)^c \neq \emptyset$.
(2) $\sum_{i\in S} r_i = 1$ *if and only if* $C_L(K, R) \subset \mathcal{F}$

Remark. By [15, Theorem 3.2], it follows that $\sum_{i\in S} r_i \ge 1$. (We can prove this fact by using (9.8) below as well. Let f be a nontrivial V_0-harmonic function. Then $f \in C_L(K, R) \cap \mathcal{F}$ and $\mathcal{E}_m(f, f) = \mathcal{E}(f, f) > 0$. By (9.8), $\sum_{i\in S} r_i \ge 1$.) Also by [15, Theorem 3.2] (or [16, Theorem 4.1.2]), $\sum_{i\in S} r_i = 1$ if and only if the Hausdorff dimension of (K, R) is one.

Proof. (1) Let u be given by (9.7). Set $\beta_i = N^{-1/2}$ for all $i \in S$, where N is the number of elements in S. Since $0 < r_i < 1$ for any $i \in S$, $\sum_{i\in S} r_i < N$. This fact along with Corollary 9.13 implies that $u \in \mathcal{F} \cap C_L(K, R) \cap (\mathcal{D}^L)^c$.
(2) First assume $\sum_{i\in S} r_i > 1$. Let u be given by (9.7) with $\beta_i = (\sum_{i\in S} r_i)^{-1/2}$. Then by Corollary 9.13, $u \in \mathcal{F}^c \cap C_L(K, R)$.

Next we show the converse. Define $L(f) = \sup_{x,y\in K} |f(x) - f(y)|/R(x, y)$ for $f \in C_L(K, R)$. Note that there exists $c > 0$ such that

$$\mathcal{E}_0(h, h) \le c \max_{p,q\in V_0} |h(p) - h(q)|^2$$

for any $h \in \ell(V_0)$. Using this and Theorem A.1, we see that

$$\begin{aligned}\mathcal{E}_m(f, f) &= \sum_{w\in W_m} \frac{1}{r_w} \mathcal{E}_0(f \circ F_w, f \circ F_w) \\ &\le c \sum_{w\in W_m} \frac{1}{r_w} \max_{p,q\in V_0} |f(F_w(p)) - f(F_w(q))|^2 \\ &\le cL(f) \sum_{w\in W_m} \frac{1}{r_w} \max_{p,q\in V_0} R(F_w(p), F_w(q))^2 \le c'L(f)\big(\sum_{i\in S} r_i\big)^m\end{aligned} \tag{9.8}$$

for any $m \ge 0$ and any $f \in C_L(K, R)$, where $c' = c\max_{p,q\in V_0} R(p, q)^2$. If $\sum_{i\in S} r_i = 1$, then (9.8) implies that $\mathcal{E}_m(f, f)$ is bounded and hence $f \in \mathcal{F}$. □

Example 9.15. Let $K = [0, 1]$. Define $R(x, y) = |x - y|$ for any $x, y \in K$. Choose $r_1, r_2 > 0$ with $r_1 + r_2 = 1$. Set $F_1(x) = r_1 x$ and $F_2(x) = r_2 x + r_1$. Then $K = F_1(K) \cup F_2(K)$ and $F_1(K) \cap F_2(K) = \{r_1\}$. We see that

$(K, \{1,2\}, \{F_1, F_2\})$ is a p. c. f. self-similar structure. Let $D = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ and let $\mathbf{r} = (r_1, r_2)$. Then $(D, \mathbf{r})$ is a regular harmonic structure. If $(\mathcal{E}, \mathcal{F})$ is the resistance form corresponding this harmonic structure, then $\mathcal{F} = H^1([0,1]) = \{f | f' \in L^2([0,1], dx)\}$, where f' is the derivative of f in the generalized sense, and $\mathcal{E}(f,g) = \int_0^1 f'(x)g'(x)dx$. Also the corresponding effective resistance is R. In this case,

$$\mathcal{D}^L = \{f | f' \text{ is bounded variation}\},$$
$$C_L(K,R) = \{f | f' \text{ is bounded}\}$$

and $C_L(K,R) \subset \mathcal{F}$.

The rest of this section is devoted to proving Theorem 9.12.

Proof of Theorem 9.12. (1) By [16, Proposition 3.2.19] and Lemma 9.11, $u \in \mathcal{F}$ if and only if $\sum_{m \geq 0} \sum_{w \in W_m} r_w (H_w u, -X^{-1} H_w u)_{V_1 \backslash V_0}$. Since $-X^{-1}$ is positive definite, this is equivalent to (9.3).
(2) Let $u_m = \sum_{p \in V_m \backslash V_{m-1}} \alpha_p \psi_p$ for $m \geq 0$, where $V_{-1} = \emptyset$. Then $u = \sum_{m \geq 0} u_m$. Note that $u_m \in C(K)$. By Lemma 9.11 and (9.4), it follows that $\sum_{m \geq 0} u_m$ is uniformly convergent on K as $m \to \infty$.
(3) If $u \in \mathcal{D}^L$, Theorem 7.2 implies that $\sup_{m \geq 0} \sum_{p \in V_m} |(H_m u)(p)| < \infty$. Therefore, we obtain (9.5)
(4) Let $u_m = \sum_{p \in V_m \backslash V_{m-1}} \alpha_p(u) \psi_p$ for $m \geq 0$. ($u_0 = \sum_{p \in V_0} u(p)\psi_p$.) Then $u = \sum_{m \geq 0} u_m$. Define $\epsilon_m \max_{w \in W_{m-1}} |H_w u|$. By (9.6), $\sum_{m \geq 0} \epsilon_m < \infty$. Next we consider the Lipschitz constant of u_m.
Case 1: Suppose that $x, y \in K_w$ for some $w \in W_{m-1}$. Set $x_1 = (F_w)^{-1}(x)$ and $y_1 = (F_w)^{-1}(y)$. Note that ψ_q is uniformly Lipschitz continuous for any $q \in V_1 \backslash V_0$. By Theorem A.1, we see

$$\begin{aligned} |u_m(x) - u_m(y)| &\leq \sum_{q \in V_1 \backslash V_0} |\alpha_{F_w(q)}(u)||\psi_q(x_1) - \psi_q(y_1)| \leq c_1 |\alpha^w(u)| R(x_1, y_1) \\ &\leq c_2 |\alpha^w(u)| (r_w)^{-1} R(x,y) \leq c_3 \epsilon_m R(x,y), \end{aligned}$$

where c_1, c_2 and c_3 are independent of x, y, w and m.
Case 2: Suppose that $w, w' \in W_{m-1}, w \neq w', x \in K_w$ and $y \in K'_w$. Then, for any $z \in F_w(V_0)$, the result of Case 1 along with Lemma 3.3 implies that

$$|u_m(x)| \leq c_3 \epsilon_m R(x,z) \leq c_4 \epsilon_m R(x, F_w(V_0)),$$

where $c_4 = c_3 \#(V_0)$. In the same manner, $|u_m(y)| \leq c_4 \epsilon_m R(y, F_{w'}(V_0))$. Note that $R(x, F_w(V_0)) + R(y, F_{w'}(V_0)) = R^V(x,y) \leq R(x,y)$, where $V = F_w(V_0) \cup F_{w'}(V_0)$. Hence $|u_m(x) - u_m(y)| \leq |u_m(x)| + |u_m(y)| \leq c_4 \epsilon_m R(x,y)$.

The above two cases implies that $|u_m(x) - u_m(y)| \leq c_4 \epsilon_m R(x,y)$ for any $x, y \in K$. Since $u = \sum_{m \geq 0} u_m$, we see $|u(x) - u(y)| \leq c_4 \sum_{m \geq 0} \epsilon_m R(x,y)$ for any $x, y \in K$. □

Appendix A

Assume the same situation as in the last section: $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is a p. c. f. self-similar structure, $(D, \mathbf{r})$ is a regular harmonic structure on $\mathcal{L}$, and $(\mathcal{E}, \mathcal{F})$ and R are the corresponding resistance form and the resistance metric respectively.

In this appendix, we show that F_i is asymptotically a similitude with a contraction ratio r_i under R. Precisely we have the following theorem.

Theorem A.1. *There exists c_1 such that*

$$c_1 r_w R(x, y) \leq R(F_w(x), F_w(y)) \leq r_w R(x, y)$$

for any $w \in W_*$ and any $x, y \in K$.

The upper estimate of $R(F_w(x), F_w(y))$ can be found in [16, Lemma 3.3.5]. So what really matters here is the lower estimate. We will do this in several steps.

First, we prove the following result on resistance forms. It shows that if one moves a resistor, the effective resistance between the new terminals of the resistor become smaller than before.

Theorem A.2. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X and let R be the resistance metric associated with $(\mathcal{E}, \mathcal{F})$. Let $x, x_1, \ldots, x_n, y, y_1, \cdots, y_n \in X$ and assume that $x \neq y$. For $r_1, \ldots, r_n > 0$, define*

$$\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + \sum_{i=1}^{n} \frac{1}{r_i}(u(x_i) - u(y_i))(v(x_i) - v(y_i))$$
$$\mathcal{E}_2(u, v) = \mathcal{E}(u, v) + \big(\sum_{i=1}^{n} \frac{1}{r_i}\big)(u(x) - u(y))(v(x) - v(y))$$

for any $u, v \in \mathcal{F}$. Then $(\mathcal{E}_1, \mathcal{F})$ and $(\mathcal{E}_2, \mathcal{F})$ are resistance forms on X. Moreover, if R_1 and R_2 are the resistance metrics associated with $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively, then

$$R_1(x, y) \geq R_2(x, y) = \big(\frac{1}{R(x, y)} + \sum_{i=1}^{n} \frac{1}{r_i}\big)^{-1}.$$

Proof. We may assume that $n = 1$, because the general case easily follows by induction. Without loss of generality, we may also suppose that $X = \{x, y, x_1, y_1\}$. We write $p_1 = x, p_2 = y, p_3 = x_1$ and $p_4 = y_1$. (X is assumed to contain exactly four points; otherwise the following discussion is much easier.) Let

$$E(u, v) = \sum_{1\leq i<j\leq 4} \frac{1}{r_{ij}}(u(p_i) - u(p_j))(v(p_i) - v(p_j)).$$

We denote $a = (r_{14})^{-1}, b = (r_{13})^{-1}, c = (r_{24})^{-1}, d = (r_{24})^{-1}$ and $e = (r_{34})^{-1}$. Writing $h = (r_1)^{-1}$ and letting $H_2(h) = R_2(x, y)^{-1}$, we see that $H_2(h) =$

$R(x,y)^{-1}+h$. On the other hand, by an elementary calculation, it follows that

$$H_1(h) = \frac{1}{r_{12}} + \frac{(a+b)(c+d)(h+e)+(ac(b+d)+db(a+c))}{(a+c)(b+d)+(h+e)(a+b+c+d)},$$

where $H_1(h) = R_1(x,y)^{-1}$. Therefore,

$$\frac{\partial H_1}{\partial h} = \frac{(ad-bc)^2}{((a+c)(b+d)+(h+e)(a+b+c+d))^2} \leq 1.$$

Since $H_1(0) = H_2(0) = R(x,y)^{-1}$, we obtain $H_1(h) \geq H_2(h)$ for any $h > 0$. This immediately implies the desired result. □

Define

$$\Lambda(r) = \{\tau = \tau_1\tau_2\cdots\tau_m | r_{\tau_1\tau_2\cdots\tau_{m-1}} > r \geq r_{\tau_1\tau_2\cdots\tau_m}\}$$

for $0 < r < 1$. It is known that $\Lambda(r)$ is a partition of Σ. (See [16, Chapter 1].) We write $\Lambda = \Lambda(r_w)$, $\Lambda_w = \{\tau|\tau \in \Lambda, \tau \neq w, K_w \cap K_\tau \neq \emptyset\}$ and $\Lambda'_w = \{\tau|\tau \in \Lambda, K_w \cap K_\tau = \emptyset\}$. Then we have the following facts.

Lemma A.3 ([16, Lemma 4.2.3]). *There exists $B > 0$ such that $\#(\Lambda_w) \leq B$ for any $w \in W_*$.*

Lemma A.4 ([16, (3.3.1)]). *For any $u, v \in \mathcal{F}$,*

$$\mathcal{E}(u,v) = \sum_{\tau\in\Lambda} \frac{1}{r_\tau}\mathcal{E}(u\circ F_\tau, u\circ F_\tau).$$

Next define $V_\Lambda = \cup_{\tau\in\Lambda}F_\tau(V_0)$, $V = V_\Lambda \cup \{F_w(x), F_w(y)\}$ and $U = V_0 \cup \{x,y\}$. Then $V = \cup_{\tau\in\Lambda_w\cup\Lambda'_w}F_\tau(V_0) \cup F_w(U)$. By Proposition 2.10, there exist $H_V \in \mathcal{LA}(V)$ such that $R_{H_V} = R|_{V\times V}$. In the same way, we have $H_U \in \mathcal{LA}(U)$. Then Lemma A.4 implies the following fact. Note that $D = H_{V_0}$.

Lemma A.5. *For any $u, v \in \ell(V)$,*

$$\mathcal{E}_{H_V}(u,v) = \frac{1}{r_w}\mathcal{E}_{H_U}(u\circ F_w, v\circ F_w) + \sum_{\tau\in\Lambda_w\cup\Lambda'_w} \frac{1}{r_\tau}\mathcal{E}_D(u\circ F_\tau, v\circ F_\tau).$$

Note that $R(F_w(x), F_w(y)) = R_{H_V}(F_w(x), F_w(y))$.

Proof of Theorem A.1. Let $V_0 = \{p_1, \cdots, p_l\}$. If $I = \{(i,j)|i<j, D_{p_ip_j} \neq 0\}$ and $r_{ij} = (D_{p_ip_j})^{-1}$ for $(i,j) \in I$, it follows that

$$\mathcal{E}_D(u,v) = \sum_{(i,j)\in I} \frac{1}{r_{ij}}(u(p_i)-u(p_j))(v(p_i)-v(p_j)).$$

We consider the $V\backslash F_w(U)$-shorted resistance form of $(\mathcal{E}_{H_V}, \ell(V))$ and denote it by $(\mathcal{E}', \mathcal{F}')$, where $\mathcal{F}' = \ell(F_w(U)\cup\{b\})$. (The one point b represents $V\backslash F_w(U)$.)

Since the points in $F_\tau(V_0)$ for $\tau \in \Lambda'_w$ contracts to the single point b, the part of $\mathcal{E}_{H_V}$ coming from Λ'_w, $\sum_{\tau\in\Lambda' w}(r_\tau)^{-1}\mathcal{E}_D(u\circ F_\tau, v\circ F_\tau)$, vanishes in the shorted form $\mathcal{E}'$. The part coming from Λ_w becomes

$$\sum_{k=1}^{m}\frac{1}{r'_k}(u(x_k)-u(y_k))(v(x_k)-v(y_k)),$$

where $(x_k, y_k) \in F_w(V_0)\times(F_w(V_0)\cup\{b\})$, $m \le B\#(I)$ and $r'_k = r_\tau r_{ij}$ for some $\tau \in \Lambda_w$ and some $(i,j) \in I$. Hence,

$$\mathcal{E}'(u,v) = \frac{1}{r_w}\mathcal{E}_{H_u}(u\circ F_w, v\circ F_w) + \sum_{k=1}^{m}\frac{1}{r'_k}(u(x_k)-u(y_k))(v(x_k)-v(y_k)).$$

Denote $\mathcal{E}'$ by $\mathcal{E}'_1$ and define $\mathcal{E}'_2$ by

$$\mathcal{E}'_2(u,v) = \frac{1}{r_w}\mathcal{E}_{H_U}(u\circ F_w, v\circ F_w) + \sum_{k=1}^{m}\frac{1}{r'_k}(u(F_w(x))-u(F_w(y)))(v(F_w(x))-v(F_w(y))).$$

The effective resistance between $F_w(x)$ and $F_w(y)$ with respect to the form $(r_w)^{-1}\mathcal{E}_{H_U}(u\circ F_w, v\circ F_w)$ is $r_w R(x,y)$. By this fact along with Theorem A.2,

$$R(F_w(x), F_w(y)) \ge R'_1(F_w(x), F_w(y)) \ge R'_2(F_w(x), F_w(y)) = \Big(\frac{1}{r_w R(x,y)} + \sum_{k=1}^{m}\frac{1}{r'_k}\Big)^{-1},$$

where R'_i and R'_2 corresponds to the effective resistance with respect to $\mathcal{E}'_i$ for $i = 1, 2$. Note that if $c = (\min_{i\in S} r_i)(\min_{(i,j)\in I} r_{ij})$, then $r'_k \ge c r_w$ for any k. Hence,

$$R'_2(F_w(x), F_w(y)) \ge \Big(\frac{1}{r_w R(x,y)} + \frac{m}{c r_w}\Big)^{-1} \ge \frac{1}{2} r_w \min\{R(x,y), A\},$$

where $A = c(B\#(I))^{-1}$. Now if $r_w R(x,y) \le A$, then $R(x,y) \ge r_w R(x,y)/2$. Otherwise, let $d = \sup_{p,q\in K} R(p,q)$. Then $A =\ge R(x,y)A/d$. This implies $R(x,y) \ge r_w R(x,y) A(2d)^{-1}$. Combining these, we see that $R(F_w(x), F_w(y)) \ge c_1 r_w R(x,y)$, where $c_1 = A(2d)^{-1}$. □

Appendix B

In this appendix, we show a relation between the hitting time and the effective resistance by using the definition of the Green function, Definition 4.1. This relation is an extension of Theorem 4.27 and Corollary 4.28 in [1], which was originally given in [7]. Let $(\mathcal{E}, \mathcal{F})$ be a resistance form on X and let R be the associated resistance metric on X. Also we assume that $(\mathcal{E}, \mathcal{F} \cap L^2(X, \mu))$ is a regular Dirichlet form on $L^2(X, \mu)$, where μ is assume to be a σ-finite Borel regular measure on X. Let $(\{X_t\}_{t>0}, \{P_x\}_{x \in X})$ be the Hunt process associated with the Dirichlet form $(\mathcal{E}, \mathcal{F} \cap L^2(X, \mu))$. See [9] about the relation between Dirichlet forms and Hunt processes. For a subset $A \subseteq X$, define $T_A = \inf\{t | t > 0, X_t \in A\}$. Then we see that

Lemma B.1. *For any finite subset $B \subset X$ and any $x \in X$,*

$$E_x(T_B) = \int_X g_B(x, y)\mu(dy).$$

This lemma along with the definition of the Green function, Definition 4.1, immediately imply the following theorem.

Theorem B.2. (1) *For any $x, y \in X$,*

$$R(x, y) = E_x(T_y) + E_y(T_x).$$

(2) *For any finite subset $B \subset X$ and any $x \in X$,*

$$R(x, B) \geq E_x(T_B).$$

Note that $E_x(T_B) = R(x, B) \int_X \psi_x^{B \cup x}(y)\mu(dy)$.

The above theorem holds for non-regular harmonic structures on p. c. f. self-similar sets as well. More precisely, let $(K, S, \{F_i\}_{i \in S})$ be a p. c. f. self-similar structure and let $(D, \mathbf{r})$ be a harmonic structure on it. If $(D, \mathbf{r})$ is not regular (i.e. $r_i \geq 1$ for some $i \in S$), then Ω is identified with a proper subset of K, where Ω is defined in Theorem 9.6. Hence in such a case, R is not a distance on K. However, it has been shown in [23] that the B-harmonic functions are naturally extended to continuous functions on K if B is a finite subset of Ω. Therefore, Theorem B.2 remains true if x, y and B belong to Ω.

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