BACKWARD REPRESENTATION OF ROUGH INTEGRAL: AN APPROACH BASED ON FRACTIONAL CALCULUS

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Abstract. On the basis of fractional calculus, the integral of controlled paths along Hölder rough paths is given explicitly as Lebesgue integrals for fractional derivative operators, without using any arguments from a discrete approximation. In this paper, we introduce a backward version of the integral and provide fundamental relations between both integrals from the perspective of the backward representation of the rough integral.

1. Introduction

On the basis of fractional calculus, Hu and Nualart [6] introduced an approach for rough path analysis. One of the notable features of this approach is the concept of integrals with respect to rough paths, called rough integrals. The usual rough integral is based on arguments from a discrete approximation and is given as the limit of the compensated Riemann–Stieltjes sums as follows:

$$\int_s^t (Y, Y')\, d(X, \mathbb{X}) := \lim_{|\mathcal{P}_{s,t}|\to 0} \sum_{i=0}^{n-1} Y_{t_i}(X_{t_{i+1}} - X_{t_i}) + Y'_{t_i}\mathbb{X}_{t_i,t_{i+1}}$$

for $0 \le s < t \le T$. Here, $(X, \mathbb{X}) \in \Omega_\beta([0,T], \mathbb{R}^{d_1})$ is a β-Hölder rough path and $(Y, Y') \in \mathcal{Q}_X^\beta([0,T], L(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}))$ is an X-controlled path with $\beta \in (1/3, 1/2]$. (See Section 2.2 for the basic concepts of rough path analysis.) This limit is taken over all finite partitions $\mathcal{P}_{s,t} := \{t_0, t_1, \dots, t_n\}$ of the interval $[s,t]$ such that $s = t_0 < t_1 < \cdots < t_n = t$ and $|\mathcal{P}_{s,t}| := \max_{0\le i\le n-1}|t_{i+1} - t_i|$. However, as defined in Definition 2.2, the rough integral of the approach based on fractional calculus is given explicitly in terms of Lebesgue integrals for fractional derivative operators without using any arguments from the discrete approximation. The integral $I_{(X,\mathbb{X})}(Y, Y')$ in Definition 2.2 was introduced by Hu and Nualart [6] and the author [10], and can be regarded as an extension of the (forward) integral of Y with respect to X in the integration theory developed by Zähle [18]. It follows from [10, Theorem 2.5] that $I_{(X,\mathbb{X})}(Y, Y')_{s,t}$ is consistent with the usual rough integral; specifically, the identity $I_{(X,\mathbb{X})}(Y, Y')_{s,t} = \int_s^t (Y, Y')\, d(X, \mathbb{X})$ holds. (See [7–9, 11, 12] for related studies.) Such explicit expressions directly lead to a quantitative estimation of the rough integrals and solutions to differential equations driven by a rough path (e.g., [1, 2, 16]).

Date: February 28, 2022.
2010 *Mathematics Subject Classification.* Primary 26A42; Secondary 26A33, 60H05.
Key words and phrases. Stieltjes integral, Fractional derivative, Rough path.

In this paper, we introduce a backward version of the integral $I_{(X,\mathbb{X})}(Y,Y')$ and consider the relations between these integrals from the perspective of the backward representation of the rough integral in [3] as follows: Let $(\overleftarrow{X},\overleftarrow{\mathbb{X}})$ and $(\overleftarrow{Y},\overleftarrow{Y'})$ be the (time T) time reversal of $(X,\mathbb{X})$ and (Y,Y'), respectively. Namely, $(\overleftarrow{X},\overleftarrow{\mathbb{X}})$ is a β-Hölder rough path defined by $\overleftarrow{X}_t = X_{T-t}$ for $t \in [0,T]$ and $\overleftarrow{\mathbb{X}}_{s,t} = -\mathbb{X}_{T-t,T-s} + (X_{T-s} - X_{T-t}) \otimes (X_{T-s} - X_{T-t})$ for $0 \le s \le t \le T$, and $(\overleftarrow{Y},\overleftarrow{Y'})$ is an $\overleftarrow{X}$-controlled path defined in the same way as $\overleftarrow{X}$. Then, the equalities

$$\begin{aligned}\int_s^T (Y,Y')\,d(X,\mathbb{X}) &= \lim_{|\mathcal{P}_{s,T}|\to 0} \sum_{i=0}^{n-1} \Big\{ Y_{t_{i+1}}(X_{t_{i+1}} - X_{t_i}) \\ &\qquad + Y'_{t_{i+1}}(\mathbb{X}_{t_i,t_{i+1}} - (X_{t_{i+1}} - X_{t_i}) \otimes (X_{t_{i+1}} - X_{t_i}))\Big\} \\ &= -\int_0^{T-s} (\overleftarrow{Y},\overleftarrow{Y'})\,d(\overleftarrow{X},\overleftarrow{\mathbb{X}}) \qquad (1.1)\end{aligned}$$

hold for $0 \le s < T$ and are called backward representation of the rough integral (cf. [3, Proposition 5.12]). The purpose of this paper is to clarify the structures of rough integrals based on fractional calculus from the perspective of the fundamental property (1.1).

In Section 2.4, we introduce a backward version of $I_{(X,\mathbb{X})}(Y,Y')$, denoted by $J_{(X,\mathbb{X})}(Y,Y')$ in Definition 2.2. Note that $J_{(X,\mathbb{X})}(Y,Y')$ can be regarded as an extension of the backward integral of Y with respect to X in [18] (see (1) of Remark 2.3). In Section 3, using the basic formulas of fractional calculus, we describe the derivation of $J_{(X,\mathbb{X})}(Y,Y')$ from the usual rough integral. Because we are studying the backward version, it will be useful to keep these aspects in mind. In Section 2.5, we focus on the backward representation (1.1). First, we show that the equalities

$$J_{(X,\mathbb{X})}(Y,Y')_{s,t} = -I_{(\overleftarrow{X},\overleftarrow{\mathbb{X}})}(\overleftarrow{Y},\overleftarrow{Y'})_{T-t,T-s}$$

and

$$I_{(X,\mathbb{X})}(Y,Y')_{s,t} = -J_{(\overleftarrow{X},\overleftarrow{\mathbb{X}})}(\overleftarrow{Y},\overleftarrow{Y'})_{T-t,T-s}$$

hold for $0 \le s < t \le T$ (Theorem 2.5). To prove these equalities, we use Lemma 2.4, which provides the relations between the fractional derivative operators and time reversals. Next, using Theorem 2.5 and the results of the previous study [10, Theorem 2.5] mentioned in the first paragraph, we show that $J_{(X,\mathbb{X})}(Y,Y')$ is also consistent with the usual rough integral; namely, the identity $J_{(X,\mathbb{X})}(Y,Y')_{s,t} = \int_s^t (Y,Y')\,d(X,\mathbb{X})$ holds (Theorem 2.6). Therefore, from Theorems 2.5 and 2.6, we obtain the backward representation (1.1) and see that $J_{(X,\mathbb{X})}(Y,Y')_{s,t}$ gives an explicit expression on the right-hand side of the first equality of (1.1). Although the definition of $J_{(X,\mathbb{X})}(Y,Y')$ is more complicated than that of $I_{(X,\mathbb{X})}(Y,Y')$, the proofs of Lemma 2.4, Theorems 2.5, and 2.6 in Section 2 are obtained through straightforward computations.

The remainder of this paper is organized as follows: In Section 2, we introduce the definition of the integrals and the main theorems. We also recall some basic components of rough path analysis and fractional calculus. In Section 3, we provide another proof for Theorem 2.6 without using [10, Theorem 2.5].

2. Framework and Main Theorems

In this section, we introduce the definitions of integrals and the main theorems. We also briefly review concepts such as rough paths, controlled paths, and fractional integral and derivative operators. We follow the standard treatments for rough path analysis [3–5, 13–15] and fractional calculus [17, 18].

2.1. **Notation.** Let V and W be finite-dimensional normed spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. Although the fundamental theories of rough paths and controlled paths are valid for suitable infinite-dimensional Banach spaces, we consider only finite-dimensional cases in this paper to avoid technical difficulties that are not relevant to our theme. Let $L(V,W)$ denote the set of all linear maps from V to W. Let T denote a positive constant that is fixed throughout this paper. Let $a, b \in \mathbb{R}$ with $0 \le a < b \le T$. Simplex $\{(s,t) \in \mathbb{R}^2 : a \le s \le t \le b\}$ is denoted by $\triangle_{a,b}$ and is a closed subset of $\mathbb{R}^2$. Let $C([a,b],V)$ and $C(\triangle_{a,b},V)$ denote the spaces of all V-valued continuous functions on the interval $[a,b]$ and $\triangle_{a,b}$, respectively. For $\psi \in C([a,b],V)$, we define $\delta\psi \in C(\triangle_{a,b},V)$ as $\delta\psi_{s,t} := \psi_t - \psi_s$ for $(s,t) \in \triangle_{a,b}$. Let $\lambda \in (0,1]$. We then set

$$\|\psi\|_{\lambda;[a,b]} := \sup_{a \le s < t \le b} \frac{\|\psi_t - \psi_s\|_V}{(t-s)^\lambda} \quad \text{and} \quad \|\Psi\|_{\lambda;[a,b]} := \sup_{a \le s < t \le b} \frac{\|\Psi_{s,t}\|_V}{(t-s)^\lambda}$$

for $\psi \in C([a,b],V)$ and $\Psi \in C(\triangle_{a,b},V)$. We omit $[a,b]$ from the notation if there is no ambiguity; that is, we write $\|\psi\|_\lambda$ and $\|\Psi\|_\lambda$ instead of $\|\psi\|_{\lambda;[a,b]}$ and $\|\Psi\|_{\lambda;[a,b]}$. We set $\mathcal{C}^\lambda([a,b],V) := \{\psi \in C([a,b],V) : \|\psi\|_{\lambda;[a,b]} < \infty\}$ and $\mathcal{C}^\lambda(\triangle_{a,b},V) := \{\Psi \in C(\triangle_{a,b},V) : \|\Psi\|_{\lambda;[a,b]} < \infty\}$. Hereafter, d_1 and d_2 denote positive integers; E and F denote the Euclidean spaces $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively; and $|\cdot|$ denotes the Euclidean norms of E, F, and their tensor spaces. Let k be a positive integer. For $\alpha \in \mathbb{R}$ and $a \in E^{\otimes k}$, we set $(-1)^\alpha := e^{i\pi\alpha}$ and $|(-1)^\alpha a| := |a|$. Let β denote a real number with $1/3 < \beta \le 1/2$ and let γ denote a real number with $(1-\beta)/2 < \gamma < \beta$. These numbers are fixed throughout this paper.

2.2. **Rough paths and controlled paths.** To describe the main theorems of this paper, we slightly reformulate the definitions of both Hölder rough paths and controlled paths. The usual definitions correspond to the case $a = 0$, $b = T$. Let $a, b \in \mathbb{R}$ with $0 \le a < b \le T$. We say that pair $(X, \mathbb{X})$ is a β-Hölder rough path on $[a,b]$ with values in E if $(X,\mathbb{X})$ satisfies the following two conditions:

(1) $X \in \mathcal{C}^\beta([a,b],E)$ and $\mathbb{X} \in \mathcal{C}^{2\beta}(\triangle_{a,b}, E \otimes E)$;
(2) for $s,t,u \in [a,b]$ with $s \le u \le t$,

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = (X_u - X_s) \otimes (X_t - X_u). \tag{2.1}$$

The space of all β-Hölder rough paths on $[a,b]$ with values in E is denoted by $\Omega_\beta([a,b],E)$. For $(X,\mathbb{X}) \in \Omega_\beta([a,b],E)$, we set $\overleftarrow{X}_t := X_{a+b-t}$ for $t \in [a,b]$ and

$$\overleftarrow{\mathbb{X}}_{s,t} := -\mathbb{X}_{a+b-t,a+b-s} + (X_{a+b-s} - X_{a+b-t}) \otimes (X_{a+b-s} - X_{a+b-t})$$

for $(s,t) \in \triangle_{a,b}$. Then, the pair $(\overleftarrow{X}, \overleftarrow{\mathbb{X}})$ is a β-Hölder rough path on $[a,b]$ with values in E. When $a = 0$, $b = T$, $(\overleftarrow{X}, \overleftarrow{\mathbb{X}})$ is called the (time T) time reversal of $(X,\mathbb{X}) \in \Omega_\beta([0,T],E)$.

Given $X \in \mathcal{C}^\beta([a,b],E)$, we say that pair (Y,Y') is an X-controlled path on $[a,b]$ with values in F if (Y,Y') satisfies the following two conditions:

(1) $Y \in \mathcal{C}^\beta([a,b],F)$ and $Y' \in \mathcal{C}^\beta([a,b],L(E,F))$;

(2) $R^Y \in \mathcal{C}^{2\beta}(\triangle_{a,b},F)$, where $R^Y_{s,t} := Y_t - Y_s - Y'_s(X_t - X_s)$ for $(s,t) \in \triangle_{a,b}$.

The space of all X-controlled paths on $[a,b]$ with values in F is denoted by $\mathcal{Q}^\beta_X([a,b],F)$. For $(Y,Y') \in \mathcal{Q}^\beta_X([a,b],F)$, we set $\overleftarrow{Y}_t := Y_{a+b-t}$ and $\overleftarrow{Y'}_t := Y'_{a+b-t}$ for $t \in [a,b]$. Then, the pair $(\overleftarrow{Y}, \overleftarrow{Y'})$ is an $\overleftarrow{X}$-controlled path on $[a,b]$ with values in F, and

$$R^{\overleftarrow{Y}}_{s,t} = -R^Y_{a+b-t,a+b-s} + (Y'_{a+b-s} - Y'_{a+b-t})(X_{a+b-s} - X_{a+b-t})$$

holds for $(s,t) \in \triangle_{a,b}$. For further details and examples of Hölder rough paths and controlled paths, see [3–5].

2.3. **Fractional integrals and derivatives.** Let a and b be real numbers with $a < b$. For $p \in [1,\infty)$, let $L^p(a,b)$ denote the complex L^p-space on the interval $[a,b]$ with respect to the Lebesgue measure. Let $f \in L^1(a,b)$ and $\alpha \in (0,\infty)$. The left- and right-sided Riemann–Liouville fractional integrals of f of order α are defined for almost all $t \in (a,b)$ by

$$I^\alpha_{a+} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds$$

and

$$I^\alpha_{b-} f(t) := \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\, ds,$$

respectively, where Γ denotes the gamma function. For $p \in [1,\infty)$, let $I^\alpha_{a+}(L^p)$ and $I^\alpha_{b-}(L^p)$ denote the images of $L^p(a,b)$ by operators I^α_{a+} and I^α_{b-}, respectively. In addition, let $f \in I^\alpha_{a+}(L^1)$ (resp. $I^\alpha_{b-}(L^1)$) with $0 < \alpha < 1$, the Weyl–Marchaud fractional derivative of f of order α is defined for almost all $t \in (a,b)$ by

$$D^\alpha_{a+} f(t) := \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^\alpha} + \alpha \int_a^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}}\, ds \right)$$

for the left-sided version, and

$$D^\alpha_{b-} f(t) := \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(b-t)^\alpha} + \alpha \int_t^b \frac{f(t) - f(s)}{(s-t)^{\alpha+1}}\, ds \right)$$

for the right-sided version. Here, the integrals above are well-defined for almost all $t \in (a, b)$. The inversion formulas, i.e.,

$$I^\alpha_{a+}(D^\alpha_{a+} f) = f \quad (\text{resp. } I^\alpha_{b-}(D^\alpha_{b-} f) = f) \tag{2.2}$$

for $f \in I^\alpha_{a+}(L^1)$ (resp. $I^\alpha_{b-}(L^1)$) with $0 < \alpha < 1$, and

$$D^\alpha_{a+}(I^\alpha_{a+} f) = f, \quad D^\alpha_{b-}(I^\alpha_{b-} f) = f \tag{2.3}$$

for $f \in L^1(a, b)$ and $0 < \alpha < 1$ are fundamental and are used in the proof of Lemma 2.1. The following two formulas are used in the proof of Lemma 3.1 in Section 3. The first is the composition formula:

$$D^\alpha_{a+}(D^\beta_{a+} f) = D^{\alpha+\beta}_{a+} f \quad (\text{resp. } D^\alpha_{b-}(D^\beta_{b-} f) = D^{\alpha+\beta}_{b-} f) \tag{2.4}$$

for $f \in I^{\alpha+\beta}_{a+}(L^1)$ (resp. $I^{\alpha+\beta}_{b-}(L^1)$), $0 < \alpha < 1$, and $0 < \beta < 1$, with $\alpha + \beta < 1$. The second is the integration by parts formula of order α:

$$(-1)^\alpha \int_a^b D^\alpha_{a+} f(t) g(t)\, dt = \int_a^b f(t) D^\alpha_{b-} g(t)\, dt \tag{2.5}$$

for $f \in I^\alpha_{a+}(L^p)$, $g \in I^\alpha_{b-}(L^q)$, $0 < \alpha < 1$, $1 \leq p < \infty$, and $1 \leq q < \infty$, with $1/p + 1/q \leq 1 + \alpha$. The following statements about the Hölder continuous functions are used in some of the discussions in this paper: Let f be a real-valued Hölder continuous function of order $\lambda \in (0, 1]$ on the interval $[a, b]$. Then, $f - f(a) \in I^\alpha_{a+}(L^p)$ and $f - f(b) \in I^\alpha_{b-}(L^p)$ hold for $\alpha \in (0, \lambda)$ and $p \in [1, \infty)$. Lemma 2.1 is a backward version of [18, Proposition 2.2].

Lemma 2.1. *Let f be a real-valued Hölder continuous function of order $\lambda \in (0, 1]$ on the interval $[a, b]$. Then, for $x, y \in [a, b]$ with $x < y$ and $\alpha \in (1 - \lambda, 1)$,*

$$f(y) - f(x) = (-1)^{-\alpha} \int_a^b D^\alpha_{b-} 1_{[x,y)}(t) D^{1-\alpha}_{a+}(f - f(a))(t)\, dt, \tag{2.6}$$

where $1_{[x,y)}$ denotes the indicator function of the interval $[x, y)$.

Proof. From the Hölder continuity of f and (2.2),

$$\begin{aligned} f(y) - f(x) &= (f(y) - f(a)) - (f(x) - f(a)) \\ &= I^{1-\alpha}_{a+} D^{1-\alpha}_{a+}(f - f(a))(y) - I^{1-\alpha}_{a+} D^{1-\alpha}_{a+}(f - f(a))(x) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_a^y (y - s)^{-\alpha} D^{1-\alpha}_{a+}(f - f(a))(s)\, ds \\ &\quad - \frac{1}{\Gamma(1-\alpha)} \int_a^x (x - s)^{-\alpha} D^{1-\alpha}_{a+}(f - f(a))(s)\, ds \\ &= \frac{1}{\Gamma(1-\alpha)} \int_a^b \left(\frac{1_{[a,y)}(s)}{(y - s)^\alpha} - \frac{1_{[a,x)}(s)}{(x - s)^\alpha} \right) D^{1-\alpha}_{a+}(f - f(a))(s)\, ds. \end{aligned}$$

It therefore suffices to show that

$$D^\alpha_{b-} 1_{[x,y)}(t) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left(\frac{1_{[a,y)}(t)}{(y - t)^\alpha} - \frac{1_{[a,x)}(t)}{(x - t)^\alpha} \right) \tag{2.7}$$

for $t \in (a,b)$. Let $\varphi(t)$ denote the right-hand side of (2.7). Note that $\varphi \in L^p(a,b)$ if and only if $\alpha p < 1$. For $t \in [a,x)$,

$$\begin{aligned} &I^\alpha_{b-}\varphi(t) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\left(\int_t^y (s-t)^{\alpha-1}(y-s)^{-\alpha}\,ds - \int_t^x (s-t)^{\alpha-1}(x-s)^{-\alpha}\,ds\right) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}(B(\alpha,1-\alpha)-B(\alpha,1-\alpha)) = 0, \end{aligned}$$

where B denotes the beta function. Similarly, for $t \in [x,y)$,

$$I^\alpha_{b-}\varphi(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_t^y (s-t)^{\alpha-1}(y-s)^{-\alpha}\,ds = \frac{B(\alpha,1-\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)} = 1.$$

Obviously, $I^\alpha_{b-}\varphi(t) = 0$ for $t \in [y,b]$. Therefore, $I^\alpha_{b-}\varphi(t) = 1_{[x,y)}(t)$ for $t \in [a,b]$. Hence, from (2.3), we obtain (2.7). The proof is therefore finished. $\square$

2.4. **Rough integrals via fractional calculus.** To describe our integration, we introduce a slight generalization of Weyl–Marchaud fractional derivatives. Let $a,b \in \mathbb{R}$ with $0 \le a < b \le T$. Let $\Psi \in \mathcal{C}^\lambda(\triangle_{a,b},V)$ with $\lambda \in (0,1]$ and $\alpha \in (0,\lambda)$. We define $\mathcal{D}^\alpha_{s+}\Psi$ with $s \in [a,b)$ and $\mathcal{D}^\alpha_{t-}\Psi$ with $t \in (a,b]$ by

$$\begin{aligned} \mathcal{D}^\alpha_{s+}\Psi(s) &:= 0, \\ \mathcal{D}^\alpha_{s+}\Psi(u) &:= \frac{1}{\Gamma(1-\alpha)}\left(\frac{\Psi_{s,u}}{(u-s)^\alpha} + \alpha\int_s^u \frac{\Psi_{v,u}}{(u-v)^{\alpha+1}}\,dv\right) \end{aligned}$$

for $u \in (s,b]$ and

$$\begin{aligned} \mathcal{D}^\alpha_{t-}\Psi(t) &:= 0, \\ \mathcal{D}^\alpha_{t-}\Psi(u) &:= \frac{(-1)^{1+\alpha}}{\Gamma(1-\alpha)}\left(\frac{\Psi_{u,t}}{(t-u)^\alpha} + \alpha\int_u^t \frac{\Psi_{u,v}}{(v-u)^{\alpha+1}}\,dv\right) \end{aligned}$$

for $u \in [a,t)$. If $\Psi \in \mathcal{C}^\lambda(\triangle_{a,b},V)$ is of the form $\Psi = \delta\psi$ for some $\psi \in \mathcal{C}^\lambda([a,b],V)$, $\mathcal{D}^\alpha_{s+}\Psi = D^\alpha_{s+}(\psi-\psi_s)$ and $\mathcal{D}^\alpha_{t-}\Psi = D^\alpha_{t-}(\psi-\psi_t)$ hold, by definition, for $\alpha \in (0,\lambda)$. The following estimates are used in some of the discussions in this paper:

$$\|\mathcal{D}^\alpha_{s+}\Psi(u)\|_V \le \frac{1}{\Gamma(1-\alpha)}\frac{\lambda}{\lambda-\alpha}\|\Psi\|_{\lambda;[s,u]}(u-s)^{\lambda-\alpha} \tag{2.8}$$

for $u \in [s,b]$ and

$$\|\mathcal{D}^\alpha_{t-}\Psi(u)\|_V \le \frac{1}{\Gamma(1-\alpha)}\frac{\lambda}{\lambda-\alpha}\|\Psi\|_{\lambda;[u,t]}(t-u)^{\lambda-\alpha} \tag{2.9}$$

for $u \in [a,t]$. These can be proved through a straightforward computation. We now introduce our definition of the integral of (Y,Y') along $(X,\mathbb{X})$. Note that $I_{(X,\mathbb{X})}(Y,Y')_{s,t}$ in Definition 2.2 was introduced by Hu and Nualart [6] and the author [10]. Recall that β is a real number with $1/3 < \beta \le 1/2$, and γ is a real number with $(1-\beta)/2 < \gamma < \beta$.

Definition 2.2. Let $a, b \in \mathbb{R}$ with $0 \le a < b \le T$. Let $(X, \mathbb{X}) \in \Omega_\beta([a,b], E)$ and $(Y, Y') \in \mathcal{Q}_X^\beta([a,b], L(E,F))$. For $(s,t) \in \triangle_{a,b}$, we define $I_{(X,\mathbb{X})}(Y,Y')_{s,t} \in F$ by

$$\begin{aligned} I_{(X,\mathbb{X})}(Y,Y')_{s,t} &:= Y_s(X_t - X_s) + Y'_s\mathbb{X}_{s,t} \\ &\quad + (-1)^{1-\gamma}\int_s^t \mathcal{D}_{s+}^{1-\gamma}R^Y(u)D_{t-}^\gamma(X - X_t)(u)\,du \\ &\quad + (-1)^{1-2\gamma}\int_s^t D_{s+}^{1-2\gamma}(Y' - Y'_s)(u)D_{t-}^\gamma(\mathcal{D}_{t-}^\gamma\mathbb{X})(u)\,du. \end{aligned}$$

For $(s,t) \in \triangle_{a,b}$, we define $J_{(X,\mathbb{X})}(Y,Y')_{s,t} \in F$ by

$$\begin{aligned} &J_{(X,\mathbb{X})}(Y,Y')_{s,t} \\ &:= Y_t(X_t - X_s) + Y'_t(\mathbb{X}_{s,t} - (X_t - X_s) \otimes (X_t - X_s)) \\ &\quad + (-1)^{\gamma-1}\int_s^t \mathcal{D}_{t-}^{1-\gamma}(R^Y - \delta Y'\delta X)(u)D_{s+}^\gamma(X - X_s)(u)\,du \\ &\quad + (-1)^{2\gamma-1}\int_s^t D_{t-}^{1-2\gamma}(Y' - Y'_t)(u)D_{s+}^\gamma(\mathcal{D}_{s+}^\gamma(\mathbb{X} - \delta X \otimes \delta X))(u)\,du. \end{aligned}$$

It follows from $(1-\beta)/2 < \gamma < \beta$ that the fractional derivatives in Definition 2.2 are well-defined; in fact, $\mathcal{D}_{s+}^{1-\gamma}R^Y$ and $\mathcal{D}_{t-}^{1-\gamma}(R^Y - \delta Y'\delta X)$ are well-defined from $1-\gamma < 2\beta$, $D_{s+}^{1-2\gamma}(Y' - Y'_s)$ and $D_{t-}^{1-2\gamma}(Y' - Y'_t)$ from $1 - 2\gamma < \beta$, and $D_{t-}^\gamma(X - X_t)$ and $D_{s+}^\gamma(X - X_s)$ from $\gamma < \beta$. For $D_{t-}^\gamma(\mathcal{D}_{t-}^\gamma\mathbb{X})$ and $D_{s+}^\gamma(\mathcal{D}_{s+}^\gamma(\mathbb{X} - \delta X \otimes \delta X))$, see (2) of Remark 2.3.

Remark 2.3. Let us make some comments about Definition 2.2.

(1) If $Y \in \mathcal{C}^{2\beta}([a,b], L(E,F))$ and $Y' \equiv 0$, i.e., Y' is identically zero, then, for $(s,t) \in \triangle_{a,b}$,

$$I_{(X,\mathbb{X})}(Y,Y')_{s,t} = Y_s(X_t - X_s) + (-1)^{1-\gamma}\int_s^t D_{s+}^{1-\gamma}(Y - Y_s)(u)D_{t-}^\gamma(X - X_t)(u)\,du$$

and

$$J_{(X,\mathbb{X})}(Y,Y')_{s,t} = Y_t(X_t - X_s) + (-1)^{\gamma-1}\int_s^t D_{t-}^{1-\gamma}(Y - Y_t)(u)D_{s+}^\gamma(X - X_s)(u)\,du.$$

The right-hand sides of these equalities belong to the category of the integration theory by Zähle [18]. The former (resp. the latter) is called the forward integral (resp. backward integral) of Y with respect to X.

(2) We note that $D_{t-}^\gamma(\mathcal{D}_{t-}^\gamma\mathbb{X})$ and $D_{s+}^\gamma(\mathcal{D}_{s+}^\gamma(\mathbb{X} - \delta X \otimes \delta X))$ are well-defined because $\mathcal{D}_{t-}^\gamma\mathbb{X}$ is β-Hölder continuous on $[a,t]$ and $\mathcal{D}_{t-}^\gamma\mathbb{X}(t) = 0$, and $\mathcal{D}_{s+}^\gamma(\mathbb{X} - \delta X \otimes \delta X)$ is β-Hölder continuous on $[s,b]$ and $\mathcal{D}_{s+}^\gamma(\mathbb{X} - \delta X \otimes \delta X)(s) = 0$. For proofs, see, e.g., [6, Lemma 6.3] and (2.16). Furthermore, it follows from [10, Lemma 3.6] that

$$D_{t-}^\gamma(\mathcal{D}_{t-}^\gamma\mathbb{X})(u) = \mathcal{D}_{t-}^{2\gamma}\mathbb{X}(u) - \mathcal{D}_{t-}^\gamma(\delta X \otimes D_{t-}^\gamma(X - X_t))(u) \tag{2.10}$$

holds for $u \in [a,t]$. Here,

$$\mathcal{D}^{\gamma}_{t-}(\delta X \otimes D^{\gamma}_{t-}(X - X_t))(u) := \frac{(-1)^{1+\gamma}\gamma}{\Gamma(1-\gamma)} \int_u^t \frac{\delta X_{u,v} \otimes D^{\gamma}_{t-}(X - X_t)(v)}{(v-u)^{\gamma+1}}\, dv.$$

Similarly,

$$D^{\gamma}_{s+}(\mathcal{D}^{\gamma}_{s+}(\mathbb{X} - \delta X \otimes \delta X))(u) = \mathcal{D}^{2\gamma}_{s+}(\mathbb{X} - \delta X \otimes \delta X)(u) - \mathcal{D}^{\gamma}_{s+}(\delta X \otimes D^{\gamma}_{s+}(X - X_s))(u) \tag{2.11}$$

holds for $u \in [s,b]$. Here,

$$\mathcal{D}^{\gamma}_{s+}(\delta X \otimes D^{\gamma}_{s+}(X - X_s))(u) := \frac{\gamma}{\Gamma(1-\gamma)} \int_s^u \frac{\delta X_{v,u} \otimes D^{\gamma}_{s+}(X - X_s)(v)}{(u-v)^{\gamma+1}}\, dv.$$

(3) From (2.8), (2.9), (2.10), and (2.11), it is easy to see that there exists a positive constant C, depending only on β and γ, such that, for $(s,t) \in \triangle_{a,b}$,

$$\begin{aligned}&|I_{(X,\mathbb{X})}(Y,Y')_{s,t} - (Y_s(X_t - X_s) + Y'_s\mathbb{X}_{s,t})| \\ &\le C\{\|R^Y\|_{2\beta}\|X\|_{\beta} + \|Y'\|_{\beta}(\|\mathbb{X}\|_{2\beta} + \|X\|^2_{\beta})\}(t-s)^{3\beta}\end{aligned}$$

and

$$\begin{aligned}&|J_{(X,\mathbb{X})}(Y,Y')_{s,t} - (Y_t(X_t - X_s) + Y'_t(\mathbb{X}_{s,t} - (X_t - X_s) \otimes (X_t - X_s)))| \\ &\le C\{\|R^Y\|_{2\beta}\|X\|_{\beta} + \|Y'\|_{\beta}(\|\mathbb{X}\|_{2\beta} + \|X\|^2_{\beta})\}(t-s)^{3\beta}.\end{aligned}$$

2.5. **Main Theorems.** We here introduce the main theorems of this paper (Theorems 2.5 and 2.6). Using Lemma 2.4, we prove Theorem 2.5. See Section 2.2 for the definitions of $(\overleftarrow{X}, \overleftarrow{\mathbb{X}}) \in \Omega_{\beta}([a,b],E)$ and $(\overleftarrow{Y}, \overleftarrow{Y'}) \in \mathcal{Q}^{\beta}_{\overleftarrow{X}}([a,b],F)$.

Lemma 2.4. *Let $a,b \in \mathbb{R}$ with $0 \le a < b \le T$. Set $u' = a+b-u$ for $u \in [a,b]$. Let $(X,\mathbb{X}) \in \Omega_{\beta}([a,b],E)$ and $(Y,Y') \in \mathcal{Q}^{\beta}_{X}([a,b],F)$. Then,*

$$D^{1-2\gamma}_{t-}(Y' - Y'_t)(u) = (-1)^{1-2\gamma} D^{1-2\gamma}_{t'+}(\overleftarrow{Y'} - \overleftarrow{Y'}_{t'})(u'), \tag{2.12}$$

$$\mathcal{D}^{1-\gamma}_{t-}(R^Y - \delta Y'\delta X)(u) = (-1)^{1-\gamma}\mathcal{D}^{1-\gamma}_{t'+}R^{\overleftarrow{Y}}(u') \tag{2.13}$$

hold for $t \in (a,b]$ and $u \in [a,t]$, and

$$D^{\gamma}_{s+}(X - X_s)(u) = (-1)^{-\gamma} D^{\gamma}_{s'-}(\overleftarrow{X} - \overleftarrow{X}_{s'})(u'), \tag{2.14}$$

$$D^{\gamma}_{s+}(\mathcal{D}^{\gamma}_{s+}(\mathbb{X} - \delta X \otimes \delta X))(u) = (-1)^{-2\gamma} D^{\gamma}_{s'-}(\mathcal{D}^{\gamma}_{s'-}\overleftarrow{\mathbb{X}})(u') \tag{2.15}$$

hold for $s \in [a,b)$ and $u \in [s,b]$.

Proof. We first prove (2.12) and (2.13). Fix $u \in [a,t]$. By using $Y'_u = \overleftarrow{Y'}_{u'}$ and the change of variable $v' = a+b-v$ in the definition of $D^{1-2\gamma}_{t-}(Y' - Y'_t)(u)$,

$$\begin{aligned}
&D^{1-2\gamma}_{t-}(Y' - Y'_t)(u)\\
&= \frac{(-1)^{1-2\gamma}}{\Gamma(1-(1-2\gamma))}\left(\frac{\overleftarrow{Y'}_{u'} - \overleftarrow{Y'}_{t'}}{(u'-t')^{1-2\gamma}} + (1-2\gamma)\int_{t'}^{u'} \frac{\overleftarrow{Y'}_{u'} - \overleftarrow{Y'}_{v'}}{(u'-v')^{(1-2\gamma)+1}}\,dv'\right)\\
&= (-1)^{1-2\gamma} D^{1-2\gamma}_{t'+}(\overleftarrow{Y'} - \overleftarrow{Y'}_{t'})(u').
\end{aligned}$$

Similarly, by using the relation $R^Y_{u,t} - \delta Y'_{u,t}\delta X_{u,t} = -R^{\overleftarrow{Y}}_{t',u'}$ and the change of variable $v' = a+b-v$ in the definition of $\mathcal{D}^{1-\gamma}_{t-}(R^Y - \delta Y'\delta X)(u)$,

$$\begin{aligned}
&\mathcal{D}^{1-\gamma}_{t-}(R^Y - \delta Y'\delta X)(u)\\
&= \frac{(-1)^{(1-\gamma)+1}}{\Gamma(1-(1-\gamma))}\left(\frac{-R^{\overleftarrow{Y}}_{t',u'}}{(u'-t')^{1-\gamma}} + (1-\gamma)\int_{t'}^{u'} \frac{-R^{\overleftarrow{Y}}_{v',u'}}{(u'-v')^{(1-\gamma)+1}}\,dv'\right)\\
&= (-1)^{1-\gamma}\mathcal{D}^{1-\gamma}_{t'+}R^{\overleftarrow{Y}}(u').
\end{aligned}$$

Next, we prove (2.15) and omit the proof of (2.14) because it is similar to (2.12). Fix $u \in [s,b]$. By using $\mathbb{X}_{s,u} - \delta X_{s,u}\otimes\delta X_{s,u} = -\overleftarrow{\mathbb{X}}_{u',s'}$, the change of variable $v' = a+b-v$ in the definition of $\mathcal{D}^{\gamma}_{s+}(\mathbb{X} - \delta X\otimes\delta X)(u)$, and $-1 = (-1)^{-\gamma}(-1)^{1+\gamma}$,

$$\mathcal{D}^{\gamma}_{s+}(\mathbb{X} - \delta X\otimes\delta X)(u) = (-1)^{-\gamma}\mathcal{D}^{\gamma}_{s'-}\overleftarrow{\mathbb{X}}(u'). \tag{2.16}$$

Therefore, using the same arguments as above,

$$\begin{aligned}
&D^{\gamma}_{s+}(\mathcal{D}^{\gamma}_{s+}(\mathbb{X} - \delta X\otimes\delta X))(u)\\
&= \frac{(-1)^{-\gamma}}{\Gamma(1-\gamma)}\left(\frac{\mathcal{D}^{\gamma}_{s'-}\overleftarrow{\mathbb{X}}(u')}{(s'-u')^{\gamma}} + \gamma\int_{u'}^{s'} \frac{\mathcal{D}^{\gamma}_{s'-}\overleftarrow{\mathbb{X}}(u') - \mathcal{D}^{\gamma}_{s'-}\overleftarrow{\mathbb{X}}(v')}{(v'-u')^{\gamma+1}}\,dv'\right)\\
&= (-1)^{-2\gamma}D^{\gamma}_{s'-}(\mathcal{D}^{\gamma}_{s'-}\overleftarrow{\mathbb{X}})(u').
\end{aligned}$$

In the last equality, we used $(-1)^{-\gamma} = (-1)^{-2\gamma}(-1)^{\gamma}$. The proof is thus finished. □

Theorem 2.5. *Let $a, b \in \mathbb{R}$ with $0 \le a < b \le T$. Let $(X, \mathbb{X}) \in \Omega_\beta([a,b], E)$ and $(Y, Y') \in \mathcal{Q}^\beta_X([a,b], L(E,F))$. Then, for $(s,t) \in \triangle_{a,b}$,*

$$J_{(X,\mathbb{X})}(Y,Y')_{s,t} = -I_{(\overleftarrow{X},\overleftarrow{\mathbb{X}})}(\overleftarrow{Y},\overleftarrow{Y'})_{a+b-t,a+b-s} \tag{2.17}$$

and

$$I_{(X,\mathbb{X})}(Y,Y')_{s,t} = -J_{(\overleftarrow{X},\overleftarrow{\mathbb{X}})}(\overleftarrow{Y},\overleftarrow{Y'})_{a+b-t,a+b-s}. \tag{2.18}$$

Proof. Set $t' = a + b - t$ and $s' = a + b - s$. Then, by using Lemma 2.4 and the change of variable $u' = a + b - u$ in the definition of $J_{(X,\mathbb{X})}(Y, Y')_{s,t}$,

$$
\begin{aligned}
&J_{(X,\mathbb{X})}(Y, Y')_{s,t} \\
&= -\overleftarrow{Y}_{t'}(\overleftarrow{X}_{s'} - \overleftarrow{X}_{t'}) - \overleftarrow{Y'}_{t'}\overleftarrow{\mathbb{X}}_{t',s'} \\
&\quad + (-1)^{\gamma-1}\int_{t'}^{s'} (-1)^{1-\gamma}\mathcal{D}_{t'+}^{1-\gamma} R^{\overleftarrow{Y}}(u')(-1)^{-\gamma} D_{s'-}^{\gamma}(\overleftarrow{X} - \overleftarrow{X}_{s'})(u')\, du' \\
&\quad + (-1)^{2\gamma-1}\int_{t'}^{s'} (-1)^{1-2\gamma} D_{t'+}^{1-2\gamma}(\overleftarrow{Y'} - \overleftarrow{Y'}_{t'})(u')(-1)^{-2\gamma} D_{s'-}^{\gamma}(\mathcal{D}_{s'-}^{\gamma}\overleftarrow{\mathbb{X}})(u')\, du' \\
&= -I_{(\overleftarrow{X},\overleftarrow{\mathbb{X}})}(\overleftarrow{Y}, \overleftarrow{Y'})_{t',s'}.
\end{aligned}
$$

Hence, (2.17) holds. From $(X, \mathbb{X}) = (\overleftarrow{\overleftarrow{X}}, \overleftarrow{\overleftarrow{\mathbb{X}}})$, $(Y, Y') = (\overleftarrow{\overleftarrow{Y}}, \overleftarrow{\overleftarrow{Y'}})$, and (2.17),

$$
I_{(X,\mathbb{X})}(Y, Y')_{s,t} = I_{(\overleftarrow{\overleftarrow{X}},\overleftarrow{\overleftarrow{\mathbb{X}}})}(\overleftarrow{\overleftarrow{Y}}, \overleftarrow{\overleftarrow{Y'}})_{a+b-s',a+b-t'} = -J_{(\overleftarrow{X},\overleftarrow{\mathbb{X}})}(\overleftarrow{Y}, \overleftarrow{Y'})_{t',s'}.
$$

Therefore, (2.18) holds. The proof is finished. □

Theorem 2.6 justifies the interpretation of $I_{(X,\mathbb{X})}(Y, Y')$ and $J_{(X,\mathbb{X})}(Y, Y')$ as the integrals of (Y, Y') along $(X, \mathbb{X})$.

Theorem 2.6. *Let $(X, \mathbb{X}) \in \Omega_\beta([0, T], E)$ and $(Y, Y') \in \mathcal{Q}_X^\beta([0, T], L(E, F))$. Then, $I_{(X,\mathbb{X})}(Y, Y')$ and $J_{(X,\mathbb{X})}(Y, Y')$ coincide with the rough integral of (Y, Y') along $(X, \mathbb{X})$. Specifically, for $(s, t) \in \triangle_{0,T}$,*

$$
I_{(X,\mathbb{X})}(Y, Y')_{s,t} = \lim_{|\mathcal{P}_{s,t}|\to 0} \sum_{i=0}^{n-1} Y_{t_i}(X_{t_{i+1}} - X_{t_i}) + Y'_{t_i}\mathbb{X}_{t_i,t_{i+1}} \tag{2.19}
$$

and

$$
J_{(X,\mathbb{X})}(Y, Y')_{s,t} = \lim_{|\mathcal{P}_{s,t}|\to 0} \sum_{i=0}^{n-1} Y_{t_i}(X_{t_{i+1}} - X_{t_i}) + Y'_{t_i}\mathbb{X}_{t_i,t_{i+1}}, \tag{2.20}
$$

where the limits are taken over all finite partitions $\mathcal{P}_{s,t} = \{t_0, t_1, \dots, t_n\}$ of the interval $[s, t]$ such that $s = t_0 < t_1 < \cdots < t_n = t$ and $|\mathcal{P}_{s,t}| = \max_{0\le i\le n-1}|t_{i+1} - t_i|$.

Proof. We omit the proof of (2.19) becasue it follows from [10, Theorem 2.5]. Using (2.19) and (2.17), we prove (2.20). (In Section 3, we provide another proof of (2.20) without using (2.19).) We fix $(s, t) \in \triangle_{0,T}$ with $s < t$ because (2.20) obviously holds when $s = t$. For $x, y \in [s, t]$ with $x < y$, we set $x' = T - x$ and $y' = T - y$. We then have

$$
\begin{aligned}
&\overleftarrow{Y}_x(\overleftarrow{X}_y - \overleftarrow{X}_x) + \overleftarrow{Y'}_x\overleftarrow{\mathbb{X}}_{x,y} + Y_{y'}(X_{x'} - X_{y'}) + Y'_{y'}\mathbb{X}_{y',x'} \\
&\quad = -R^Y_{y',x'}\delta X_{y',x'} - \delta Y'_{y',x'}(\mathbb{X}_{y',x'} - \delta X_{y',x'} \otimes \delta X_{y',x'})
\end{aligned}
$$

and therefore,

$$\begin{aligned}&|\overleftarrow{Y}_x(\overleftarrow{X}_y - \overleftarrow{X}_x) + \overleftarrow{Y'}_x \overleftarrow{\mathbb{X}}_{x,y} + Y_{y'}(X_{x'} - X_{y'}) + Y'_{y'}\mathbb{X}_{y',x'}| \\ &\quad \le (\|R^Y\|_{2\beta}\|X\|_\beta + \|Y'\|_\beta(\|\mathbb{X}\|_{2\beta} + \|X\|_\beta^2))(y-x)^{3\beta}. \qquad (2.21)\end{aligned}$$

By using (2.17), (2.19), (2.21) with $x = s_j$ and $y = s_{j+1}$, and $3\beta > 1$, we have

$$\begin{aligned}J_{(X,\mathbb{X})}(Y,Y')_{s,t} &= -I_{(\overleftarrow{X},\overleftarrow{\mathbb{X}})}(\overleftarrow{Y},\overleftarrow{Y'})_{t',s'} \\ &= -\lim_{|\mathcal{P}_{t',s'}|\to 0} \sum_{j=0}^{m-1} \overleftarrow{Y}_{s_j}(\overleftarrow{X}_{s_{j+1}} - \overleftarrow{X}_{s_j}) + \overleftarrow{Y'}_{s_j}\overleftarrow{\mathbb{X}}_{s_j,s_{j+1}} \\ &= \lim_{|\mathcal{P}_{t',s'}|\to 0} \sum_{j=0}^{m-1} Y_{s'_{j+1}}(X_{s'_j} - X_{s'_{j+1}}) + Y'_{s'_{j+1}}\mathbb{X}_{s'_{j+1},s'_j},\end{aligned}$$

where the limits are taken over all finite partitions $\mathcal{P}_{t',s'} = \{s_0, s_1, \ldots, s_m\}$ of the interval $[t', s']$ such that $t' = s_0 < s_1 < \cdots < s_m = s'$. Because $\{s'_0, s'_1, \ldots, s'_m\}$ is a partition of the interval $[s,t]$ such that $s = s'_m < s'_{m-1} < \cdots < s'_0 = t$, we obtain the statement of (2.20). The proof is thus finished. □

Using Theorems 2.5 and 2.6, we describe the backward representation (1.1) as follows: Under the assumptions and the notation of Theorem 2.6, for $s \in [0,T)$,

$$\begin{aligned}\int_s^T (Y,Y')\,d(X,\mathbb{X}) &= J_{(X,\mathbb{X})}(Y,Y')_{s,T} \\ &= -I_{(\overleftarrow{X},\overleftarrow{\mathbb{X}})}(\overleftarrow{Y},\overleftarrow{Y'})_{0,T-s} \\ &= -\int_0^{T-s} (\overleftarrow{Y},\overleftarrow{Y'})\,d(\overleftarrow{X},\overleftarrow{\mathbb{X}})\end{aligned}$$

and for $\mathcal{P}_{s,T} = \{t_0, t_1, \ldots, t_n\}$ such that $s = t_0 < t_1 < \cdots < t_n = T$,

$$\begin{aligned}&\sum_{i=0}^{n-1}\Big\{Y_{t_{i+1}}(X_{t_{i+1}} - X_{t_i}) + Y'_{t_{i+1}}(\mathbb{X}_{t_i,t_{i+1}} - (X_{t_{i+1}} - X_{t_i}) \otimes (X_{t_{i+1}} - X_{t_i}))\Big\} \\ &= -\sum_{i=0}^{n-1}\Big\{\overleftarrow{Y}_{t'_{i+1}}(\overleftarrow{X}_{t'_i} - \overleftarrow{X}_{t'_{i+1}}) + \overleftarrow{Y'}_{t'_{i+1}}\overleftarrow{\mathbb{X}}_{t'_{i+1},t'_i}\Big\} \\ &\to -I_{(\overleftarrow{X},\overleftarrow{\mathbb{X}})}(\overleftarrow{Y},\overleftarrow{Y'})_{0,T-s}\end{aligned}$$

as $|\mathcal{P}_{s,T}| \to 0$. Therefore, we obtain the equalities of (1.1).

3. Another proof of Theorem 2.6

In this section, we provide another proof for (2.20) without using (2.19). The outline of the proof is based on that of (2.19), described in the author's previous study [10]. Lemma 3.1, Propositions 3.2, and 3.3 below are regarded as backward versions of Lemmas 3.5, 3.7, and Proposition 2.4, respectively, in [10]. Recall that $1_{[x,y)}$ is the indicator function of the interval $[x,y) \subset \mathbb{R}$.

Lemma 3.1. *Let* $(X,\mathbb{X}) \in \Omega_\beta([0,T],E)$ *and* $(s,t) \in \triangle_{0,T}$ *with* $s<t$*. Then, for* $x,y \in [s,t]$ *with* $x<y$*,*

$$\mathbb{X}_{x,y} = (-1)^{\gamma-1}\int_s^t D_{t-}^{1-\gamma}1_{[x,y)}(u)(X_u - X_x)\otimes D_{s+}^{\gamma}(X-X_s)(u)\,du \\ + (-1)^{2\gamma-1}\int_s^t D_{t-}^{1-2\gamma}1_{[x,y)}(u)D_{s+}^{\gamma}(\mathcal{D}_{s+}^{\gamma}(\mathbb{X}-\delta X\otimes\delta X))(u)\,du. \tag{3.1}$$

Proof. From the definition of $(X,\mathbb{X})$ and (2.6),

$$\begin{aligned}\mathbb{X}_{x,y} &= \mathbb{X}_{s,y} - \mathbb{X}_{s,x} - (X_x - X_s)\otimes(X_y - X_x)\\ &= (-1)^{\gamma-1}\int_s^t D_{t-}^{1-\gamma}1_{[x,y)}(u)D_{s+}^{\gamma}(\mathbb{X}_{s,} - \mathbb{X}_{s,s})(u)\,du\\ &\quad - (-1)^{\gamma-1}\int_s^t D_{t-}^{1-\gamma}1_{[x,y)}(u)(X_x - X_s)\otimes D_{s+}^{\gamma}(X-X_s)(u)\,du.\end{aligned}$$

From $\mathbb{X}_{s,s}=0$ and (2.1), for $u\in[s,T]$,

$$\begin{aligned}&D_{s+}^{\gamma}(\mathbb{X}_{s,} - \mathbb{X}_{s,s})(u)\\ &= \frac{1}{\Gamma(1-\gamma)}\left(\frac{\mathbb{X}_{s,u}-\mathbb{X}_{s,s}}{(u-s)^{\gamma}} + \gamma\int_s^u \frac{\mathbb{X}_{s,u}-\mathbb{X}_{s,v}}{(u-v)^{\gamma+1}}\,dv\right)\\ &= \mathcal{D}_{s+}^{\gamma}\mathbb{X}(u) + \frac{\gamma}{\Gamma(1-\gamma)}\int_s^u \frac{(X_v - X_s)\otimes(X_u - X_v)}{(u-v)^{\gamma+1}}\,dv\\ &= \mathcal{D}_{s+}^{\gamma}\mathbb{X}(u) - \mathcal{D}_{s+}^{\gamma}(\delta X\otimes\delta X)(u) + (X_u - X_s)\otimes D_{s+}^{\gamma}(X-X_s)(u).\end{aligned}$$

Therefore, from (2.4) and (2.5) with $\alpha=\gamma$, $1/\gamma \le p < \infty$, and $1\le q < 1/(1-\gamma)$,

$$\begin{aligned}\mathbb{X}_{x,y} &= (-1)^{\gamma-1}\int_s^t D_{t-}^{1-\gamma}1_{[x,y)}(u)\\ &\quad\times((X_u - X_s)-(X_x - X_s))\otimes D_{s+}^{\gamma}(X-X_s)(u)\,du\\ &\quad + (-1)^{\gamma-1}\int_s^t D_{t-}^{1-\gamma}1_{[x,y)}(u)\mathcal{D}_{s+}^{\gamma}(\mathbb{X}-\delta X\otimes\delta X)(u)\,du\\ &= (-1)^{\gamma-1}\int_s^t D_{t-}^{1-\gamma}1_{[x,y)}(u)(X_u - X_x)\otimes D_{s+}^{\gamma}(X-X_s)(u)\,du\\ &\quad + (-1)^{2\gamma-1}\int_s^t D_{t-}^{1-2\gamma}1_{[x,y)}(u)D_{s+}^{\gamma}(\mathcal{D}_{s+}^{\gamma}(\mathbb{X}-\delta X\otimes\delta X))(u)\,du.\end{aligned}$$

We note that the conditions for the uses of (2.4) and (2.5) are satisfied, from (a) $\mathcal{D}_{s+}^{\gamma}(\mathbb{X}-\delta X\otimes\delta X)$ is β-Hölder continuous on the interval $[s,T]$ and $\mathcal{D}_{s+}^{\gamma}(\mathbb{X}-\delta X\otimes\delta X)(s)=0$, and (b) $1_{[x,y)}\in I_{t-}^{1-\gamma}(L^q)$ if and only if $(1-\gamma)q<1$. The proof is thus finished. □

Proposition 3.2. *Let* $(X,\mathbb{X}) \in \Omega_\beta([0,T],E)$ *and* $(Y,Y') \in \mathcal{Q}_X^\beta([0,T],L(E,F))$. *Then, for* $(s,t) \in \triangle_{0,T}$ *with* $s<t$,

$$J_{(X,\mathbb{X})}(Y,Y')_{s,t} = (-1)^{\gamma-1}\int_s^t \hat{\mathcal{D}}_{t-}^{1-\gamma}(R^Y - \delta Y'\delta X)(u) D_{s+}^\gamma (X - X_s)(u)\,du + (-1)^{2\gamma-1}\int_s^t D_{t-}^{1-2\gamma} Y'(u) D_{s+}^\gamma(\mathcal{D}_{s+}^\gamma(\mathbb{X} - \delta X\otimes\delta X))(u)\,du,$$

where

$$\hat{\mathcal{D}}_{t-}^{1-\gamma}(R^Y - \delta Y'\delta X)(u) := \frac{(-1)^{(1-\gamma)+1}}{\Gamma(1-(1-\gamma))}\left(\frac{-Y_u}{(t-u)^{1-\gamma}} + (1-\gamma)\int_u^t \frac{R_{u,v}^Y - \delta Y'_{u,v}\delta X_{u,v}}{(v-u)^{(1-\gamma)+1}}\,dv\right)$$

for $u \in [0,t)$.

Proof. From (2.6) and (3.1) with $x=s$ and $y=t$, we have

$$\begin{aligned} &Y_t(X_t - X_s) + Y'_t(\mathbb{X}_{s,t} - (X_t - X_s)\otimes(X_t - X_s)) \\ &= (-1)^{\gamma-1}\int_s^t D_{t-}^{1-\gamma}((Y_t + Y'_t(X_u - X_t))1_{[s,t)})(u) D_{s+}^\gamma(X - X_s)(u)\,du \\ &\quad + (-1)^{2\gamma-1}\int_s^t D_{t-}^{1-2\gamma}(Y'_t 1_{[s,t)})(u) D_{s+}^\gamma(\mathcal{D}_{s+}^\gamma(\mathbb{X} - \delta X\otimes\delta X))(u)\,du. \end{aligned}$$

Therefore, from the definition of $J_{(X,\mathbb{X})}(Y,Y')_{s,t}$, we have

$$\begin{aligned} &J_{(X,\mathbb{X})}(Y,Y')_{s,t} \\ &= (-1)^{\gamma-1}\int_s^t (\mathcal{D}_{t-}^{1-\gamma}(R^Y - \delta Y'\delta X)(u) + D_{t-}^{1-\gamma}((Y_t + Y'_t(X_u - X_t))1_{[s,t)})(u)) \\ &\quad \times D_{s+}^\gamma(X - X_s)(u)\,du \\ &\quad + (-1)^{2\gamma-1}\int_s^t (D_{t-}^{1-2\gamma}(Y' - Y'_t)(u) + D_{t-}^{1-2\gamma}(Y'_t 1_{[s,t)})(u)) \\ &\qquad \times D_{s+}^\gamma(\mathcal{D}_{s+}^\gamma(\mathbb{X} - \delta X\otimes\delta X))(u)\,du. \end{aligned}$$

From the equality

$$D_{t-}^{1-k\gamma}1_{[s,t)}(u) = \frac{(-1)^{1-k\gamma}}{\Gamma(1-(1-k\gamma))}\frac{1}{(t-u)^{1-k\gamma}}$$

for $k=1,2$ and $u \in (s,t)$ (see (2.7) with $x=a$ and $y=b$),

$$\begin{aligned} &\mathcal{D}_{t-}^{1-\gamma}(R^Y - \delta Y'\delta X)(u) + D_{t-}^{1-\gamma}((Y_t + Y'_t(X_u - X_t))1_{[s,t)})(u) \\ &= \hat{\mathcal{D}}_{t-}^{1-\gamma}(R^Y - \delta Y'\delta X)(u), \\ &D_{t-}^{1-2\gamma}(Y' - Y'_t)(u) + D_{t-}^{1-2\gamma}(Y'_t 1_{[s,t)})(u) = D_{t-}^{1-2\gamma}Y'(u) \end{aligned}$$

hold for $u \in (s,t)$. Therefore, we obtain the statement of the proposition. □

Proposition 3.3. *Let* $X \in \mathcal{C}^\beta([0,T],E)$ *and* $(Y,Y') \in \mathcal{Q}_X^\beta([0,T],F)$*. Then, for* $(s,t) \in \triangle_{0,T}$ *with* $s < t$*,*

$$\lim_{|\mathcal{P}_{s,t}|\to 0} \int_s^t |D_{t-}^{1-\gamma}(\sum_{i=0}^{n-1}(Y_{t_i} + Y'_{t_i}(X_u - X_{t_i}))1_{[t_i,t_{i+1})})(u) \\ - \hat{\mathcal{D}}_{t-}^{1-\gamma}(R^Y - \delta Y'\delta X)(u)|\, du = 0 \tag{3.2}$$

and

$$\lim_{|\mathcal{P}_{s,t}|\to 0} \int_s^t |D_{t-}^{1-2\gamma}(\sum_{i=0}^{n-1} Y'_{t_i}1_{[t_i,t_{i+1})})(u) - D_{t-}^{1-2\gamma}Y'(u)|\, du = 0. \tag{3.3}$$

Proof. We omit the proof of (3.3) because it is similar to (3.2). In fact, formally taking $Y' \equiv 0$ and $\alpha = 1 - 2\gamma$ in the proof of (3.2) provides a proof of (3.3). Set $\alpha = 1 - \gamma$. Note that $\alpha < 2\beta$. For $u \in (s,t)$, we set

$$\begin{aligned}
&(D_{t-}^{\alpha}(\sum_{i=0}^{n-1}(Y_{t_i} + Y'_{t_i}(X_u - X_{t_i}))1_{[t_i,t_{i+1})})(u) - \hat{\mathcal{D}}_{t-}^{\alpha}(R^Y - \delta Y'\delta X)(u)) \\
&\quad \times (-1)^{-\alpha}\Gamma(1-\alpha) \\
&= (\sum_{i=0}^{n-1}(Y_{t_i} + Y'_{t_i}(X_u - X_{t_i}))1_{[t_i,t_{i+1})}(u) - Y_u)(t-u)^{-\alpha} + \alpha \int_u^t \frac{\Psi_{u,v}^{\mathcal{P}}}{(v-u)^{\alpha+1}}\, dv \\
&=: S_{\mathcal{P}}^1(u) + S_{\mathcal{P}}^2(u).
\end{aligned}$$

Here,

$$\Psi_{u,v}^{\mathcal{P}} := \sum_{i=0}^{n-1}(Y_{t_i} + Y'_{t_i}(X_u - X_{t_i}))(1_{[t_i,t_{i+1})}(u) - 1_{[t_i,t_{i+1})}(v)) + R_{u,v}^Y - \delta Y'_{u,v}\delta X_{u,v}$$

for $(u,v) \in \triangle_{s,t}$. It therefore suffices to show that

$$\lim_{|\mathcal{P}_{s,t}|\to 0} \int_s^t |S_{\mathcal{P}}^l(u)|\, du = 0 \tag{3.4}$$

holds for $l = 1, 2$. First, from the equality

$$(Y_{t_i} + Y'_{t_i}(X_u - X_{t_i}))1_{[t_i,t_{i+1})}(u) - Y_u = -R_{t_i,u}^Y$$

for $u \in [t_i, t_{i+1})$, we have

$$\begin{aligned}\int_s^t |S^1_{\mathcal{P}}(u)|\,du &= \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} |S^1_{\mathcal{P}}(u)|\,du\\ &= \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} |R^Y_{t_i,u}|(t-u)^{-\alpha}\,du\\ &\le \|R^Y\|_{2\beta}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(u-t_i)^{2\beta}(t-u)^{-\alpha}\,du\\ &\le \|R^Y\|_{2\beta}|\mathcal{P}_{s,t}|^{2\beta}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(t-u)^{-\alpha}\,du\\ &= \|R^Y\|_{2\beta}|\mathcal{P}_{s,t}|^{2\beta}(1-\alpha)^{-1}(t-s)^{1-\alpha}.\end{aligned}$$

Therefore, (3.4) holds for $l = 1$. Next, using the equality

$$S^2_{\mathcal{P}}(u) = \alpha\int_u^{t_{i+1}} \frac{\Psi^{\mathcal{P}}_{u,v}}{(v-u)^{\alpha+1}}\,dv + \alpha\sum_{j=i+1}^{n-1}\int_{t_j}^{t_{j+1}}\frac{\Psi^{\mathcal{P}}_{u,v}}{(v-u)^{\alpha+1}}\,dv$$

for $u \in [t_i, t_{i+1})$, we have

$$\begin{aligned}\int_s^t |S^2_{\mathcal{P}}(u)|\,du &= \int_{t_{n-1}}^{t_n}|S^2_{\mathcal{P}}(u)|\,du + \sum_{i=0}^{n-2}\int_{t_i}^{t_{i+1}}|S^2_{\mathcal{P}}(u)|\,du\\ &\le \alpha\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_u^{t_{i+1}}\frac{|\Psi^{\mathcal{P}}_{u,v}|}{(v-u)^{\alpha+1}}\,dv\,du\\ &\quad + \alpha\sum_{i=0}^{n-2}\int_{t_i}^{t_{i+1}}\sum_{j=i+1}^{n-1}\int_{t_j}^{t_{j+1}}\frac{|\Psi^{\mathcal{P}}_{u,v}|}{(v-u)^{\alpha+1}}\,dv\,du\\ &=: A_1 + A_2.\end{aligned}$$

When $t_i \le u \le v < t_{i+1}$, $\Psi^{\mathcal{P}}_{u,v} = R^Y_{u,v} - \delta Y'_{u,v}\delta X_{u,v}$. Therefore,

$$\begin{aligned}A_1 &\le (\|R^Y\|_{2\beta} + \|Y'\|_\beta\|X\|_\beta)\alpha\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_u^{t_{i+1}}(v-u)^{2\beta-\alpha-1}\,dv\,du\\ &= (\|R^Y\|_{2\beta} + \|Y'\|_\beta\|X\|_\beta)\alpha(2\beta-\alpha)^{-1}(2\beta-\alpha+1)^{-1}\sum_{i=0}^{n-1}(t_{i+1}-t_i)^{2\beta-\alpha+1}\\ &\le (\|R^Y\|_{2\beta} + \|Y'\|_\beta\|X\|_\beta)\alpha(2\beta-\alpha)^{-1}(2\beta-\alpha+1)^{-1}|\mathcal{P}_{s,t}|^{2\beta-\alpha}(t-s).\end{aligned}$$

When $t_i \le u < t_{i+1} \le t_j \le v < t_{j+1}$,

$$\begin{aligned}\Psi^{\mathcal{P}}_{u,v} &= (Y_{t_i} + Y'_{t_i}(X_u - X_{t_i})) - (Y_{t_j} + Y'_{t_j}(X_u - X_{t_j})) + R^Y_{u,v} - \delta Y'_{u,v}\delta X_{u,v}\\ &= -R^Y_{t_i,u} + R^Y_{t_j,v} - (Y'_v - Y'_{t_j})(X_v - X_u).\end{aligned}$$

Therefore,

$$
\begin{aligned}
A_2 \le \|R^Y\|_{2\beta}\alpha \sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}(u-t_i)^{2\beta}(v-u)^{-\alpha-1}\,dv\,du \\
+ \|R^Y\|_{2\beta}\alpha \sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}(v-t_j)^{2\beta}(v-u)^{-\alpha-1}\,dv\,du \\
+ \|Y'\|_{\beta}\|X\|_{\beta}\alpha \sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}(v-t_j)^{\beta}(v-u)^{\beta-\alpha-1}\,dv\,du \\
=: A_{21}+A_{22}+A_{23}.
\end{aligned}
$$

Through a straightforward computation,

$$
\begin{aligned}
A_{21} &= \|R^Y\|_{2\beta}\alpha \sum_{i=0}^{n-2}\int_{t_i}^{t_{i+1}}(u-t_i)^{2\beta}\int_{t_{i+1}}^{t}(v-u)^{-\alpha-1}\,dv\,du \\
&\le \|R^Y\|_{2\beta}\sum_{i=0}^{n-2}\int_{t_i}^{t_{i+1}}(u-t_i)^{2\beta}(t_{i+1}-u)^{-\alpha}\,du \\
&= \|R^Y\|_{2\beta}B(2\beta+1,1-\alpha)\sum_{i=0}^{n-2}(t_{i+1}-t_i)^{2\beta-\alpha+1} \\
&\le \|R^Y\|_{2\beta}B(2\beta+1,1-\alpha)|\mathcal{P}_{s,t}|^{2\beta-\alpha}(t-s),
\end{aligned}
$$

where B denotes the beta function. For A_{22} and A_{23}, we provide the following estimate. For $\lambda\in(0,\infty)$ and $\mu\in(0,1)$ with $\lambda+\mu>1$,

$$
\begin{aligned}
&\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}(v-t_j)^{\lambda}(v-u)^{\mu-2}\,dv\,du \\
&\le \sum_{j=1}^{n-1}\sum_{i=0}^{j-1}(t_{j+1}-t_j)^{\lambda}\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}(v-u)^{\mu-2}\,dv\,du \\
&= \sum_{j=1}^{n-1}(t_{j+1}-t_j)^{\lambda}(1-\mu)^{-1}\mu^{-1}\{(t_{j+1}-t_j)^{\mu}+(t_j-s)^{\mu}-(t_{j+1}-s)^{\mu}\} \\
&\le (1-\mu)^{-1}\mu^{-1}|\mathcal{P}_{s,t}|^{\lambda+\mu-1}(t-s).
\end{aligned}
$$

Letting $(\lambda,\mu)=(2\beta,1-\alpha),(\beta,1+\beta-\alpha))$,

$$
\begin{aligned}
A_{22}+A_{23} \le \{&\|R^Y\|_{2\beta}(1-\alpha)^{-1} \\
&+\|Y'\|_{\beta}\|X\|_{\beta}\alpha(\alpha-\beta)^{-1}(1+\beta-\alpha)^{-1}\}|\mathcal{P}_{s,t}|^{2\beta-\alpha}(t-s).
\end{aligned}
$$

Hence, from the estimates of A_1 and A_2, (3.4) holds for $l=2$. Therefore, we obtain the statement of (3.2). □

We now provide another proof of (2.20) in Theorem 2.6.

Proof. We fix $(s,t)\in\triangle_{0,T}$ with $s<t$ because (2.20) obviously holds when $s=t$. From (2.6) and (3.1) with $x=t_i$ and $y=t_{i+1}$,

$$\begin{aligned}&\sum_{i=0}^{n-1} Y_{t_i}(X_{t_{i+1}}-X_{t_i})+Y'_{t_i}\mathbb{X}_{t_i,t_{i+1}}\\&=(-1)^{\gamma-1}\int_s^t D_{t-}^{1-\gamma}(\sum_{i=0}^{n-1}(Y_{t_i}+Y'_{t_i}(X_u-X_{t_i}))1_{[t_i,t_{i+1})})(u)D_{s+}^{\gamma}(X-X_s)(u)\,du\\&\quad+(-1)^{2\gamma-1}\int_s^t D_{t-}^{1-2\gamma}(\sum_{i=0}^{n-1}Y'_{t_i}1_{[t_i,t_{i+1})})(u)D_{s+}^{\gamma}(\mathcal{D}_{s+}^{\gamma}(\mathbb{X}-\delta X\otimes\delta X))(u)\,du.\end{aligned}$$

Therefore, from Proposition 3.2,

$$\begin{aligned}&|\sum_{i=0}^{n-1} Y_{t_i}(X_{t_{i+1}}-X_{t_i})+Y'_{t_i}\mathbb{X}_{t_i,t_{i+1}}-J_{(X,\mathbb{X})}(Y,Y')_{s,t}|\\&\le\int_s^t |D_{t-}^{1-\gamma}(\sum_{i=0}^{n-1}(Y_{t_i}+Y'_{t_i}(X_u-X_{t_i}))1_{[t_i,t_{i+1})})(u)-\hat{\mathcal{D}}_{t-}^{1-\gamma}(R^Y-\delta Y'\delta X)(u)|\,du\\&\quad\times\|D_{s+}^{\gamma}(X-X_s)\|_\infty\\&\quad+\int_s^t |D_{t-}^{1-2\gamma}(\sum_{i=0}^{n-1}Y'_{t_i}1_{[t_i,t_{i+1})})(u)-D_{t-}^{1-2\gamma}Y'(u)|\,du\\&\qquad\times\|D_{s+}^{\gamma}(\mathcal{D}_{s+}^{\gamma}(\mathbb{X}-\delta X\otimes\delta X))\|_\infty,\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the supremum norm on $[s,t]$. Therefore, from Proposition 3.3, we obtain the statement of (2.20). ☐

Acknowledgements. The author thanks Professor Masanori Hino for his valuable advice, which was one of the main factors behind this study. This work was supported by JSPS KAKENHI Grant Numbers JP18K13431 and JP21H00988.

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