Resistance forms, quasisymmetric maps and heat kernel estimates

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1 Introduction

(X, d, μ): a metric measure space
X: a set, d: a metric on X, μ: a Borel regular measure on (X, d)

$$\text{A heat equation on } X\text{: } \frac{\partial u}{\partial t} = Lu, \ L \text{ is a "Laplacian" on } X$$

$$\Downarrow$$

$$\text{Heat kernel: } p(t, x, y), \ t > 0, x, y \in X.$$

$$u(t, x)$$
$$||$$
$$\int_X p(t, x, y) u_0(y) \mu(dy)$$
$$||$$
$$E_x(u_0(X_t))$$

$$\text{Transition density: } p(t, x, y)$$

$$\Uparrow$$

$$(\{X_t\}_{t>0}, \{P_x\}_{x\in X})\text{: a Markov process (Hunt process) on } X$$

$$\Uparrow$$

A regular Dirichlet form $(\mathcal{E}, \mathcal{F})$: a quadratic form on $L^2(X, \mu)$ with the "Markov" property

$$\mathcal{E}(u, v) = -\int_X u(Lv) d\mu$$

Heat kernel estimates:

(1) **Brownian motion on** $\mathbb{R}^n$ $\leftrightarrow$ the heat equation: $\dfrac{\partial u}{\partial t} = c \sum_{i=1}^{n} \dfrac{\partial^2 u}{\partial {x_i}^2}$

$$\text{Gaussian}: \quad p(t,x,y) = \frac{c_1}{t^{n/2}} \exp\left(-c_2 \frac{|x-y|^2}{t} \right)$$

(2) **Riemannian manifold**: Li-Yau(1986)

(X, d): complete Riemannan manifold with the Ricci curvature ≥ 0

$$p(t,x,y) \approx \frac{c_1}{V_d(x,t^{1/2})} \exp\left(-c_2 \frac{d(x,y)^2}{t} \right),$$

where $V_d(x,r)$: the volume of a Ball $= \mu(B_d(x,r))$,
$B_d(x,r) = \{y | d(x,y) < r\}$.

(3) **Brownian motions on Fractals**:

Sierpinski gasket (Barlow-Perkins), Sierpinski carpet (Barlow-Bass)

$$\text{sub-Gaussian}: \quad p(t,x,y) \approx \frac{c_1}{t^{\alpha/\beta}} \exp\left(-c_2 \left(\frac{d(x,y)^\beta}{t} \right)^{1/(\beta-1)} \right)$$

$\beta > 2$: the walk dimension, α: the Hausdorff dimension

(4) **the Li-Yau type sub-Gaussian (LY)**:

$$p(t,x,y) \approx \frac{c_1}{V_d(x,t^{1/\beta})} \exp\left(-c_2 \left(\frac{d(x,y)^\beta}{t} \right)^{1/(\beta-1)} \right)$$

General "desirable" estimate for diffusion processes

(5) α**-stable process on** $\mathbb{R}^n$: $\alpha \in (0,2)$

↑

Jump process (Pathes of the process are not continuous.)

$$\mathcal{E}^{(\alpha)}(u,u) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+\alpha}} dxdy = \int_{\mathbb{R}^n} u(x)\big((-\Delta)^{\alpha/2}u\big)(x)dx$$

Laplacian $L = -(-\Delta)^{\alpha/2}$: not a local operator

$$p(t,x,y) \approx \min\left\{t^{-n/\alpha}, \frac{t}{|x-y|^{n+\alpha}}\right\}$$

Convension: $f,g : X \to [0,\infty)$. $f \asymp g \underset{\text{def}}{\Leftrightarrow} \exists c_1, c_2 > 0$ such that

$$c_1 f(x) \le g(x) \le c_2 f(x)$$

Aim of Study 1: Intrinsic meatic

The original metric is not always the best.

"Good" heat kernel esimate may not always hold under the original metric d. There may exist a metric which is suitable for describing asymptotic behaviors of the heat kernel.

When and How can we find such a metric?

Aim of Study 2: Regulation of Jumps

If the process is not diffusion, the jumps may cause troubles to describe asymptotic behaviors.

How can we regulate Jumps?

We will study those problems in the case of

Hunt processes associated with **resistance forms**.

$\uparrow$

strongly recurrent Hunt process

Capacity of a point is positive

Definition of resistance forms

Definition 1.1. X: a set.
$(\mathcal{E}, \mathcal{F})$ is called a **resistance form** on $X \underset{\text{def}}{\Leftrightarrow}$ (RF1) through (RF5) hold.
(RF1) $\mathcal{F}$ is a linear subspace of $\ell(X)$, $1 \in \mathcal{F}$,
$\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$, non-negative symmetric
$\mathcal{E}(u, u) = 0$ if and only if u is constant on X.
(RF2) Let $\sim$ be an equivalent relation on $\mathcal{F}$ defined by $u \sim v$ if and only if $u - v$ is constant on X. Then $(\mathcal{F}/\sim, \mathcal{E})$ is a Hilbert space.
(RF3) $x \neq y \Rightarrow \exists u \in \mathcal{F}$ such that $u(x) \neq u(y)$.
(RF4) For any $x, y \in X$,

$$R_{(\mathcal{E},\mathcal{F})}(x, y) = \sup \left\{ \frac{|u(x) - u(y)|^2}{\mathcal{E}(u, u)} : u \in \mathcal{F}, \mathcal{E}(u, u) > 0 \right\} < +\infty$$

(RF5) Markov property: Define $\overline{u}$ by

$$\overline{u}(p) = \begin{cases} 1 & \text{if } u(p) \geq 1, \\ u(p) & \text{if } 0 < u(p) < 1, \\ 0 & \text{if } u(p) \leq 0. \end{cases}$$

Then $\overline{u} \in \mathcal{F}$ and $\mathcal{E}(\bar{u}, \bar{u}) \leq \mathcal{E}(u, u)$ for any $u \in \mathcal{F}$.

$R_{(\mathcal{E},\mathcal{F})}(x, y)$: the **resistance metric** on X associated with $(\mathcal{E}, \mathcal{F})$

Theorem 1.2. *$R_{(\mathcal{E},\mathcal{F})}(\cdot, \cdot)$ is a meric on X. For any $u \in \mathcal{F}$,*

$$|u(x) - u(y)|^2 \leq R_{(\mathcal{E},\mathcal{F})}(x, y)\mathcal{E}(u, u)$$

for any $x, y \in X$.

For simplicity, we use $R(x, y)$ instead of $R_{(\mathcal{E},\mathcal{F})}(x, y)$.

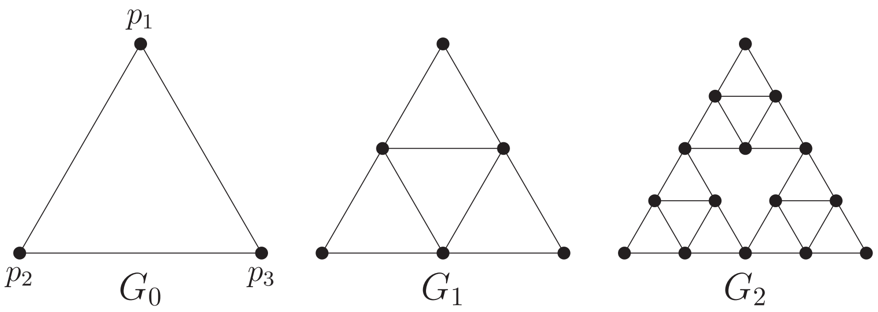

Figure 1: Approximation of the Sierpinski gasket by graphs G_m

Examples of resistance forms

(1) 1-dim. Brownian motion:

$$\mathcal{E}(u,v) = \int_{\mathbb{R}} \frac{du}{dx}\frac{dv}{dx}dx$$

$$\mathcal{F} = \{u|\mathcal{E}(u,u) < +\infty\} = H^1(\mathbb{R})$$

$$R(x,y) = |x-y|$$

(2) Standard resistance form on the Sierpinski gasket: For $i = 1,2,3$,

$$F_i(z) = (z-p_i)/2 + p_i$$

K: the Sierpinski gasket

$$K = F_1(K) \cup F_2(K) \cup F_3(K)$$

$$V_0 = \{p_1, p_2, p_3\}$$

$$V_{m+1} = F_1(V_m) \cup F_2(V_m) \cup F_3(V_m)$$

$$\mathcal{E}_m(u,u) = \frac{1}{2} \sum_{(p,q) \text{ is an edge of the Graph } G_m} \left(\frac{5}{3}\right)^m (u(p)-u(q))^2$$

$$\mathcal{F} = \{u| \lim_{m\to\infty} \mathcal{E}_m(u,u) < +\infty\}$$

$$\mathcal{E}(u,v) = \lim_{m\to\infty} \mathcal{E}_m(u,v)$$

$(\mathcal{E},\mathcal{F})$: the standard resistance form on K

$$R(x,y) \asymp |x-y|^{(\log 5 - \log 3)/\log 2}$$

(3) Random walks on (weighted) graphs (V, C):
V: a countable set,
$\{C(x,y)\}_{x,y\in V}$: the conductances, $C(x,y) = C(y,x) \geq 0$, $C(x,x) = 0$
Assume that
Locally finite: $\{y|C(x,y) > 0\}$ is finite
Connected(irreducible): For any $x, y \in V$, $\exists\{x_1, \ldots, x_n\}$ such that $x_1 = x, x_n = y$ and $C(x_i, x_{i+1}) > 0$ for any i

Random walk associated with (V, C):

$$C(x) = \sum_y C(x,y)\text{: the weight of } x$$

$$P(x,y) = \frac{C(x,y)}{C(x)}\text{: the transition probability from } x \text{ to } y$$

$$P^n(x,y) = \sum_{z\in V} P^{n-1}(x,z)P(z,y)\text{: the transition probability at the time } n$$

$P^n(x,y)$: the "heat kernel" associated with the random walk

Resistance form associated with (V, C):

$$\mathcal{F} = \{u|u : V \to \mathbb{R}, \sum_{x,y} C(x,y)(u(x) - u(y))^2 < +\infty\}$$

$$\mathcal{E}(u,v) = \frac{1}{2}\sum_{x,y} C(x,y)(u(x) - u(y))(v(x) - v(y)).$$

$(\mathcal{E}, \mathcal{F})$ is a resistance form on V.

Barlow-Coulhon-Kumagai: relations between the heat kernel estimate and the geometric property of the resistance metric

Plan: to find a metric d which satisfies $(\mathrm{RVD})_\beta$:

$$\textbf{Resistance} \times \textbf{Volume} \asymp (\textbf{Distance})^\beta,$$
$$\Downarrow$$
"good" heat kernel estimate

(This "principle" is known to work well for other situations as well.)
To preserve some desirable properties of the resistance form, we require

d: **quasisymmetric** with respect to R.

Quasisymmetric maps (QS maps for short): Tukia & Väisälä
↑
a generalization of quasiconformal functions on $\mathbb{C}$

Definition 1.3. (X, d) and (X, ρ): metric spaces
ρ: **quasisymmetric**, or QS for short, with respect to $d \underset{\text{def}}{\Leftrightarrow}$
$\exists$ a homeomorphism $h : [0, \infty) \to [0, \infty)$ such that $h(0) = 0$ and

$$d(x, z) < td(x, y) \Rightarrow \rho(x, z) < h(t)\rho(x, y)$$

We write $\rho \underset{\text{QS}}{\sim} d$.

Fact: $\rho \underset{\text{QS}}{\sim} d \Leftrightarrow d \underset{\text{QS}}{\sim} \rho$.

Regulation of Jumps: Annulus comparable condition

$(\mathcal{E}, \mathcal{F})$: a resistance form on X

$(\mathcal{E}, \mathcal{F})$ is **local** $\underset{\text{def}}{\Leftrightarrow}$ $\mathcal{E}(u, v) = 0$ if $\inf\{R(x, y) | x \in \text{supp}(u), y \in \text{supp}(v)\} > 0$

$$\Downarrow$$

No Jumps

Definition 1.4. $(\mathcal{E}, \mathcal{F})$ satisfies the **Annulus Comparable Condition, (ACC)** for short, $\underset{\text{def}}{\Leftrightarrow}$ (X, R) is uniformly perfect and $\exists\ \epsilon > 0$ such that

$$R(x, B_R(x, r)^c) \asymp \boxed{R\big(x, \overline{B_R(x, (1+\epsilon)r)} \cap B_R(x, r)^c\big)} \tag{1.1}$$

$\uparrow$
Annulus

for any $x \in X$ and any $r > 0$ with $B_R(x, r) \neq X$.

$$R(A, B) = \Big(\inf\{\mathcal{E}(u, u) | u|_A \equiv 1, u|_B \equiv 0, u \in \mathcal{F}\}\Big)^{-1}$$

Local $\Rightarrow$ Equality in (1.1) $\Rightarrow$ (ACC)

Volume doubling property

Definition 1.5. (X, d, μ): a measure metric space
μ is **volume doulbing** with respect to d, $(\mathrm{VD})_d$ for short $\underset{\mathrm{def}}{\Leftrightarrow}$
$\exists\ c > 0$ such that

$$\mu(B_d(x, 2r)) \leq c\mu(B_d(x, r))$$

for any $r > 0$ and any $x \in X$.

(QS) preserves the volume doubling prperty: if $d \underset{\mathrm{QS}}{\sim} \rho$, then

$$\mu \text{ is } (\mathrm{VD})_d \Leftrightarrow \mu \text{ is } (\mathrm{VD})_\rho$$

Conclusion: (ACC) and μ is $(\mathrm{VD})_R$

$$\Updownarrow$$

(ACC) and $\exists d : d \underset{\mathrm{QS}}{\sim} R$, $\exists \beta > 0$ such that

$$p(t, x, x) \asymp \frac{1}{V_d(x, t^{1/\beta})} : \text{the diagonal heat kernel estimate } (\mathrm{DHK})_\beta,$$

and

$$p(t, x, x) \leq Cp(2t, x, x) : \text{the kernel doubling property (KD)}$$

Resistance forms

Quasisymmetric maps

Heat kernel estimate

2 Resistance forms

$(\mathcal{E}, \mathcal{F})$: a resistance form on a set X

2.1 Topology given by a resistance form

$B \subseteq X$. Define

$$\mathcal{F}(B) = \{u | u \in \mathcal{F}, u|_B \equiv 0\}$$
$$B^{\mathcal{F}} = \{x | x \in X, u(x) = 0 \text{ for any } u \in \mathcal{F}(B)\}$$

$\mathcal{C}_{\mathcal{F}} = \{B | B \subseteq X, B^{\mathcal{F}} = B\}$ satisfies the axiom of closed sets.

$||$

$\mathcal{F}$-topology

$\updownarrow$

R-topology: the topology given by the resistance meric R.

Proposition 2.1. *(1) $\mathcal{F}$-closed $\Rightarrow$ R-closed*
(2) If (X, R) is compact, then the converse of (1) is also true.
(3) In general, the converse of (1) is not true.

Notation.
$C(X) = \{u | u \text{ is continuous with respect to } R\text{-topology}\}$
$C_0(X) = \{u | u \in C(X), \text{supp}(u) \text{ is } R\text{-compact.}\}$

Definition 2.2. $(\mathcal{E}, \mathcal{F})$ is **regular** $\underset{\text{def}}{\Leftrightarrow}$
$\mathcal{F} \cap C_0(X)$ is dense in $C_0(X)$ in the sense of $||u||_\infty = \sup_{x \in X} |u(x)|$.

Theorem 2.3. *$(\mathcal{E}, \mathcal{F})$: regular $\Leftrightarrow$ $\mathcal{F}$-toplogy $=$ R-topology*
In particular, (X, R): compact $\Rightarrow$ $(\mathcal{E}, \mathcal{F})$: regular

Hereafter, we always assume that $(\mathcal{E}, \mathcal{F})$ is regular.

2.2 Green's function

Theorem 2.4. *$B \subseteq X$: closed*
$\exists$ Unique $g_B : X \times X \to [0, +\infty)$ with
(GF) *Define $g_B^x(y) = g_B(x, y)$. Then $g_B^x \in \mathcal{F}(B)$. For any $u \in \mathcal{F}(B)$ and any $x \in X$,*

$$\mathcal{E}(g_B^x, u) = u(x)$$

$g_B(x, y)$: the **Green function** with the boundary B
or the B-Green function

$$g_B(x, x) \geq g_B(x, y) \geq 0$$
$$g_B(x, y) = g_B(y, x)$$
$$g_B(x, x) > 0 \Leftrightarrow x \notin B$$
$$|g_B(x, y) - g_B(x, z)| \leq R_B(y, z) \leq R(y, z)$$

Moreover, define

$$\mathcal{F}^B = \mathcal{F}(B) + \mathbb{R} = \{u | u \in \mathcal{F}, u \text{ is constant on } B\}$$
$$X_B = (X \backslash B) \cup \{B\} : \text{shrinking } B \text{ into a point}$$

Then $(\mathcal{E}, \mathcal{F}^B)$ is a resistance form on X_B.
$R_B(\cdot, \cdot)$: associated resistance metric on X_B.
Then, (due to Metz in case $B = \{z\}$),

$$g_B(x, y) = \frac{R_B(x, B) + R_B(y, B) - R_B(x, y)}{2}$$

↑
Gromov product of the metric R_B

If $B = \{z\}$, then $R_B(x, y) = R(x, y)$.

In general,
(X, d): a metric space. Define

$$k(x, y) = \frac{d(x, z) + d(y, z) - d(x, y)}{2}$$
$$(Au)(x) = \int_X k(x, y) f(y) \mu(dy)$$

What is A?

2.3 Harmonic functions and Traces

$B \subseteq X$: closed
Define

$$\mathcal{F}|_B = \{u|_B : u \in \mathcal{F}\}.$$

Proposition 2.5. *For any $\varphi \in \mathcal{F}|_B$, $\exists$ unique $f \in \mathcal{F}$ such that $f|_B = \varphi$ and*

$$\mathcal{E}(f, f) = \min_{u \in \mathcal{F}, u|_B = \varphi} \mathcal{E}(u, u)$$

f: the **harmonic function** with boundary value φ on the boundary B
or the B-harmonic function with boundary value φ.
Define $f = h_B(\varphi)$ and $\mathcal{H}_B = h_B(\mathcal{F}|_B)$. Then

$$h_B : \mathcal{F}|_B \to \mathcal{H}_B \subseteq \mathcal{F} \text{ is linear.}$$
$$\mathcal{F} = \mathcal{H}_B \oplus \mathcal{F}(B) \quad (**)$$
$$\uparrow$$
$$\mathcal{E}(u, v) = 0 \text{ if } u \in \mathcal{H}_B \text{ and } v \in \mathcal{F}(B).$$

In the case of Dirichlet forms, analogous decomposition as (**) is known.
See Fukushima-Oshima-Takeda.
Define

$$\mathcal{E}|_B(\varphi, \psi) = \mathcal{E}(h_B(\varphi), h_B(\psi))$$

for any $\varphi, \psi \in \mathcal{F}|_B$. Then

Proposition 2.6. *$(\mathcal{E}|_B, \mathcal{F}|_B)$ is a resistance form on B.*
The corresponding resistance metric $= R|_{B \times B}$.
$(\mathcal{E}, \mathcal{F})$: regular $\Rightarrow$ $(\mathcal{E}|_B, \mathcal{F}|_B)$: regular

$(\mathcal{E}|_B, \mathcal{F}|_B)$: the **trace** of $(\mathcal{E}, \mathcal{F})$ on B.

2.4 Dirichlet form associated with $(\mathcal{E}, \mathcal{F})$

Assume that
μ: a Radon measure on (X, R)
$0 < \mu(B_R(x, r)) < +\infty$ for any $x \in X$ and any $r > 0$.
Define

$$\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + \int_X uvd\mu$$
$$\mathcal{D} = \mathcal{E}_1\text{-closure of } \mathcal{F} \cap C_0(X).$$

Theorem 2.7.
$(\mathcal{E}, \mathcal{F})$: regular $\Rightarrow$ $(\mathcal{E}, \mathcal{D})$: a regular Dirichlet form on $L^2(X, \mu)$
Moreover, $(\mathcal{E}, \mathcal{F})$: local $\Rightarrow$ $(\mathcal{E}, \mathcal{D})$: local.

a regular Dirichlet form
$\downarrow$
a Hunt process, i.e. a strong Markov process with right continuous pathes

local $\Rightarrow$ pathes are continuous. (Diffusion)

Definition 2.8 (**Capacity**)**.** (1) $U \subseteq X$: open,

$$\mathrm{CapU} = \inf\{\mathcal{E}_1(\mathrm{u}, \mathrm{u}) | \mathrm{u} \in \mathcal{F}, \mathrm{u} \geq 1 \ \text{ on } U\}$$

(2) $A \subseteq X$,

$$\mathrm{CapA} = \inf\{\mathrm{CapU} | U\text{: open}, \mathrm{A} \subseteq \mathrm{U}\}$$

Fact For any $x \in X$, $\exists c_x > 0$ such that, for any $u \in \mathcal{D}$,

$$|u(x)| \leq c_x \sqrt{\mathcal{E}_1(u, u)}$$
$$\Downarrow$$
$$K \subseteq X\text{: compact, } 0 < \inf_{x \in K} \mathrm{Cap}\{\mathrm{x}\}.$$
$$\Downarrow$$

the Hunt process is determined for all $x \in X$.

In general, the Hunt process associated with a regular Dirichlet form is determined up to “excpetionl sets”.

2.5 Transition density/Heat kernel

μ: a Radon measure on (X, R), $0 < \mu(B_R(x, r)) < +\infty$
$(\mathcal{E}, \mathcal{F})$: a regular resistance form on X.
$\downarrow$
$(\mathcal{E}, \mathcal{D})$: a regular Dirichlet form on $L^2(X, \mu)$
$\downarrow$
$(\{X_t\}_{t>0}, \{P_x\}_{x\in X})$: a Hunt process on X
(defined for every $x \in X$)

Theorem 2.9. *Assume that $\overline{B_R(x, r)}$ is compact for any $x, \in X$ and $r > 0$. Then there exists $p(t, x, y) : (0, \infty) \times X \times X \to [0, \infty)$, continuous with*
(TD1) $p^{t,x} \in \mathcal{D}$, *where* $p^{t,x}(y) = p(t, x, y)$.
(TD2) $p(t, x, y) = p(t, y, x)$
(TD3) *For any mesurable* $u \geq 0$,

$$E_x(u(X_t)) = \int_X p(t, x, y)u(y)\mu(dy).$$

(TD4)

$$p(t + s, x, y) = \int_X p(t, x, z)p(s, z, y)\mu(dz)$$

$p(t, x, y)$: the **transition density/heat kernel**

Existence and continuity of the transition density
Chen et al: general regular Dirichlet forms, ultarcontractive $\Rightarrow$ quasicontinuous
Grigor'yan: general regular Dirichlet, locally ultracontractive $\Rightarrow$ quasicontinuous
Croydon: resistance forms, ultracontractive $\Rightarrow$ continuous

Proposition 2.10. *Without any further assumption,*

$$p(r\mu(B_R(x,r)),x,x) \leq \frac{2+\sqrt{2}}{\mu(B_R(x,r))}$$

3 Goemetry and analysis on (X, R) via quasisymmetric maps

3.1 Exit time, resistance and annulus comparablity

Definition 3.1. (X, d): a metric space
(X, d): **uniformly perfect** $\underset{\text{def}}{\Leftrightarrow}$ $\exists \epsilon > 0$ such that
$B_d(x, (1+\epsilon)r) \backslash B_d(x, r) \neq \emptyset$ for any $x \in X$ and $r > 0$ with $X \backslash B_d(x, r) \neq \emptyset$.

Hereafter, $(\mathcal{E}, \mathcal{F})$: a regular resistance form on X
μ: a randon measure on (X, R)
$\overline{B_R(x, r)}$: compact
$\downarrow$
$(\mathcal{E}, \mathcal{D})$ a regular Dirichlet form on $L^2(X, \mu)$
$(\{X_t\}_{t>0}, \{P_x\}_{x>0})$: regular Hunt process
$p(t, x, y)$: the transition density

For simplicity, we only give statements the case where (X, R) is not bounded.

Recall the Annulus comparable condition (ACC): $\exists\epsilon > 0$ such that

$$R(x, B_R(x,r)^c) \asymp R(x, \overline{B_R(x,(1+\epsilon)r)} \cap B_R(x,r)^c).$$

Definition 3.2 (**Exit time**). $A \subseteq X$,

$$\tau_A = \inf\{t > 0 | X_t \notin A\}.$$

Proposition 3.3.

$$E_x(\tau_A) = \int_X g_{A^c}(x,y)\mu(dy) = \int_A \frac{R_{A^c}(x,A^c) + R_{A^c}(y,A^c) - R_{A^c}(x,y)}{2}\mu(dy)$$

Theorem 3.4. *Assume*
μ: $(\mathrm{VD})_R$, *i.e. volume doublig with repsect to R,*
(X,R): *uniformly perfect*
d: *a metric on X, $d \underset{\mathrm{QS}}{\sim} R$, i.e. d is quasisymmetric with respect to R.*
Then

(ACC)

$\Updownarrow$

Exit time estimate $(\mathrm{Exit})_d$: $E_x(\tau_{B_d(x,r)}) \asymp \overline{R}_d(x,r)V_d(x,r)$

$\Updownarrow$

Resistance estimate $(\mathrm{Res})_d$: $R(x, B_d(x,r)^c) \asymp \overline{R}_d(x,r)$,

where $\overline{R}_d(x,r) = \sup_{y\in B_d(x,r)} R(x,y)$.

Exit time estimate: Resistance $\times$ Volume $\asymp$ Exit time

Assume that (X, R) is uniformly perfect.

Theorem 3.5. *If* $d \underset{\mathrm{QS}}{\sim} R$, (ACC) *holds,* μ*:* $(\mathrm{VD})_{\mathrm{R}}$, *then*

$$p(\overline{R}_d(x,r)V_d(x,r), x, x) \asymp \frac{1}{V_d(x,r)} : \textit{Diagonal estimate}$$

and

$$p(\overline{R}_d(x,r)V_d(x,r), x, y) \geq \frac{c}{V_d(x,r)} : \textit{Near diagonal lower estimate}$$

for $x, y \in X$ *with* $d(x,y) \leq cr$*.*

In particular, if $d = R$, then

$$p(rV_R(x,r), x, x) \asymp \frac{1}{V_R(x,r)}$$

X is a graph, random walk, $d = R$: Barlow-Coulhon-Kumagai
$d = R$, continuous: Kumagai

Observation: In the diagonal estimate, if $\overline{R}_d(x,r)V_d(x,r) \asymp r^\beta$, then

$$p(t, x, x) \asymp \frac{1}{V_d(x, t^{1/\beta})}.$$

Find $d \underset{\mathrm{QS}}{\sim} R$ such that $\boxed{\text{Resistance} \times \text{Volume} = (\text{Distance})^\beta}$: $(\mathrm{RVD})_\beta$!!

3.2 Construction of quasisymmetric metric

(X, ρ, μ): a metric measure space
Assume that (X, ρ) is uniformly perfect.

Theorem 3.6. *Fix $a \geq 0$.*
If μ: $(\mathrm{VD})_\rho$ then, for sufficiently large $\beta > 0$, $\exists$ d: a metric on X such that $\rho \underset{\mathrm{QS}}{\sim} d$ and

$$\rho(x,y)^a V_d(x, d(x,y)) \asymp d(x,y)^\beta \tag{M}$$

(M) is a natural analogue of $(\mathrm{RVD})_\beta$.

Remark. In the case $a = 0$, the above theorem recovers the following famous result:
If (X, ρ) is uniformly perfect and μ is $(\mathrm{VD})_\rho$, then there exists a metric d on X such that $d \underset{\mathrm{QS}}{\sim} \rho$ and μ is Ahlfors regular, i.e.

$$\mu(B_d(x,r)) \asymp r^\beta$$

For $\gamma > 0$, define the condition $(\mathrm{SD})_\gamma$: **slow decay of volume**
$\exists \eta : (0,1] \to (0,\infty)$, $\eta(\lambda) \downarrow 0$ as $\lambda \downarrow 0$ monotonically, and, for any $\lambda \in (0,1]$, any $x, y \in X$,

$$\frac{V_d(x, \lambda d(x,y))}{V_d(x, d(x,y))} \geq \frac{\lambda^\gamma}{\eta(\lambda)}$$

Theorem 3.7. *Fix $a > 0$. Assume that (X, d) is uniformly perfect. Then*

$$(\mathrm{SD})_\beta \wedge (\mathrm{M}) \Leftrightarrow \rho \underset{\mathrm{QS}}{\sim} d \wedge (\mathrm{M})$$
$$\Downarrow$$
$$\mu \text{ is } (\mathrm{VD})_\rho \text{ and } (\mathrm{VD})_{\mathrm{d}}.$$

4 Heat kernel estimate

4.1 Main Theorems

List of Conditions:

$(\mathrm{DHK})_{d,\beta}$: the diagonal heat kernel estimate

$$p(t,x,x) \asymp \frac{1}{V_d(x,t^{1/\beta})}$$

(KD): kernel doubling, $\exists c > 0$,

$$p(t,x,x) \leq cp(2t,x,x)$$

$(\mathrm{RVD})_{d,\beta}$: Resistance $\times$ Volume = Distance$^\beta$

$$R(x,y)V_d(x,d(x,y)) \asymp d(x,y)^\beta$$

$(\mathrm{SD})_{d,\beta}$: slow decay of volume
$\exists \eta : (0,1] \to (0,+\infty)$, $\eta(\lambda) \downarrow 0$ as $\lambda \downarrow 0$ monotonically and

$$\frac{V_d(x,\lambda d(x,y))}{V_d(x,d(x,y))} \geq \frac{\lambda^\beta}{\eta(\lambda)}$$

Theorem 4.1. *Assume that (X, R) is uniformly perfect. Then*

$$\boxed{(X,d)\text{: \textit{uniformly perfect}} \wedge (\mathrm{SD})_{d,\beta} \wedge (\mathrm{RVD})_{d,\beta}}$$
$$\Updownarrow$$
$$\boxed{d \underset{\mathrm{QS}}{\sim} R \wedge (\mathrm{RVD})_{d,\beta}}$$
$$\Updownarrow$$
$$\boxed{d \underset{\mathrm{QS}}{\sim} R \wedge (\mathrm{DHK})_{d,\beta} \wedge (\mathrm{KD})}$$

Moreover, if $(\mathcal{E}, \mathcal{F})$ is **local**, *then the above set of conditioins implies*

$$p(t,x,y) \leq \frac{c_1}{V_d(x,t^{1/\beta})} \exp\left(-c_2 \left(\frac{d(x,y)^\beta}{t} \right)^{1/(\beta-1)} \right)$$

If $(\mathcal{E}, \mathcal{F})$ is **local** *and d is* **geodesic**, *then*

$$\frac{c_3}{V_d(x,t^{1/\beta})} \exp\left(-c_4 \left(\frac{d(x,y)^\beta}{t} \right)^{1/(\beta-1)} \right) \leq p(t,x,y)$$

Theorem 4.2. *Assume (X, R) is uniformly perfect. Then*

$$\boxed{\mu\text{: } (\mathrm{VD})_R \wedge (\mathrm{ACC})}$$

$$\Updownarrow$$

$$\boxed{\mu\text{: } (\mathrm{VD})_R \wedge R(x, B_R(x,r)^c) \asymp r}$$

$$\Updownarrow$$

$$\boxed{(\mathrm{ACC}) \wedge \exists d \text{ and } \beta > 0 \text{ such that } d \underset{\mathrm{QS}}{\sim} R,\ (\mathrm{DHK})_{d,\beta} \text{ and } (\mathrm{KD})}$$

Remark. **local** $\Rightarrow$ (ACC) and/or $R(x, B_R(x,r)^c) \asymp r$

4.2 Applicaition to traces

Assume that (X, R) is uniformly perfect.

$B \subseteq X$: closed
Consider the trace $(\mathcal{E}|_B, \mathcal{F}|_B)$ of $(\mathcal{E}, \mathcal{F})$ on B.
Recall that

$$(\mathcal{E}, \mathcal{F}): \text{ regular} \Rightarrow (\mathcal{E}|_B, \mathcal{F}|_B): \text{ regular}$$

Theorem 4.3. *Assume that $(B, R|_B)$ is uniformly perfect.*

$$\text{(ACC) for } (\mathcal{E}, \mathcal{F}) \Rightarrow \text{(ACC) for } (\mathcal{E}|_B, \mathcal{F}|_B).$$

Assumptions:
(X, R) and $(B, R|_B)$: uniformly perfect
$(\mathcal{E}, \mathcal{F})$: regular
(ACC) holds for $(\mathcal{E}, \mathcal{F})$.
$B_R(x, r)$: compact

$$\nu: \text{ a Radon measure on } (B, R|_B)$$
$$\downarrow$$
$$(\mathcal{E}|_B, \mathcal{D}_B): \text{ a regular Dirichlet form on } L^2(B, \nu).$$
$$\downarrow$$
$$\text{Transition density: } p^B_\nu(t, x, y) \text{ on } B$$

Theorem 4.4. *Assume that $d \underset{\mathrm{QS}}{\sim} R$ and $(\mathrm{DHK})_{d,\beta}$.*
If $\exists \gamma > 0$ such that

$$\mu(B_d(x, r)) \asymp r^\gamma \nu(B_d(x, r) \cap B),$$

then $\beta > \gamma$ and

$$p^B_\nu(t, x, x) \asymp \frac{1}{\nu(B_d(x, t^{1/(\beta-\gamma)}) \cap B)}$$

Moreover, if $\mu(B_d(x, r)) \asymp r^\alpha$, then

$$p^B_\nu(t, x, x) \asymp t^{\frac{\alpha-\gamma}{\beta-\gamma}}.$$

4.3 Examples

α-stable process on $\mathbb{R}^1$: $\alpha \in (1,2]$

$$\mathcal{E}^{(\alpha)}(u,v) = \int_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^{1+\alpha}} dx$$
$$\mathcal{F}^{(\alpha)} = \{u | u \in C(\mathbb{R}), \mathcal{E}^{(\alpha)}(u,u) < +\infty\}$$
$$R^{(\alpha)}(x,y) = c|x-y|^{\alpha-1}$$

for $\alpha \in (1,2)$. For $\alpha = 2$, it corresponds to the Brownian motion on $\mathbb{R}^1$.

(ACC) is OK.

Case 1: $\mu = dx$ the Lebesgue measure. Then μ is $(\mathrm{VD})_R$.

$$p(t,x,x) \asymp \frac{1}{t^{1/\alpha}}.$$

Case 2: $\mu = x^\delta dx$ for $\delta > -1 \Rightarrow \mu$ is $(\mathrm{VD})_{\mathrm{R}}$.

$$p_\mu(t,0,0) \asymp t^{-\tau} : \tau = \frac{\delta+1}{\delta+\alpha}$$

Case 3: Trace onto the middle 3rd Cantor set K:
ν: the $\log 3/\log 2$-dim. Hausdorff measure on K. Let μ_* be the Lebesgue measure.

$$\mu_*(B_R(x,r)) \asymp r^{\frac{\log 2}{(\alpha-1)\log 3}} \nu(B_R(x,r))$$
$$p_\nu^K(t,x,x) \asymp t^{-\eta} : \eta = \frac{\log 2}{(\alpha-1)\log 3 + \log 2}$$

The standard resistance form on the Sierpinski gasket

Natural measure $\mu =$ the $\log 3/\log 2$-dim. Hausdorff measure.

$$p(t,x,y) \approx \frac{c_1}{t^{\alpha/\beta}} \exp\left(-c_2 \left(\frac{d(x,y)^\beta}{t}\right)^{1/(\beta-1)}\right),$$

where $\alpha = \dfrac{\log 3}{\log 2}, \beta = \dfrac{\log 5}{\log 2}$ and $d(x,y) = |x-y| = R(x,y)^{\frac{\log 2}{\log 5 - \log 2}}$.

Case 1: Change the measure μ:

Case 2: Trace onto an Ahlfors δ-regular set B:
$\exists \nu$ on Y such that

$$\nu(B_d(x,r) \cap B) \asymp r^\delta$$

Then

$$p_\nu^B(t,x,x) \asymp t^{-\eta} : \eta = \frac{\delta \log 2}{\log 5 - \log 3 + \delta \log 2}$$

In particular, $B =$ the line segment of the outer triangle: $\delta = 1$
Characterization of $\mathcal{F}|_B$ as a Besov space: Alf Jonsson

$$\alpha = \frac{\log 5 - \log 3 + \log 2}{\log 2}$$

$$\downarrow$$

$$\mathcal{F}|_B = \mathcal{F}^{(\alpha)} = \text{the domain for the } \alpha\text{-stable process on } \mathbb{R}.$$

$$\mathcal{E}|_B(u,u) \asymp \mathcal{E}^{(\alpha)}(u,u)$$

But...................